(phi, Gamma)-modules on analytic, adic, and perfectoid spaces

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1 Introduction

This talk is based on joint work with Ruochuan Liu, some of which is published in "Relative p-adic Hodge theory: Foundations" and future papers.

Let F be a perfect field of characteristic p. Consider the category of continuous representations of G_F on finite generated $\mathbb{Z}_p$-modules. This has the structure of a tensor category.

Theorem 1.1. *There is an equivalence of categories*

$$\left\{\begin{matrix} \textit{contravariant representations} \\ \textit{of } G_F \textit{ on finitely} \\ \textit{generated } \mathbb{Z}_p\textit{-modules} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{finitely generated} \\ W(F)\textit{-modules} \\ \textit{with } \varphi^* D \cong D \end{matrix}\right\}$$

sending $V \mapsto D(V) := (V \otimes_{\mathbb{Z}_p} W(\overline{F}))^{G_F}$ *and* $V(D) := (D \otimes_{W(F)} W(\overline{F}))^{\varphi} \leftarrow D$.

We think of the left hand side as the "étale side" and the right hand side as the "coherent cohomology" side. The theorem is giving some equivalence between Galois cohomology and the coherent cohomology of the complex $D \xrightarrow{\varphi-1} D$.

There are extensions of this result.

- One can replace F by K, a complete non-archimedean field containing $\mathbb{Q}_p$.
- One can replace K by a rigid analytic space over K.
- One can replace $\mathbb{Z}_p$ by $\mathbb{Q}_p$, which may seem obvious but is subtle.

Those were the short term goals. But let me also touch on the broader goals.

Let X be a rigid space over K. I want to relate étale local systems on X (with the pro-étale topology) with "algebraic" sheaves on $X_{\text{proét}}$, and we want this association to be compatible with higher direct images along smooth proper morphisms. This will lead to comparison isomorphisms with coefficients.

2 Perfectoid fields

How can I systematically increase the level of generality? The first thing I can do is try to replace a field of characteristic p with a field of characteristic 0.

If K is a *perfectoid field* (i.e. K not discrete and Frobenius $O_K/(p) \to O_K/(p)$ is surjective) then $K^\flat$ is a perfect non-archimedean field of characteristic p and $G_K \cong G_{K^\flat}$, which is a reflection of an equivalence between the finite étale sites of K and $K^\flat$.

Definition 2.1. $K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$ or $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$.

If K is perfectoid, then we can trade $G_{K^\flat}$ for G_K in the theorem. You can then formally promote this to non-perfectoid fields, by finding a (Galois) algebraic extension L/K such that $\widehat{L}$ is perfectoid (but it will still be the case that $G_L \cong G_{\widehat{L}}$ by Krasner's Lemma). Then you have an equivalence of categories

$$\left\{\begin{matrix}\text{contravariant representations}\\ \text{of } G_F \text{ on finitely}\\ \text{generated } \mathbb{Z}_p\text{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finitely generated}\\ W(\widehat{L}^\flat)\text{-modules}\\ \text{with } \varphi\text{-action}\\ \textit{plus } G_{L/K}\textit{-action}\end{matrix}\right\}$$

The additional data of the $G_{L/K}$-action is to recover the specifics of K. You get this for *any* L: you could take L to be the algebraic closure of K, but then you wouldn't get that much information. If you take L to be pretty small, then you would get a lot.

This is basically Fontaine's equivalence, except he only phrased it in the special case of K being a finite extension of $\mathbb{Q}_p$ and $L = K(\mu_{p^\infty})$.

3 Pro-étale formulation

Let $X = \mathrm{Spa}(K, \theta_K)K$. We consider the pro-étale topology on X. I won't define this, but you can think of it being similar to the étale site except certain infinite étale towers are allowed. There is a notion of constant sheaf $\underline{\mathbb{Z}_p}$ on $X_{\text{proét}}$ - this is like the constant sheaf, but not literally. Then there is an equivalence

$$\left\{\begin{matrix}\text{locally finite}\\ \text{sheaves of } \mathbb{Z}_p\text{-modules}\\ \text{on } X_{\text{proét}}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{locally finitely presented}\\ \text{sheaves of } \mathbb{A}\text{-modules on } X_{\text{proét}}\\ \text{plus } \varphi\text{-action}\end{matrix}\right\}.$$

Here $\mathbb{A}(L) = W(\widehat{L^G})$. Here we don't need any additional $G_{L/K}$ structure because we are dealing with sheaves. This is nothing more than a reformulation of the earlier theorem. The maps are by $V \to V \otimes_{\mathbb{Z}_p} \mathbb{A}$ and $D^\varphi \leftarrow D$.

Remark 3.1. It doesn't seem like the right hand side is obviously more tractable than the left hand side, unlike in the earlier theorem. However, it is. Thinking of the right hand side coherent sheaves, the important point is that they are determined on "affines." In this case, an object of the right hand side can be determined its evaluation on *any* L/K such that $\widehat{L}$ is perfectoid plus descent data to K. (Thus, such extensions play the role of "affines" here.)

Let X be a rigid space over K (viewed as an adic space). Then we get the same conclusion. Note that $X_{\text{proét}}$ contains enough "perfectoid subdomains."

Example 3.2. If $X = \mathrm{Spa}(A, A^+)$ then we can find a tower of finite étale extensions

$$A \to A_1 \to A_2 \to \dots$$

such that $(\varinjlim A_n)^\wedge$ is perfectoid. This is an example of a perfectoid subdomain.

If G_K is compact group, then continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces have G_K-stable lattices. However, if X is connected, then an analogue of G_K is the étale fundamental group (de Jong), which is *not compact*.

Example 3.3. The uniformization of a Tate elliptic curve $\mathbb{G}_m \xrightarrow{\mathbb{Z}} \mathbb{G}_m/q^{\mathbb{Z}}$ has the discrete group $\mathbb{Z}$ as its deck transformations.

This means that we can't simply "invert p" in the integral correspondence to obtain a $\mathbb{Q}_p$-correspondence. The solution is to replace the sheaf $\mathbb{A}$. We have a map $\mathbb{A} \to \mathbb{B} = \mathbb{A}[1/p] \hookleftarrow \mathbb{B}^\dagger$. Rather than give the definition, we merely give an analogy.

- $\mathbb{A} \leftrightarrow \mathbb{Z}_p[[t]]^{\wedge(p)}$. The elements of this ring are double sums, but the series may not converge in any range because we cannot control the coefficients in both directions.
- $\mathbb{B}^+ \leftrightarrow$ the subring where the series converges on some range $* < |t| < 1$. Then $\mathbb{B}$ is obtained by inverting p.
- $\mathbb{C} \leftrightarrow$ dropping the restriction of bounded coefficients, which is called the "Roble ring."

Theorem 3.4. *The category*

$$\left\{ \begin{matrix} \textit{locally finite free} \\ \underline{\mathbb{Q}_p}\textit{-modules on } X_{\textit{proét}} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{locally finite free } \mathbb{C}\textit{-modules} \\ \textit{plus } \varphi\textit{-action} \\ \textit{which locally descend to } \mathbb{A}^\dagger \end{matrix} \right\}.$$

The left hand side is thought of as (pro)étale $\mathbb{Q}_p$-local systems. The descent condition is like picking out the subcategory of *semistable* vector bundles (this is literally true when we're working over a point).

The $\mathbb{C}$ satisfies Tate/Kiehl type acyclic/gluing properties. That gluing doesn't really work for $\mathbb{A}$, which is why we needed to replace it by this slightly mysterious ring.