Matching and orbital integrals

Zhiyu Zhang

1 Motivation

Consider a tulple of reductive groups over a number field F_0

$$H_1 \hookrightarrow G \hookleftarrow H_2$$

and choose a good test function $f = \prod_v f_v \in C_c^\infty(G(\mathbb{A}))$. Main part of **RTF** is an equality

$$\text{"Spectral Side"} = \text{"Geometric Side"}$$

$$\sum_{\pi \text{ irr cusp auto rep of } G} (...) = \sum_{\gamma \in H_1(F_0)\backslash G(F_0)/H_2(F_0)} Vol_\gamma \int_{H_1^{-1}\gamma H_2(\mathbb{A})} f dh_1 dh_2$$

Idea: For any matrix $A = (a_{ij})_{n\times n}$, $\sum \lambda_i = \sum a_{ii}$.
Variants: twist by a character η, **action of H on a good G-variety** X (e.g a symmetric space), (semi-)linearization...
Warning: There are big convergence issues. This is why we like regular semisimple orbits (as the orbit is closed, the restriction of f is still compactly supported, and the volume factor is easy to compute).
Slogan. Comparison of RTFs is a very useful tool (JL, base change, GGP...).
How ?

- Match regular semisimple orbits (study the orbit space)
- Match orbit integrals (existence of transfer)
- Fundamental lemma
- Choose good test functions to separate terms (density, base change, multiplicity one)

When ?
Why do we expect such comparison on the geometric side? One importance case is **the twist case**: the action of H on X and H' on X' over F_0 are different, but become the same after base change to large field. And we get untwist/matching after taking quotient by showing the twist does not change the orbits.

2 Jacquet-Rallis case

Twist of conjugation action of GL_{n-1} on GL_n and on $\begin{bmatrix} GL_{n-1} & * \\ * & 0 \end{bmatrix}$.

(Introduce the notations as in the Jacquet-Rallis setting, F/F_0 separable quadratic extension...)

So we have

- GL_n side: $H' = GL_{n-1}$ acts on
$$S_n = \{\gamma \in Res_{F/F_0} GL_n | \bar{\gamma}\gamma = 1\}$$
and on
$$S_{n-1} \times V'_{n-1} \hookrightarrow Res_{F/F_0} \begin{bmatrix} GL_{n-1} & * \\ * & 0 \end{bmatrix}$$
by conjugation.

- U_n side: $H = U(V)$ acts on
$$G = U(V^{\#}) = \{g \in Res_{F/F_0} GL_n | {}^t\bar{g}Jg = J\}$$
and on
$$U(V) \times V \hookrightarrow Res_{F/F_0} \begin{bmatrix} GL_{n-1} & * \\ * & 0 \end{bmatrix}$$
by conjugation.

Theorem 1. There is a natural bijection of regular semisimple orbits
$$\coprod_V [U(V^{\#})(F_0)]_{rs} \cong [S_n(F_0)]_{rs} \quad (1)$$
and
$$\coprod_V [(U(V) \times V)(F_0)]_{rs} \cong [(S_{n-1} \times V'_{n-1})(F_0)]_{rs} \quad (2)$$
where V runs over $\mathbf{Herm}_{n-1}$, the set of isomorphic classes of $n-1$ dimensional non-degenerate F/F_0 Hermitian spaces.

Before giving a proof, let's make some observations. If $F = F_0 \times F_0$ is split, then the action at both sides becomes the standard conjugation action of GL_{n-1}. The theorem is obvious (even without rs assumption) in this case.

Remark 1. One will see later the rs assumption is neccesary in the proof for general case.

In general, as $F \otimes_{F_0} F = F \times F$, after a base change from F_0 to F we arrive at the split case. And the embeddings e.g $S_n(F_0) \hookrightarrow GL_n(F)$ can be thought as embeddings of F_0-points to F-points.

Use the standard pairing on F^{n-1} given by $(x, y) = \sum_i x_i \bar{y}_i$ we get the trivial Hermitian space V_0 and $V_0^{\#}$. Compare $U(V_0^{\#}) = \{{}^t\bar{g}g = 1\}$ with $S_n = \{\bar{\gamma}\gamma = 1\}$ (more precisely, compare the actions), we see

Proposition 1. The action of H' on S_n and $H = U(V_0)$ on $G = U(V_0^\#)$ is F/F_0-twist of each other, the twist is given by the transpose anti-involution on GL_n/F. Similar result holds for the variant version.

Then let's recall the notation of regular semisimpleness. Let a reductive group H act on a smooth affine variety X over F_0, we say $x \in X(F_0)$ is **regular semisimple** if Hx is Zariski closed and H_x is trivial. This condition satisfies faithful flat descent, so we expect it's representable.
Fact: There exists an open subscheme X_{rs} of X parametrizing rs points. In practice (which is true in our case), X_{rs} is non-empty, affine and dense.

Then one may imagine X_{rs}/H (which exists as a scheme) parametrizing rs orbits. But it's a general phenomenon that $(X/H)(F_0) \neq X(F_0)/H(F_0)$ for non-algebraically closed field F_0, and one has to consider H-torsors. By definition,

$$(X/H)(F_0) = \coprod_{\alpha \in H^1(F_0, H)} X_\alpha(F_0)/H_\alpha(F_0)$$

where T_α is the H-torsor corresponding to α, $H_\alpha = Aut(T_\alpha)$, $X_{\alpha,rs} = (X \times T_\alpha)/H$.

Proposition 2. $H^1(F_0, GL_{n-1}) = 1$, and $H^1(F_0, U(V_0))$ is in bijection with isomorphism classes of $n-1$-dimensional F/F_0-hermitian spaces.

Proof. The first one is Hilbert Satz 90, the second proof is similar to how one identifies GL_n torsors with rank n vector bundles. □

Return to the theorem, one gets that LHS of (1) is $(U(V_0^\#)_{rs}//U(V_0))(F_0)$, and RHS is $((S_n)_{rs}//GL_{n-1})(F_0)$. To finish the proof, we use the following proposition which says the twist is trivial on the quotient:

Proposition 3. $x \to {}^t x$ is identity on $(GL_n)_{rs}//GL_{n-1}$. Therefore,

$$U(V_0^\#)_{rs}//U(V_0) \cong (S_n)_{rs}//GL_{n-1}.$$

Proof. For our purpose, we only need to look at field-valued points. This reduces to checking that for any rs matrix $g \in GL_n(E)$ (E can be any field), g is $GL_{n-1}(E)$ conjugate to ${}^t g$, which will be done in next section. □

The proof of (2) in the theorem is similar.

3 Concrete matching of elements

The above conceptual explanation indicates that to prove mathching of orbits, it's useful to consider the embedding

$$U(V)(F_0) \times V(F_0) \hookrightarrow \begin{bmatrix} GL_{n-1} & * \\ * & 0 \end{bmatrix}(F) \hookleftarrow S_{n-1}(F_0) \times V'_{n-1}(F_0)$$

. Note the stabilizer of a rs element is trivial hence two rs elements are $H(F) = GL_{n-1}(F)$-conjugated iff they are $H(F_0)$-conjugated. So we have embedded LHS and RHS of (1) and (2) into a common large orbit space, and do matching there.

Definition 1. (g, u) and (γ, u_1, u_2) is matched iff they are conjugated by $GL_{n-1}(F)$ in $M_{n\times n}(F)$.

The geometry of GL_{n-1} action on GL_n is summarized as the following theorem (the variant version is similar).

Theorem 2. Let E be any field, $g = \begin{bmatrix} A & u \\ v & d \end{bmatrix} \in GL_n(E)$. Then

1. g is regular semisimple iff $e, ge, \ldots, g^{n-1}e$ **form a basis of** E^n **and** $e^*, e^*g, \ldots, e^*g^{n-1}$ **form a basis of** $(E^n)^*$
   iff $u, Au, \ldots, A^{n-2}u$ form a basis of E^{n-1} and $v, vA, \ldots, vA^{n-2}$ form a basis of $(E^{n-1})^*$
   iff $det((vA^{i+j}u)_{0\le i,j\le n-2}) \neq 0$ (so rs elements form an non-empty affine open subset).

2. For regular semisimple g, define $inv(g)$ as the data $det(\lambda I + A) \in E[\lambda]$, vA^iu ($i = 0, \ldots, n-2$) and d. Then for regular semisimple g_1, g_2, $g_1 \sim g_2$ iff $inv(g_1) = inv(g_2)$.

Proof. We give a sketch. If $n = 2$, the action is

$$\begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & u \\ v & d \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & a^{-1}u \\ av & d \end{bmatrix}$$

Let " $t \to 0$ or ∞ ", we see the orbit is closed iff $uv \neq 0$ or $u = v = 0$ (if $uv = t \neq 0$ then the orbit is defined by $\{uv = t\}$ hence is closed), rs iff $uv \neq 0$, so the theorem is true.

The proof for general case is similar. (1.) is easy except the first equivalence: for one side e.g if $e, ge, \ldots, g^{n-1}e$ does not form a basis of E^n, then g has a proper invariant subspace, so g look like $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ under another basis, then choose scalar matrix t as above and let $t \to \infty$, the limit point is fixed by all $t \neq 0$, so the orbit is not closed or the stabilizer is not trivial. For another side, if $hg = gh$ for $h \in GL_{n-1}(E)$, as $he = e$ we know $hg^ie = g^ie$, but g^ie form a basis, so $h = id$ hence the stablizer of h is trivial. For the closedness, one need to use limit argument to classify all closed orbits.

The proof of (2.) is easy: if $inv(g_1) = inv(g_2)$, define $h \in GL_{n-1}$ by sending $A_1^iu_1$ to $A_2^iu_2$ ($i = 0, \ldots, n-2$. As they are both basis of E^{n-1}, this is well-defined, use the equality of invariants to show $hg_1h^{-1}(g_1^ie) = g_2(g_2^ie)$ hence $hg_1h^{-1} = g_2$.

□

Corollary 1. For any rs matrix $g \in GL_n(E)$, g is $GL_{n-1}(E)$ conjugate to ${}^tg$.

Remark 2. This is the analog of the classical result that any $n \times n$ matrix is conjugated to its transpose. Here the result is not true for any matrix: consider $\begin{bmatrix} 1 & 1 \\ 0 & 1` \end{bmatrix}$.

Remark 3. One can prove the matching concretely. For example, take $\gamma \in S_n(F_0)_{rs}$, as $\gamma \sim {}^t\gamma$, there is a $x \in GL_{n-1}(F)$ s.t $x\gamma x^{-1} = {}^t\gamma$. Applying the conjugation we get $\bar{x}\bar{\gamma}\bar{x}^{-1} = {}^t\bar{\gamma}$, use $\bar{\gamma}\gamma = 1$ we get $\bar{x}\gamma\bar{x}^{-1} = {}^t\gamma$ hence $\bar{x} = x$ as $Stab(\gamma) = 1$. Similarly, ${}^tx = x$ so $x \in \mathbf{Herm}_{n-1}$. Therefore,

$${}^t\bar{\gamma}x\gamma x^{-1} = {}^t\bar{\gamma}{}^t\gamma = 1$$

so $\gamma \in U(x \oplus 1)_{rs}$.

In conclusion, we get the matching of rs orbits, and it's time to discuss the matching of orbit integral.

4 Smooth transfer

Recall

Theorem 3. (classification of n-dimensional non-degenerate F/F_0-Hermitian spaces over local and global fields)

- (Split case) only the trivial one, $U(\langle,\rangle) = GL_n$;
- ($\mathbb{C}/\mathbb{R}$) any V is isomorphic to $V_{p,q}$ defined by $diag 1_p, -1_q$ where p, q are two natural number with $p+q=n$. $V_{p,q}$ are not isomorphic to each other, but $U(p,q) := U(V_{p,q}) \cong U(q,p)$.
- (p-adic field) $\det : \mathbf{Herm}_n \cong F_0^\times / NF^\times \cong \mathbb{Z}/2 = \{0,1\}$. For n odd, $U(V_0) \cong U(V_1)$ are quasi-split. For n even, $U(V_0) \ncong U(V_1)$ and only $U(V_0)$ is quasi-split.
- (totally real field) certain local-global principle holds.

For the proof in the p-adic case, one firstly checks $n=1$ and $n=2$, and use that any V with dimension ≤ 3 has isotropic vectors to do induction.

(define orbit integrals as in the Jacquet-Rallis setting...)

Note the twist by η on the GL_n side, it's necessary to have the transfer factor $\omega(\gamma)$ in the definition. And the product of all local transfer factors is 1, hence does not effect the global matching.

Definition 2. A function $f' \in \mathcal{S}(S_n(F_0))$ and a pair of functions $(f_0, f_1) \in S(U(V_0^{\#})(F_0)) \times S(U(V_1^{\#})(F_0))$ are transfers of each other if for each $i \in \{0,1\}$ and each $g \in U(V_i^{\#})(F_0)_{rs}$, we have

$$Orb(g, f_i) = Orb(\gamma, f')$$

whenever $\gamma \in S_n(F_0)_{rs}$ matches g.

The variant version is defined similarly, so is the Lie-algebra version.

Theorem 4. In the p-adic case, the smooth transfer always exists.

The idea is to firstly reduce to the Lie-algebra version using Cayley map, then because of the local constancy of orbit integral (which is one feature of p-adic fields), one only need to prove the existence around every points. Use Harish-Chandra's semisimple descent (understanding orbital integrals in terms of slice representations) and induction, one gets the existence away from the center. Finally the compatibility of transfer and Fourier transform solves the remaining case. The $n=1$ case is explicit and important for induction.

The fundamental lemma will be discussed next time .