AUTOMORPHIC FORMS AND MOTIVIC COHOMOLOGY III

LECTURES BY AKSHAY VENKATESH,
NOTES BY TONY FENG

Contents

1. Recap

Let G be a reductive group over $\mathbf{Q}$ (with no split torus in its center), and X a locally symmetric space attached to G. Let π be a cohomological (tempered at ∞) automorphic representation. We consider the cohomology $H^*(X, \mathbf{Q})_\pi$. Now, π is associated to a coadjoint motive $M/\mathbf{Q}$ with $\dim M = \dim G$, and we set

$$\Lambda_{\text{mot}} := H^1_{\text{mot}}(\mathbf{Q}, \mathbf{Q}(1))_{\text{int}}.$$

This is a $\mathbf{Q}$-vector space related to $L(\text{Ad}, \pi, 1)$.

The conjecture predicts that there is an action of $\wedge^* \Lambda_{\text{mot}}^\vee$ on $H^*(X, \mathbf{Q})_\pi$.

We construct this action at the level of Betti/étale realizations. Namely, we have regulator maps

$$\begin{array}{ccc} & & \Lambda_{\text{mot},\mathbf{C}} \\ & \nearrow & \\ \Lambda_{\text{mot}} & & \\ & \searrow & \\ & & \Lambda_{\text{mot},\mathbf{Q}_p} \end{array}$$

Last time we constructed an action of $\wedge^* \Lambda_{\text{mot},\mathbf{C}}$ on $H^*(X, \mathbf{C})_\pi$ and gave evidence that it preserves the $\mathbf{Q}$-structures. This time we want to construct an action of $\wedge^*(\Lambda^\vee_{\text{mot},\mathbf{Q}_p})$ on $H^*(X, \mathbf{Q}_p)_\pi$ and give evidence that it preserves the $\mathbf{Q}$-structures.

2. Derived Hecke algebra

2.1. **Summary.** Let $S = \mathbf{Z}/p^n\mathbf{Z}$. Pick another prime $q \neq p$. Let $G_q = G(\mathbf{Q}_q)$ and $K_q = G(\mathbf{Z}_q)$. The derived Hecke algebra is

$$\text{DHA}_{q,S} := \text{Ext}^*_{SG_q}(S[G_q/K_q], S[G_q/K_q])$$

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where the Ext is taken in the category of smooth G_q-modules. In degree 0 this is the usual Hecke algebra. This acts on $H^*(X, S)$. Last time we explained concretely what the operators look like. We have

$$\mathrm{Ext}^*_{SG_q}(S[G_q/K_q], S[G_q/K_q]) \cong \bigoplus_{g_i \in K_q \backslash G_q / K_q} H^*(K_q \cap g_i K_q g_i^{-1}, S).$$

An element $(g_i, \alpha \in H^*(K_q \cap g_i K_q g_i^{-1}, S))$ acts on $H^*(X, S)$ by sending $x \in H^*(X, S)$ to $\pi_{2*}(\widetilde{\alpha} \smile \pi_1^*(x))$, where $\widetilde{\alpha}$ is the pullback of α to $X(K_q \cap g_i K_q g_i^{-1})$:

$$\begin{array}{ccccc} & X(K_q \cap g_i K_q g_i^{-1}) & \xrightarrow{\smile \widetilde{\alpha}} & X(K_q \cap g_i K_q g_i^{-1}) & \\ \swarrow^{\pi_1} & & & & \searrow^{\pi_2} \\ X & & & & X \end{array}$$

Finally, there is a Satake isomorphism for $q \equiv 1 \pmod{p^n}$,

$$\mathrm{DHA}_{q, \mathbf{Z}/p^n}(G) \cong \mathrm{DHA}_{q, \mathbf{Z}/p^n}(T)^W.$$

2.2. **Key idea.** We want to summarize the key idea for analyzing derived Hecke operators. To be explicit, we consider the example $G = \mathrm{Res}_{L/\mathbf{Q}} \mathrm{PGL}_2$. (Then δ is the number of complex places of L.)

Let q be a prime of L and

$$\alpha \colon \mathbf{F}_q^\times \to \mathbf{Z}/p^n$$

a homomorphism.

Then $T_{q,\alpha}$ comes from the diagram

$$\begin{array}{ccccc} & X_0(q) & \xrightarrow{\smile \widetilde{\alpha}} & X_0(q) & \\ \swarrow_{\pi_1} & & & & \searrow^{\pi_2} \\ X & & & & X \end{array}$$

Here $\widetilde{\alpha}$ comes from the composite homomorphism

$$\Gamma_0(q) \to \mathbf{F}_q^\times \xrightarrow{\alpha} \mathbf{Z}/p^n$$

sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \pmod{q}.$$

Then we obtain a derived Hecke operator

$$T_{q,\alpha} \colon H^j(X, S) \mapsto H^{j+1}(X, S).$$

How do you know that it is non-zero? (The issue is the step where you cup with $\widetilde{\alpha}$.) Note that after passing to a characteristic 0 field, the derived Hecke operators must vanish for weight reasons, since they are Galois-equivariant.

2.2.1. *Abstract situation.* Suppose you have a topological space Y with a free action of a p-group G (imagine $G = \mathbf{Z}/p^n$). You can consider the action of $S[G]$ on $H_*(Y, S)$. Also, there is a map $H^*(G, S) \to H^*(Y/G, S)$, so $H^*(G, S)$ acts on $H^*(Y/G, S)$. The two actions are sort of dual. If Y is contractible, then the first action is trivial while the second is free. If Y is G, then the first action is free and the second is trivial. Roughly, if one action is "large" then the other one is "small".

Example 2.1. In our example $G = \mathbf{Z}/p^n$ and T is an intermediate covering of $X_1(q)$ so that $Y/G = X_0(q)$.

$$X_1(q) \to Y \xrightarrow{\mathbf{Z}/p^n} X_0(q).$$

Call $Y = X_1(q)^*$. By the principle above, to show that $\smile \widetilde{\alpha}$ (which is like the second action) is non-zero, you need to know that the action of $S[\mathbf{Z}/p^n]$ on $H_*(X_1(q)^*, S)$ is far from free.

This issue comes up in another context, namely the Taylor-Wiles method. This works by adding the same auxiliary level structure. In the original incarnation you want to prove that the same action *is* free; those settings all involve $\delta = 0$. Calegari-Geraghty found a way to adapt this method when $\delta > 0$, and it involves understanding exactly how the action fails to be free. You need a way to "certify" that the action is not free, and a way to get that is to factor the action through a Galois deformation ring. In concrete terms, this means that if you have a Galois representation with nebentypus, then you can determine the nebentypus locally. You can understand the Galois deformation ring using Galois cohomology, and choose q so that the action is not very free. This is the key point that lets you show that the operators are non-zero, and the input is existence of Galois representations with expected local properties.

2.3. **Statement of main theorem.** We now give a precise statement of our main theorem. We make the following assumptions:

- We assume that G is simply-connected and split. Let T be a maximal split torus for G and $T^\vee$ be a maximal split torus for $G^\vee$.
- Assume that $H^*(X, \mathbf{Z}_p)$ is torsion-free. (Later we want to say something about torsion. When $\delta > 0$ there is torsion that doesn't lift to characteristic 0.)
- "There are no congruences between π and other forms of the same level". More precisely, let $\mathbf{T}$ be the Hecke algebra for X. The $\mathbf{T}$-action on π give a homomorphism $\mathbf{T} \to \mathbf{Z}$. Let $\mathfrak{m} = \ker(\mathbf{T} \to \mathbf{Z}_p)$. We assume that $\mathbf{T}_\mathfrak{m} \cong \mathbf{Z}_p$.
- There exist Galois representations for cohomology class for G with the expected properties. For π, the representation is of the form
$$\rho\colon G_\mathbf{Q} \to G^\vee(\mathbf{Z}_p).$$
- ρ should have large image (large enough for the Taylor-Wiles method).
- All local deformation rings for $\overline{\rho}$ at bad places are formally smooth.

The image of the p-adic regulator should be

$$\Lambda_{\text{mot},\mathbf{Z}_p} = H^1_f(\mathbf{Q}, \operatorname{Ad}^* \rho(1)).$$

This is a $\mathbf{Z}_p$-module.

Definition 2.2. For a fixed n, we define a subspace

$$\widetilde{\mathbf{T}}_{\mathbf{Z}/p^n} \subset \operatorname{End}(H^*(X, \mathbf{Z}/p^n))$$

by taking the subgroup generated by $T_{q,\alpha}$ for all q and α. We then define

$$\widetilde{\mathbf{T}} \subset \operatorname{End}(H^*(X, \mathbf{Z}_p)_\pi)$$

to be the subspace which at each n lands in $\widetilde{\mathbf{T}}_{\mathbf{Z}/p^n}$.

The main theorem is then:

Theorem 2.3. *Under all our assumptions, $H^*(X, \mathbf{Z}_p)_\pi$ is free over $\widetilde{\mathbf{T}}$, and there is an identification $\widetilde{\mathbf{T}} \cong \wedge^*(\Lambda^\vee_{\text{mot},\mathbf{Z}_p})$, so both are free modules of rank δ over $\mathbf{Z}_p$.*

Actually a crucial part of the theorem is the following characterization of the isomorphism.

Let $q \equiv 1 \pmod{p^n}$ be a prime. Let $T_q = T(\mathbf{Z}/q)$. Suppose $\overline{\rho}(\mathrm{Frob}_q)$ is conjugate to a regular element of $\widehat{T}(\mathbf{Z}_p)$. Choose such an element $t_q \in \widehat{T}(\mathbf{Z}_p)$. From the data of q and t_q, we can define maps

$$\begin{array}{ccc} & & (\mathrm{DHA}_q, \mathbf{Z}/p^n)^{(1)}_{\mathfrak{m}} \\ & \overset{H}{\nearrow} & \\ \mathrm{Hom}(T_q, \mathbf{Z}/p^n) & & \\ & \underset{G}{\searrow} & \\ & & \Lambda^{\vee}_{\mathrm{mot},\mathbf{Z}_p}/p^n \end{array}$$

The characterization is: for all $n \geq 1$, there exists $N \gg n$ such that for $q \equiv 1 \pmod{p^N}$, the isomorphism is given by explicit maps H and G in the above diagram. (We think that we can take $N = n$.)

The map H comes from the Satake isomorphism. What is the map G? We can conjugate ρ so that $\rho(\mathrm{Frob}_q) = t_q$. This gives a class in

$$\frac{H^1(\mathbf{Q}_q, \mathrm{Ad}\,\rho/p^n)}{H^1_{\mathrm{unr}}(\mathbf{Q}_q, \mathrm{Ad}\,\rho/p^n)} \cong \mathrm{Hom}(T_q, \mathbf{Z}/p^n).$$

For $\alpha \in \mathrm{Hom}(T_q, \mathbf{Z}/p^n)$, we get $G(\alpha) \in \Lambda^{\vee}_{\mathrm{mot},\mathbf{Z}_p}/p^n$ by restricting to $\mathbf{Q}_q$ and then using local duality: $G(\alpha)$ sends $\beta \in H^1(\mathbf{Q}_q, \mathrm{Ad}^*\rho(1))$ to $\langle \beta|_{\mathbf{Q}_q}, \text{image of } \alpha \rangle$.

The main point of the proof is as follows. Let's consider $G = \mathrm{PGL}_2$, so we have the diagram

$$\begin{array}{ccccc} & X_0(q) & \longrightarrow & X_0(q) & \\ \swarrow & & & & \searrow \\ X & & & & X \end{array}$$

We have $X_0(q) = X_1(q)^*/\Delta_q$, where $\Delta_q \cong \mathbf{Z}/p^n$. We need to understand the action of $H^*(\Delta_q, S)$ on $H^*(X_0(q), S)$. We can view

$$H^*(X_0(q), S) = \mathrm{Hom}_{S[\Delta_q]}(\mathrm{Chains}(X_1(q)^*), S).$$

In these terms, the action is the natural one of $\mathrm{Ext}_{S[\Delta_q]}(S, S)$ on $\mathrm{Hom}_{S[\Delta_q]}(\mathrm{Chains}(X_1(q)^*), S)$. The Taylor-Wiles-Calegari-Geraghty method shows how to choose q such that the action of $S[\Delta_q]$ on $\mathrm{Chains}(X_1(q)^*), S)$ "approaches a limit" (the acting thing becomes a polynomial algebra, and the chains become concentrated in 1 degree).

3. Weight one forms

3.1. **Numerics.** We have constructed an action of $\Lambda^{\vee}_{\mathrm{mot},\mathbf{Q}_p}$ on $H^*(X, \mathbf{Q}_p)_\pi$. We want to give evidence that it preserves the $\mathbf{Q}$-structure.

Let $X = X_1(N)$, viewed as a scheme over $\mathbf{Z}[1/N]$. Let ω be the weight one line bundle, so $\omega^2(-\text{cusps}) \cong K_X$. Let $g = \sum a_n q^n$ be a weight one cuspidal Hecke eigenform for X. The derived Hecke algebra consists of operations as follows: if $q \equiv 1 \pmod{p^n}$, and $\alpha\colon \mathbf{F}_q^* \to \mathbf{Z}/p^n$, then we get a derived Hecke operator

$$T_{q,\alpha}\colon H^0(X_{\mathbf{Z}/p^n}, \omega) \to H^1(X_{\mathbf{Z}/p^n}, \omega)$$

as explained last time.

Let $g' = \sum a_n \chi(n)^{-1} q^n$ where χ is the Nebentypus of g. There is a pairing

$$H^0(X_{\mathbf{Z}/p^n}, \omega(\text{cusps})) \times H^1(X_{\mathbf{Z}/p^n}, \omega) \to \mathbf{Z}/p^n.$$

So I can form

$$\langle T_{q,\alpha} g, g' \rangle \in \mathbf{Z}/p^n.$$

The conjecture makes a prediction for these numbers. Let $E = \mathbf{Q}(a_n)$. There is a Galois representation

$$\rho_g \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(E)$$

attached to g. From this we get a p-adic representation

$$\rho_{g,p} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(E_p).$$

Then $\Lambda_{\mathrm{mot},\mathbf{Q}_p} = H^1_f(\mathbf{Q}, \mathrm{Ad}\,\rho_{g,p}(1))$. The representation is trivialized by restricting to $H/\mathbf{Q}$. Then

$$H^1_f(\mathbf{Q}, \mathrm{Ad}\,\rho_{g,p}(1)) = (\mathcal{O}_H^* \otimes \mathrm{Ad}\,\rho_{g,p})^{\mathrm{Gal}(H/\mathbf{Q})}.$$

The motivic rational structure is

$$\Lambda_{\mathrm{mot}} = (\mathcal{O}_H^* \otimes \mathrm{Ad}\,\rho_g)^{\mathrm{Gal}(H/\mathbf{Q})}$$

which is a rank 1 E-vector space, say Eu. (The rank 1 is because this behaves like the situation $\delta = 1$.)

The conjecture (that $\wedge^{\vee}_{\mathrm{mot},\mathbf{Q}_p}$ preserves $\mathbf{Q}$-structures) implies that (in a rough form)

$$\langle T_{q,\alpha} g, g' \rangle \sim \alpha(\text{reduction of } u \bmod q).$$

We have a numerical verification joint with M. Harris [HV], and now a proof joint with Darmon, Harris and Rotger.

We describe the statement more precisely, in a special case: suppose g arises from a complex cubic field L. In this case $\zeta_L(s) = \zeta(s) L(s,g)$, $\Lambda_{\mathrm{mot}} = \mathcal{O}_L^{\times} \otimes \mathbf{Q}$. Fix $u \in \mathcal{O}_L^*$. The conjecture implies that for $q \equiv 1 \pmod{p^n}$ inert in $\mathbf{Q}(\sqrt{\mathrm{disc}\,L})$, so there is a unique degree 1 prime of L above q, which we denote $\mathfrak{q}$,

$$\langle T_{q,\alpha} g, g' \rangle = \mathbf{Q}^* \cdot \alpha(u \mod \mathfrak{q}).$$

The rational number is independent of p and q.

3.2. **Proof.** We can prove the prediction for all g with dihedral Galois representation, assuming $n = 1$ and "the Merel constant for u is non-zero mod p." In other words, under these assumptions we exhibit a unit satisfying this condition.

The Merel constant is:

$$u := \prod_{i=1}^{(q-1)/2} i^{-8i} \in (\mathbf{Z}/q)^*.$$

We want to produce some unit such that

$$\langle T_{q,\alpha} g, g' \rangle \sim \alpha(\text{unit mod } q).$$

We consider the modular curve $X_1(N)$, and let $X_1(N,q)$ be the modular curve obtained by adding $\Gamma_0(q)$ structure to $X_1(N)$. The homomorphism $\alpha \colon \Gamma_0(q) \to \mathbf{Z}/p$ induces a cohomology class in $H^1_{\text{ét}}(X_0(q), \mathbf{Z}/p)$ which we push forward to $H^1(X_0(q)_{\mathbf{Z}/p}, \mathcal{O})$. We want to compute the (Serre duality) pairing

$$\langle \underbrace{\pi_{2*}(\pi_1^* g \smile \alpha)}_{\in H^1(\omega)}, g' \rangle_{X_1(N)} = \langle \pi_1^* g \smile \alpha, \pi_2^* g' \rangle_{X_1(N,q)}.$$

Now,

$$\langle \underbrace{\pi_1^* g \smile \alpha}_{H^1(\omega)}, \underbrace{\pi_2^* g'}_{H^0(\omega(-\text{cusps}))} \rangle_{X_1(N,q)} = \langle \underbrace{\pi_1^* g \smile \pi_2^* g'}_{\text{weight 2 cusp form for } X_1(N,q)}, \underbrace{\widetilde{\alpha}}_{H^1(\mathcal{O})} \rangle \in \mathbf{F}_p.$$

Finally, since $\widetilde{\alpha}$ is pulled back from $X_0(q)$, the above is

$$\langle \mathrm{proj}_{X_0(q)}^{X_1(N,q)}(\pi_1^* g \smile \pi_2^* g'), \underbrace{\widetilde{\alpha}}_{H^1(\mathcal{O})} \rangle$$

Merel showed that if G is the reduction mod p of the weight 2 Eisenstein series on $X_0(q)$, which becomes a cusp form, then

$$\langle G, \widetilde{\alpha} \rangle = \alpha(v)$$

where $v = \prod_{i=1}^{(q-1)/2} i^{-8i}$ is the Merel constant.

In conclusion we want

$$\langle \mathrm{proj}_{X_0(q)}^{X_1(N,q)}(\pi_1^* g \smile \pi_2^* g'), \underbrace{\widetilde{\alpha}}_{H^1(\mathcal{O})} \rangle \sim \alpha(\text{unit mod } q).$$

The proof is by a theta correspondence. It is an important aspect that Merel's constant shows up on the right hand side. (The proof was inspired by ideas in Lecouturier's thesis, although it doesn't directly use any of the results.)

Remark 3.1. The $\widetilde{\alpha}$ is an Eisenstein class, meaning that it is killed by the Eisenstein ideal (concretely, this means that every Hecke operator acts by its degree).

(Assume $p \geq 5$.) Let SS_q be the set of supersingular elliptic curves over $\overline{\mathbf{F}}_q$. Let X be the set of functions $SS_q \to \mathbf{Z}$. It has Hecke operators T_ℓ, and there is an inner product

$$\langle \cdot, \cdot \rangle \colon X \otimes X \to \mathbf{Z}[1/6]$$

which is self-adjoint for this Hecke action:

$$\langle T_\ell f, g \rangle = \langle f, T_\ell g \rangle.$$

Let $M_2(q)$ and $S_2(q)$ be the spaces of weight 2, respectively weight 2 cuspidal, modular forms for $\Gamma_0(q)$. These have q-expansions of the form

$$\sum a_n q^n, \quad a_0 \in \frac{1}{2}\mathbf{Z}, a_i \in \mathbf{Z}.$$

The Hecke algebras for X and $M_2(q)$ are isomorphic. There is a map

$$\Theta \colon X \otimes_T X \to M_2(q)$$

sending

$$f \otimes g \mapsto \frac{\langle f, 1 \rangle \langle g, 1 \rangle}{2} + \sum_{m \geq 1} \langle T_m f, g \rangle q^m.$$

This is a Hecke-equivariant isomorphism after inverting 2 and 3.

The basic point is that the class $\widetilde{\alpha}$ will become more comprehensible when transported to $X \otimes_T X$. There is a way of making a function on CM points, which becomes a function in $X \otimes_T X$ supported on the reduction mod q of CM points, and makes the pairing into

$$\langle \Theta \begin{pmatrix} \text{explicit element supported} \\ \text{on reduction of CM points} \end{pmatrix}, \widetilde{\alpha} \rangle = \langle \begin{pmatrix} \text{explicit element supported} \\ \text{on reduction of CM points} \end{pmatrix}, \Theta^* \widetilde{\alpha} \rangle.$$

Here Θ^* is the dual map

$$(S_2(q)/p)^\vee \to [(X \otimes_T X)^0/p]^\vee.$$

The claim is that $\widetilde{\alpha}$ is sent to the element $\mathcal{E} \otimes 1 + 1 \otimes \mathcal{E}$ where $\mathcal{E}$ is a specific element of X (identified with its own dual via the canonical pairing), namely the function from supersingular elliptic curves to $\mathbf{F}_{q^2}^*$ given by

$$\mathcal{E} = \frac{\Delta^{q+1}}{E_{q+1}^{12}}.$$

Morally this is "Δ evaluated on the supersingular points". This relates the pairing to a pairing between reduction of CM points mod q and something in Δ (modular units), hence the reduction of elliptic units mod q. The calculation involves the product over all supersingular elliptic curves of $\frac{\Delta^{q+1}}{E_{q+1}^{12}}$, and this is the Merel constant v.

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