Cohomology Theories and Commutative Rings

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1 Spectra

1.1 Generalized cohomology theories

Algebraic topology is the study of algebraic spaces by the mean of algebraic variants. For example, we have the cohomology groups $H^n(X; A)$, which we abbreviate $H^n(X)$. There are many constructions of cohomology, and for sufficiently nice spaces they all give the same answers. We're always going to assume that we are in this setting (e.g. CW complexes). Motivated by this, Eilenberg and Steenrod formalized an *axiomatic* definition of cohomology:

1. Functoriality
2. Homotopy invariance
3. Multiplicativity
4. Suspension
5. Excision
6. *Dimension axiom*

Eilenberg and Steenrod proved that these axioms characterize $H^n(X; A)$. However, people discovered many cohomology-like theories that satisfied all the axioms except the last one.

Definition 1.1. We say that a sequence of functors $\{E^n\}_{n\in\mathbb{Z}}$ from spaces to abelian groups is a *generalized cohomology theory* if it satisfies axioms 1-5. Henceforth we may drop the "generalized."

Example 1.2. The set of complex vector bundles on X modulo isomorphism forms a commutative monoid under $\oplus$. The associated abelian group $K^0(X)$ is the called the complex K-theory of X.

The definition can be extended to $K^n(X)$ for all integers n and all spaces X, which satisfies 1-5 but not the dimension axiom. In fact,

$$K^n(*) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

1.2 Spectra

Theorem 1.3 (Brown representability). *Let $\{E^n\}$ be a cohomology theory. Then each E^n is representable, i.e. there are spaces $\{Z(n)\}_{n\in\mathbb{Z}}$ such that*

$$E^n(X) = \{\textit{maps from } X \textit{ into } Z(n)\}/\textit{homotopy}.$$

The suspension axioms gives homotopy quivalences $Z(n) \cong \Omega Z(n+1)$.

Definition 1.4. A *spectrum* is a a sequence of spaces $\{Z(n)\}_{n\in\mathbb{Z}}$ together with the data of homotopy equivalence $Z(n) \cong \Omega Z(n+1)$.

So a spectrum is the same as cohomology theory, but it takes a different point of view. We're used to thinking of a cohomology theory as algebraic, whereas a spectrum is something more like a topological space. In particular we can talk about maps between spectra, etc.

Example 1.5. Let A be an abelian group. Then

$$H^n(X;A) \cong \{ \text{ maps } f\colon X \to K(A,n)\}/\text{homotopy}.$$

Here $K(A,n)$ is an *Eilenberg-MacLane space* characterized by

$$\pi_* K(A,n) = \begin{cases} A & * = n \\ 0 & \text{otherwise} \end{cases}.$$

1.3 Analogies

I want to throw out some analogies. I want to think of topological spaces as *enlarging* the space of objects from algebra. We've been used to thinking of cohomology as a tool for studying spaces, but we now we can turn this around and view "topological spaces" as a tool for generalizing algebraic constructions.

Classical Algebra	Homotopy-Theoretic Algebra
Set	Space
Abelian group	Spectrum
Tensor product $\otimes$	Smash product $\wedge$
Associative ring	Associative ring spectrum
Commutative ring	Commutative ring spectrum

2 Ring Spectra

Definition 2.1. An *associative (or A_∞) ring spectrum* is a spectrum E equipped with a multiplication $E \wedge E \to E$ satisfying a suitable associative law.

Strictly speaking, what I'm defining is a "structured ring spectrum."

This subject has a reputation for being technical, I think because there are many incorrect ways to define things. This is unfair, because there are also many correct ways to define them.

Think of a spectrum as a cohomology theory, which assigns to every topological space a graded abelian group. A ring spectrum assigns a graded ring instead. The idea of a structured ring spectrum is that associativity holds at the *cocycle level*. There is also the notion of a commutative ring spectrum, or E_∞ ring spectrum.

Example 2.2. If R is associative, then HR (the usual cohomology with coefficients in R) is an associative ring specturm.

Example 2.3. If R is commutative, then HR is a commutative ring spectrum.

Example 2.4. Complex K-theory is represented bya commutative ring spectrum K. What does this mean? Complex K-theory is described by vector bundles. There is a multiplication on vector bundles given by $\otimes$, and the commutativity comes from the existence of a *canonical isomorphism* $E \otimes F \cong F \otimes E$.

This is the idea reflected by a commutive ring theory. It reflects the idea that there is a canonical commutivity at the level of the representing objects.

In E_∞, the E stands for "everything" and the ∞ emphasizes that we really meant "everything." Every diagram that you write down can be expected to commute, etc.

3 Field spectra

3.1 Fields in classical algebra

For the rest of the talk, I want to explore what happens when you take seriously the idea that commutative ring spectra are generalizations of rings.

The easiest rings are fields. A (skew) field can be described the equivalenc conditions:

1. Every non-zero element is invertible.
2. Every module is free.

What about ring spectra? If E comes from a ring, you can recover the ring by taking the cohomology of a point. Now the following conditions are equivalent:

1. Every non-zero homogeneous element of $E^*(\{x\})$ is invertible.
2. Every graded module over $E^*(\{x\})$ is free.

It follows from the second condition that every E-module (in spectra) is free. If these conditions are satisfied, then we say that E is a *cohomological field*.

Example 3.1. Let k be a (skew) field. Then Hk is a cohomological field.

Example 3.2. Complex K-theory is not a field. Indeed, $K^0(*)$ is $\mathbb{Z}$ in degree 0, which certainly has non-invertible elements. There is a fix: one can construct K/p, which is "complex K-theory modulo p" (the cofiber of the map $p\colon K \to K$). Then

$$(K/p)^*(\{x\}) \cong (\mathbb{Z}/p\mathbb{Z})[t^{\pm 1}].$$

This *is* a cohomological field.

You might think that the "field" K/p looks like $\mathbb{Z}/p$, maybe with some small twist. I want to convince you that it is very different.

3.2 Characteristic of a field

If I started a talk with "let k be a field" then you might have wondered if I was talking about a field of characteristic 0 or p. These are very different; the former conjures geometric intuition about $\mathbb{C}$, while the latter is associated with things like Frobenius, etc. So the characteristic is the most important invariant of a field.

What does it mean for two fields to have the same characteristic? Here is one characterization.

Theorem 3.3. *Let k and k' be (skew) fields. Then k and k' have the same characteristic if and only if $k \otimes k' \neq 0$.*

This motivates:

Definition 3.4. Let E and E' be cohomological fields. Then E and E' have the same characteristic if and only if $E \wedge E' \neq 0$.

We have to tell you what the smash product of spectra is. Without defining it, we'll say that the smash product has the property that if you smash E with a ring spectrum, then you get a module over that ring. Thus, if E and E' are cohomological fields then $E \wedge E'$ is a free E-module and a free E'-module. This means that we can find E as a direct summand in a sum of free E'-modules, and vice versa.

How can we tell if two cohomological fields don't have the same characteristic? In particuliar, they are cohomology theories: you feed in a topological space, and get an answer.

Let p be a prime number and X the classifying space of $G = \mathbb{Z}/p\mathbb{Z}$ (i.e. $\pi_1(X)$ is G, and all others are 0).

- If k is a field of characteristic not p, then $H^n(X;k) \cong 0$ for $n > 0$.
- If k has characteristic p then the answer looks very different: $H^n(X;k) \cong k$ for all $n \geq 0$.
- Now I want to compare this with the mod p K-theory. From every G-representation, we get a local system on X, hence a vector bundle. This construction gives a map from the representation ring of G into the complex K-theory of X.

$$(K/p)^n(X) = \begin{cases} \mathrm{Rep}(G) \otimes (\mathbb{Z}/p\mathbb{Z}) & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

These examples yield modules of ranks $1, \infty, p$ over the cohomology of a point.

Corollary 3.5. *K/p is not of the same characteristic as any ordinary field.*

This turns out to be a pretty powerful method.

3.3 A field guide to fields

Here is a quick tutorial on how to find the characteristic of a cohomological field. Let E be a cohomological field. Then $k = E^0(\{x\})$ is a classical field.

- Does k have characteristic 0? If so, then $E \sim H\mathbb{Q}$.

If not, let p be the characteristic of k and let X be the classifying space of $\mathbb{Z}/p\mathbb{Z}$.

- Does $E^*(X)$ have infinite rank over $E^*(\{x\})$? If so, then $E \sim H\mathbb{F}_p$.

- If not, then one can show that rank $E^*(X)$ over $E^*(\{x\})$ is p^n for $0 < n < \infty$. Then we say that E has *height n*.

Remark 3.6. If two have the same characteristic, then one is a direct sum of shifts of another.

Theorem 3.7. *For every prime p and every positive n, there exists a cohomological field E of height n such that $E^0(\{x\})$ has characteristic p. All such fields are of the same characteristic.*

This cohomological field is called the nth *Morava K-theory* and denoted by $K(n)$.

Remark 3.8. Since this is only up to characteristic, you can get any feild by taking cohomology of a point.

Morava proved the existence part. The uniqueness is a consequence of the nilpotence theorem of Ravenel-Hopkins-Smith.

Example 3.9. $K(1) = K/p$. That's why these are called K-theories.

Caveats.

- One can extend the definition of $K(n)$ to $0 \leq n \leq \infty$ by setting $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$.

- The Morava K-theories $K(n)$ are associative ring spectra. They are *not* commutative for $0 < n < \infty$.

- The $K(n)$ are equivalence classes; only the characteristic of $K(n)$ is well-defined. (The literature is not uniformon this point).

3.4 Summary

For a fixed prime p, we introduced an infinite family of cohomological fields

$$K(1) \quad K(2) \quad K(3) \quad \ldots$$

These interpolate between fields of characteristic zero ($K(0) \cong H\mathbb{Q}$) and fields of characteristic p ($K(\infty) \cong H\mathbb{F}_p$).

In the remaining lectures, we'll talk about what life looks like in these "intermediate fields." In particular, what does algebra look like?

In the second lecture, we'll discuss the behiavor of represntation theory over these fields of "intermediate characteristic." For instance, is it more like characteristic 0, or is it more like characteristic p?

In the third lecture, we'll study some rudimentary algebraic geometry in these settings, focusing in particular on roots of unity.