FACTORIZATION ALGEBRAS (OCT 8, 2020)

TALK BY JACOB LURIE,
NOTES BY TONY FENG

Contents

1. Affine Grassmannian

Let G be a reductive algebraic group over $\mathbf{C}$. For now we will assume $G = \mathrm{GL}_n$, and later on we'll switch to $G = \mathrm{SL}_n$.

Definition 1.1. The *affine Grassmannian* for $G = \mathrm{GL}_n$ is the quotient

$$\mathrm{Gr} := \mathrm{GL}_n(\mathbf{C}((t))/\mathrm{GL}_n(\mathbf{C}[[t]])$$

So far we're just thinking of this as a set. But we're going to explain that it has the structure of an increasing union of algebraic varieties.

Fix the trivial n-dimensional vector space $\mathbf{C}((t))^n$. Consider the lattice $L_0 = \mathbf{C}[[t]]^n \subset \mathbf{C}((t))^n$. Clearly $\mathrm{GL}_n(\mathbf{C}((t)))$ acts transitively on the set of lattices in $\mathbf{C}((t))^n$, and $\mathrm{GL}_n(\mathbf{C}[[t]])$ is the stabilizer of L_0.

Fix an integer $d \geq 0$ an define $\mathrm{Gr}(d)$ to be the set of lattices L such that $t^d L_0 \subseteq L \subseteq t^{-d} L_0$. Then $\mathrm{Gr} = \varinjlim \mathrm{Gr}(d)$. Now, the $\mathrm{Gr}(d)$ are finite-dimensional. To analyze it, note that $L \in \mathrm{Gr}(d)$ is determined by its image in $t^{-d}L_0/t^d L_0$. So $\mathrm{Gr}(d)$ maps injectively into the set of complex subspaces of $V := t^{-d}L_0/t^dL_0$, which is $\coprod_{0 \leq m \leq 2nd} \mathrm{Gr}(m, V)$.

The image of $\mathrm{Gr}(d)$ is the collection of complex subspaces of $V = t^{-d}L_0/t^dL_0$ which are $\mathbf{C}[[t]]$-submodules, i.e. stable under t. This is a closed condition, so $\mathrm{Gr}(d)$ is a projective variety.

2. The Beilinson-Drinfeld Grassmannian

We fix X to be a smooth algebraic curve over $\mathbf{C}$.

Let

$$\mathrm{Gr}_x := \left\{ (\mathcal{P}, \gamma) \colon \begin{matrix} \mathcal{P} = \text{ rank } n \text{ vector bundle on } X \\ \gamma = \text{ trivialization of } \mathcal{P}|_{X-x} \end{matrix} \right\} /\text{isom.}$$

Fix $x \in X$ and a local coordinate t at x, which induces an isomorphism $\widehat{X}_x \cong \mathrm{Spf}\,\mathbf{C}[[t]]$. We can write

$$X = (X - \{x\}) \cup_{\widehat{X}_x - x} \widehat{X}_x.$$

Using this to get a "gluing" description of vector bundles on X, we get an identification (depending on the choice of local coordinate) $\mathrm{Gr}_x \cong \mathrm{Gr}$.

Now if R is a $\mathbf{C}$-algebra, let $\mathrm{Gr}_x(R)$ be the set of pairs $(\mathcal{P}, \gamma)$, where $\mathcal{P}$ is a rank n vector bundle on $X \times \mathrm{Spec}\, R$ and γ is a trivialization on $(X - \{x\}) \times \mathrm{Spec}\, R$, up to isomorphism.

Example 2.1. Let's compare this to the "naïve definition" $\mathrm{GL}_n(R((t)))/\mathrm{GL}_n(R[[t]])$. This is contained in $\mathrm{Gr}_x(R)$, and is bijective when R is local, or more generally when every projective R-module is free, but not in general.

We will introduce an object Gr_X, the *Beilinson-Drinfeld Grassmannian*, which puts the Gr_x into a family over X. First we define

$$\mathrm{Gr}_X(\mathbf{C}) := \left\{ (\mathcal{P}, x, \gamma) \colon \begin{array}{c} \mathcal{P} = \text{ rank } n \text{ vector bundle on } X \\ x \in X(\mathbf{C}) \\ \gamma = \text{ trivialization of } \mathcal{P}|_{X-x} \end{array} \right\} / \sim .$$

More generally, we define $\mathrm{Gr}_X(R)$.

$$\mathrm{Gr}_X(R) := \left\{ (\mathcal{P}, x, \gamma) \colon \begin{array}{c} \mathcal{P} = \text{ rank } n \text{ vector bundle on } X_R \\ x \in X(R) \\ \gamma = \text{ trivialization of } \mathcal{P}|_{X_R - \Gamma_x} \end{array} \right\} / \sim .$$

This allows us to view Gr_X as a stack. We have a map $\mathrm{Gr}_X \to X$ sending $(\mathcal{P}, \gamma, x) \mapsto x$, whose fiber over $\{x\}$ is Gr_x.

There is a version $\mathrm{Gr}_{X^2}(R)$, parametrizing $\{(\mathcal{P}, x, y, \gamma)\}$ where $\mathcal{P}$ is a vector bundle of rank n on X_R, $x, y \in X(R)$, and γ is a trivialization of $\mathcal{P}_{X_R - \Gamma_x - \Gamma_y}$. There is a map $\mathrm{Gr}_{X^2} \to X^2$, whose fiber over (x, y) is Gr_x if $x = y$, and $\mathrm{Gr}_x \times \mathrm{Gr}_y$ if $x \neq y$.

$$\begin{array}{ccccc} (\mathrm{Gr}_X \times \mathrm{Gr}_X) \times_{X^2} (X^2 - \Delta) & \hookrightarrow & \mathrm{Gr}_{X^2} & \hookleftarrow & \mathrm{Gr}_X \\ \downarrow & & \downarrow & & \downarrow \\ X^2 - \Delta & \hookrightarrow & X^2 & \hookleftarrow & \Delta \end{array}$$

More generally, there's a space $\mathrm{Gr}_{X^n} \to X^n$ whose fiber over $(x_1, \dots, x_n)$ is $\prod_{y \in \{x_1, \dots, x_n\}} \mathrm{Gr}_y$.

2.1. **What just happened?** This was an example of "factorization". The slogan is that "Gr is a factorizable ind-scheme".

3. The determinant bundle

3.1. **Determinant bundle on the affine Grassmannian.** Fix integers $0 \leq d \leq m$. We have a Grassmannian $\mathrm{Gr}(d, m)$. It has a Plucker embedding into $\mathbb{P}^{\binom{m}{d}-1}$, by sending $V \subset \mathbf{C}^m$ to $\wedge^d V \subset \wedge^d \mathbf{C}^m$.

Consider $f^*\mathcal{O}(1)$, an ample line bundle on $\mathrm{Gr}(d, m)$. Now, we had $\mathrm{Gr} = \varinjlim \mathrm{Gr}(d)$. On $\mathrm{Gr}(d)$ we have a line bundle, sending $V \subset t^{-d}L_0/t^d L_0$ to $(\wedge^{\mathrm{top}} V)^{-1} = (\det V)^{-1}$. In other words, it takes $L \mapsto \det(L/t^d L_0)^{-1}$. We would like to normalize this to be independent of d, so we tensor with $\det(L_0/t^d L_0)$. Now, these line bundles are comapatible as d varies and so define a line bundle $\mathcal{L}_{\det}$ on Gr.

3.2. **Determinant bundle on the BD Grassmannian.** Now let X be a *complete* smooth algebraic curve. We define $\mathrm{Gr}_X = \{(\mathcal{P}, x, \gamma)\}/\sim$ as before. Given $(\mathcal{P}, x, \gamma)$, we can make a 1-dimensional $\mathbf{C}$-vector space by the global version of the preceding construction:

$$\frac{\det H^1(X,\mathcal{P}) \otimes \det H^0(X,\mathcal{P})^{-1}}{\det H^1(X,\mathcal{O}^n) \otimes \det H^0(X,\mathcal{O}^n)^{-1}}.$$

Warning 3.1. The determinant of a vector space is regarded as a $\mathbf{Z}$-graded vector space, where the grading is the Euler characteristic. You need to keep track of this (at least mod 2).

More generally, given a vector bundle $\mathcal{P}$ on $X \times \operatorname{Spec} R$, we form

$$\frac{\det R\Gamma(X_R, \mathcal{P})}{\det R\Gamma(X_R, \mathcal{O}^n)^{-1}}.$$

This defines a line bundle on the moduli stack of G-bundles, and we are pulling it back to Gr_X.

We claim that the determinant line bundle $\mathcal{L}_{\mathrm{Det},X^n}$ on Gr_{X^n} "factorizes". For example, on Gr_{X^2} there is a line bundle $\mathcal{L}_{\det,X^2}$.

$$\begin{array}{ccccc}
(\mathrm{Gr}_X \times \mathrm{Gr}_X) \times_{X^2} (X^2 - \Delta) & \longrightarrow & \mathrm{Gr}_{X^2} & \longleftarrow & \mathrm{Gr}_X \\
\downarrow & & \downarrow & & \downarrow \\
X^2 - \Delta & \longrightarrow & X^2 & \xleftarrow[\Delta]{} & X
\end{array}$$

We claim that $\mathcal{L}_{\det}$ on Gr_{X^2} restricts to $\mathcal{L}_{\det} \boxtimes \mathcal{L}_{\det}$ on $X^2 - \Delta$ and $\mathcal{L}_{\det}$ on X.

Why? An R-point of Gr_{X^2} is $(\mathcal{P}, x, y, \gamma)$. We have $\gamma\colon \mathcal{O}^n \to \mathcal{P}$ away from x and y. Suppose γ happens to extend over x and y (not necessarily isomorphically). Then

$$\det(R\Gamma(X_R, \mathcal{P})) \cong \det(R\Gamma(X_R, \mathcal{O}^n)) \otimes \det(R\Gamma(X_R, \mathcal{P}/\mathcal{O}^n))$$

and since $\mathcal{P}/\mathcal{O}^n$ is supported on the distinct points x, y, the $\det(R\Gamma(X_R, \mathcal{P}/\mathcal{O}^n))$ factors into contributions from x, y.

3.3. **Cohomology of the determinant bundle.** From now on, fix $\ell \geq 0$. We contemplate "$H^0(\mathrm{Gr}, \mathcal{L}_{\det}^{\otimes \ell})$". What does this even mean? We have $\mathrm{Gr} = \varinjlim_d \mathrm{Gr}(d)$, so it's reasonable to define $H^0(\mathrm{Gr}, \mathcal{L}_{\det}^{\otimes \ell}) = \varprojlim H^0(\mathrm{Gr}(d), \mathcal{L}_{\det}^{\otimes \ell})$. Since the $\mathrm{Gr}(d)$ are algebraic varieties, each $H^0(\mathrm{Gr}(d), \mathcal{L}_{\det}^{\otimes \ell})$ is a finite-dimensional vector space over $\mathbf{C}$. But the inverse limit is infinite-dimensional, and has an inverse limit topology.

We don't want to deal with topological vector spaces, so instead we contemplate

$$\mathcal{A} := \varinjlim_d H^0(\mathrm{Gr}(d), \mathcal{L}_{\det}^{\otimes \ell})^{\vee}.$$

This is now a $\mathbf{C}$-vector space of countable dimension. It is an example of a factorization algebra.

More generally, for each $n \geq 0$, we have $\mathrm{Gr}_{X^n} \to X^n$ and there is a determinant bundle $\mathcal{L}_{\det,X^n}$ on Gr_{X^n}. We define $\mathcal{A}_{X^n}$ to be the "pre-dual" of $\pi_* \mathcal{L}_{\det,X^n}^{\otimes \ell}$.

What does this mean more concretely, e.g. in terms of finite-dimensional algebraic geometry? We have $\mathrm{Gr}_{X^n} = \varinjlim_d \mathrm{Gr}_{X^n}(d)$, and $\pi\colon \mathrm{Gr}_{X^n}(d) \to X^n$. Take $\pi_*(\mathcal{L}_{\det,X^n}^{\otimes \ell}|_{\mathrm{Gr}_{X^n}(d)})$. Questions:

- Does this give a vector bundle?
- Is it compatible with the construction over a point?

It turns out that the answer is *yes* (to both questions) for well-chosen approximations $\mathrm{Gr}_{X^n}(d)$. (They should be flat over X^n, and the content is that the higher cohomology groups vanish.)

Now we can formally define $\mathcal{A}_{X^n}$. Let $\mathcal{A}_{X^n} = \varinjlim_d \left(\pi_* (\mathcal{L}_{\det, X^n}^{\otimes \ell}|_{\mathrm{Gr}_{X^n}(d)})^\vee \right)$. This is a direct limit of vector bundles, so a flat quasicoherent sheaf on X^n. Its fiber at x looks like $\mathcal{A} = H^0(\mathrm{Gr}, \mathcal{L}_{\det}^{\otimes \ell})$.

Now consider how it looks like on restricting: a formal consequence of what we have said is that $\mathcal{A}_{X^2}|_{X^2 - \Delta} \cong (\mathcal{A}_X \boxtimes \mathcal{A}_X)|_{X^2-\Delta}$, while $\mathcal{A}_{X^2}|_{\Delta X} \cong \mathcal{A}_X$.

$$\begin{array}{ccccc} (\mathcal{A}_X \boxtimes \mathcal{A}_X)|_{X^2-\Delta} & & \mathcal{A}_{X^2} & & \mathcal{A}_X \\ | & & | & & | \\ (X^2 - \Delta) & \xrightarrow{j} & X^2 & \longleftarrow & X \end{array}$$

$\mathcal{A}_X$ is an example of a (super) factorization algebra on X. We are going to axiomatize the structure we see here to give the definition of factorization algebras. Actually, at this point we change GL_n to SL_n (because of Warning 3.1) and then the analogous $\mathcal{A}_X$ is an honest factorization algebra on X.

4. Factorization algebras

Let X be a smooth algebraic curve over $\mathbf{C}$.

Definition 4.1. A *non-unital factorization algebra* on X is a rule which assigns to every commutative $\mathbf{C}$-algebra R and every finite subset $S \subset X(R)$, a flat R-module $\mathcal{A}_S$ plus the following data:

(1) For any $\mathbf{C}$-algebra homomorphism $R \to R'$, letting $S' \subset X(R')$ be the image of $S \subset X(R)$, then we have an isomorphism $\mathcal{A}_{S'} \xrightarrow{\sim} R' \otimes_R \mathcal{A}_S$.
(2) If $S = \coprod_{i \in I} S_i$ which are "geometrically disjoint" (i.e. disjoint after any base change to any R-algebra), then we have $\mathcal{A}_S \xrightarrow{\sim} \bigotimes_{S_i} \mathcal{A}_{S_i}$.
(3) Compatibility isomorphisms for the above data.

Assume for simplicity that $X = \mathrm{Spec}\ A$ is affine. We can take $R = A$, $S = \{\mathrm{Id}\} \colon \mathrm{Spec}\ R \to X$ the identity map. Then $\mathcal{A}_{\{\mathrm{Id}\}}$ is $\mathcal{A}_X$.

Next consider $R = A \otimes_{\mathbf{C}} A$ and $\pi_1, \pi_2 \colon \mathrm{Spec}\ R \rightrightarrows X$. Then $A_{\{\pi_1, \pi_2\}} =: A_{X^2}$ is a quasicoherent sheaf on X^2. How are these related?

- Consider $\Delta \colon X \to X^2$. This corresponds to $A \otimes_{\mathbf{C}} A \xrightarrow{m} A$. The points $\pi_1, \pi_2 \in X(A \otimes_{\mathbf{C}} A)$ are both sent to Id in $X(A)$. Compatibility with base change says that we have an isomorphism $\Delta^* \mathcal{A}_{X^2} \xrightarrow{\sim} \mathcal{A}_X$.
- Inside $X^2 - \Delta =: \mathrm{Spec}\ B$, the points π_1, π_2 become disjoint. So we should have $\mathcal{A}_{X^2}|_{X^2-\Delta} \cong (\mathcal{A}_X \boxtimes \mathcal{A}_X)|_{X^2-\Delta}$.

In fact, we claim that all the structure required to build the factorization algebra are recovered from the data here.

Say $\mathcal{A}$ is a non-unital factorization algebra. We have $\mathcal{A}_X$ flat quasicoherent on X, and $\mathcal{A}_{X^2}$ flat quasicoherent on X^2. The flatness implies that for $j \colon (X^2 - \Delta) \hookrightarrow X^2$,

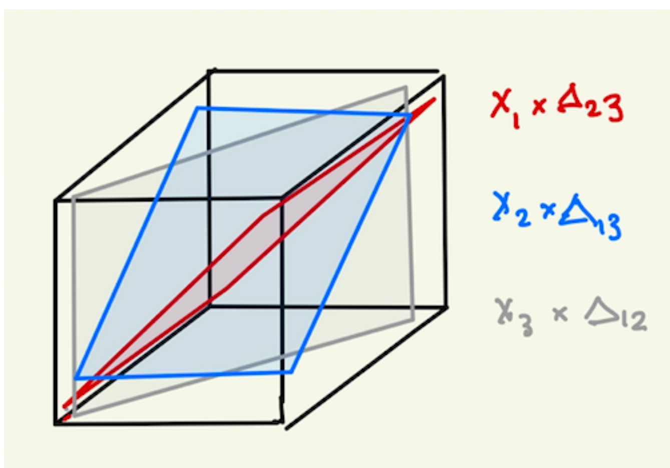
$$\mathcal{A}_{X^2} \hookrightarrow j_*(\mathcal{A}_X \boxtimes \mathcal{A}_X|_{X^2-\Delta}).$$

Last time we said that $\mathcal{A}_X$ is a sheaf whose sections could be considered as "observables". Note that $\mathcal{A}_X \boxtimes \mathcal{A}_X \hookrightarrow j_*(\mathcal{A}_X \boxtimes \mathcal{A}_X|_{X^2-\Delta})$ as well. The properness of this inclusion is saying that you can't observe two observables at the same point. The sheaf $\mathcal{A}_{X^2}$ is telling what regularization is needed to do this.

This makes it seem like you have to specify a whole lot of data in order to get a factorization algebra. In fact, the claim is that everything is determined by the data:

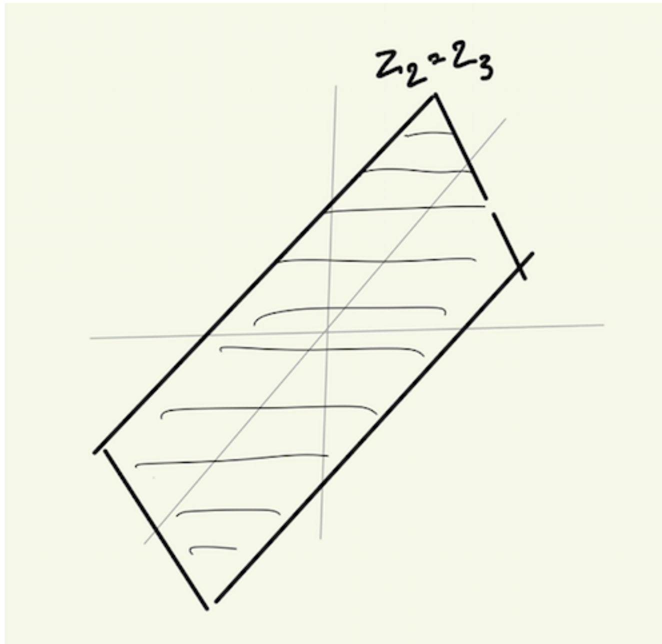
- $\mathcal{A}_X$,
- $\mathcal{A}_{X^2} \hookrightarrow j_*(\mathcal{A}_X \boxtimes \mathcal{A}_X|_{X^2-\Delta})$,
- $e\colon \mathcal{A}_{X^2}|_\Delta \cong \mathcal{A}_X$.

(However there are conditions.) We explain this through a representative example. Suppose you want to understand $\mathcal{A}_{X^3}$ as a sheaf on X^3.



Picture: three divisors $X_1 \times \Delta_{23}, X_2 \times \Delta_{13}, X_3 \times \Delta_{12}$. They meet in the diagonal copy of X. Now, $\mathcal{A}_{X^3}$ is flat on the smooth variety X^3. So removing a codimension ≥ 2 subset doesn't change anything, so we can remove the diagonal copy of X. Generically it looks like $\mathcal{A}_X \boxtimes \mathcal{A}_X \boxtimes \mathcal{A}_X$. On $X_1 \times \Delta$, it looks like $\mathcal{A}_X \boxtimes \mathcal{A}_{X^2}$.

To convince you that this works, we will give explicit formulas for $X = \mathbb{A}^1 = \operatorname{Spec} \mathbf{C}[z]$. Write $X^3 = \operatorname{Spec} \mathbf{C}[z_1, z_2, z_3]$. We will explain how to reconstruct $\mathcal{A}_{X^3}$. By the flatness, $\mathcal{A}_{X^3} \subset \mathcal{A}_X \boxtimes \mathcal{A}_X \boxtimes \mathcal{A}_X[(z_1 - z_2)^{-1}, (z_2 - z_3)^{-1}, (z_3 - z_1)^{-1}]$. It is cut out by the condition of extending over the three divisors $\{z_1 = z_3\}$, $\{z_1 = z_2\}$, and $\{z_2 = z_3\}$.



The condition of extending over the divisor $\{z_2 = z_3\}$ is that it lie in

$$\mathcal{A}_X \boxtimes \mathcal{A}_{X^2}[(z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}] \subset \mathcal{A}_X \boxtimes \mathcal{A}_X \boxtimes \mathcal{A}_X[(z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}].$$

Therefore, $\mathcal{A}_{X^3}$ is the intersection inside $\mathcal{A}_X^{\boxtimes 3}[(z_1 - z_2)^{-1}, (z_2 - z_3)^{-1}, (z_3 - z_1)^{-1}]$ of $\mathcal{A}_{X_1} \boxtimes \mathcal{A}_{X_2 \times X_3}[(z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}]$ and the two other submodules obtained by permuting coordinates.

5. Unital factorization algebras[1]

We have just defined non-unital factorization algebras. Now we will define a *unital* factorization algebra. In particular we ask for the same data as before, *plus* functoriality in S. That is:

Definition 5.1. A *unital factorization algebra* is a rule which assigns to every commutative $\mathbf{C}$-algebra R and every finite subset $S \subset X(R)$, a flat R-module $\mathcal{A}_S$ plus the following data:

(1) For any $\mathbf{C}$-algebra homomorphism $R \to R'$, letting $S' \subset X(R')$ be the image of $S \subset X(R)$, then we have an isomorphism $\mathcal{A}_{S'} \xrightarrow{\sim} R' \otimes_R \mathcal{A}_S$.
(2) If $S = \coprod_{i \in I} S_i$ which are "geometrically disjoint" (i.e. disjoint after any base change to any R-algebra), then we have $\mathcal{A}_S \xrightarrow{\sim} \bigotimes_{S_i} \mathcal{A}_{S_i}$.
(3) Compatibility isomorphisms for the above data.
(4) (Unitality) For all $S \subset S' \subset X(R)$, a map $\mathcal{A}_S \to \mathcal{A}_{S'}$.

Example 5.2. Continuing our example from last time, we had a non-unital factorization algebra given by the rule

$$x \in X(\mathbf{C}) \mapsto \text{cosections of } \mathcal{L}_{\det}^{\otimes \ell} \text{ on } \mathrm{Gr}_x.$$

More generally, for $S \subset X(\mathbf{C})$ we have $\mathcal{A}_S = \bigotimes_{x \in S} \mathcal{A}_x$.

[1]Continued on Oct. 15.

Now suppose we have $S \subset S'$. Now Gr_S parametrizes G-bundles on X trivialized outside S, so it admits a map to $\mathrm{Gr}_{S'}$ by restricting the domain of the trivialization. That induces a map from cosections of $\mathcal{L}_{\det}^{\otimes \ell}$ on Gr_S to cosections of $\mathcal{L}_{\det}^{\otimes \ell}$ on $\mathrm{Gr}_{S'}$. So this gives unitality data.

Example 5.3. Consider $S = \emptyset$ and $S' = \{x\}$. Then $\mathrm{Gr}_S = \mathrm{Spec}\ \mathbf{C}$ and $\mathrm{Gr}_{S'} = \mathrm{Gr}_x$. The map $\mathrm{Gr}_S \to \mathrm{Gr}_{S'}$ is the basepoint at the trivial bundle with its tautological trivialization.

Let $\mathcal{A}$ be a non-unital factorization algebra. Recall that we had isomorphisms $\mathcal{A}_{X^2}|_\Delta \xrightarrow{\sim} \mathcal{A}_X$ and $\mathcal{A}_{X^2}|_{X^2-\Delta} \xrightarrow{\sim} \mathcal{A}_X \boxtimes \mathcal{A}_X|_{X^2-\Delta}$. This allowed to reconstruct the rest of the data of a factorization algebra (but there are conditions).

What does it take to make $\mathcal{A}$ into a *unital* factorization algebra? We need to give $\mathcal{A}_\emptyset \to \mathcal{A}_{\{x\}}$ for $x \in X(R)$. This glues into a section $\mathbb{1} \colon \mathcal{O}_X \to \mathcal{A}_X$. What conditions does it satisfy?

Given R and $x, y \in X(R)$, we consider $\{x\} \subset \{x, y\}$. As x, y vary, this glues to a map of quasicoherent sheaves $f \colon \pi^*\mathcal{A}_X \to \mathcal{A}_{X^2}$. The compatibility conditions imply that:

- on $X^2 - \Delta$, f restricts to a map
$$\mathcal{A}_X \boxtimes \mathcal{O}_X|_{X^2-\Delta} \cong (\pi^*\mathcal{A}_X)|_{X^2-\Delta} \to (\mathcal{A}_{X^2})|_{X^2-\Delta} = \mathcal{A}_X \boxtimes \mathcal{A}_X|_{X^2-\Delta}$$
which must be identified with $\mathrm{Id}_{\mathcal{A}_X} \boxtimes \mathbb{1}$.
- On Δ, f restricts to a map
$$\mathcal{A}_X \cong \Delta^*\pi^*\mathcal{A}_X \to \Delta^*\mathcal{A}_{X^2} \xrightarrow{\sim} \mathcal{A}_X$$
which must be identified with $\mathrm{Id}_{\mathcal{A}_X}$.

Remark 5.4. Think about sections as being observables. We can't multiply such things in general. But $\mathbb{1}$ is allowed to be multiplied with other observables.

6. Connection structure

The data of the unit gives a lot of extra structure: it turns the quasicoherent sheaves into quasicoherent sheaves with a flat connection as a quasicoherent $\mathcal{O}_{X^n}$-module.

This means that if is a unital factorization algebra, then we get
$$\nabla \colon \mathcal{A}_{X^n} \to \mathcal{A}_{X^n} \boxtimes \Omega^1_{X^n}$$
satisfying some vanishing condition on curvature. Another way of expressing this is by saying that $\mathcal{A}_{X^n}$ has the structure of an algebraic $\mathcal{D}_{X^n}$-module.

We will adopt a different perspective, modeled on the notion of parallel transport. We say that $x, y \in X(R)$ are *infinitesimally close* if they have the same image in $X(R^{\mathrm{red}} := R/\sqrt{R})$. The following definition is due to Grothendieck.

Definition 6.1 (Flat connection as crystal on the infinitesimal site)**.** If $\mathcal{E}$ is a quasicoherent sheaf on X, then a *flat connection* is a rule which assigns to every pair $x, y \in X(R)$ which are infinitesimally close, isomorphisms of R-modules $x^*\mathcal{E} \xrightarrow{\sim} y^*\mathcal{E}$.

Let A be a unital factorization algebra and $x, y \in X(R)$. Then $\mathcal{A}_{\{x\}} = x^*\mathcal{A}_X$ and $\mathcal{A}_{\{y\}} = y^*\mathcal{A}_X$. Consider $(x, y) \colon \mathrm{Spec}\ R \to X^2$. So we have maps

$$\begin{array}{ccccc} \mathcal{A}_{\{x\}} & \longrightarrow & \mathcal{A}_{\{x,y\}} & \longleftarrow & \mathcal{A}_{\{y\}} \\ \| & & \| & & \| \\ x^*\mathcal{A}_X & & (x,y)^*\mathcal{A}_X & & y^*\mathcal{A}_X \end{array}$$

We claim that these are isomorphisms. Indeed, x, y have the same image in $X(R/I)$ for I a nilpotent ideal. So we have maps of flat R-modules which become isomorphisms after quotienting out by a nilpotent ideal – then they must already be isomorphisms of R-modules.

Example 6.2. In the case of $\mathcal{A}$ being cosections of the determinant line bundle on Gr, Gr_x parametrizes G-bundles on X plus a trivialization on $X - x$. We even have an isomorphism $\mathrm{Gr}_x \cong \mathrm{Gr}_y$ for infinitesimally close x, y. So there is a "connection" on the spaces, even before we take cosections – this is one advantage of Grothendieck's formulation of a connection (that it makes sense even for non-linear objects).

7. Commutative factorization algebras

We give an example of a "commutative factorization algebra". Let X be an algebraic curve over $\mathbf{C}$ and $Y = \mathrm{Spec}\ B$ be a smooth affine variety. Given a point $x \in X$, we can form the formal completion $\widehat{X}_x = \mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$. Then $\mathrm{Map}(\widehat{X}_x, Y) = \mathrm{Hom}_{\mathbf{C}}(B, \widehat{\mathcal{O}}_{X,x})$.

Example 7.1. Suppose $Y = \mathbb{A}^1$, so $B = \mathbf{C}[t]$. Then $\mathrm{Map}(\widehat{X}_x, Y) = \widehat{\mathcal{O}}_{X,x} = \varprojlim \widehat{\mathcal{O}}_{X,x}/\mathfrak{m}^n$. Each $\widehat{\mathcal{O}}_{X,x}/\mathfrak{m}^n$ is an affine space. So this looks like an inverse limit of copies of affine space, with transition maps given by forgetting higher order terms. So in this case the mapping space $\mathrm{Map}(\widehat{X}_x, Y)$ is $\mathbb{A}^\infty \cong \mathrm{Spec}\ \mathbf{C}[a_0, a_1, a_2, \ldots]$.

Given Y, we can make a factorization algebra $\mathcal{A}$ on X such that $\mathcal{A}_x$ is the coordinate ring of $\mathrm{Map}(\widehat{X}_x, Y)$.

More generally, if R is $\mathbf{C}$-algebra and $S \subset X(R)$ is a finite set, $\mathcal{A}_S$ is even a commutative R-algebra, and its spectrum is

$$\mathrm{Spec}\,(\mathcal{A}_S) = \mathrm{Map}((X \times \mathrm{Spec}\ R)^\wedge_{\Gamma_S}, Y)$$

Note that if the points S are disjoint, then the formal completion is a disjoint union. So $\mathcal{A}_S$ factors as a tensor product. This corresponds to the factorization axiom.

The multiplication on $\mathcal{A}_x$ is the same as provided by the factorization structure.