VIRASORO ALGEBRA AND UNIVERSAL FACTORIZATION ALGEBRAS (NOV 20, 2020)

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Contents

Let k be a field of characteristic 0. Previously we had discussed the notion of factorization algebra on a smooth algebraic curve X. This was a sequence of quasicoherent sheaves on powers of X. In particular, for each $x \in X$, we get a vector space $\mathcal{A}_x$ on X. The idea we want to articulate in this talk is that $\mathcal{A}_x$ depends only on $\widehat{X}_x = \mathrm{Spf}(k[[t]])$, and not on the global structure of X.

1. Strictly positive part of the Virasoro algebra

1.1. Automorphisms of the disk.

Definition 1.1. Let R be a commutative ring. The *standard formal disk* over R is $\widehat{D}_R := \mathrm{Spf}\, R[[x]]$. There is an obvious zero section $\mathrm{Spec}\, R \to \mathrm{Spf}\, R[[x]]$.

We will study the group of automorphisms of $\widehat{D} := \mathrm{Spf}\, k[[x]]$ which fix the basepoint.

Definition 1.2. Let $\mathrm{Aut}_*(\widehat{D})$ be the automorphism group of $\widehat{D}$ as a pointed formal scheme. This is the group scheme with R-valued points being automorphisms of $\mathrm{Spf}(R[[x]])$ preserving the locus where $x = 0$, as a formal scheme over R.

What does $\mathrm{Aut}_*(\widehat{D})$ look like? An automorphism of $\mathrm{Spf}(R[[x]])$ is (tautologically) the same as an automorphism of $R[[x]]$ which preserves the topology, but that is automatic under the additional condition that it preserves the ideal (x).

Any such automorphism has the form $x \mapsto f(x) = \sum a_n x^n \in R[[x]]$. The condition that it preserves (x), and is an automorphism, is equivalent to $a_0 = 0$ and $a_1 \in R^\times$. So

$$\mathrm{Aut}_*(\widehat{D})(R) = \left\{ \sum a_n x^n : a_0 = 0, a_1 \in R^\times \right\}.$$

Hence $\mathrm{Aut}_*(\widehat{D}) = \mathrm{Spec}\,(k[a_1^{\pm}, a_2, a_3, \ldots])$. This is a pro-algebraic group over k.

What is the structure of $\mathrm{Aut}_*(\widehat{D})$? We have a map

$$\mathrm{Aut}_*(\widehat{D}) \twoheadrightarrow \mathbf{G}_m$$

sending $f(x) \mapsto \frac{\partial f}{\partial x}|_{x=0} = a_1$.

The kernel is a pro-unipotent group scheme that we'll call U. It has a filtration with subquotients being $\mathbf{G}_a$. So all in all, we have an extension

$$1 \to \underbrace{U}_{\text{pro-unipotent}} \to \mathrm{Aut}_*(\widehat{D}) \to G_m \to 1.$$

1.2. **Lie algebra.** Recall that we are in characteristic 0. Let's consider $\mathrm{Lie}(\mathrm{Aut}_*(\widehat{D}))$. By definition, this is

$$\mathrm{Lie}(\mathrm{Aut}_*(\widehat{D})) := \ker(\mathrm{Aut}_*(\widehat{D})(k[\epsilon]/(\epsilon^2)) \to \mathrm{Aut}_*(\widehat{D})(k)).$$

We will call this Vir_{++}, for "strictly positive part of the Virasoro algebra". This is an inverse limit of Lie algebras. Informally, it is "vector fields on $\widehat{D}$ which vanish at the origin". Such a vector field can be written as a formal expression

$$\sum_{n>0} b_n x^n \frac{\partial}{\partial x}.$$

So Vir_{++} has a topological basis given by vector fields $\{L_n = -x^{n+1}\frac{\partial}{\partial x}\}_{n\geq 0}$. (This is "half" the Virasoro algebra.)

Exercise 1.3. Show that the Lie bracket on Vir_{++} is given by $[L_m, L_n] = (m-n)L_{m+n}$.

Because we're in characteristic 0, there is a fully faithfully embedding from continuous representations of $\mathrm{Aut}_*(\widehat{D})$ to continuous representations of Vir_{++}. Let's explicate what this means. Continuity for a representation of Vir_{++} means that

(A) For all $v \in V$, $L_m v = 0$ for $m \gg 0$.

What representations are we missing? We had an exact sequence.

$$0 \to U \to \mathrm{Aut}_*(\widehat{D}) \to \mathbf{G}_m \to 0.$$

Hence at the level of Lie algebras, we get an exact sequence

$$0 \to \langle L_1, L_2, \ldots \rangle \to \mathrm{Vir}_{++} \to \langle L_0 \rangle \to 0.$$

We have a splitting $\mathrm{Aut}_x(\widehat{D}) = \mathbf{G}_m \rtimes U$. That gives a splitting $\mathrm{Vir}_{++} = \langle L_0, L_1, L_2, \ldots \rangle$.

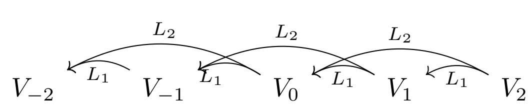
The essential image of $\mathrm{Rep}\,\mathrm{Aut}_*(\widehat{D}) \to \mathrm{Rep}\,\mathrm{Vir}_{++}$ are the "integrable" representations. For $\mathbf{G}_m$, this means

(C) The action of L_0 is diagonalizable with integer eigenvalues.

For L_n, it means

(B) Each L_n acts locally nilpotently for $n > 0$.

If V is an integrable representation, then $V = \bigoplus_{n\in\mathbf{Z}} V_n$ with L_0 acting by n on V_n. Recalling that $[L_0, L_n] = -nL_n$, this implies that L_n carries each V_i into V_{i-n}.



Remark 1.4. Conditions (A) and (B) are automatic if $V_n = 0$ for $n << 0$ (but not conversely), which is a condition that often arises in practice.

Example 1.5. Let's classify irreducible integrable representations of Vir_{++}. Any irreducible representation is semisimple, hence restricts to a semisimple representation on the pro-unipotent radical. Hence it is trivial. So the irreducible representations of Vir_{++} are all inflated from $\mathbf{G}_m$.

1.3. Moving disks.

Definition 1.6. Let R be a k-algebra. A *(pointed) formal disk over* R is a formal R-scheme $\mathcal{X}$ with a point $\eta\colon \mathrm{Spec}\,(R) \to \mathcal{X}$ which is (Zariski/étale/fpqc) locally isomorphic to $(\mathrm{Spf}\, R[[x]], \text{zero section})$.

Every such object has the form $\mathrm{Spf}(\prod_{n\geq 0} L^{\otimes n})$, where L is an invertible R-module. Geometrically, L corresponds to some line bundle $\mathcal{L}^{-1} \to \mathrm{Spec}\ R$, and the pointed formal disk is the formal completion along a section.

In other words, a pointed formal disk over Spec R is an $\mathrm{Aut}_x(\widehat{D})$-torsor over Spec R, which is classified by $H^1_{\text{ét}}(\mathrm{Spec}\ R, \mathrm{Aut}_x(\widehat{D}))$. Note that we get the same notion for any of the usual topologies, using the structure of $\mathrm{Aut}_x(\widehat{D})$ as an extension of $\mathbf{G}_m$'s and $\mathbf{G}_a$'s and the corresponding fact for those groups.

2. Localization from $\mathrm{Rep}(\mathrm{Vir}_{++})$ to universal quasicoherent sheaves

Let X be an algebraic curve over k. Take the formal completion of $X \times_k X$ along the diagonal divisor $X \hookrightarrow X \times_k X$, and call it $\mathcal{X}$.

The projections give this the structure

$$\begin{array}{ccc} & \mathcal{X} & \\ \swarrow & & \searrow \\ X & & X \end{array}$$

making $\mathcal{X}$ a formal disk over X, pointed by the diagonal section.

This construction supplies an $\mathrm{Aut}_*(\widehat{D})$-torsor over the curve X. Twisting by this torsor gives a ("localization") functor $\mathrm{Rep}(\mathrm{Aut}_*(\widehat{D})) \to \mathrm{QCoh}(X)$. It is in fact a tensor functor, and the tensor functor allows to recover the torsor.

The category $\mathrm{Rep}(\mathrm{Aut}_*(\widehat{D}))$ has nothing to do with X. So it deserves to be thought of as "the quasicoherent sheaves which live naturally on all X's".

Example 2.1. Take k, thought of as the trivial representation of $\mathrm{Aut}_*(\widehat{D})$. It is the unit, so its image is $\mathcal{O}_X$.

Example 2.2. Let k be the irreducible (hence 1-dimensional) representation of $\mathrm{Aut}_*(\widehat{D})$ where L_0 acts by n. This is sent to $\Omega_X^{\otimes -n}$.

Example 2.3. Let V be the 2-dimensional representation

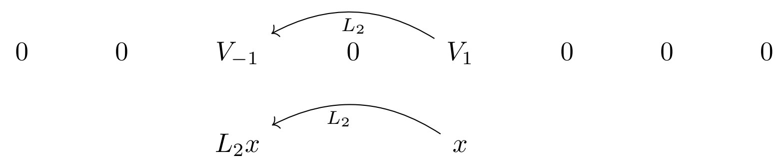
$$0 \qquad 0 \qquad 0 \qquad V_0 \xleftarrow{L_1} V_1 \qquad 0 \qquad 0 \qquad 0$$

$$L_1 x \xleftarrow{L_1} x$$

This gives a quasicoherent sheaf on X. It is an extension $0 \to \mathcal{O}_X \to ? \to T_X \to 0$. We claim that the extension is $\mathcal{D}_X^{\leq 1}$ as a right $\mathcal{O}_X$-module. (Note that it splits canonically as a left $\mathcal{O}_X$-module.)

These examples fit with the philosophy that the image of localization are quasicoherent sheaves that one can write down canonically on any curve.

Exercise 2.4. There is a representation



What quasicoherent sheaf does it correspond to?

2.1. **Summary and what's next.** We introduced Vir_{++}, the topological Lie algebra generated by $\{L_n\}_{n\geq 0}$. We saw that integrable representations of Vir_+ correspond to "quasicoherent sheaves that live naturally on any algebraic curve".

We are eventually going to introduce an object which will similarly correspond to "factorization algebras that live naturally on any curve".

Recall that for a unital factorization algebra, the underlying quasicoherent sheaf has a connection. We will first write down the thing which corresponds to "vector bundles with connection (i.e. $\mathcal{D}$-modules) that live naturally on any curve".

3. Positive part of the Virasoro algebra

We will proceed as before, but now without the basepoint.

We consider $\mathrm{Aut}(\widehat{D}) \supset \mathrm{Aut}_*(\widehat{D})$. We have $\mathrm{Aut}(\widehat{D})(R)$ are automorphisms of $\mathrm{Spf}(R[[x]])$ as a formal scheme, i.e. continuous automorphisms of $R[[x]]$. These are given by a power series $f(x) = \sum a_n x^n$, such that a_0 is nilpotent (continuity), and $a_1 \in R^\times$ (automorphism).

Warning 3.1. $\mathrm{Aut}(\widehat{D})$ is not a scheme. But it is a formal scheme. We have $\mathrm{Aut}(\widehat{D}) = \mathrm{Spf}(k[a_1^{\pm 1}, a_2, \ldots][[a_0]])$. More precisely, it is an affine formal group scheme.

Again we can think about the Lie algebra of $\mathrm{Aut}(\widehat{D})$. It is

$$\mathrm{Vir}_+ := \mathrm{Lie}(\mathrm{Aut}(\widehat{D})) = \ker(\mathrm{Aut}(\widehat{D})(k[\epsilon]/\epsilon^2) \to \mathrm{Aut}(\widehat{D})(k)).$$

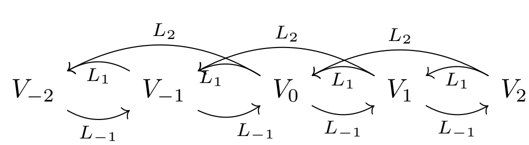
Intuitively, these are all vector fields on $\widehat{D}$ (not just the ones vanishing at the origin). These are expressions of the form $\sum_{n\geq 0} b_n x^n \frac{\partial}{\partial x^n}$.

Now, Vir_+ has a topological basis given by $\{L_n = -x^{n+1}\frac{\partial}{\partial x}\}_{n\geq -1}$. The Lie bracket is given by $[L_m, L_n] = (m-n)L_{m+n}$. Note although there is now an L_{-1}, the bracket doesn't allow to generate L_{-2}.

As in the previous hour, we can consider representations of $\mathrm{Aut}(\widehat{D})$. These are the integrable representations of Vir_+. This is the same as integrable representations of Vir_{++}.

(A) For all $v \in V$, $L_m v = 0$ for $m \gg 0$.
(B) Each L_n acts locally nilpotently for $n > 0$.
(C) The action of L_0 is diagonalizable with integer eigenvalues.

Note that we are *not* imposing any condition on L_{-1}.



What do representations look like?

Example 3.2. Let the trivial representation k of Vir_{++} extends to the trivial representation of Vir_{+}. However, none of the other irreducible representations of Vir_{++} extend to a representation of Vir_{+}.

Example 3.3. Consider induced representations $V \otimes_{U(\mathrm{Vir}_{++})} U(\mathrm{Vir}_{+})$. As a vector space, they are $V \oplus L_{-1}V \oplus L_{-1}^2 V \oplus \dots$.

4. Localization from $\mathrm{Rep}(\mathrm{Vir}_{+})$ to universal $\mathcal{D}$-modules

Definition 4.1. A *formal disk* over R is a formal R-scheme which is locally isomorphic to $\mathrm{Spf}(R[[x]])$. If k has characteristic 0, then it doesn't matter which topology we mean here. But if k has characteristic p, then it might matter.

Example 4.2. Suppose k has characteristic p, and $t \in k$ has no pth root. We have $\mathrm{Spf}(\widehat{k[x]}_{(x^p-t)})$. Then fpqc locally it is a formal disk, but Zariski locally it is not (as the residue field got enlarged).

Since we are in characteristic 0, we can ignore these subtleties. Formal disks over R are equivalent to $\mathrm{Aut}(\widehat{D})$-torsors over $\mathrm{Spec}\ R$.

Example 4.3. Let X be an algebraic curve. Let $\mathcal{X}$ be the formal completion of $X \times X$ along the diagonal. Either projection map to X gives $\mathcal{X}$ the structure of a formal disk over X.

Definition 4.4 (Simpson-Teleman/Grothendieck)**.** Let X be a smooth k-scheme. The *de Rham space* X_{dR} of X is the functor $R \mapsto X(R^{\mathrm{red}})$. This is a surjection if X is smooth.

If X is smooth, we have $X \to X_{\mathrm{dR}}$. The formal disk $\mathcal{X} \to X$ is the pullback of an object over X_{dR}.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_{\mathrm{dR}} \end{array}$$

Lemma 4.5. *"X is a formal disk over X_{dR}".*

What does this mean precisely? Given an R-valued point of X_{dR}, we can form $X \times_{X_{\mathrm{dR}}} \mathrm{Spec}\ R$. Claim: it is representable by a formal disk over R. Proof: by smoothness $\mathrm{Spec}\ R \to X_{\mathrm{dR}}$ lifts to X, i.e. $\mathrm{Spec}\ R \to X_{\mathrm{dR}}$ factors through X. So $X \times_{X_{\mathrm{dR}}} \mathrm{Spec}\ R = (\underbrace{X \times_{X_{\mathrm{dR}}} X}_{\mathcal{X}}) \times_X \mathrm{Spec}\ R$.

So $X \twoheadrightarrow X_{\mathrm{dR}}$ can be viewed as an $\mathrm{Aut}(\widehat{D})$-torsor over X_{dR}. We thus have a localization functor from representations of $\mathrm{Aut}(\widehat{D})$ to quasicoherent sheaves on X_{dR}, which is Grothendieck's conception of quasicoherent sheaves on X equipped with a (flat) connection.

Heuristic: integrable representations of Vir_{+} are "D-modules that live functorially on any curve".

Example 4.6. Let k have trivial Vir_{+}-action. This gives $\mathcal{O}_X$ with the usual connection.

Example 4.7. Let V be an integrable representation of Vir_{++}. What D-module corresponds to $W = \mathrm{Ind}_{\mathrm{Vir}_{++}}^{\mathrm{Vir}_{+}}(V)$? We have quasicoherent sheaves $\mathcal{E}_V$ and $\mathcal{E}_W$, where now $\mathcal{E}_W$ has a connection. There is a map $V \to W$ as represention[s] of Vir_{++}, so we have an $\mathcal{O}_X$-module map $\mathcal{E}_V \to \mathcal{E}_W$. So we have a map of $\mathcal{D}_X$-modules $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}_V \to \mathcal{E}_W$. We claim that it is an isomorphism.

We define a filtration $W = W_{\leq n} = V \oplus L_{-1}V \oplus L_1^2 V \oplus \ldots \oplus L_{-1}^n V$. We have

$$0 \to W_{\leq n-1} \to W_{\leq n} \to L_{-1}^n V \to 0.$$

This is an exact sequence of Vir_{++}-representations. We claim that it localizes to

$$0 \to \mathcal{D}_X^{\leq n-1} \otimes_{\mathcal{O}_X} \mathcal{E}_V \to \mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{E}_V \to T_X^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{E}_V \to 0$$

This will help to check the claim in Example 4.7.

Example 4.8. Let $X = \mathbb{A}^1$. Note that by functoriality, $V \mapsto \mathcal{E}_V$ produces translation-invariant $\mathcal{O}$-modules on $\mathbb{A}^1$ and translation-invariant $\mathcal{D}$-modules on $\mathbb{A}^1$.

Indeed, the formal completion $(\mathbb{A}^1 \times \mathbb{A}^1)^\wedge_\Delta$ is the standard formal disk on $\mathbb{A}^1$. We have $\mathbb{A}^1 \times \mathbb{A}^1 = \mathrm{Spec}\; k[x,y] = \mathrm{Spec}\; k[x, y-x]$. So we can write this as $\mathrm{Spf}(k[x][[y-x]])$, and notice that this is translation invariant on $\mathbb{A}^1$ (translating x, y both by δ doesn't do anything).

So there is a functor from representations of Vir_{++} to quasicoherent sheaves on $\mathbb{A}^1$, sending $V \mapsto V \otimes_k \mathcal{O}_X$. That is, this construction produces translation invariant quasicoherent sheaves on $\mathbb{A}^1$, which is identified with Vect (the category of vector spaces). In this case the localization is the forgetful functor.

Representations of Vir_+ produce translation-invariant $\mathcal{D}$-modules on $\mathbb{A}^1$. This is equivalent to the category of k-vector spaces with an endomorphism T. As an $\mathcal{O}$-modules, it is $V \otimes \mathcal{O}_X$. The operator L_{-1} is T.

4.1. **Summary and what's next.** We discussed that representations of $\mathrm{Aut}_*(\widehat{D})$ are continuous integrable reps of Vir_{++}, which can be interpreted as "quasicoherent sheaves living on every algeraic curve X", e.g. $X \mapsto \omega_X^{\otimes n}$.

We discussed that reps of $\mathrm{Aut}(\widehat{D})$ are continuous integrable reps of Vir_+, which can be interpreted as "D-modules living on every algebraic curve X", e.g. $X \mapsto \mathcal{D}_X$.

What if we want representations of something which can be interpreted as "factorization algebras living on every curve"?

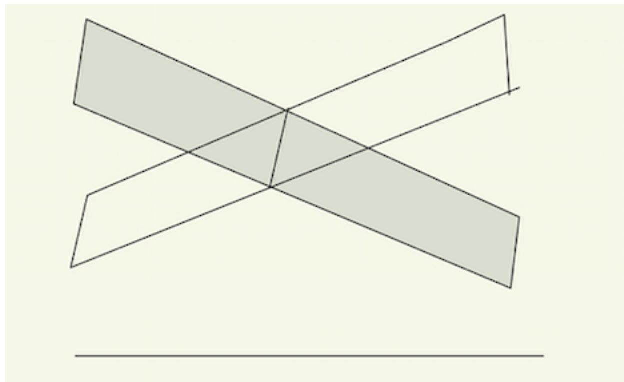
5. Formal multidisks

Definition 5.1. Let R be a $\mathbf{C}$-algebra. A *formal multidisk* over R is a formal R-scheme X which looks (locally) like the formal completion of $\mathbb{A}^1_R = \mathrm{Spec}\; R[x]$ along a divisor which is finite flat over R.

Example 5.2. We could have replaced $\mathbb{A}^1$ by any smooth algebraic curve. If X is any smooth curve, $D \subset X \times \mathrm{Spec}\; R$ a divisor, $\widehat{X}_D$ is a formal multidisk over R.

In particular, any formal disk $\mathrm{Spf}\, R[[x]]$ is a formal multidisk, as it is such a formal completion along a section. Any disjoint union of formal multidisks is a formal multidisk.

If $R = \mathbf{C}$, then all formal multidisks are disjoint unions of $\mathrm{Spf}\; \mathbf{C}[[x]]$. In general, a formal disk over Spec R is a family of such objects, but the point is that the number of connected components is not constant.

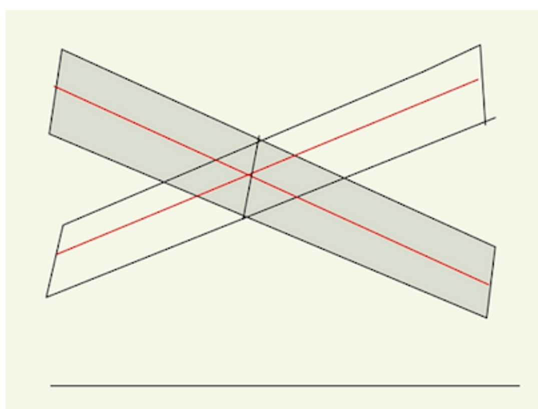


Definition 5.3. Let I be a finite set. Then an I*-pointed* formal multidisk (over R) is a pair $(\mathcal{X}, f)$ where $\mathcal{X}$ is a formal multidisk over R, and $f: I \to \mathcal{X}(R)$ such that

$$\coprod_I \operatorname{Spec} R \to \mathcal{X}$$

is set-theoretically surjective.

Example 5.4. Let X be a smooth algebraic curve (over $\mathbf{C}$). Let $f: I \to X(R)$ and $\mathcal{X}$ be the formal completion of $X \times \operatorname{Spec} R$ along $D = \sum_{i \in I} \operatorname{Im}(f(i))$. This is an I-pointed formal multidisk.



Definition 5.5. Let MDisk^I be the "moduli stack of I-pointed formal multidisks", so $\mathrm{MDisk}^I(R)$ is the groupoid of I-pointed formal multidisks over R.

Let X be a non-empty algebraic curve over $\mathbf{C}$. The previous construction gives a map $X^I \to \mathrm{MDisk}^I$. The LHS is a scheme, and the RHS is a stack. We claim that this map is a pro-smooth surjection.

Indeed, a map $\operatorname{Spec} \mathbf{C} \to \mathrm{MDisk}^I$ amounts to choosing $(\mathcal{X}, I \to \mathcal{X}(\mathbf{C}))$. Consider the fibered product

$$\begin{array}{ccc} & & \operatorname{Spec} \mathbf{C} \\ & & \downarrow \\ X^I & \longrightarrow & \mathrm{MDisk}^I \end{array}$$

The fiber over X^I parametrizes embeddings $\mathcal{X} \hookrightarrow X$. What is this? It's open in the space of all maps $\mathcal{X} \to X$, which is some jet space, affine if X is affine, and you can write it as an inverse limit of smooth varieties along smooth transition maps.

Example 5.6. Jets from $\mathcal{X} = \operatorname{Spf} \mathbf{C}[[x]]$ into $X = \mathbb{A}^1$ are parametrized by the coefficients of the power series, so $\mathbb{A}^\infty$. The embeddings are the subspace where the linear term is non-zero.

So the upper horizontal arrow in the diagram below is flat surjective, affine if X is affine.

$$\begin{array}{ccc} \operatorname{Inj}(X, \mathcal{X}) & \longrightarrow & \operatorname{Spec} \mathbf{C} \\ \downarrow & & \downarrow \\ X^I & \longrightarrow & \operatorname{MDisk}^I \end{array}$$

Definition 5.7. A *flat quasi-coherent sheaf* $\mathcal{F}$ *on* MDisk^I is a rule which assigns to each $(\mathcal{X}, f) \in \operatorname{MDisk}^I(R)$ a flat R-module $\mathcal{F}_{(\mathcal{X},f)}$ functorial in R and in $(\mathcal{X}, f)$.

The fact that MDisk^I is close to being an algebraic stack means that the notion of quasi-coherent sheaf is reasonably behaved.

6. Universal factorization algebras

Definition 6.1. A *universal non-unital factorization algebra* is a rule which associates to every I-pointed formal multidisk $\mathcal{X}$ over any R a flat R-module $\mathcal{A}(\mathcal{X}, I \to \mathcal{X}(R))$ such that

(1) It is functorial in R:
$$\mathcal{A}(\mathcal{X}, I \to \mathcal{X}(R)) \otimes_R S \cong \mathcal{A}(\mathcal{X}, I \to \mathcal{X}(S)).$$
(2) It is functorial for isomorphisms in $\mathcal{X}$: any $\mathcal{X} \xrightarrow{\sim} \mathcal{Y}$ gives
$$\mathcal{A}(\mathcal{X}, I) \cong \mathcal{A}(\mathcal{Y}, I).$$
(3) For every $I \twoheadrightarrow J \to \mathcal{X}(R)$, we get an isomorphism $\mathcal{A}(\mathcal{X}, I) \xrightarrow{\sim} \mathcal{A}(\mathcal{X}, J)$ (i.e. $\mathcal{A}$ only depends on the image of I in $\mathcal{X}(R)$)
(4) (Factorization) $\mathcal{A}(\coprod_{j \in J} \mathcal{X}_j, \coprod_{j \in J} I_j) \cong \bigotimes_{j \in J} \mathcal{A}(\mathcal{X}_j, I_j)$.

Note that if X is an algebraic curve over k, any universal factorization algebra $\mathcal{A}$ determines a factorization $\mathcal{A}_X$ on X, namely for $f \colon I \to X(R)$ we set
$$\mathcal{A}_{X^I} := \mathcal{A}((X \times \operatorname{Spec} R)^\wedge_{\operatorname{Im}(f)}, I).$$

Also, if $\mathcal{A}$ is a universal factorization algebra, then $\mathcal{A}$ determines a quasicoherent sheaf on $\mathcal{A}^I$ on MDisk^I for all I.

Example 6.2. Note that $\operatorname{MDisk}^{\{1\}} = B \operatorname{Aut}_*(\widehat{D})$. So quasi-coherent sheaves on $\operatorname{MDisk}^{\{1\}}$ are identified with $\operatorname{Rep}(\operatorname{Aut}_*(\widehat{D}))$, i.e. integrable representations of Vir_{++}.

Remark 6.3. Let A be an associative algebra over $\mathbf{C}$ and $\mathfrak{g}$ is a Lie algebra on $\mathbf{C}$. There are two notions of "$\mathfrak{g}$ acting on A".

One: give a map of Lie algebra $\mathfrak{g} \to \operatorname{Der}(A)$. This is the infinitesimal version of a group acting on A.

Or, you could give a map of Lie algebras $\mathfrak{g} \to A$, i.e. an associative algebra map $U(\mathfrak{g}) \to A$.

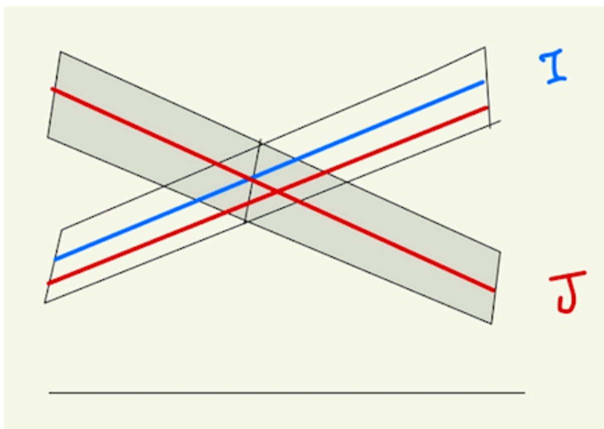
We are talking about the first type. The action of the full Virasoro algebra is like the second type.

7. Variants of universal factorization algebras

Now we discuss the *unital* versions of universal factorization algebras. Let R be a **C**-algebra. Recall that a *formal multidisk* over R is a formal R-scheme $\mathcal{X}$ that looks locally like the formal completion of $\mathbb{A}^1 \times \operatorname{Spec} R$ along a divisor.

Definition 7.1 (Universal factorization algebra, version 1)**.** A *universal factorization algebra* is a rule which assigns to each **C**-algebra R plus a formal multidisk $\mathcal{X}/R$ plus a finite set $I \to \mathcal{X}(R)$ which is "topologically surjective" (i.e. $\coprod_I \operatorname{Spec} R \to \mathcal{X}$ is a set-theoretic surjection) a flat R-module $\mathcal{A}_{\mathcal{X}}$ which is

(1) functorial in R:
$$\mathcal{A}_{\mathcal{X} \times_R S} \cong S \otimes_R \mathcal{A}_{\mathcal{X}}$$
(2) factorizes:
$$\mathcal{A}_{\mathcal{X} \coprod \mathcal{Y}} = \mathcal{A}_{\mathcal{X}} \otimes_R \mathcal{A}_{\mathcal{Y}}.$$
(3) (unitality) functorial in I: suppose we have $I \to J \to \mathcal{X}(R)$. The map from J could be surjective, but the map from I not.



Then we have a map of formal multidisks
$$\begin{array}{ccc} I & \longrightarrow & J \\ \downarrow & & \downarrow \\ \mathcal{X}^{\wedge}_{\operatorname{Im}(I)} & \hookrightarrow & \mathcal{X} \end{array}$$
We ask for a map $\mathcal{A}_{\mathcal{X}^{\wedge}_{\operatorname{Im}(I)}} \to \mathcal{A}_{\mathcal{X}}$, which is an isomorphism if $I \to J$ is surjective.

(4) Finally, we say that a bunch of diagrams commute.

Example 7.2. $\emptyset$ is a formal multidisk. The factorization property will imply $\mathcal{A}_{\emptyset} \cong R$ as a degenerate case.

For any I, we have a map $\emptyset \to I$ and a diagram
$$\begin{array}{ccc} \emptyset & \longrightarrow & I \\ \downarrow & & \downarrow \\ \emptyset(R) & \longrightarrow & \mathcal{X}(R) \end{array}$$
This gives a section $R \to \mathcal{A}_{\mathcal{X}}$.

As we said, a factorization algebra with a unit is automatically equipped with a connection. How would we build a connection into this story?

Definition 7.3. Let $\mathcal{X}$ be a formal multidisk over R and I a finite set. A *weak I-pointing* of $\mathcal{X}$ is a map $I \to \mathcal{X}(R^{\mathrm{red}})$ which is "topologically surjective." (Informally, $\mathcal{X}(R^{\mathrm{red}})$ is the quotient of $\mathcal{X}(R)$ up to the equivalence of being "infinitesimally close".)

In the previous lecture we introduced $\mathrm{MDisk}^I(R)$, the category of I-pointed formal multidisks over $\operatorname{Spec} R$. We can analogously define $\mathrm{MDisk}^I_{\mathrm{wk}}(R)$, the category of weakly I-pointed formal multidisks over $\operatorname{Spec} R$

A way to get points of MDisk^I was to start with a smooth algebraic curve and formally complete it at a divisor. This gives a map $X^I \to \mathrm{MDisk}^I$, which we argued was faithfully flat (even pro-smooth), affine if X is affine. The diagram below is cartesian.

$$\begin{array}{ccc} X^I & \twoheadrightarrow & \mathrm{MDisk}^I \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ X^I_{\mathrm{dR}} & \longrightarrow & \mathrm{MDisk}^I_{\mathrm{wk}} \end{array}$$

Moreover we know that $X^I \twoheadrightarrow X^I_{\mathrm{dR}}$ is flat, by Lemma 4.5. Hence we formally deduce that $X^I_{\mathrm{dR}} \twoheadrightarrow \mathrm{MDisk}^I_{\mathrm{wk}}$ is faithfully flat, affine if X is affine.

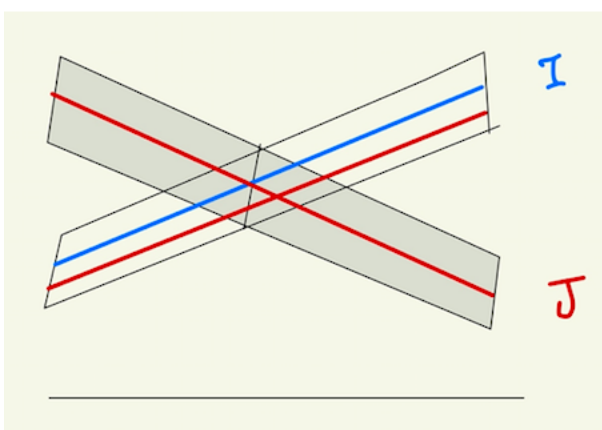
Example 7.4. For $I = \{1\}$, $\mathrm{MDisk}^I = B\operatorname{Aut}_*(\widehat{D})$. Analogously, $\mathrm{MDisk}^I_{\mathrm{wk}} = B\operatorname{Aut}(\widehat{D})$. Note that $\operatorname{Aut}(\widehat{D})$ is a group object in formal schemes.

Definition 7.5 (Universal factorization algebra, version 2)**.** A *universal factorization algebra* is a rule which assigns to each $\mathbf{C}$-algebra R plus a formal multidisk $\mathcal{X}/R$ plus a weak I-pointing $I \to \mathcal{X}(R^{\mathrm{red}})$ which is "topologically surjective", a flat R-module $\mathcal{A}_{\mathcal{X}}$ which is

(1) functorial in R:
$$\mathcal{A}_{\mathcal{X}\times_R S} \cong S \otimes_R \mathcal{A}_{\mathcal{X}}$$

(2) factorizes:
$$\mathcal{A}_{\mathcal{X} \coprod \mathcal{Y}} = \mathcal{A}_{\mathcal{X}} \otimes_R \mathcal{A}_{\mathcal{Y}}.$$

(3) (unitality) functorial in I: suppose we have $I \to J \to \mathcal{X}(R)$. The map from J could be surjective, but the map from I not.



Then we have a map of formal multidisks

$$\begin{array}{ccc} I & \longrightarrow & J \\ \downarrow & & \downarrow \\ \mathcal{X}^{\wedge}_{\mathrm{Im}\,(I)} & \hookrightarrow & \mathcal{X} \end{array}$$

We ask for a map $\mathcal{A}_{\mathcal{X}^{\wedge}_{\mathrm{Im}\,(I)}} \to \mathcal{A}_{\mathcal{X}}$, which is an isomorphism if $I \to J$ is surjective.

(4) Finally, we say that a bunch of diagrams commute.

Recall that a representation of $\mathrm{Aut}_*(\widehat{D})$ gave a universal quasicoherent sheaf, and a representation of $\mathrm{Aut}(\widehat{D})$ gave a univeral D-module. A universal factorization algebra in the second version pulls back to a $\mathcal{D}$-module on X^I for any curve X. Indeed, in Grothendieck's point of view a $\mathcal{D}$-module on X^I is a quasicoherent sheaf on X^I_{dR}, and you can just pull back to X^I_{dR}.

Note that in the world of factorization algebras on a curve, once we talk about unital factorization algebras it doesn't matter whether you talk about $\mathcal{O}$-modules or $\mathcal{D}$-modules. This statement has an analog here:

Lemma 7.6. *Definitions 1 and 2 are equivalent.*

Proof. Clearly, any universal factorization algebra A in the sense of Definition 2 determines a universal factorization algebra in the sense of Definition 1. Conversely, we need to show that if $\mathcal{A}$ is a UFA in the sense of Definition 1, it canonically promotes to a UFA in the sense of Definition 2. Let R be a $\mathbf{C}$-algebra, $\mathcal{X}$ a formal multidisk over R. Suppose $\varphi\colon I \to \mathcal{X}(R^{\mathrm{red}})$ is topologically surjective. We want a flat R-module $\mathcal{A}_{\mathcal{X},I}$. Since X is smooth, $\mathcal{X}(R) \twoheadrightarrow \mathcal{X}(R^{\mathrm{red}})$. So we can refine the weak I-pointing to an I-pointing. That is, we can find some $\widetilde{\varphi}\colon I \to \mathcal{X}(R^{\mathrm{red}})$.

$$\begin{array}{ccc} I & \xrightarrow{\quad\varphi\quad} & \mathcal{X}(R^{\mathrm{red}}) \\ & {\scriptstyle\widetilde{\varphi}}\searrow \quad \nearrow & \\ & \mathcal{X}(R) & \end{array}$$

Try taking $\mathcal{A}_{\mathcal{X},\varphi} = \mathcal{A}_{\mathcal{X},\widetilde{\varphi}}$. Is it well-defined? Suppose you had two different lifts $\widetilde{\varphi}$ and $\widetilde{\varphi}'$. Then the unit gives

$$\mathcal{A}_{\widetilde{\varphi}} \to \mathcal{A}_{X,\widetilde{\varphi}\coprod\widetilde{\varphi}'} \leftarrow \mathcal{A}_{\mathcal{X},\widetilde{\varphi}'}$$

We claim that both maps are isomorphisms. Now it suffices to check this claim when $\widetilde{\varphi} = \widetilde{\varphi}'$. In that case the map $I \coprod I \to \mathcal{X}(R)$ factors through $\widetilde{\varphi}\colon I \to \mathcal{X}(R)$. That gives a section of the maps, which is an isomorphism by axiom (3).

$$\begin{array}{ccccc} \mathcal{A}_{\widetilde{\varphi}} & \longrightarrow & \mathcal{A}_{\widetilde{\varphi}\sqcup\widetilde{\varphi}'} & \longleftarrow & \mathcal{A}_{\widetilde{\varphi}} \\ & \searrow & \downarrow{\scriptstyle\sim} & \swarrow & \\ & & \mathcal{A}_{\widetilde{\varphi}} & & \end{array}$$

∎

Remark 7.7. There's a more general statement. Let A be a UFA in either sense. Suppose

$$\begin{array}{ccc} I & \longrightarrow & J \\ & & \downarrow \\ & & \mathcal{X}(R) \end{array}$$

In this situation if the map from J is topologically surjective, but the map from I may not be. However suppose it is (both topologically surjective). That's what happened in the argument above. Then we get a map $\mathcal{A}_{\mathcal{X},I} \to \mathcal{A}_{\mathcal{X},J}$ and we claim that it's an isomorphism. Almost the same proof works.

Definition 7.8 (Universal factorization algebra, version 3)**.** A *universal factorization algebra* is a rule which assigns to each $\mathbf{C}$-algebra R plus a formal multidisk $\mathcal{X}/R$ a flat R-module $\mathcal{A}_{\mathcal{X}}$ which is

(1) functorial in R:
$$\mathcal{A}_{\mathcal{X}\times_R S} \cong S \otimes_R \mathcal{A}_{\mathcal{X}}$$
(2) factorizes:
$$\mathcal{A}_{\mathcal{X} \coprod \mathcal{Y}} = \mathcal{A}_{\mathcal{X}} \otimes_R \mathcal{A}_{\mathcal{Y}}.$$
(3) If $\mathcal{X} \hookrightarrow \mathcal{Y}$ is a formal completion of $\mathcal{Y}$ along a subset, then we have an induced map $\mathcal{A}_{\mathcal{X}} \to \mathcal{A}_{\mathcal{Y}}$.
(4) Finally, we say that a bunch of diagrams commute.

Definition 1 is equivalent to Definition 3. The point is as follows. For a formal multidisk $\mathcal{X}$ over R and $\{\mathcal{A}_{\mathcal{X}\leftarrow I}\}$ a universal factorization algebra in the sense of Definition 1. Then $\mathcal{A}_{\mathcal{X}} := \mathcal{A}_{\mathcal{X}\leftarrow I}$ is well-defined locally on Spec R independent of the choice of I.

Note: for any smooth curve X over $\mathbf{C}$, we have a diagram

$$\begin{array}{ccc} & \{\text{universal factorization algebras}\} & \\ \swarrow & & \searrow \\ \left\{\text{Representations of } \mathrm{Aut}(\widehat{D})\right\} & & \{\text{factorization algebras on } X\} \\ \searrow & & \swarrow \\ & \{\mathcal{D}\text{-modules on } X\} & \end{array}$$

Conversely, we can recover the UFA $\mathcal{A}$. Take $\mathcal{A}_{X^I} = \pi^*\mathcal{A}_I$. Let $\mathcal{A}_X$ be the induced factorization algebra on X. There is a faithfully flat map $\pi \colon X^I \to \mathrm{MDisk}^I$, affine if X is affine. Suppose we know the $\mathcal{D}$-module $\pi^*\mathcal{A}_I$. What is $\mathcal{A}_I$? We have

$$\mathcal{A}_I \to \pi_*\pi^*\mathcal{A}_I \to \pi_*\mathcal{A}_{X^I}.$$

We have

$$\mathcal{A}_I \to \pi_*\mathcal{A}_{X^I}. \to K$$

for some K which is a flat sheaf on MDisk^I. It suffices to understand this for 1 copy of the curve.