EXERCISES

1. Lecture 1

Exercise 1.1. Given an n-pointed proper, nodal curve $(C, p_1, \dots, p_n)$ and a map $f : C \to X$, check that finiteness of the automorphism group $\mathrm{Aut}(C, p_1, \dots, p_n, f)$ is equivalent to the property that any contracted irreducible component C_0 of genus 0 has at least three marked points or nodes and any contracted irreducible component of genus 1 has at least one marked point or node.

Exercise 1.2. Consider the moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2[\text{line}])$. We can stratify this space based on whether the domain is smooth and whether the stable map is injective. Describe the fibers of the morphism to the space of conics $\mathbb{P}^5 = |\mathcal{O}(2)|$ as well as the isotropy groups of various strata. Do the same for degree 3 maps to $\mathbb{P}^2$. (In the degree 2 case, one can also define a morphism to the space of dual conics by associating $[C, f]$ the locus of lines ℓ such that $f^{-1}(\ell)$ is singular or positive-dimensional; one can show that the coarse space of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to its image in $\mathbb{P}^5 \times (\mathbb{P}^5)^\vee$.)

Exercise 1.3. Assume that the virtual class is compatible with forgetful morphism, i.e.

$$\mathrm{forg}_{n+1}^*[\overline{\mathcal{M}}_{0,n}(X, \beta)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,n+1}(X, \beta)]^{\mathrm{vir}},$$

prove the string and divisor equations: if $n \geq 3$ or $\beta \neq 0$, then

$$\langle 1, \gamma_1, \dots, \gamma_n \rangle_{0,\beta}^X = 0,$$
$$\langle D, \gamma_1, \dots, \gamma_n \rangle_{0,\beta}^X = (D \cdot \beta) \langle \gamma_1, \dots, \gamma_n \rangle_{0,\beta}^X$$

where $D \in H^2(X, \mathbb{Q})$.

Exercise 1.4. Let $X = \mathbb{P}^n$ and let $H_i \in H^{2i}(X, \mathbb{Q}), i = 0, \dots n$ denote the fundamental class of a linear subspace of codimension i. Using the dimension constraint, calculate the matrix for quantum multiplication by H_1 in the basis H_k. More generally, show that $H_i \circ H_j = H_{i+j}$ if $i + j \leq n$ and $H_i \circ H_j = qH_{i+j-n-1}$ otherwise.

Exercise 1.5. Check that the quantum connection, defined (hopefully in the first lecture) as

$$\nabla_\lambda = \partial_\lambda - \lambda \circ$$

is a flat connection.