Twisted sheaves, ten years later

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1 Applications of twisted sheaves

We're first going to give a laundry list of results which have been proven (perhaps not originally) using the theory of twisted sheaves.

Theorem 1.1 (Gabber)**.** *If X is a quasicompact separated scheme with an ample invertible sheaf, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$ *(the second is the cohomological Brauer group* $H^2(X_{\acute{e}t}, \mathbb{G}_m)$*).*

Theorem 1.2 (de Jong)**.** *Let K/k have transcendence degree 2 over an algebraically closed field $k = \overline{k}$. Then for all $\alpha \in \mathrm{Br}(K)$, the index of α equals the period of α.*

Theorem 1.3 (Lieblich)**.** *Let K/k have transcendence degree 2 and k be a finite field. Then for all $\alpha \in \mathrm{Br}(K)$, the index of α divides the square of the period of α.*

Theorem 1.4 (Libliech-Panimala-Smesh)**.** *If Colliot-Thélène's conjecture on 0-cycles of degree 1 holds for geometrically rationally connected varieties, then any K/k of transcendence degree 1, where k is a totally imaginary number field, has finite μ-invariant.*

Theorem 1.5 (Lieblich-Maulik-Snowden)**.** *Given a finite field k, the Tate conjecture for K3 surfaces over finite extensions of k is equivalent to the statement that for all L/k finite, there exist only finitely many K3 surfaces over L up to isomorphism.*

Theorem 1.6 (Lieblich)**.** *A general point on the Ogus space of marked supersingular K3 surfaces lies on infinitely many pairwise distinct rational curves.*

Theorem 1.7 (Lieblich-Olsson)**.** *In characteristic p, if X is a K3 surface and Y is a variety such that $D(X) \cong D(Y)$ (the bounded derived categories of coherent cohomology) then Y is a moduli space of sheaves on X.*

This was known in characteristic 0 using Hodge structures, but this is obviously unavailable in characteristic p. Our method is totally different.

2 Twisted sheaves

2.1 The fundamental asymmetry

What gives birth to all of this is an asymmetry in how we're taught to think about the moduli space of sheaves, which we can cure and thus allow information to flow in two directions instead of one. I'll try to explain what I mean by this.

Let X/k be a variety. Then we can consider moduli spaces parametrizing sheaves on X, e.g.

- the Picard space $\mathrm{Pic}_{X/k}$,
- $M^s_{X/k}(c)$, the moduli spaces of simple (stable) sheaves with fixed invariants (Chern classes, ...)
- $P^s_{X/k}(c)$, the moduli space of perfect complexes.

This looks like a machine where we put in a variety and get out some scheme. However, that's wrong. What we really have is an asymmetric sheaf which takes in a variety and spits out a *gerbe*, not a scheme. This asymmetry in some sense blocks information from flowing back the other way.

Twisted sheaves are a symmetric version of this machine through which one feeds in a gerbe and gets out a gerbe. At least superficially, one can imagine that this is reversible.

2.2 Gerbes

Everything we say is only "approximately correct."

Definition 2.1. A $\mathbb{G}_m$-*gerbe* is an algebraic stack $\mathcal{X}$ such that for any object of $\mathcal{X}$ over T, the automorphism sheaf is functorially identified with $(\mathbb{G}_m)_T$.

Thus a $\mathbb{G}_m$-gerbe is a stack that is "mildly stacky" - we know precisely what its automorphism groups are.

Example 2.2. Examples of $\mathbb{G}_m$-gerbes:

- $B\mathbb{G}_m$.
- $B\mathbb{G}_m(T)$, the groupoid of invertible sheaves on T.
- $\mathcal{P}ic_{X/k}(T)$, the moduli space of invertible sheaves on X_T.
- $\mathcal{M}^s_{X/k}(T)$, the moduli space of simple sheaves on X_T.

A strong consequence of being a $\mathbb{G}_m$-gerbe is that it can be comapred with $B\mathbb{G}_m$. That is, given a $\mathbb{G}_m$-gerbe $\mathcal{X}$, we can produce a map

$$\mathcal{X} \to X$$

where X is the sheafification of the presheaf of isomorphism classes (an algebraic space), and there exists an étale surjection $U \to X$ such that the pullback is isomorphic to $B\mathbb{G}_m \times U$:

$$\begin{array}{ccc} B\mathbb{G}_m \times U \cong \mathcal{X} \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & X \end{array}$$

One of the nice features that a gerbe has is that we can understand quasicoherent sheaves on it in a very simple way. Any quasicoherent sheaf $\mathcal{F}R$ on a $\mathbb{G}_m$-gerbe $\mathcal{F}$ can be decomposed canonically as

$$\mathcal{F} = \bigoplus \mathcal{F}_i$$

where $\mathcal{F}_i$ is an eigensheaf, on which the $\mathbb{G}_m$ action is given by $\alpha \cdot f = \alpha^i f$. (There is a subtle sign change going on, which comes from the fact that we are passing from a right action to a left action.)

2.3 Twisted sheaves

Definition 2.3. An *$\mathcal{X}$-twisted quasicoherent sheaf* is a quasi-coherent sheaf $\mathcal{F}$ such that the action $\mathbb{G}_m \times \mathcal{F} \to \mathcal{F}$ by the stabilizer is scalar multiplication, i.e. $\mathcal{F} = \mathcal{F}_1$. We denote the category of twisted sheaves by $QCoh^1(\mathcal{X})$.

One fun fact is that there is a classification of twisted sheaves in terms of cohomology. This says that $\mathbb{G}_m$-gerbes $\mathcal{X} \to X$ are classified up to isomorphism by $H^2(X, \mathbb{G}_m)$.

Example 2.4. Knowing this, you can ask what is the cohomology class of $\mathcal{P}ic_{X/k} \to \mathrm{Pic}_{X/k}$. The answer is that it is the "universal obstruction" to specifying a point of $\mathrm{Pic}_{X/k}$ by an invertible sheaf.

2.4 Symmetrization

Observation: there is a canonical equivalence $QCoh(X) \to QCoh^1(B\mathbb{G}_m \times X)$ induced by $F \mapsto \chi \boxtimes F$ where χ is an invertible twisted sheaf on $B\mathbb{G}_m$ corresponding to standard representation $B\mathbb{G}_m \to B\mathbb{G}_m$.

Using this observation, we can rephrase all of the classical problems by replacing quasicoherent sheaves with twisted sheaves on the trivial gerbe.

We have a universal sheaf $\mathcal{L}$ on $(B\mathbb{G}_m \times X) \times \mathcal{P}ic_X$. Now $\mathcal{L}$ is "bi-twisted" because there is an action from each factor.

Phrased this way, there is no reason to require the left hand side to be the trivial gerbe. In the general case, we would be interested in a product of $\mathbb{G}_m$-gerbes $\mathcal{X} \times \mathcal{Y}$ and F a universal "bitwisted sheaf" on the product.

2.5 Case study: K3 surfaces

Fix a K3 surface X/k. A common occurrence in this situation is that

> *Many moduli spaces of sheaves on X are themselves K3 surfaces.*

What's the symmetric form of this statement? We have two *gerbes* $\mathcal{X}$ and $\mathcal{Y}$, and a universal (bitwisted) sheaf $\mathcal{F}$ on $\mathcal{X} \times \mathcal{Y}$.

$$\begin{array}{ccccc} \mathcal{X} & \xleftarrow{p} & \mathcal{X} \times \mathcal{Y} & \xrightarrow{q} & \mathcal{Y} \\ \downarrow & & & & \downarrow \\ X & & & & Y \end{array}$$

The classical information flow is $X \rightsquigarrow (Y, \beta)$ with $\beta \in H^2(Y, \mathbb{G}_m)$, i.e. a $\mathbb{G}_m$-gerbe over Y. Using the symmetric apparatus of twisted sheaves, we can go backwards: $(X, \alpha) \rightsquigarrow Y$. There are two uses of this:

1. Fix X and choose a sequence of $\alpha_n \in H^2(X, \mathbb{G}_m) \rightsquigarrow Y_n$. This is a machine $\mathrm{Br}(X) \rightsquigarrow$ points in the space of K3 surfaces.

2. If $X \to \operatorname{Spec} k$ is supersingular, then a theorem of Artin says that $R^2 f_* \mu_p$ is a group scheme with connected component $\mathbb{G}_a$.

   If $X \to \mathbb{P}^1$ is an elliptic structure, then from $\alpha \in \mathrm{Br}(X \times \mathbb{G}_a)$ we can produce a *continuous* family of K3 surfaces $Y \to \mathbb{P}^1$ corresponding to a map from $\mathbb{G}_a$ to the moduli space of supersingular K3 surfaces.

2.6 Weird meta-theorem

Question: Let X/k be a variety over a field. Does the stack of simple sheaves on X contain a geometrically integral locally closed substack?

The meta-answer is no. The reason is that any universal argument would work for gerbes, which wouldn't be true in that generality.