# Roots of Unity in Intermediate Characteristic

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## 1 Roots of unity

Let $k$ be an algebraically closed field. The difference in roots of unity in $k$ can be traced to a difference in the $p$th power map: if $k$ has characteristic $p$ then there is only one $p$th root of unity; otherwise there are $p$ of them. So again, there are two different phenomena in characteristic 0 and $p$, and we would like to know what goes on in the Morava $K$-theories interpolating between them.

**Question:** What do roots of unity look like in the intermediate fields $K(n)$?

*Warning:* For $0 < n < \infty$, the fields $K(n)$ are not commutative. So roots of unity in the fields $K(n)$ themselves are not the right thing to look at. Instead, we should look at something called $K(n)$-*local homotopy theory.*

*Definition* 1.1. A spectrum is $K(n)$-*acyclic* if $X \wedge K(n) \cong 0$. Thus

$$\{K(n)\text{-local spectra}\} = \{\text{spectra}\}/\{K(n)\text{-acyclic spectra }\}.$$

The map $\{\text{spectra}\} \to \{K(n)\text{-local spectra}\}$ is called "$K(n)$-localization."

*Example* 1.2. To get a feel for this, let's consider some extreme cases. The $K(0)$-local spectra are chain complexes over $\mathbb{Q}$, and you should think of $L_{K(0)}$ as "tensoring with $\mathbb{Q}$."

$K(\infty)$-local spectra are like "$p$-adically complete spectra." You can think of $L_{K(\infty)}$ as being like $p$-adic completion. We note, however, that in the setting of spectra, there is essentially one thing that you could mean by "working rationally." However, there are several things that you could mean by "working $p$-adically," and this is one of them.

In general, you can think of $K(n)$-localization is a mixture of these two procedures. Think to the algebro-geometric situation: if you want to form the completed local ring at a point, you have to first localize, and then complete. Analogously, $K(n)$-localization is built out of first taking a localization and then a completion.

We introduced this notion because $K(n)$ wasn't commutative. Now, there are many $K(n)$-local spectra which are commutative. In fact, $K(n)$-localization takes ring spectra to ring spectra. However, be warned that $L_{K(n)}$ annihilates all ordinary rings, so the commutative algebra that we are investigating is in some sense orthogonal to the usual one.

Our goal is to do some algebraic geometry with these objects. By algebraic geometry, we only mean algebra (i.e. affine schemes). There turn out to be more of these than you think.

# 2 Algebraic geometry over spectra

## 2.1 Algebraic geometry over $\mathbb{C}$

Let $S$ be a finite set. Then $S$ "is" an affine algebraic variety. One could say $S = \operatorname{Spec} \mathbb{C}^S$. The left hand side is a set while the right hand side is an algebraic variety, so what do we really mean by this?

What do we really mean by this?

The first interpretation is literal. There is a canonical bijection from $S$ to the set of points of the affine scheme $\operatorname{Spec} \mathbb{C}^S$, sending $s \mapsto \mathfrak{p}_s := \{f : S \to \mathbb{C} \mid f(s) = 0\}$.

This isn't the interpretation I want to pursue. A second *sheaf-theoretic* interpretation begins by what the "quasicoherent sheaves" are on the two spaces. On the one hand, quasicoherent sheaves on $\operatorname{Spec} \mathbb{C}^S$ are just $\mathbb{C}^S$-modules. On the other hand, quasicoherent sheaves on the topological space $S$ should be $S$-graded vector spaces, i.e. a local systems of vector spaces on $S$. The statement $S = \operatorname{Spec} \mathbb{C}^S$ is reflecting an equivalence of categoreis

$$\{\text{local systems of vector spaces on } S\} \cong \{\mathbb{C}^S \text{ modules}\}.$$

## 2.2 Spectra

We can do homotopy theory with this interpretation. Let $X$ be a space and $R$ a ring spectrum. If $\mathcal{L}$ is a local system of $R$-modules on $X$, then $C^*(X; \mathcal{L})$ is an $R$-module. But in fact we see more structure: it is a module over the "function spectrum" $R^X := C^*(X; R)$. This is another ring spectrum. You can think of this as the ring spectrum for which the $R^X$-cohomology on $Y$ is the $R$-cohomology on $X \times Y$. Or, you can think of it as the global sections of the constant local system $R^X$.

Thus, $L \rightsquigarrow C^*(X; \mathcal{L})$ induces a map

$$\{\text{local systems of } R\text{-modules on } X\} \to \{R^X\text{-modules}\}.$$

In general, you don't expect this to be much like an equivalence of categories. If $X = BG/\mathbb{C}$, then the left hand side is the category of representations of $G$ over $\mathbb{C}$ and the right hand side is the category of $\mathbb{C}$-vector spaces, and this functor assigns to a representation its invariant subspace. This is far from being an equivalence, as it annihilates all the non-trivial irreducible representations.

However, we want to restrict ourselves to the $K(n)$-local setting.

*Definition* 2.1. We say that a space $X$ is *n-truncated* if $\pi_* X \cong 0$ for $* > n$.

**Theorem 2.2** (Hopkins-Lurie)**.** *If $X$ is a p-finite n-truncated space, then the global sections functor is an equivalence*

$$\{\textit{local systems of } R\textit{-modules on } X\} \to \{R^X - \textit{modules}\}.$$

This is a reformulation of the unipotence result of the previous lecture, which said that all local systems could be built from constant ones.

# 3 Functor of points

## 3.1 A third interpretation

There is a third interpretation of the equality $S$ " = " Spec $\mathbb{C}^S$ via the functor of points. For this I would like to think about schemes in terms of their functor of points. If $A$ is a $\mathbb{C}$-algebra, identify Spec $A$ with the functor $\mathbb{C}$-algebras to sets which assigns to $B \rightsquigarrow \mathrm{Hom}(A, B)$.

We want to say that applied to $\mathbb{C}^S$ you get $S$, so lets compare this functor with the constant functor $B \mapsto S$, denoted $\underline{S}$. There is a natural map

$$\underline{S} \to \mathrm{Spec}\ \mathbb{C}^S$$

sending $s \in \underline{S}(B) \mapsto (\mathbb{C}^S \to \mathbb{C}^{\{s\}} \to B)$, where the second map is the algebra structure. This isn't an isomorphism; if it were, then it would be saying that any map from Spec $B$ to a finite set is constant, but actually it's only locally constant. They are almost the same, and one way to articulate this is to say that Spec $\mathbb{C}^S$ is the sheafification of $\underline{S}$ for the Zariski topology (or étale topology, or fppf, fpqc...)

Now let's go to homotopy theory. I want to talk about affine schemes in this setting. We work over a commutative ring spectrum $R$ which is $K(n)$-local. Every $R$-algebra $A$ represents a functor

$$\mathrm{Spec}\ A \colon \{R\text{-algebras}\} \to \{\text{spaces}\}$$

sending $B \mapsto \mathrm{Hom}_R(A, B)$. Now, we're working in homotopy theory so the space of maps is not just a set but a space.

*Example* 3.1. $\mu_p(R) = \mathrm{Spec}\ R[\mathbb{Z}/p\mathbb{Z}]$. As a spectrum, $R[\mathbb{Z}/p\mathbb{Z}]$ is a disjoint union of $p$ copies of $R$, with some group structure.

What about affine schemes associated to a space? For any $X$, we have a map from the constant functor

$$\underline{X} \to \mathrm{Spec}\ R^X.$$

This sends $x \in \underline{X}(B) \mapsto (R^X \to R^{\{x\}} \to B)$.

You could ask naïvely if this is an isomorphism. We don't expect this to be true, as it isn't even true in ordinary algebra. However, what was true in ordinary algebra was that the right hand side was the sheafification of the left hand side for some topology. So we could ask if we could define a Grothendieck topology on $K(n)$-local commutative rings which makes this true.

*Definition* 3.2. A map $f\colon A \to B$ of $K(n)$-local commutative rings is a *covering* if

1. The construction $M \mapsto B \wedge_A M$ (extension of scalars) preserves inverse limits.

2. $(M \wedge_A B \cong 0) \implies (M \cong 0)$.

The first condition is very strong. In normal commutative algebra, tensoring doesn't preserve inverse limits. So this is like demanding finite and flat. The second condition is like demanding faithful, so you think of this as analogous to the fppf topology.

This condition is so strong that you might think that there aren't many examples. However that's wrong.

**Theorem 3.3.** *Let $f\colon X \to Y$ be a map of $n$-truncated $p$-finite spaces. If $\pi_0 X \twoheadrightarrow \pi_0 Y$ then $f^*\colon R^Y \to R^X$ is a covering.*

Why? You have to think about what extension of scalars does. It's a functor from $R^Y$-modules to $R^X$ modules. But we know that for $n$-truncated $p$-finite spaces, these can be thought of equivalently as local systems on $Y$ and $X$, respectively. Then extension of scalars corresponds to pulling back. The theorem follows from the fact that pulling back plays well with global sections.

**Corollary 3.4.** *Let $X$ be a $p$-finite $n$-truncated space. Then* Spec $R^X$ *is the sheafification of* $\underline{X}$.

Suppose you have a $B$-valued point $R^X \to B$. We would like this to come from a $B$-valued point of the constant factor, i.e. factor through $R^{\{x\}}$. We don't expect to be able to do this on the nose; we only expect it to do it after passing a covering. If $X$ is connected, the obvious thing to do is to form the pushout

$$\begin{tikzcd} R^X \arrow[r] \arrow[d] & B \arrow[d] \\ R^{\{x\}} \arrow[r] & B \wedge_{R^X} R^{\{x\}}. \end{tikzcd}$$

If $X$ is connected, then the theorem says that the left vertical map is a covering, so the right is too. If $X$ is not conencted, we have to choose a point of each component and a covering for each.

How should you think of this?

**Slogan:** if $X$ is a $p$-finite space $n$-truncated space, then $X$ is affine.

*Example* 3.5. Let $G$ be a finite $p$-group. Then $BG$ is affine.

What does this mean? You can consider $G$-torsors, and there is a universal $G$-torsor which lives over an affine scheme. This is contrary to algebraic geometry, where $BG$ lives in the world of DM stacks and is decidedly not affine.

## 3.2 A picture of Spec $R^X$

One might ask, what does Spec $R^X$ look like? Well, what does $R^X$ look like? Let's consider the example of $X = K(\mathbb{Z}/p\mathbb{Z}, m)$. In other words, $\pi_* X$ is non-zero only in dimension $m$, where it is $\mathbb{Z}/p\mathbb{Z}$. This is the basic building block out of which all spaces are made. The Morava $K$-theories are fields, so in some sense basic ring spectra. Thus, this is a very elemental calculation.

**Theorem 3.6** (Ravenel-Wilson)**.**

$$K(n)^* X = \begin{cases} K(n)^*(\mathrm{pt}) & m > n \\ \textit{something computable} & m < n \\ K(n)^*(\mathrm{pt})[\mathbb{Z}/p\mathbb{Z}] & m = n \end{cases}$$

Actually, this requires $K(n)$ to be "sufficiently large." Otherwise, there may be a Galois twist.

I wanted to know about $R^X$ when $R$ is a commutative $K(n)$-local ring spectrum.

**Corollary 3.7.** *For $R$ sufficiently large, if $X = K(\mathbb{Z}/p\mathbb{Z}, n)$ then $R^X = R[\mathbb{Z}/p\mathbb{Z}]$.*

Sufficiently large means that $R$ has to in some sense "contain the roots of unity." Taking Spec , we find (since $X$ is a $n$-truncated $p$-finite space)

**Corollary 3.8.** *For $R$ sufficiently large, $\mu_p$ is the constant sheaf with value $K(\mathbb{Z}/p\mathbb{Z}, n)$.*

*Remark* 3.9. For large characteristic 0, we have $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$. In characteristic 0, it's not constant. We're finding something in between in these intermediate characteristics.

Let me make a more concrete statement by evaluating my functor on $R$. Then we get a space, and let's see what we learn about the homotopy groups.

**Corollary 3.10.** *For $R$ sufficiently large*

$$\pi_* \mu_p(R) \cong H^{n-*}(\mathrm{Spec}\ R; \mathbb{Z}/p\mathbb{Z}).$$

Since the condition "sufficiently large" is satisfied locally, an unconditional statement is

**Corollary 3.11.** $\pi_* \mu_p(R) \cong H^{n-*}(\mathrm{Spec}\ \mathbb{R}; \mathbb{Z}/p\mathbb{Z}(1))$.

You can read this in two ways. One is that it gives you information about the homotopy groups of $\mu_p(R)$ in terms of cohomology, but that cohomology is not easily computed because the Grothendieck topology has many (unexpected?) covers. The other way is that it gives you information about the cohomology using homotopy groups. This is analogous to étale cohomology, where many computations are done by reducing to something about Galois cohomology using the fact that the $\mu_p$ is constant. So you can think of this corollary as giving you a tool that may play you a similar role in this setting.

Maybe you didn't like our Grothendieck topology very much. We only used two things about our topology:

1. it had enough covers (a map of $n$-truncated $p$-finite spaces surjective on $\pi_0$ was a covering)
2. there aren't too many covers: the topological is subcanonical, so the functors Spec $R$ are sheaves.

*Example* 3.12. Let's see what happens when $n = 1$. We can take $R$ to be complex $K$-theory ($p$-adically completed, since we're working $K(n)$-locally).

Let $G = \mu_p(\mathbb{C})$ be the cyclic group generated by $e^{2\pi/p}$.

The standard representation $V$ of $G$ determines a local system on $BG$, which represents a class in $K^0(BG)$. This is special, because it comes from a line bundle (so it's a unit in $K$-theory), and its $p$th power is canonically trivialized. So you can think of these as a $p$th root of unity in $K$ theory. Therefore, it is represented by a map $BG \to \mu_p(R)$. This is the map that the theorem is telling you should be there: if you look $K(n = 1)$-locally, then you see a constant sheaf.

So what is this story telling you? It is telling you that with respect to the story that I've defined, the functor $\mu_p(R)$ is the constant sheaf: you should see a $K(\mathbb{Z}/p, n)$, plus maybe some other stuff that should disappear as you localize. You might think that if you work locally enough, then you get exactly $K(\mathbb{Z}/p, n)$.

### 3.3 Lubin-Tate spectra

Let $K$ be a perfet field of characteristic $p$ and $\mathbb{G}_0 \to \operatorname{Spec} k$ a formal group of height $n$ and dimension 1. Then there is a universal deformation (Lubin-Tate) $\mathbb{G} \to \operatorname{Spec} A$, and $A \cong W(k)[[v_1, \ldots, v_{n-1}]]$.

A theorem of Ladweber, Morava, Goerss-Hopkins-Miller says that there is an essentially unique even periodic cohomology thtoery attached to this situation. The theory is represented a $K(n)$-local commutative ring spectrum and is functorial.

**Slogan:** $K(n)$ is the "residue field" of $E$, a complete local field.

Then you can ask what does $\mu_p(E)$ look like?

**Theorem 3.13** (Hopkins-Lurie)**.** *If k is algebraically closed, then*

$$\mu_p(E) \cong K(\mathbb{Z}/p\mathbb{Z}, n).$$

Let's discuss some fun consequences. Suppose we wanted to talk about the entire multiplicative group, not just the roots of unity. The group $\mathbb{C}^\times$ is big:

$$0 \to \text{roots of unity} \to \mathbb{C}^\times \to \{\text{junk}\} \to 0.$$

Similarly, the ring spectrum $E$ has an associated multiplicative group $E^\times$ (which is really a topological space). In degree 0, $\pi_* E^\times$ is enormous if $* = 0$ or if $* \geq 2$ is even, and it vanishes in odd degrees. This looks ugly: everything is 0 or enormous. What we would like to do, analogously to the situation above, is cut out the uninteresting "junk."

| Sculpture | Characteristic 0 | $K(n)$-local |
|---|---|---|
| Marble | $\mathbb{C}^\times$ | $E^\times$ |
| Chisel | $\mu_p$ | $\mu_p$ |
| Angel | $\mu_{p^\infty}(\mathbb{C}^\times) \cong \mathbb{Q}_p/\mathbb{Z}_p$ | ?? |

What do we mean by the "interesting part" of $E^\times$? Define $X = \varinjlim X_\alpha$ where the direct limit is taken over all $p$-finite spaces $X_\alpha$ equipped with a map (of infinite loop spaces) to $E^\times$. Control over $\mu_p(E)$ gives us control over $X$. If you can undersatnd the $X_\alpha$, you should be able to figure out what this direct limit looks like.

**Corollary 3.14.** $\pi_* X = \begin{cases} \mathrm{Hom}(\pi^s_{n-*}, \mathbb{Q}_p/\mathbb{Z}_p) & 0 \le * \le n \\ 0 & * > n. \end{cases}$

Here $\pi^s_{n-*}$ are the stable homotopy groups of spheres. You might think that this isn't surprising, because we're doing homotopy theory anyway. But actually I think that this is rather surprising, and it's pointing to some deeper reason that we don't understand yet.