Quantum Cohomology and Symplectic Resolutions

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1 Introduction

1.1 The objects of interest

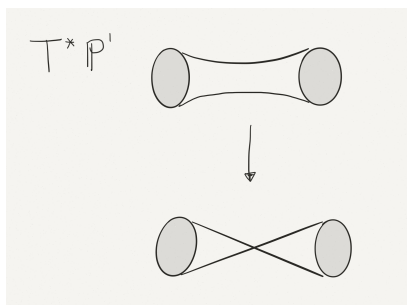
This lecture series is about quantum cohomology and symplectic resolutions, so I'll start with some remarks about what these are.

Quantum cohomology. Let me start off by describing the idea of quantum cohomology. If I have a smooth variety $X/\mathbb{C}$, then very roughly the quantum cohomology $QH^\bullet(X)$ is a ring deformation of the usual cohomology ring $H^\bullet(X)$ where the structure constants are given by "counting" rational curves on X.

This is very hard in general, but the theme of the lectures is that for certain families of varieties the answer is extremely well-behaved. That brings us to the next point.

Symplectic resolutions. A symplectic resolution is a map from $X \to X_0$, where X_0 is affine singular and X is holomorphic symplectic.

Example 1.1. We have seen that $T^*\mathbb{P}^1$ maps to the singular quadric cone in $\mathbb{C}^3$ as the blowup of the cone point.



In these cases the quantum cohomology $QH^\bullet(X)$ is comparatively well-behaved. More precisely, $H^\bullet(X)$ carries an action of an explicit non-commutative algebra, we expect to realize $QH^\bullet(X)$ within this non-commutative algebra.

The goal is to give examples of this phenomenon, and explain the proofs in those cases. Then we'll indicate what's known and not known in general.

1.2 Motivation

Why do we care?

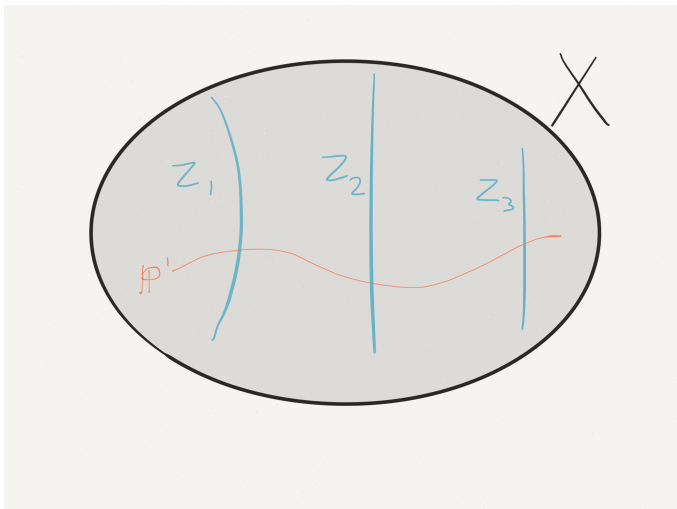
1. There is an expectation that $QH^\bullet(X)$ is related to non-commutative deformations of X. In particular, there are explicit properties of the deformations that can be read off from the quantum cohomology.

2. The symplectic resolutions are a gateway for understanding curve counting on more complicated geometries.

2 Quantum Cohomology and Quantum Connections

2.1 Counting rational curves on X

For now, X is a smooth projective variety. What do we mean by "counting" curves on X?



Given X and some cycles $Z_1, Z_2, Z_3 \subset X$, we are interested in counting the number of $\mathbb{P}^1$s of degree d passing through Z_1, Z_2, Z_3. We would like to model this problem by doing intersection theory on some parameter space. The important thing is to build the right parameter space. The one which we shall eventually use, the *moduli space of stable maps*, has a lot of "recursive structure" on it that is responsible for the algebraic structure in quantum cohomology.

The idea of the moduli space of stable maps is to view the situation not as a curve sitting in X, but as a map $f\colon \mathbb{P}^1 \to X$ from an *abstract* $\mathbb{P}^1$ up to reparametrization of the domain.

Attempt 1. Let's first try to define the moduli space to be

$$\mathcal{M}_{0,0}(X,\beta) = \left\{(C,f) \mid \begin{smallmatrix} C \text{ smooth, connected rational curve} \\ f\colon C \to X, \text{ such that } f_*[C]=\beta\in H^2(X) \end{smallmatrix}\right\} / \sim$$

where $\sim$ means equivalence up to a reparametrization of C, i.e. two maps f_1, f_2 are considered equivalent if there is some φ such that

$$\begin{tikzcd} C_1 \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "f_1"'] & & C_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

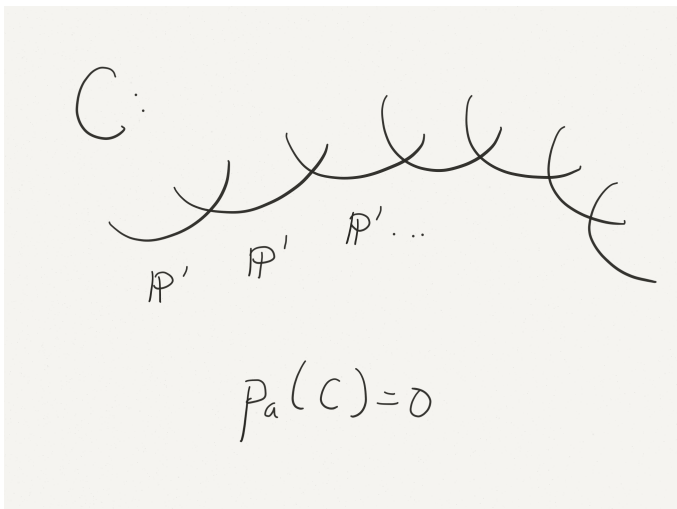
Remark 2.1. The first 0 in $\mathcal{M}_{0,0}$ refers to the fact that we are considering C with (arithmetic) genus 0, and the second refers to the fact that we are not considering any marked points yet.

The problem with this definition of $\mathcal{M}_{0,0}(X,\beta)$ is that it is not compact. We can *compactify* it by allowing C to be nodal.

Attempt 2. As our next attempt, we consider the space

$$\overline{\mathcal{M}_{0,0}}(X,\beta) = \left\{(C,f) \mid \begin{smallmatrix} C \text{ connected, nodal curve of genus } 0 \\ f\colon C\to X, \text{ such that } f_*[C]=\beta\in H^2(X) \end{smallmatrix}\right\} / \sim .$$

Unfortuantely, this is still not quite right because it won't be finite type. This space contains curves that become arbitrarily "complicated," such as an arbitrarily long tail of $\mathbb{P}^1$ joined at nodes.



The problem can be solved by imposing *finiteness of automorphisms*.

Definition 2.2. We define the *moduli space of stable maps* to be

$$\overline{\mathcal{M}_{0,0}}(X,\beta) = \left\{(C,f) \mid \begin{smallmatrix} C \text{ connected, nodal curve of genus } 0 \\ f\colon C\to X, \text{ such that } f_*[C]=\beta\in H^2(X) \\ \#\operatorname{Aut}(C,f)<\infty \end{smallmatrix}\right\} / \sim .$$

The condition that $\operatorname{Aut}(C,f)$ be finite is called *stability*.

Exercise 2.3. Check that stability is equivalent to every contracted irreducible component having at least 3 nodes.

Actually, we want a more refined variation of this where we can keep track of curves meeting specified *cycles* on X. To do that, we add some decorations on the curves to track their intersections, i.e. we allow *marked points*.

Definition 2.4. We define the moduli space

$$\overline{\mathcal{M}_{0,n}}(X,\beta) = \left\{(C,f,p_1,\ldots,p_n) \mid \begin{smallmatrix} C \text{ connected, nodal curve of genus } 0 \\ p_1,\ldots,p_n\in C^{\text{smooth}} \\ f\colon C\to X, \text{ such that } f_*[C]=\beta\in H^2(X) \\ \#\operatorname{Aut}(C,f,p_1,\ldots,p_n)<\infty \end{smallmatrix}\right\} / \sim .$$

The data $(C,f,p_1,\ldots,p_n)$ is called *stable* if $\operatorname{Aut}(C,f,p_1,\ldots,p_n)$ is finite.

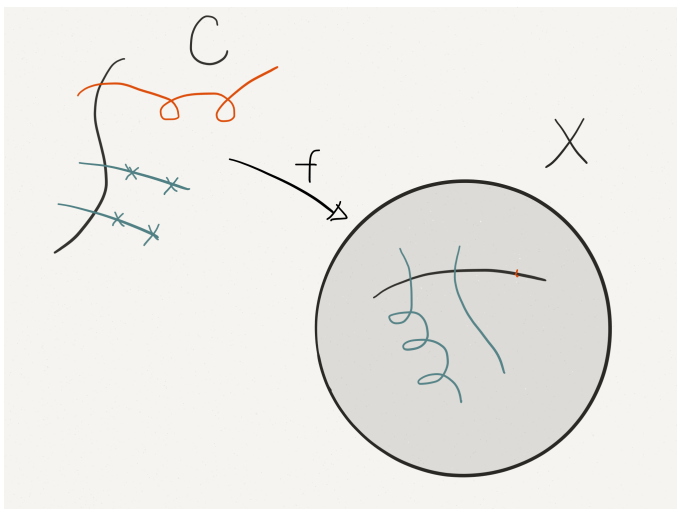
Exercise 2.5. Check that stability is equivalent to every contracted irreducible component having at least 3 nodes or marked points.

Theorem 2.6. *$\overline{\mathcal{M}_{0,n}}(X,\beta)$ is a proper Deligne-Mumford stack of finite type.*

We needed to include the nodal curves in order to achieve properness.

Remark 2.7. We started off thinking about embedded rational curves in X, but we ended up with a definition that doesn't have any embedding condition. If we want to keep C nodal, then we need to allow f not to be an immersion.

Example 2.8. A stable map:



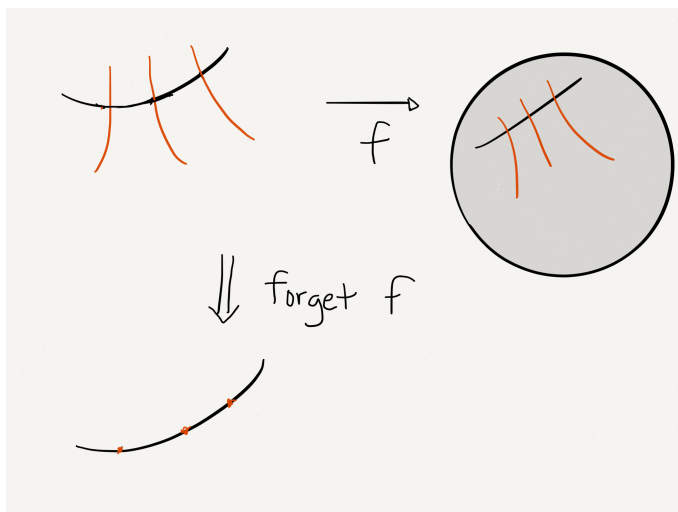
The red curve is contracted. The marked points on the blue components ensure stability. The map f need not be injective: it contracts the red component, and identifies some pairs of points on a blue component.

2.2 Properties of the moduli spaces

Forgetful maps. There are various "forgetful maps" obtained by forgetting part of the data of a stable map.

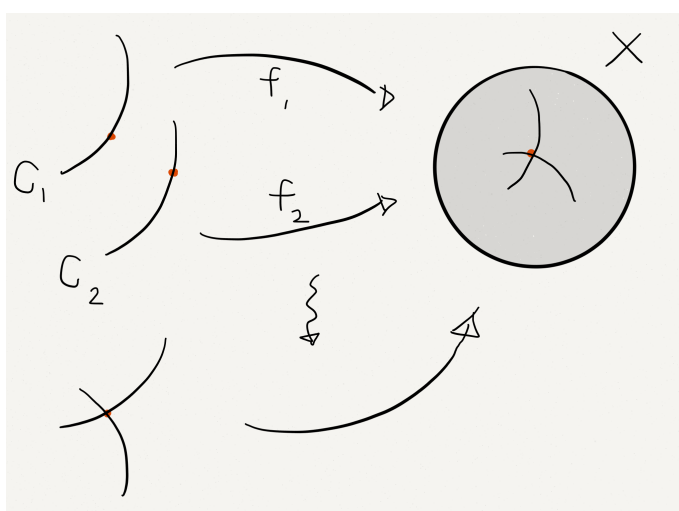
- A point of $\overline{\mathcal{M}_{0,n}}(X,\beta)$ consists of $(C, f, p_1, \ldots, p_n)$. Forgetting everything except p_i induces a map
$$\overline{\mathcal{M}_{0,n}}(X,\beta) \xrightarrow{\mathrm{ev}_i} X$$
sending $(C, f, p_1, \ldots, p_n) \mapsto f(p_i)$. Notice that if we want to encode the condition of the ith point passing through a cycle β, then we can take the pre-image of β under ev_i. The subspace $\mathrm{ev}_i^{-1}(\beta)$ parametrizes precisely those stable maps $(C, f, p_1, \ldots, p_n)$ in which p_i is sent to β.

- Forgetting f induces a map
$$\overline{\mathcal{M}_{0,n}}(X,\beta) \to \overline{\mathcal{M}_{0,n}} = \overline{\mathcal{M}_{0,n}}(\mathrm{pt})$$
sending $(C, f, p_1, \ldots, p_n) \mapsto (C, p_1, \ldots, p_n)$. However, forgetting the map might cause some (uncontracted) components to become unstable, so we also need to contract those components.



- Forgetting the marked point p_{n+1} induces a map
$$\overline{\mathcal{M}_{0,n+1}}(X,\beta) \to \overline{\mathcal{M}_{0,n}}(X,\beta)$$
$(C, f, p_1, \ldots, p_{n+1}) \mapsto (C, f, p_1, \ldots, p_n)$. Again, we have to contract any unstable irreducible components.

Gluing maps. We can take two stable maps and glue them together at points have a coming image.



More precisely, if we have two maps $f_1 \colon C_1 \to X$ and $f_2 \colon C_2 \to X$ such that $f_1(p) = f_2(q)$, then we can glue C_1 and C_2 by identifying p and q, and take the union

$$f_1 \cup_{p\sim q} f_1 \colon C_1 \cup_{p\sim q} C_2 \to X.$$

If $(f_1)_*[C] = \beta_1$ and $(f_2)_*[C] = \beta_2$, then $f_1 \cup f_2[C] = \beta_1 + \beta_2$.

Therefore, if we partition $\{1, \dots, n\} = A \sqcup B$ then this construction induces a map

$$\begin{tikzcd} \overline{\mathcal{M}_{0,\{p\}\cup A}}(X,\beta_1) \times_X \overline{\mathcal{M}_{0,\{q\}\cup B}}(X,\beta_2) \arrow[d, "\text{glue}"'] \\ \overline{\mathcal{M}_{0,n}}(X,\beta_1+\beta_2) \end{tikzcd}$$

(The fibered product over X is via the ev_p and ev_q, ensuring that $f_1(p) = f_2(q)$.) The image of glue lies entirely in the nodal part, so you can view this as giving a boundary divisor.

2.3 The virtual fundamental class

A natural question to ask is: what's the dimension of the space $\overline{\mathcal{M}_{0,n}}(X,\beta)$?

We can try to estimate the dimension as follows.

	Domain	Map
Expected Dimension	$n-3$	$h^0(C, f^*T_X) - h^1(C, f^*T_X)$

We expect $n - 3 = \dim \mathcal{M}_{0,n} = \{C\}$ dimensions coming from varying the domain C.

How many degrees of freedom do we have in varying the map $f \colon C \to X$? By deformation theory,the space of first-order deformations is parametrized by $H^0(C, f^*T_X)$. Why? You can think of this as giving a tangent vector on X for every point of C, which tells you to first order how to deform the map. Of course, not all the tangent directions will be realized by an honest deformation: some will be obstructed. The obstruction group is parametrized by $h^1(C, f^*T_X)$. (There are no higher obstruction groups on a curve, since they would be described by higher cohomology groups.)

The heuristic calculation gives

$$\begin{aligned} \text{expected dimension} &= n - 3 + h^0(C, f^*T_X) - h^1(C, F^*T_X) \\ &= \chi(C, f^*T_X) + n - 3 \\ &= \boxed{-K_X \cdot \beta + \dim X + n - 3.} \end{aligned}$$

This doesn't give the dimension, but it turns out that it does give a correct lower bound:

Theorem 2.9. *We have* $\dim_{[C,f]} \mathcal{M} \geq$ *expected dimension.*

Ideally, $\overline{\mathcal{M}_{0,n}}(X,\beta)$ is smooth and has dimension equal to the expected dimension (but this doesn't happen for any of the examples that will occur in these lectures). Then if we

want to count curves passing through cycles $Z_1, \ldots, Z_n$ on X, we could try to model this via intersection theory.

Namely, the cycles $Z_1, \ldots, Z_n$ give (by Poincaré duality on their fundamental classes) cohomology classes $\gamma_1, \ldots, \gamma_n \in H^\bullet(X)$, so we can pull these back to $\overline{\mathcal{M}_{0,n}}(X,\beta)$ via the evaluation maps and consider

$$\mathrm{ev}_1^* \gamma_1 \smile \ldots \smile \mathrm{ev}_n^* \gamma_n \in H^\bullet \overline{\mathcal{M}_{0,n}}(X,\beta)$$

If we then cap this with the fundamental class $[\mathcal{M}]$, then we should get the desired count. (In terms of homology, think of this as the cap product of the fundamental classes, which should correspond to their intersection.) This can be interpreted as a count of the stable maps where $f(p_1) \in Z_1, \ldots, f(p_n) \in Z_n$.

In order to obtain a finite answer, we need to demand that

$$\sum \operatorname{codim} Z_i = 2(\text{expected dimension})$$

since $[\mathcal{M}] \in H_{2(\text{expected dimension})}(\overline{\mathcal{M}_{0,n}}(X,\beta))$.

That was an idealized discussion. In the real world, the moduli space $\overline{\mathcal{M}_{0,n}}(X,\beta)$ will *not* be ideal. So instead we'll work with a replacement. We'll define a *virtual class* $[\overline{\mathcal{M}_{0,n}}(X,\beta)]^{\mathrm{vir}} \in H_{2(\text{expected dimension})}(\overline{\mathcal{M}_{0,n}}(X,\beta), \mathbb{Q})$ which *always* lives in this dimension, even if $\overline{\mathcal{M}_{0,n}}(X,\beta)$ doesn't have the expected dimension. We'll use this as a replacement for the fundamental class in usual intersection theory. The point is that this virtual class will have good formal properties.

Definition 2.10. The n-pointed, genus 0 *Gromov-Witten invariant* for of $\gamma_1, \ldots, \gamma_n \in H^\bullet(X)$ is defined to be

$$\begin{aligned}\langle \gamma_1, \ldots, \gamma_n \rangle &:= \int_{[\overline{\mathcal{M}_{0,n}}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \gamma_1 \smile \ldots \smile \mathrm{ev}_n^* \gamma_n \\ &:= [\overline{\mathcal{M}_{0,n}}(X,\beta)]^{\mathrm{vir}} \frown \left(\mathrm{ev}_1^* \gamma_1 \smile \ldots \smile \mathrm{ev}_n^* \gamma_n\right)\end{aligned}$$

if $\sum_i \deg \gamma_i = 2 \cdot$ expected dimension, and 0 otherwise.

This is the replacement of the original curve-counting problem.

Properties of the virtual fundamental class. Let's elaborate on the "good formal properties" of $[\overline{\mathcal{M}_{0,n}}(X,\beta)]^{\mathrm{vir}}$.

1. It is *deformation invariant*. If we have a smooth proper *family* of target varieties

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ T \end{tikzcd}$$

instead of a single variety X, and we want to see how the count changes as we move in this family, then we get a family of moduli spaces

$$\begin{array}{c} \overline{\mathcal{M}_{0,n}}(X/T,\beta) \\ \downarrow \\ T \end{array}$$

The family could be very badly behaved (e.g. it could acquire extra components over certain points), but the virtual fundamental class extends to this family, which means in particular that $\langle \gamma_1, \ldots, \gamma_n \rangle$ is *independent of* X_t.

Remark 2.11. This is a shadow of the fact that there is an alternative definition of everything using symplectic geometry, so the complex structure is actually irrelevant.

2. *Functoriality.* We have the forgetful map

$$\mathrm{forg}\colon \overline{\mathcal{M}_{0,n+1}}(X,\beta) \to \overline{\mathcal{M}_{0,n}}(X,\beta)$$

and it turns out that

$$\boxed{\mathrm{forg}^*([\overline{\mathcal{M}_{0,n}}(X,\beta)]^{\mathrm{vir}}) = [\overline{\mathcal{M}_{0,n+1}}(X,\beta)]^{\mathrm{vir}}.}$$

As a sanity check, we note that forg^* above is pulling back a *homology* class, hence increasing the degree by 1 (think of it as the set-theoretic pre-image), which is right because the expected dimension also goes up by 1 when adding in a marked point.

3. If $\overline{\mathcal{M}_{0,n}}(X,\beta)$ is smooth of the expected dimension, then the virtual fundamental class coincides with the actual fundamental class:

$$[\overline{\mathcal{M}_{0,n}}(X,\beta)]^{\mathrm{vir}} = [\overline{\mathcal{M}_{0,n}}(X,\beta)].$$

Finally, let me commently briefly on what this virtual fundamental class actually is. Supopse we have a rank r vector bundle $V \to Y$, and a smooth section $\sigma\colon Y \to V$

$$\begin{tikzcd} V \arrow[d, "\pi"'] \\ Y \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

Then we can form the zero-set of σ, $Z(\sigma) = \Gamma_\sigma \cap Y$. Let $i\colon Z(\sigma) \hookrightarrow Y$ be the inclusion map. If σ were *transverse* to the zero section, then we would have

$$i_*[Z(\sigma)] = c_{\mathrm{top}}(V) \frown [Y].$$

The point of virtual fundamental class is to restore this identification if the section is not transverse.

In the smooth world, we would just perturb the section σ until it were smooth. We can't do that in the world of algebraic geometry, so instead we construct the "refined intersection product" $[Z(r)] = [\Gamma_\sigma] \cap [Y] \in H_{2d-2r}(Z(r))$. This is a local model for the virtual fundamental class.

2.4 Quantum cohomology

One way to think about the cup product is that if we have three cycles Z_1, Z_2, Z_3 then

$$\#(Z_1 \cap Z_2 \cap Z_3) = (\gamma_1 \smile \gamma_2 \smile \gamma_3)$$

where γ_i is Poincaré dual to Z_i.

The idea of quantum cohomology is that perhaps the cycles don't meet, so instead of studying the number of *points* commong to all the Z_i, we will study the number of *rational curves* through them.

Definition 2.12. Denote by $\mathbb{Q}[[q^\beta \mid \beta \in \mathrm{Eff}(X)]]$ the ring of formal power series, where $\mathrm{Eff}(X)$ is generated by classes in $H_2(X, \mathbb{Z})$ represented by algebraic curves. The *quantum cohomology ring* is $\mathbb{Q}[[q^\beta \mid \beta \in \mathrm{Eff}(X)]] \otimes H^\bullet(X, \mathbb{Q})$, with the usual addition.

If $\gamma_1, \gamma_2 \in H^\bullet(X)$, then we define the quantum product $\gamma_1 \bullet \gamma_2$ by specifying its Poincaré pairing with any γ_3:

$$(\gamma_1 \bullet \gamma_2, \gamma_3) = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_{0,\beta} \cdot q^\beta$$

where q^β is a formal parameter that records the degree.

To make this more concrete, choose a basis $D_1, \ldots, D_r$ for $H^2(X, \mathbb{Z})$ as $D_1, \ldots, D_r$. Then the power series ring in question is $\mathbb{Q}[[q_1, \ldots, q_r]]$ and we have

$$(\gamma_1 \bullet \gamma_2, \gamma_3) = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_{0,\beta} \cdot q_1^{D_1 \cdot \beta} \cdot \ldots \cdot q_r^{D_r \cdot \beta}.$$

Remark 2.13. This is only the special case of "3-pointed moduli spaces." There is a more general theory of "n-pointed moduli spaces" for which the formulas are considerably more complicated.

Theorem 2.14. *We have the following properties of the quantum product:*

1. *$\bullet$ is associative and graded-commutative on $H^\bullet(X) \otimes \mathbb{Q}[[q_1, \ldots q_r]$, and*
2. *$\bullet$ deforms the usual cup product on $H^\bullet(X)$, and*
3. *$1 \bullet \gamma = \gamma$.*

Proof. The graded-commutativity is clear from the definition, and for associativity we want to show that

$$(\gamma_1 \bullet \gamma_2) \bullet \gamma_3 = (\gamma_2 \bullet \gamma_3) \bullet \gamma_1.$$

We can check this by pairing with some other class γ_4:

$$((\gamma_1 \bullet \gamma_2) \bullet \gamma_3, \gamma_4) = ((\gamma_2 \bullet \gamma_3) \bullet \gamma_1, \gamma_4). \tag{1}$$

Going back to the definition of the quantum product, the right hand side of (1) is

$$\begin{aligned}((\gamma_2 \bullet \gamma_3) \bullet \gamma_1, \gamma_4 &= \sum_\beta q^\beta \langle \gamma_2 \bullet \gamma_3, \gamma_1, \gamma_4 \rangle \\ &= \sum_\beta q^\beta \left\langle \sum_\gamma (\gamma_2, \gamma_3, \gamma)\gamma^\vee, \gamma_1, \gamma_4, \gamma \right\rangle \\ &= \sum_\beta q^\beta \left\langle \sum_\gamma \sum_{\beta'} q^{\beta'} (\gamma_2, \gamma_3, \gamma)\gamma^\vee, \gamma_1, \gamma_4, \gamma \right\rangle \\ &= \sum_\beta q^\beta \sum_\gamma \langle \gamma_2, \gamma_3, \gamma \rangle \langle \gamma^\vee, \gamma_1, \gamma_4 \rangle\end{aligned}$$

and the left hand side of (1) is the same with γ_1 and γ_3 switched:

$$\sum_\beta \sum_\gamma q^\beta \langle \gamma_1, \gamma_2, \gamma \rangle \langle \gamma^\vee, \gamma_3, \gamma_4 \rangle \tag{2}$$

(We have suppressed β in the notation above.) Now we interpret each of these expressions as curve counts. Recall that $\langle \gamma_1, \ldots, \gamma_n \rangle_\beta$ can be interpreted as the number of stable curves with n marked points passing through the cycles $\gamma_1, \ldots, \gamma_n$ respectively.

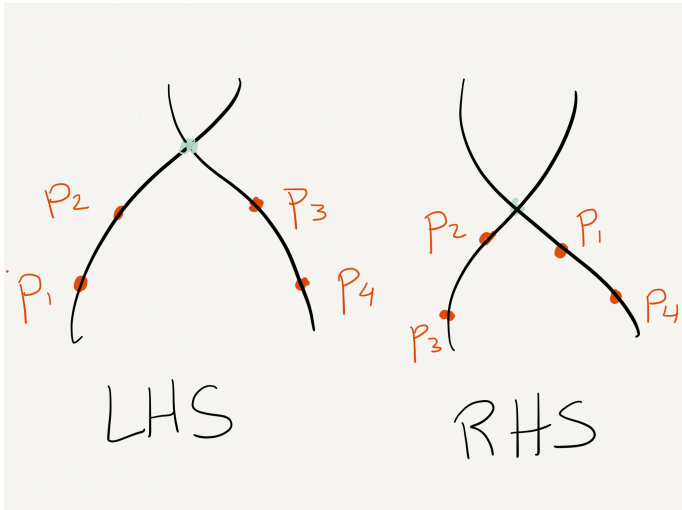
The expression (2) is counting pairs of maps of rational curves (C_1, p_1, p_2, p_3) and (C_2, q_1, q_2, q_3) into X such that $p_1 \in \gamma_1, p_2 \in \gamma_2, p_3 \in \gamma, q_1 \in \gamma, q_2 \in \gamma_2, q_3 \in \gamma_4$. Formally, that means that we are capping a cohomology and a homology class, described below.

The cohomology class. Let π denote the forgetful map

$$\begin{array}{c} \overline{\mathcal{M}_{0,4}}(X, \beta) \\ \Big\downarrow \pi \\ \overline{\mathcal{M}_{0,4}} \cong \mathbb{P}^1 \end{array}$$

The moduli space $\overline{\mathcal{M}_{0,4}}$ has a boundary divisor represented by the stable nodal curve on the

left in the diagram, with point labeled $1, 2$ on one side and $3, 4$ on the other.



The cohomology class we want to consider is the pullback of the class corresponding to this divisor. For the right hand side of (1), the count will use the pullback of the curve on the right instead. The key pont is that since $\overline{\mathcal{M}_{0,4}} \cong \mathbb{P}^1$, any two divisor classes are linearly equivalent.

The homology class. We have the gluing map

$$\begin{array}{c} \bigcup_{\beta_1+\beta_2=\beta} \overline{\mathcal{M}_{0,\{p,1,2\}}}(X,\beta_1) \times_X \overline{\mathcal{M}_{0,\{q,3,4\}}}(X,\beta_2) \\ \Big\downarrow \text{glue} \\ \overline{\mathcal{M}_{0,4}}(X,\beta) \end{array}$$

Then we get a pushforward of the virtual fundamental classes:

$$\sum_{\beta_1+\beta_2=\beta} \text{glue}_* \left([\overline{\mathcal{M}_{0,3}}(X,\beta_1)]^{\text{vir}} \times_X [\overline{\mathcal{M}_{0,3}}(X,\beta_2)]^{\text{vir}} \right).$$

Thus we see that the expressions on the two sides of (1) corresponding to capping a virtual fundamental class with two cohomology classes, which are equivalent.

♠♠♠ TONY: [I don't actually see why the "curve count" forces any gluing of two marked points - why couldn't the marked points get sent to different images in γ and $\gamma^\vee$?]

Remark 2.15. X doesn't have to be projective. We just need the evaluation maps to be proper (e.g. it suffices for X to be projective over an affine).

□

2.5 Quantum connections

Recall that we introduced the quantum cohomology ring $H^\bullet(X)[[q^\beta \mid \beta \in \mathrm{Eff}(X)]]$. You can think of q^β as a function on $H^2(X, \mathbb{C}^\times/2\pi i\mathbb{Z}) = T^\vee$ of $H^\bullet(X)$, taking $u \mapsto q^{(\beta,u)}$.

Definition 2.16. For $\lambda \in H^2(X, \mathbb{C})$, let

$$\partial_\lambda q^\beta = (\beta \cdot \lambda) q^\beta.$$

We define a *formal, flat (small) connection* on $T^\vee$ by

$$\nabla_\lambda = \partial_\lambda - \lambda \bullet .$$

Exercise 2.17. Show that ∇_λ is flat. (Essentially the same argument as for associativity.)

Example 2.18. On the problem sheet, you work out the quantum multiplication on $\mathbb{P}^n$. In terms of the standard basis, this was

$$\begin{pmatrix} 0 & & & q \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

So flatness boiled down to

$$q\frac{\partial}{\partial q}\Psi = \begin{pmatrix} 0 & & & q \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}\Psi.$$

Goal: what can we say about $QH^\bullet(X)$? What can we say about ∇ (e.g. about its monodromy)?

Whenever we have a group action of G on X, we get an action of the equivariant cohomology ring $H^\bullet_G(\mathrm{pt})$ on $H^\bullet_G(X)$.

Example 2.19. If $G = T$, then $H^\bullet_G(\mathrm{pt}) = \mathbb{C}[t_1, \ldots, t_r]$. Then G acts on $\overline{\mathcal{M}}(x, \beta)$, and there is an *equivariant* virtual fundamental class $[\overline{\mathcal{M}(x,\beta)}]^{\mathrm{vir}} \in H^G_{BM}(\overline{\mathcal{M}(x,\beta)})$.

By the definition of equivariant cohomology, we can recover the ordinary cohomology essentially just by specializing the variables $t_1, \ldots, t_r$ to 0. (Strictly speaking, there are some technical complications: perhaps we need $H^\bullet_G(X)$ to be free over $H^\bullet_G(\mathrm{pt})$.

3 Equivariant Symplectic Resolutions

3.1 Symplectic resolutions

Let X be a smooth variety and ω an algebraic symplectic form on X, i.e. $\omega \in H^0(X, \Omega^2_X)$. Let X_0 be an affine (singular) variety. A *symplectic resolution* is a proper, birational morphism

$$\begin{array}{c} (X, \omega) \\ \downarrow \\ X_0 \end{array}$$

Often we will consider the equivariant case, where we have a $\mathbb{G} = G \times \mathbb{C}^\times$ action on the source and target.

Example 3.1. The cotangent bundle $T^*\mathbb{P}^1$ (with the standard symplectic structure) is a symplectic resolution of a quadric cone in $\mathbb{C}^3$ by contracting the 0 section. Here we have an action of $\mathbb{C}^\times \times \mathbb{C}^\times$. The first $\mathbb{C}^\times$ acts on $\mathbb{P}^1$, preserving ω, and the second $\mathbb{C}^\times$ scales the fibers.

Example 3.2. More generally, if G is semisimple and simply-connected and $B \subset G$ is the Borel, then the *Springer resolution*

$$\widetilde{\mathcal{N}} = T^*(G/B) \to \mathcal{N}$$

is a symplectic resolution.

Example 3.3. Other examples include Hilbert schemes $\mathrm{Hilb}_n\mathbb{C}^2 \to \mathrm{Sym}^n \mathbb{C}^2$, Nakajima quiver varieties, hypertoric varieties, etc.

3.2 The Springer resolution

What can we say about the equivariant quantum cohomology ring of the Springer resolution, $QH_{G\times\mathbb{C}^\times}(T^*(G/B))$? For notational convenience, we set $X = \widetilde{\mathcal{N}}$ in this section.

Principle: Classical geometry determines the quantum predictions.

Preliminary facts.

1. We have a natural identification $H_2(X, \mathbb{Z}) = \mathrm{Hom}(\mathbb{C}^\times, T) = \Lambda^\vee$ (the coweight lattice) and $H^2(X, \mathbb{Z}) = \Lambda$ (the weight lattice). Then it turns out that the classes of effective curves are generated by the *positive* coroots $\alpha^\vee$.

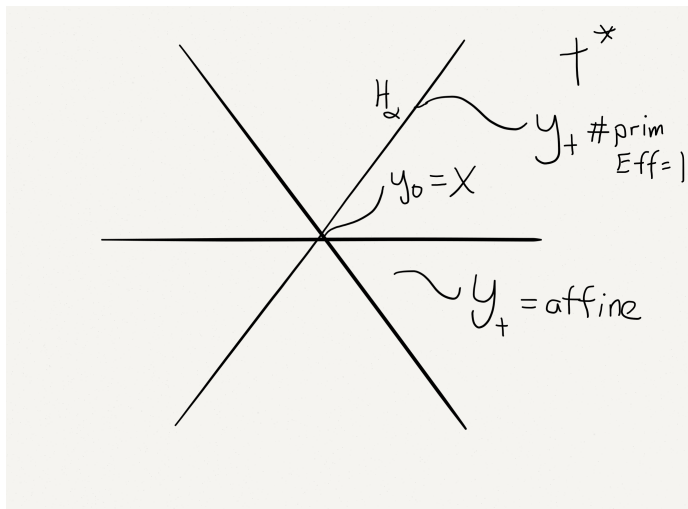
2. If $X \to \mathcal{N}$ is the springer resolution, then the *Steinberg variety*

$$Z := X \times_{\mathcal{N}} X$$

is a union of Lagrangians in $X \times X$. Then $H^{\text{top}}_{BM}(Z) \cong \mathbb{Q}[W]$ is an algebra under convolution. The equivariant homology $H^{G\times\mathbb{C}^\times}_{BM}(Z)$ is a graded affine Hecke algebra.

Geometrically, these correspondences are defined by an action of W on $H^\bullet_{G\times\mathbb{C}^\times}(X)$.

3. The last piece of structure has to do with deforming the Springer resolution. X sits in a family $\mathcal{Y} \to \mathfrak{t}^*$ of symplectic varieties, called the *Grothendieck simultaneous resolution*, where $Y_0 = X$ and $\mathcal{Y}_{\text{gen}} = G/T$ (which in particular is affine).



For every α, we get the hyperplane

$$H_\alpha = \{z \in \mathfrak{t}^* \mid \langle \alpha, z\rangle = 0\}.$$

Away from the union of these hyperplanes, the fiber $\mathcal{Y}_t$ is affine. The fiber over the intersection of more and more hyperplanes gets bigger and bigger. In particular, the fiber over a point on a single coroot hyperplane has a single effective curve class.

There is an action of $G \times \mathbb{C}^\times$ on $\mathcal{Y}$, and of $\mathbb{C}^\times$ on $\mathfrak{t}^*$ (by scaling).

These are basically the only ingredients that we needed to establish the answer to our original question.

Theorem 3.4 (Braverman, Maulik, Okounkov)**.** *Let $\hbar$ be the weight of ω with respect to the $\mathbb{C}^\times$-action. For $\lambda \in \Lambda$, let $D_\lambda \in H^2_{G\times\mathbb{C}^\times}(X, \mathbb{Z})$ be the associated divisor class. Then we have an equality of operators*

$$D_\lambda \bullet = D_\lambda \smile + \hbar \sum_{\alpha^\vee > 0} \frac{q^{\alpha^\vee}}{1 - q^{\alpha^\vee}} (\lambda, \alpha^\vee)(s_\alpha - 1).$$

Remark 3.5. We highlight some interesting consequences of the theorem, and comment on their generality:

1. The "quantum corrections" are divisible by $\hbar$. In particular, if we work non-equivariantly, or even if we merely ignore the $\mathbb{C}^\times$-action, then $D_\lambda \bullet = D \smile$.

   This is true very generally. The reason is something we said last time: quantum cohomology is deformation invariant. If we ignore the $\mathbb{C}^\times$ equivariance, then we can always deform X to a generic fiber of the universal deformation (Grothendieck's simultaneous resolution), which is affine.

2. Notice that the quantum correction terms live in the group algebra of the Weyl group, which corresponds to Lagrangian correspondences in the Steinberg variety. It is a general phenomenon that the quantum correction terms look like

$$\hbar \cdot [\text{Steinberg correspondence}].$$

3. The answer has a "root-theoretic" flavor, with the quantum correction term being a sum over roots of some *universal expression* in the root.

4. The contribution of a given root is by a *rational function*. For a typical variety, you have a formal power series and you have to think about analytic continuation problems, but here everything extends automatically.

   Unlike the previous three observations, which were quite general, this is something that we only know how to prove in an ad hoc manner by analysis of the particular symplectic singularities.

Quantum connection. Now go back to our quantum connection

$$\nabla_\lambda = \partial_\lambda - D_\lambda.$$

This is a connection with "regular singularities". We can even think of it as a *family* of connections depending on the equivariant parameters. Namely, we get an action of $\pi_1(T^\vee_{\text{reg}})$ on $H^\bullet(X)$, and it was shown by Heckman-Opdam that this representation is "periodic," i.e. there exist "intertwiner operators" relating the connection at different equivariant parameters.

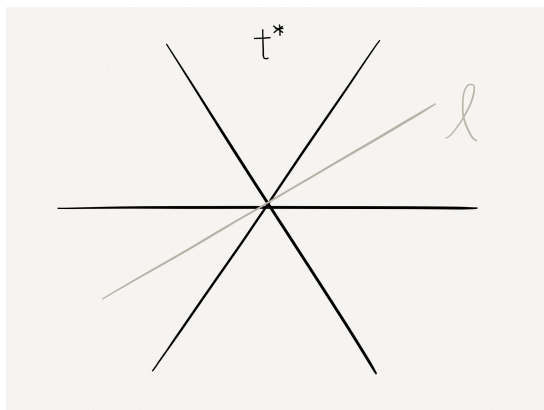
So the total package that we expect to get for any quantum resolution is: a connection with regular singularities on $T^\vee$, plus "shift operators" intertwining equivariant parameters.

Remark 3.6. The only cases where we can actually produce this package are cases where we can "dominate the whole problem" by computing everything explicitly. There is no middle ground where we can say some qualitative things without having explicit control of everything.

Sketch of Proof of Theorem 3.4. (1) We already mentioned that you do this by deforming X to the generic fiber.

(2) Take a generic line throught the origin in $\mathfrak{t}^*$, and study the restriction of the family over this line:

$$\begin{tikzcd} \mathcal{Y}_\ell \arrow[d] & X \arrow[l, hook'] \arrow[d] \\ \ell & 0 \arrow[l, hook'] \end{tikzcd}$$



Because we chose a generic line, it will be the case that

$$\overline{\mathcal{M}_{0,n}}(X,\beta) = \overline{\mathcal{M}_{0,n}}(\mathcal{Y}_\ell,\beta).$$

♠♠♠ TONY: [why??] In fact, this is not only an equality of sets but an equality of *schemes*.

However, their expected dimensions differ. Since we've gotten the same moduli problem in two different ways, we will get two different obstruction theories. This is reflected in the disparity between the two expected dimensions. Recall that the expected dimension of X is

$$\dim X + n - 3 + K_X \cdot \beta$$

and the expected dimension of $\mathcal{Y}_\ell$ is one more.

Under the induced diagram

$$\begin{tikzcd} \mathcal{M}(X,\beta) \arrow[r] \arrow[d] & \mathcal{M}(y_\ell,\beta) \arrow[d] \\ 0 \arrow[r] & \ell \end{tikzcd}$$

we have

$$\hbar[\mathcal{M}(\mathcal{Y}_0,\beta)]^{\mathrm{vir}} = [\mathcal{M}(X,\beta)]^{\mathrm{vir}}.$$

The reason is that the canonical bundle on X is trivial, and has weight $\hbar$ by definition.

This explains the divisibility by $\hbar$, but not why we get Steinberg operators. That comes from dimension reasons. Note that for $n = 2$, the expected dimension of $\overline{\mathcal{M}_{0,2}}(\mathcal{Y}_\ell,\beta)]^{\mathrm{vir}}$

is $(\dim X + 1) + 2 - 3 = \dim X$. Therefore, $[\overline{\mathcal{M}(X,\beta)}_{0,2}]^{\mathrm{vir}} \in H_{2\dim X}[\overline{\mathcal{M}(X,\beta)}_{0,2}]^{\mathrm{vir}}$. Now, consider the product of the evaluation maps

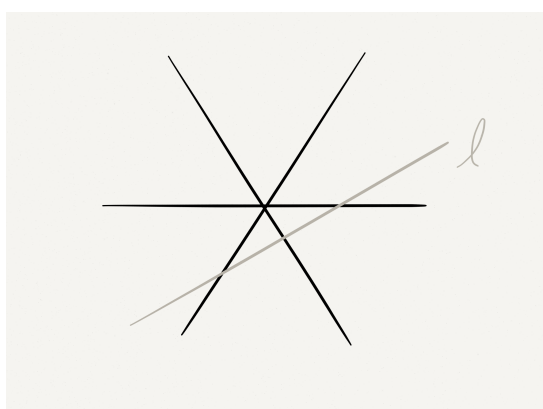
$$\mathcal{M}_{0,2}(X,\beta) \xrightarrow{\mathrm{ev}_1 \times \mathrm{ev}_2} X \times X.$$

Since $\mathcal{N}$ is affine, any map from a connected proper curve mapping to X is contracted to a point after composing with the projection from X to $\mathcal{N}$. Therefore, the map above factors through the Steinberg variety:

$$\mathcal{M}_{0,2}(X,\beta) \xrightarrow{\mathrm{ev}_1 \times \mathrm{ev}_2} X \times_{\mathcal{N}} X \hookrightarrow X \times X.$$

Since $\dim(Z = X \times_{\mathcal{N}} X) = \dim X$ (by the semi-smallness of the Springer resolution). Therefore, $\mathrm{ev}_*[\overline{\mathcal{M}(X,\beta)}_{0,2}]^{\mathrm{vir}}$ lies in the top homology class of Z, hence must be a linear combination $\sum a_i[Z_i]$ where $[Z_i]$ are fundamental classes of irreducible components, and thus correspond to an element of $\mathbb{Q}[W]$.

(3) We're going to use the same picture and deformation invariance again. Since we know that we only need to calculate some rational numbers a_i, we can use specialization. In particular, we can study $\mathcal{Y}_\ell$ non-equivariantly ♠♠♠ TONY: [I don't think I really appreciate this point]. Now that we can forget about the $\mathbb{C}^\times$-action, we can deform ℓ to *not* pass through the origin. Deformation invariance says that the Gromov-Witten theory of the fibers will be invariant: $GW(\mathcal{Y}_\ell)$ is the same as $GW(\mathcal{Y}_{\ell'})$.



Now the $\mathcal{Y}_{\ell'}$ have many curve classes. Indeed, ℓ' intersects each root hyperplane exactly once, and thus picks up one (primitive) effective curve class from each root. The fiber over a general point in H_α is a $T^*\mathbb{P}^1$-bundle over something affine:

$$\begin{tikzcd} \mathcal{Y}_t \arrow[d] & T^*\mathbb{P}^1 \arrow[l] \\ \text{affine} & \end{tikzcd}$$

So each coroot contributes a term to D_λ. What is the contribution of $\alpha^\vee$? It is the same as the contribution of $T^*\mathbb{P}^1$. That explains why every term in the sum looks basically the same: the universal expression is coming from some calculation on $T^*\mathbb{P}^1$, which is then done by bare hands.

□

4 The Hilbert Scheme $\mathrm{Hilb}_n \mathbb{C}^2$

4.1 Overview

Summary: For $X \to X_0$ an equivariant symplectic resolution, we are interested in the equivariant quantum cohomology ring $QH^\bullet_{G\times\mathbb{C}^\times}(X)$. In the previous section we talked about quantum multiplication by divisor classes. Then we explained, using the Springer resolution, a "reduction principle" to the case where X has one algebraic curve class. This was via a deformation argument, in which we deformed the resolution $T^*(G/B) \rightsquigarrow T^*\mathbb{P}^1$.

We can use this method to analyze a slightly more general class of spaces, including $T^*\mathbb{P}^1$, $T^*\,\mathrm{Gr}$, and $\mathrm{Hilb}_n(\mathbb{C}^2)$. For the rest of the lectures we're going to study this last space, the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{C}^2)$.

This was originally proved by Okounkov-Pandariphande, but we will present a more recent argument of Maulik-Okounkov.

Definition 4.1. We define $\mathrm{Hilb}_n(\mathbb{C}^2)$ to be the Hilbert scheme of length n subschemes of $\mathbb{C}^2$.

The torus action of $T = (\mathbb{C}^\times)^2$ on $\mathbb{C}^2$ induces an action of T on $\mathrm{Hilb}_n(\mathbb{C}^2)$, with the weight of ω being $t_1 + t_2$. We have a natural map

$$\mathrm{Hilb}_n \mathbb{C}^2 \to \mathrm{Sym}^n \mathbb{C}^2 = \mathbb{C}^{2n}/S_n.$$

It turns out (though it is highly non-obvious) that $\mathrm{Hilb}_n(\mathbb{C}^2)$ is smooth of dimension $2n$.

One of the first insights is to consider not just Hilb_n for a single n, but the family of all of them at once:

$$\mathrm{Hilb} = \bigsqcup_n \mathrm{Hilb}_n \mathbb{C}^2.$$

Once we do this, we have an action of the *Heisenberg algebra* $\bigoplus_{n\geq 0} H^\bullet_T(\mathrm{Hilb}_n \mathbb{C}^2)$, which was defined by Nakajima, Grojnowski, and Vasserot. The Heisenberg algebra is generated by elements $\{\alpha_k, \alpha_{-k}\}_k$ satisfying the relation $[\alpha_i, \alpha_j] = i\delta_{i,j}$. It *mixes* the grading: α_k takes $\mathrm{Hilb}_n(\mathbb{C}^2) \to \mathrm{Hilb}_{n-k}(\mathbb{C}^2)$ and α_{-k} takes $\mathrm{Hilb}_n(\mathbb{C}^2) \to \mathrm{Hilb}_{n+k}(\mathbb{C}^2)$.

Let me say briefly describe the idea underlying the analysis of $\mathrm{Hilb}_n(\mathbb{C}^2)$. There are two principles at play which allow us to understand the Hilbert scheme.

1. Let $\mathcal{M}(r,n)$ be the *moduli space of higher rank sheaves*, parametrizing

$$\left\{\begin{array}{c} \mathcal{E}= \text{torsion-free coh. sheaf on } \mathbb{P}^2 \\ \varphi\colon \mathcal{E}|_{\ell_\infty} \cong \mathcal{O}|_{\ell_\infty}^{\oplus r} \text{ (“framing”)} \\ \text{rank } \mathcal{E}=r,\ c_2(\mathcal{E})=n \end{array}\right\}$$

   If $r = 1$, then this is the Hilbert scheme. Indeed, $\mathcal{E}$ describes an ideal sheaf of $Z \subset \mathbb{C}^2 \subset \mathbb{P}^2$, such that $\mathcal{I}_Z$ is "framed" of rank 1.

   This is a symplectic resolution (of its "affinization"), and in particular $\mathcal{M}(r,n)$ is smooth of dimension $2rn$. We have an action of a torus $A = (\mathbb{C}^\times)^r$ acting on $\mathcal{M}(r,n)$ by reparametrizing the *framing*. This A preserves the symplectic form.

Why would we introduce this seemingly much more complicated space in order to analyze the Hilbert scheme? Analogously, to how it was advantageous to vary n in order to obtain an action of the Heisenberg algebra, varying r introduces extra structure. The idea is to embed the Hilbert schemes in this bigger symplectic resolution, and then study its quantum cohomology. The point is that $\mathcal{M}(r,n)$ turns out to be *easier* to study than the Hilbert scheme alone. (Actually, in the proof allowing $r = 2$ is enough.)

Example 4.2. If $r = 2$, then we have a stratification of the fixed locus of A as

$$\mathcal{M}(2,n)^A = \coprod_{a+b=n} \mathrm{Hilb}_a \times \mathrm{Hilb}_n$$

where the map from the right to the left takes $(I, J) \mapsto I \oplus J$.

2. Study "geometric shift operators" which we alluded to last time. This will be done in the next section.

4.2 Equivariant cohomology

There is no time to give a full treatment of equivariant cohomology, but we say at least a few words (describing only what is relevant to the situation at hand). Suppose we have a torus A acting on a scheme Y. Then the A-equivariant cohomology ring $H_A^\bullet(Y)$ of Y becomes an algebra over the A-equivariant cohomology ring of pt, which is $H_A^\bullet(\mathrm{pt}) \cong \mathbb{Q}[a_1, \ldots, a_r] =: R$. From now on we *assume that $H_A^\bullet(Y)$ is free as an R-module.*

Now we have an inclusion of the fixed locus

$$i\colon Y^A \hookrightarrow Y.$$

Since A acts trivially on Y^A by definition, its equivariant cohomology structure is the trivial one:

$$H_A^\bullet(Y) = H^\bullet(Y^A) \otimes R.$$

Now we have maps in both directions between $H_A^\bullet(Y)$ and $H_A^\bullet(Y^A)$:

- a pullback map $i^*\colon H_A^\bullet(Y) \to H_A^\bullet(Y^A)$,
- a pushforward map $i_*\colon H_A^\bullet(Y^A) \to H_A^{\bullet-?}(Y)$ which changes degrees.

Theorem 4.3 (Atiyah-Bott localization)**.** *Both of theses maps induce isomorphisms after tensoring with* $\mathrm{Frac}(R)$*:*

$$H_A^\bullet(Y) \otimes \mathrm{Frac}(R) \cong H_A^\bullet(Y^A) \otimes \mathrm{Frac}(R).$$

These aren't quite inverses, since they don't have the same effect on the degree, but they are almost inverse in the sense that there is a simple formula relating their composition to the identity: for all $\gamma \in H_A^\bullet(Y)$, we have

$$\gamma = \sum_{\text{fixed components } F} \frac{(i_F)_* i_F^* \gamma}{c_{\mathrm{top}}(N_F)}.$$

Applications. This is useful for doing equivariant intersection theory on Y via Y^A. If Y^A is proper, then we can define

$$\int_Y \gamma = \int_{Y^A} \text{“}(i_*)^{-1}\gamma\text{”} \in \mathrm{Frac}(R).$$

Example 4.4. Consider $A = (\mathbb{C}^\times)^2$ acting on $\mathbb{C}^2$ by scaling the x and y coordinates. Identify $H_A^\bullet(\mathrm{pt}) \cong \mathbb{C}[t_1, t_2]$. Then we have the identities

$$\begin{aligned}
\int_{\mathbb{C}^2} 1 &= \frac{1}{t_1 t_2} \\
\int_{\mathbb{C}^2} [x-\text{axis}] \smile [x-\text{axis}] &= \frac{t_2^2}{t_1 t_2} = \frac{t_2}{t_1} \\
\int_{\mathbb{C}^2} [x-\text{axis}] \smile [y-\text{axis}] &= 1 \text{ (proper intersection)}
\end{aligned}$$

♠♠♠ TONY: [but what do these calculations actually “mean?”]

If we apply this to $\mathcal{M}(2)^A = \mathrm{Hilb} \times \mathrm{Hilb}$, then we get

$$i^* \colon H^\bullet_{T\times A}(\mathcal{M}(2)) \otimes \mathrm{Frac}(R) \cong H^\bullet_{T\times A}(\mathrm{Hilb})^{\otimes 2} \otimes \mathrm{Frac}(R).$$

Here T is the rank 2 torus that acts by scaling $\mathbb{C}^2$. However, this isn't the isomorphism that we really want to work with. We would like a “more convenient” form of this isomorphism, in particular an isomorphism defined by a Lagrangian correspondence, which would take middle-dimensional classes to middle-dimensional classes (i^* doesn't have this property).

The solution to this is via the “stable envelope.” This isn't quite canonical, as it requires a choice of a generic one-parameter subgroup in A.

4.3 The stable envelope

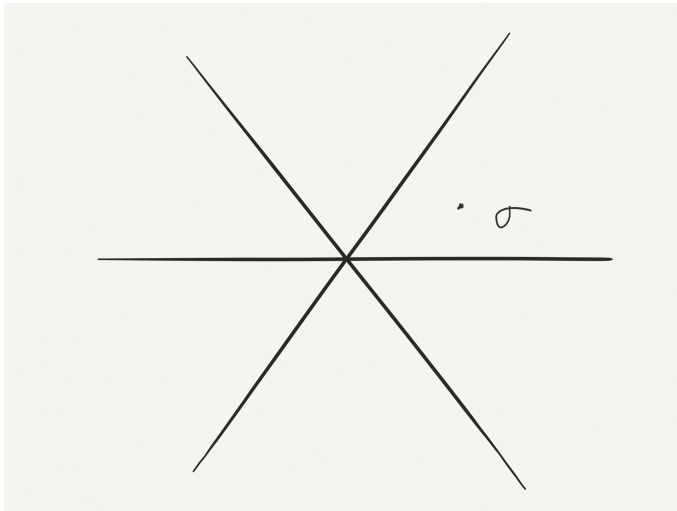
General Setup: we have a symplectic resolution

$$\begin{array}{c}
(X, \omega) \\
\downarrow \\
X_0
\end{array}$$

and a torus A acting on X, preserving ω.

Choose a one-parameter subgroup $\sigma \colon \mathbb{C}^\times \to A$ such that $X^\gamma = X^A$. Looking at the level of coweights, it suffices for this subgroup to lie in the *interior* of a Weyl chamber, i.e. avoid

any walls.

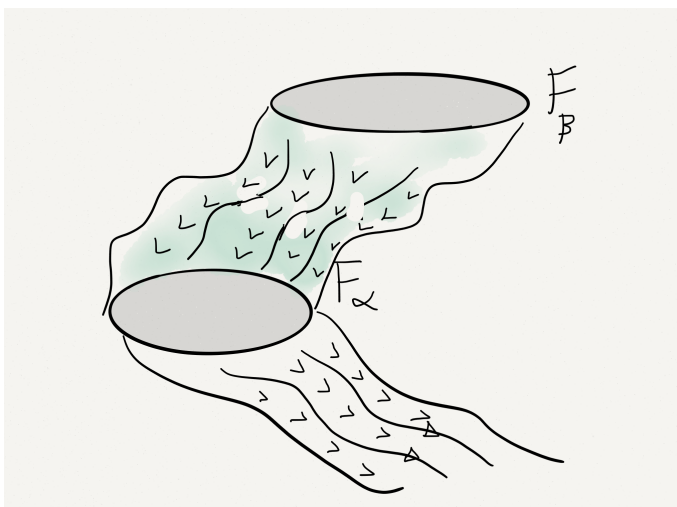


This puts a partial ordering on the fixed components.

Definition 4.5. Suppose $X^A = \bigsqcup F_\alpha$. Define the *leaf* of F_α to be

$$\mathrm{Leaf}(F_\alpha) = \{x \in X \mid \lim_{t \to 0} \sigma(t)x \in F_\alpha\}.$$

i.e. everything that "flows to F_α" as $t \to 0$.



We define a partial order by $F_\beta \leq F_\alpha$ if $F_\beta \cap \overline{\mathrm{Leaf}(F_\alpha)} \neq \emptyset$.

Notice $H_A^\bullet(F) = H^\bullet(F) \otimes \mathbb{Q}[a_1, \ldots, a_r]$. Then for $\gamma \in H_A^\bullet(F)$, we can define $\deg_A \gamma$ to be the degree in $a_1, \ldots, a_r$.

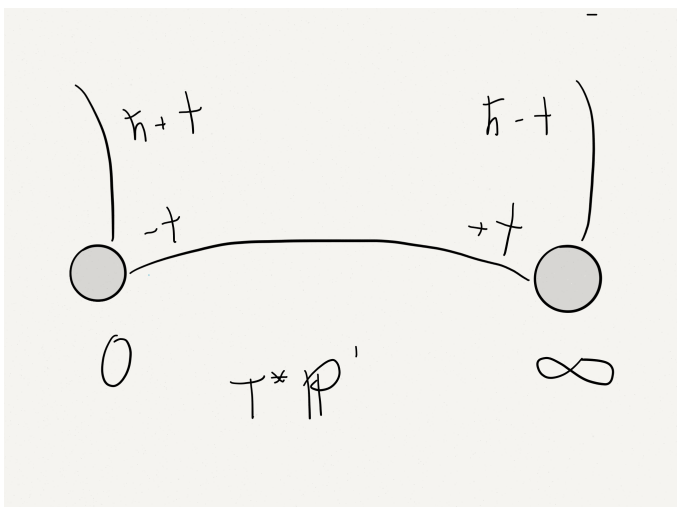
Theorem 4.6. *There exists a unique Lagrangian (hence middle dimension to middle dimension) correspondence*

$$\mathrm{Stab}_\sigma \colon H_A^\bullet(X^A) \to H_A^\bullet(X)$$

such that for $\gamma \in H_A^\bullet(F)$,

1. $\mathrm{Stab}(\gamma)$ *is supported on* $\bigcup_{F' \leq F} \mathit{Leaf}(F')$,
2. $i_F^*(\mathrm{Stab}\,\gamma) = e(N^-) \smile \gamma$, *where* N^- *is sub-bundle of the normal bundle of* F *in* X *corresponding to the "negative/unstable" directions, and*
3. *if* $F' < F$, *then* $\deg_A(i_{F'}^* \mathrm{Stab}\,\gamma) < \frac{1}{2} \operatorname{codim} F'$.

Example 4.7. For $T^*\mathbb{P}^1$, we have $A = \mathbb{C}^\times$. A generic character is positive or negative, and if it is positive at ∞ (so that ∞ is an attractor) then the flow looks like



Let F_0 be the cotangent fiber at 0. Then we have

- $\mathrm{Leaf}(0) = F_0$,
- $\overline{\mathrm{Leaf}(\infty)} = [\mathbb{P}^1]$.
- $\mathrm{Leaf}(\infty)|_0 = \hbar + t$,
- $\mathrm{Stab}([0]) = [F_0]$,
- $\mathrm{Stab}([\infty]) = [\mathbb{P}^1] + [F_0]$.

♠♠♠ TONY: [explicate this]

Now applying this to $Y = \mathrm{Hilb}$, we have a map

$$H_A^\bullet(\mathrm{Hilb} \times \mathrm{Hilb}) \to H^\bullet(\mathcal{M}(2)).$$

Remark 4.8. Let $X = T^* \mathrm{Fl}$, and A the maximal torus of G. Then $X^A = \bigsqcup_{w \in W} p_W$ (a set of points indexed by the Weyl group). If we pick σ in the interior of a Weyl chamber, then we get Schubert cells on the flag variety. What is the stable basis in this case? (For every point we get should get a Lagrangian submanifold of the cotangent bundle).

It turns out that $\mathrm{Stab}_\sigma(p_w) = (\text{characteristic cycle}) j_! \mathbb{Q}_{\Sigma_w^\circ}$, where Σ_w° is the open Schubert cell.

Definition 4.9. We define $\mathrm{UStab}_\sigma = \mathrm{Stab}_{\sigma^{-1}}$. This "reverses" all of the flows.

Then $H_A^\bullet(X^A)$ maps to $H_A^\bullet(X^A)$ in two ways, via Stab and UStab. We set

$$R_\sigma := (\mathrm{UStab}_\sigma^{-1} \otimes \mathrm{Stab})_{\mathrm{Frac}}(R) \colon H_A^\bullet(X^A) \otimes \mathrm{Frac}(R) \to H_A^\bullet(X) \otimes \mathrm{Frac}(R).$$

Exercise 4.10. Calculate what this is for $T^*\mathbb{P}^1$.

What's the point of this construction? The idea is that $\mathrm{Stab}(\gamma)$ is good for translating geometric operators on X to geometric operators on X^A. For instance, we have the following result.

Proposition 4.11. *We have*

$$(\mathrm{Stab}(\gamma), \mathrm{UStab}(\gamma'))|_X = (\gamma, \gamma')_{X^A}.$$

(You might have to use denominators in order to define the pairing, but the result has no denominators.)

Example 4.12. In particular, if γ and γ' come from different fixed loci then this tells us that $(\mathrm{Stab}(\gamma), \mathrm{UStab}(\gamma'))|_X = 0$.

Proof Sketch. We'll state the idea of this is: assume X^A is isolated. If $p, p' \in X^A$. We are interested in $(\mathrm{Stab}(p), \mathrm{UStab}(p'))$.

The first step is to show that the intersection of $\mathrm{Stab}(p)$ and $\mathrm{UStab}(p')$ is proper. This is plausible, as the flow will escape any affine parts because all the points will be stable for one flow and unstable for the other, since Stab and UStab are opposite. Since we are dealing with proper intersections, so we don't need to invert denominators. That means the pairing will actually be a polynomial.

The second step is an expected dimension calculation. Since these are both Lagrangian cycles, their intersection should have degree 0, so it is just a number $\alpha \in \mathbb{Q}$. This is something *non-equivariant*, which we can try to calculate using localization (this was also a key step to the argument from last time!)

Finally, one notes that

$$(\mathrm{Stab}(p), \mathrm{UStab}(p'))_X = \sum_{q \text{ fixed}} \frac{i_q^* \mathrm{Stab}(p) i_q^*(\mathrm{UStab}(p))}{e(N_q)}$$

where $e(N_q)$ has degree $\operatorname{codim} q = \dim X$, and $\mathrm{Stab}(p)$ has degree less than $1/2 \dim X$ if $q \neq p$, and similarly for $\mathrm{UStab}(p)$. By sending the equivariant parameters to ∞, you can argue that $\alpha = 0$.

This is a common argument: use properness to get rid of denominators, and then formal degree considerations to argue that the result must be non-equivariant, at which point you can just specialize the equivariant variables. □

4.4 Quantum computations on Hilb_n

We're going to atttempt to describe the quantum cohomology ring of Hilb_n. As we have mentioned, the key to understanding the quantum cohomology of Hilb_n is understanding the classical geometry.

It is a fact that $\mathrm{Pic}(\mathrm{Hilb}_n) = \mathbb{Z}c_1(\mathrm{Taut})$, where the tautological bundle on Hilb_n is a rank n bundle whose fiber over Z is $H^0(\mathcal{O}_Z)$. From now on we abbreviate $c_1 = c_1(\mathrm{Taut})$. We would like to understand the quantum operator $c_1\bullet$. The language in which the answer is described involves the Heisenberg algebra and its action on $\bigoplus_{n\geq 0} H_T^\bullet(\mathrm{Hilb}_n)$. We adopt the usual notational convention for the Heisenberg algebra

- Raising operators $\alpha_{-k}\colon H_T^\bullet(\mathrm{Hilb}_n) \to H_T^\bullet(\mathrm{Hilb}_{n+k})$ (geometrically, this corresponds to adding a clump of length k)
- Lowering operators $\alpha_k\colon H_T^\bullet(\mathrm{Hilb}_n) \to H_T^\bullet(\mathrm{Hilb}_{n-k})$ (essentially the adjoint of α_{-k})

Theorem 4.13 (Okounkov-Pandariphande)**.** *We have*

$$c_1\bullet = c_1 \smile +(t_1+t_2)\left(\sum_{k\geq 1} k\frac{\alpha_{-k}\alpha_k(-q)^k}{(1-(-q)^k} - \frac{-q}{1-(-q)}\sum_{k\geq 1}\alpha_{-k}\alpha_k\right)$$

Note the similarities of the features to what we found for the Springer resolution. The correction term is divisible by t_1+t_2, which is the weight of the symplectic form. The q-series also fi together into a very nice rational form.

Remark 4.14. This operator actually generates the ring.

The idea is to study $\mathcal{M}(2)$, and relate this to Hilb, via the constructions that we discussed in the previous section:

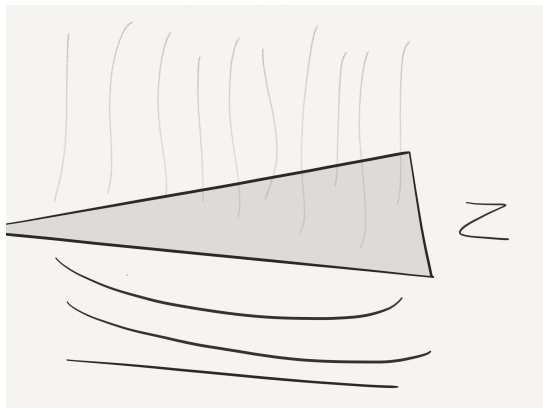
- $\mathrm{Stab}\colon H_{T\times A}^\bullet(\mathrm{Hilb})^{\otimes 2} \to H_{T\times A}^\bullet(\mathcal{M}(2))$,
- $\mathrm{UStab}\colon H_{T\times A}^\bullet(\mathrm{Hilb})^{\otimes 2} \to H_{T\times A}^\bullet(\mathcal{M}(2))$,
- $\mathcal{R}(a_1-a_2) = \mathrm{UStab}^{-1}\circ\mathrm{Stab}$ acts on $\mathrm{Hilb}\times\mathrm{Hilb}$ after localizing.

General Principle. We have a diagram

$$\begin{tikzcd}
\mathrm{Hilb}\times\mathrm{Hilb} \arrow[r, bend left, "\mathrm{Stab}"] \arrow[r, bend right, "\mathrm{UStab}"'] \arrow[d, "L^A=\mathrm{Lagrangian}^A"'] & \mathcal{M}(2) \arrow[d, "\text{Steinberg Lagrangian}"] \\
\mathrm{Hilb}\times\mathrm{Hilb} \arrow[r, bend left, "\mathrm{Stab}"] \arrow[r, bend right, "\mathrm{UStab}"'] & \mathcal{M}(2)
\end{tikzcd}$$

Then it will be the case that $\boxed{[L^A, \mathcal{R}] = 0}$ for any Steinberg class L^A. (You can prove this by the same kinds of arguments that we have been discussing.) If you have some control over $\mathcal{R}$, then you get lots of relations.

So how can we get a handle on $\mathcal{R}$? On a leaf of Z,



we have $\mathcal{R}_{ZZ} = \frac{c_{\mathrm{top}}(\mathrm{Taut}\otimes\hbar)}{c_{\mathrm{top}}(\mathrm{Taut})} \smile$, i.e. "diagonal term" of the action of $\mathcal{R}$ on Z is given by the chern classes of the tautological bundle (here $\hbar = t_1 + t_2$).

Now let's discuss some quantum computations. Denote

$$Q(2) = c_1^{\mathcal{M}(2)} \bullet (-)$$
$$Q(1) = c_1^{\mathrm{Hilb}} \bullet (-)$$

We already know on general grounds that the quantum part of this is given by Steinberg varieties.

Notation. For an operator F on $H_T^\bullet(\mathcal{M}(2))$, we denote by ΔF the operator on $H_T^\bullet(\mathrm{Hilb}_n)$ defined by

$$\Delta F = \mathrm{Stab}^{-1} \circ F \circ \mathrm{Stab}\,.$$

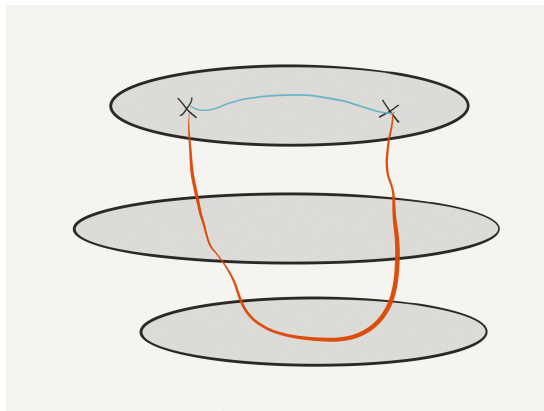
Proposition 4.15. *We have*

$$\Delta Q(2) = Q(1) \otimes 1 + 1 \otimes Q(1) + \textit{(off-diagonal terms)} + \textit{(scalar operator)}.$$

Remark 4.16. We can always figure out scalar discrepencies at the end by examining the effect on the virtual fundamental class, so we don't really care about that part right now.

Idea of proof. Say we have two points on a fixed component, and we are considering curves from one to the other. There are curves that stay on the fixed component, but also curves

that travel to other components and come back.



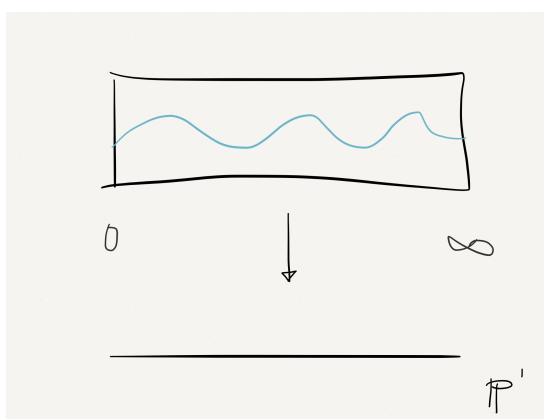
It was an exercise in the second problem sheet to show that the second kind don't contribute to $Q(2)$. The precise way to argue was to send $a_1 - a_2 \to \infty$, and see that those curves are killed. $\square$

The second big idea which goes to the proof is the (twisted) Graph construction (which were introduced in Braverman's lectures). Say you have an action of a torus T on any X, and a one-parameter subgroup $\sigma\colon \mathbb{C}^\times \to T$. Then we can construct the twisted X-bundle over $\mathbb{P}^1$,

$$\begin{array}{c} X(\sigma) \;=\; (\mathbb{C}^2 \setminus 0) \times_{\mathbb{C}^\times} X \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

Let $\widetilde{\beta} \in H_2(X(\sigma))$ be a class that projects to $[\mathbb{P}^1] \in H_2(\mathbb{P}^1)$. We consider the moduli space of stable maps with two marked points, forcing the first one to lie over 0 and the second to lie over ∞:

$$\overline{\mathcal{M}_{0,2}}(X(\sigma), \widetilde{\beta}) \supset \mathrm{ev}_1^{-1}(0) \smile \mathrm{ev}_2^{-1}(\infty).$$



Now we can study the T-equivariant Gromov-Witten theory. We have an operator S_σ on $H_T^\bullet(X) \otimes \mathrm{Frac}(R)$ determined by the property

$$\sum_{\widetilde{\beta}} q^{\widetilde{\beta}} \langle \gamma_1, \gamma_2 \rangle_{\widetilde{\beta}}^{X(\sigma)} = (\gamma_1, S(\sigma)\gamma_2).$$

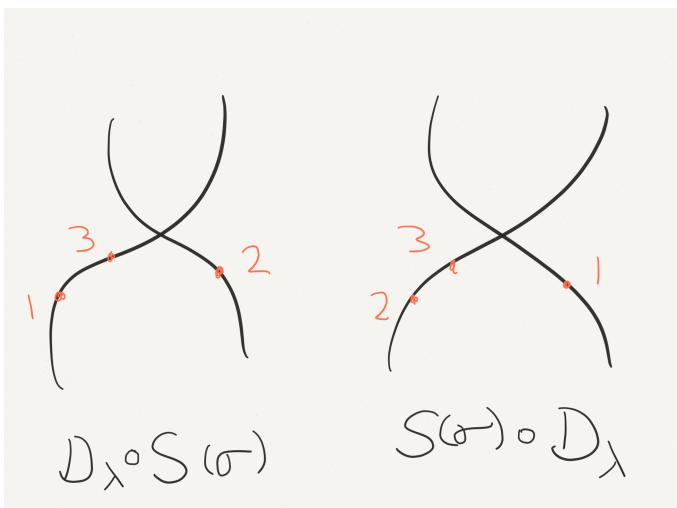
Let D be the divisor operator, defined by the property that for $D_\lambda \in H^2(X)$ corresponding to a coweight λ, $D(\lambda)$ is the operator of quantum multiplication with D_λ.

Proposition 4.17. *We have* $[D, S(\sigma)] = 0$.

Proof. We can consider $\overline{\mathcal{M}_{0,3}}(X(\sigma))$. Unraveling the definitions, you see that this boils down to an equality

$$\mathrm{ev}_1^{-1}(0) \smile \mathrm{ev}_2^{-1}(\infty) \smile \mathrm{ev}_3^{-1}(0) = \mathrm{ev}_1^{-1}(\infty) \smile \mathrm{ev}_2^{-1}(0) \smile \mathrm{ev}_3^{-1}(0)$$

i.e. an equality with intersecting a certain virtual fundamental class with the pullback of the curve divisors



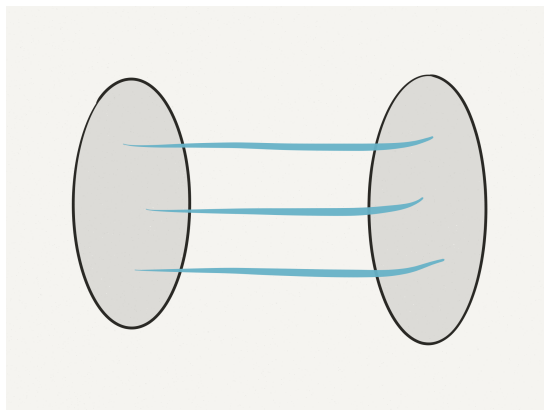
□

The point is that this introduces many more relations into the quantum cohomology ring.

Now we study equivariance with respect to the bigger group $T \times \mathbb{C}_\epsilon^\times$, with the $\mathbb{C}_\epsilon^\times$ acting on the base $\mathbb{P}^1$ via the weight ϵ. Whereas $[a] = \infty \in H^2(\mathbb{P}^1)$ in the previous example, we now have $[c] - [\infty] = \epsilon \cdot 1$, so D no longer commutes with $S(\sigma)$. The new commutation relation is

$$\boxed{\epsilon q \frac{\partial}{\partial q} S(\sigma) = D_0 S(\sigma) - S(\sigma) D_\infty.}$$

Proof sketch. Ok now we can actually sketch the argument for Theorem 4.13. Choose a generic one-parameter family $\sigma\colon \mathbb{C}^\times \to A$. Then it will be the case that $\langle \mathrm{Stab}, S(\sigma)\,\mathrm{Stab}\rangle$ is a proper intersection. Then one can make the usual argument, reducing the problem to computing non-equivariant parameters. The result is that the only contributing curves are ones like those in the picture below, with no "bubbling."



We didn't explain why this is true. It has to do with the fact that if you look at the affinization of $\mathcal{M}(2)$, then the weights are all ± 1: "the σ-weights of $H^\bullet(\mathcal{M}(2), \mathbb{C})$ are small generators." This is the key thing that is true about $\mathcal{M}(2)$ but *not* Hilb, which makes it more tractable. The upshot is that $\Delta S(\sigma) = (-q)^{|\cdot|\otimes 1}\mathcal{R}$.

Ok, let's collect the relations:

- $[\Delta S, \mathcal{R}] = 0$,
- $[\Delta Q(2)_+, \mathcal{R}] = 0$ (+ means quantum part),
- $[\Delta Q(2), (-q)^{|\cdot|\otimes 1}\mathcal{R}] = 0$,

Using an earlier formula that I didn't write down, you can deduce that

$$R(\Delta c_1)\mathcal{R}^{-1} = \Delta c_1 + (t_1 + t_2)\sum \alpha_k \otimes \alpha_{-k} - \alpha_{-k} \otimes \alpha_k$$

Putting together all of these identities determines the off-diagonal part of ΔQ_2. Denote by $\Delta_k Q(2)$ the off-diagonal part shifting the index by k. Then you get an identity

$$\sum (1 - (-q)^k)\Delta_k Q(2) = \text{left hand side of } R\Delta c_1 R^{-1}.$$

You can then solve for the off-diagonal terms to find the rational functions. A little further argument involving the tautological class pins down everything.

Of course, this was all quite vague. You should think of the point as being that the three relations are very contraining.

In general we expect the intertwining relations for $S(\sigma)$ to determine the quantum operators.

5 Further Directions

The proof we just gave was sort of "soft" in the sense that we didn't use anything deep about curves. In this last section we highlight some further directions to pursue, which are harder.

1. Quiver varieties for general quivers Q. Study the matrix elements of the "$1/u$ term" of $\mathcal{R}$.

2. What are other quantum operators, e.g. those obtained from higher degree tautological Chern classes? There is a nice conjectural answer, again in terms of the $\mathcal{R}$-matrices.

3. What is the monodromy of the quantum connections? There is an expectation of Bezrukavnikov-Okounkov for the categorification of this action. The easy case is the Springer resolution, which was done a long time ago. The case of the Hilbert scheme is hard.

4. Quantum K-theory (to be discussed next week by Okounkov). Basically none of our methods here work in that setting.

5. The relation to Donaldson-Thomas theory (curve-counting in 3-folds). If $\dim X = 3$, you can try to do virtual enumerative geometry on the Hilbert scheme of curves in X.

   If you fix a curve class β and constant term n, then there is a conjecture that $\frac{1}{2}\sum q^n[\mathrm{Hilb}_{\beta,n}(X)]$ is a rational function. This is known in many cases, most generally by work of Pandariphande-Pixton. Their argument is essentially by degeneration of hypersurfaces to a toric variety, e.g. $\mathbb{P}^3$. The toric case then degenerates to $\mathbb{C}^2\times\mathbb{P}^1$ or $T^*\mathbb{P}^1\times\mathbb{P}^1$, etc. For examples like $\mathrm{Hilb}\,\mathbb{C}^2$ or $\mathrm{Hilb}\,T^*\mathbb{P}^1$, you can trace their arguments to the calculations we described.