EXERCISES

1. Lecture 2

Exercise 1.1.

1. Show that $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d) = \overline{\mathcal{M}}_{0,0}(T^*\mathbb{P}^1, d)$ and compare the expected dimensions of the two spaces.
2. In order to compare the two virtual fundamental classes on the same space, compare the space of obstructions $H^1(C, f^*T_X)$ to extending deformations of a map $f : C \to X$. This will define a vector bundle on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$ whose Euler class gives the virtual class for $T^*\mathbb{P}^1$.
3. Suppose we work equivariantly with respect to the $\mathbb{C}^*$-action that scales the cotangent fiber with weight $\hbar$ (and acts trivially on $\mathbb{P}^1$), what is the equivariant virtual class for $\overline{\mathcal{M}}_{0,0}(T^*\mathbb{P}^1, d)$?

A similar argument can be used to relate the virtual classes for $T^*(G/B)$ to G/B; by sending $\hbar \to \infty$ (after rescaling), one can recover quantum operators on G/B from those on $T^*(G/B)$, which in lecture, we reduced to $T^*\mathbb{P}^1$ via deformation. So in a sense, embedding in the cotangent bundle allows us to reduce to a simpler geometry.

Exercise 1.2. Suppose we have the action of a torus T on $\mathbb{P}^1$ so that the tangent space at fixed point 0 has T-weight α (and the tangent space at ∞ has weight $-\alpha$. Given a T-equivariant invertible sheaf L on $\mathbb{P}^1$ such that the fibers over 0 and ∞ have weights χ_0, χ_∞.

1. Show that the degree of L is $d = \frac{\chi_0 - \chi_\infty}{\alpha}$
2. If $d \geq 0$, show that $H^0(\mathbb{P}^1, L)$ has T-weights given by $\chi_\infty, \chi_\infty + \alpha, \ldots, \chi_\infty + d\alpha = \chi_0$ (and the analogous statement for $d \leq -2$ and H^1.)
3. Let $T = (\mathbb{C}^*)^2$, acting on $X = T^*\mathbb{P}^1$ with tangent weights $t, \hbar - t$ at 0 and $-t, \hbar + t$ at ∞. Suppose we have the (unique) T-fixed map $C = \mathbb{P}^1 \to T^*\mathbb{P}^1$ of degree d; after passing to a finite cover, C inherits a T-action with tangent weights $t/d, -t/d$. Compute the T-weights of the obstruction space $H^1(C, f^*T^*\mathbb{P}^1)$ and check that $\hbar$ occurs.
4. In the case $d = 3$, write out all T-fixed stable maps in $\overline{\mathcal{M}}_{0,2}(X, d)$ which send the first marked point to 0 and the second marked point to ∞.

In calculating the equivariant Gromov-Witten invariant $\langle [0], [\infty] \rangle_{0,d}^X$ by torus localization, the contribution of each fixed locus includes the product of weights of the obstruction space at that map. Using this exercise, every non-contracted component of a fixed stable map contributes a weight of $\hbar$ to this product. Therefore, all such loci (except for the locus corresponding to irreducible domain), contribute $\hbar^2$. If we are calculating modulo $\hbar^2$, we can ignore all fixed loci except for the one corresponding to an irreducible domain.