André-Oort type conjectures for variations of Hodge structures

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1 The problem

Let S be a smooth connected quasiprojective variety over k and $f\colon \mathcal{X} \to S$ be a smooth projective family. The main problem is to characterize the points $s \in S$ where the motivic fiber $[\mathcal{X}_s]$ is "simpler" than the general fiber.

The André-Oort conjecture is the case where we consider a family of abelian varieties, and by "simpler" we mean that the fiber has CM.

Conjecture 1.1 (André-Oort)**.** *Let $f\colon \mathcal{X} \to S$ be a smooth complex family of principally polarized abelian varieties of dimension g. If S contains a Zariski-dense set of CM-points then S is a connected Shimura variety in the sense that the classifying map $S \to \mathcal{A}_g$ factors through $Sh_K(G, X)$.*

Example 1.2. Consider $S = Y_0(1) \times Y_0(1)$ and $f\colon \mathcal{X} \to S$ the universal family. The CM points of the Shimura surface S are pairs (x, y) of CM-points. On the other hand, S also contains Shimura curves:

- $Y_0 \times \{p\}$ or $\{p\} \times Y_0$ where p is special (CM).
- The image in $Y_0(1) \times Y_0(1)$ of pairs of curves related by an N-isogeny.

Conjecture 1.3 (André-Oort, special case). *An irreducible curve of $\mathbb{C} \times \mathbb{C}$ containing infinitely many CM points is one of the Shimura curves listed.*

2 Hodge theoretic version

We now consider virtual variations of Hodge structure. There are classifying spaces for these by Shimura varieties. Anyway, in this case the special locus is the *Hodge locus*, which is the set of s where exceptional Hodge classes appear in $\mathcal{V}_s^{\otimes}$.

From a Tannakian point of view, this is the set of points where the Mumford-Tate group $\mathbf{MT}_s$ is strictly smaller than the generic one.

Theorem 2.1 (Cattani-Deligne-Kaplan)**.** *The Hodge locus is a countable union of algebraic subvarieties of S.*

- A *special subvariety* is an irreducible stratum of the Hodge locus.
- A *special point* is a special subvariety of dimension 0.
- A CM-point of the Hodge locus is a special point s such that $\mathbf{MT}_s$ is a torus.

Our goal is to understand the distribution in S of special points, and especially special points. The André-Oort conjecture says that a subvariety in which CM points are dense is a special subvariety.

3 Results

The conjecture was proven by Klingler-Ullmo-Yafaev under the Generalized Riemann Hypothesis. The proof uses ideas from ergodic, algebraic, and arithmetic geometry, using effective Cebotarev bounds from GRH. The crucial arithmetic input is a good lower bound for the size of Galois orbits of CM-points, which is also provided by GRH.

Pila proved the conjecture unconditionally in the special case of $Y_0(1)^N$, using a different circle of ideas (o-minimal geometries and functional transcendence) for the geometric part, and Siegel's lower bound on class numbers for arithmetic point. Recently, Tsimerman noticed that the lower bound for the size of orbits follows from an averaged version of the Colmez conjecture, which has recently been announced. This gives an unconditional proof in the case of Hodge type.

♠♠♠ TONY: [I fell off the horse]