MODULAR SYMBOLS AND ARITHMETIC, II

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1. Recap

Let $p \geq 5$ be a prime dividing a positive integer N. In the previous talk, we constructed a map

$$\varpi\colon H_1(X_1(N); \mathbf{Z}_p)^+/I \to H^2(\mathbf{Z}[1/p, \mu_N]; \mathbf{Z}_p(2))^+.$$

Here I is the *Eisenstein ideal*, generated by $T_\ell - 1 - \ell\langle \ell \rangle$ for $\ell \nmid N$ and $U_\ell - 1$ for $\ell \mid N$. This map sent the Manin symbol $[u : v]$ to the Steinberg symbol $\{1 - \zeta_N^u, 1 - \zeta_N^v\}$.

This map was part of the general philosophy

> "The geometry of GL_n/F (near a boundary component) is related to the arithmetic of GL_{n-1}/F."

In this talk we will construct a map in the opposite direction:

$$\Upsilon\colon H^2(\mathbf{Z}[1/p, \mu_N]; \mathbf{Z}_p(2))^+ \to H_1(X_1(N); \mathbf{Z}_p)^+/I.$$

We recall the notation from last time:

$$Y = H^2(\mathbf{Z}[1/p, \mu_N]; \mathbf{Z}_p(2))^+$$
$$S = H_1(X_1(N); \mathbf{Z}_p)^+.$$

This map will not be explicit. It will be constructed out of the Galois action on $H^1_{\text{ét}}(X_1(N)_{\overline{\mathbf{Q}}}; \mathbf{Z}_p(1))$.

2. Galois representations

We recall how to construct the Galois representation attached to a newform of weight 2. Let f be a weight 2 newform of level N, with q-expansion

$$f = \sum_{n=1}^{\infty} a_n q^n.$$

To f we can attach a Galois representation $\rho_f\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_f)$, where $\mathcal{O}_f := \mathbf{Z}_p(\{a_n\})$, as follows.

The eigenform f defines a homomorphism $\mathfrak{h}_2(N, \mathbf{Z}_p) \to \mathcal{O}_f$, sending $T_n \mapsto a_n$, and we let I_f be the kernel. Let $T_f = H^1_{\text{ét}}(X_1(N)_{\overline{\mathbf{Q}}}; \mathbf{Z}_p(1))/I_f$. This turns out to have rank 2 over $\mathcal{O}_f$, and has a Galois action which furnishes the representation ρ_f.

We say f is *ordinary* if $a_p \in \mathcal{O}_f^\times$. Then T_f is ordinary, meaning there is an exact sequence

$$0 \to T_{f,\mathrm{sub}} \to T_f \to T_{f,\mathrm{quo}} \to 0$$

of $\mathcal{O}_f[G_{\mathbf{Q}_p}]$-modules where $T_{f,\mathrm{sub}}$ and $T_{f,\mathrm{quo}}$ are both rank 1 over $\mathcal{O}_f$. With respect to the corresponding basis, our representation ρ_f restricted to the inertia group I_p looks like

$$\rho_f|_{I_p} = \begin{pmatrix} \chi_p \epsilon & * \\ 0 & 1 \end{pmatrix}$$

where χ_p is the p-adic cyclotomic character and ϵ is the Nebentypus.

The Eisenstein ideal $I \subset \mathfrak{H} = \mathfrak{H}_2(N, \mathbf{Z}_p)$ acts on the space of weight 2 modular forms, $\mathcal{M} := M_2(N, \mathbf{Z}_p)$. We have an isomorphism

$$\mathfrak{H}/I \xrightarrow{\sim} \mathbf{Z}_p[\Delta]$$

where $\Delta = (\mathbf{Z}/N\mathbf{Z})^*/\pm 1$, via $T_\ell \mapsto 1 + \ell\langle \ell \rangle$ and $U_\ell \mapsto 1$. There is a surjection $\mathfrak{H} \to \mathfrak{h}$, and $\mathfrak{h}/I$ is actually *finite*: it measures congruences between the Eisenstein series $E_{2,\chi}$ and newforms.

I am going to explain the idea behind the construction of the map Υ, which goes back to Ribet in his proof of the converse to Herbrand.

Consider $\rho_f \mod I$. With respect to the basis we've already chosen we can write it as

$$\rho_f \pmod{I} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$$

It's reducible. We also know that $\overline{c}|_{G_{\mathbf{Q}_p}} = 0$. It turns out that $(\det \overline{\rho}_f^{-1})\overline{c}$ is a 1-cocycle on $G_{\mathbf{Q}}$, with the property that its restriction to $G_{\mathbf{Q}(\mu_{Np^\infty})}$ is a homomorphism that is unramified everywhere. The idea is that you can descend this extension back down to $\mathbf{Q}(\mu_N)$, which has an odd action of Δ, which gives a quotient of the p-part of $\mathrm{Cl}_{\mathbf{Q}(\mu_N)}$.

3. The Main Conjecture

Let Δ' be the prime-to-p part of Δ. Let A be a $\mathbf{Z}_p[\Delta]$-module. For any $\theta \colon \Delta' \to \overline{\mathbf{Q}}_p^*$ which is an even, prime-to-p order character of $(\mathbf{Z}/N\mathbf{Z})^\times$, we define

$$A_\theta := A \otimes_{\mathbf{Z}_p[\Delta']} \mathbf{Z}_p(\theta).$$

We have

$$A \cong \bigoplus_{[\theta]} A_\theta.$$

Let $A' = \bigoplus_{[\theta]} A_\theta$ where $[\theta]$ runs over classes such that θ has conductor Mp, and $\theta\omega^{-1}(p) \neq 1$. Here ω is the obvious composition $(\mathbf{Z}/N\mathbf{Z})^\times \to (\mathbf{Z}/p\mathbf{Z})^\times \hookrightarrow \mu_p(\mathbf{Z}_p)$, and we view $\theta\omega^{-1}$ as a primitive Dirichlet character. If $M = 1$, we also ask that $\theta \neq 1, \omega^2$.

Now, $\mathfrak{h}$ is a $\mathbf{Z}_p[\Delta]$-module. In our convention, $j \in \Delta$ acts as $\langle j \rangle^{-1}$.

For an $\mathfrak{h}$-module A, we define

$$A_{\mathfrak{m}} = \bigoplus_{[\theta]} A'_{\mathfrak{m}_\theta}$$

where $\mathfrak{m}_\theta$ is the unique maximal ideal of $\mathfrak{h}_\theta$ containing I. This is the "Eisenstein part" of A.

Theorem 3.1 (Mazur-Wiles, Wiles)**.** *We have $\mathfrak{h}_{\mathfrak{m}}/I \cong \Lambda/(\xi)$ where $\Lambda = \mathbf{Z}_p[\Delta]'$ and $\xi \in \Lambda$ has the property that for all $\chi \colon \Delta \to \overline{\mathbf{Q}}_p^\times$ which are even with prime-to-p order,*

$$\widetilde{\chi}(\xi) = L_p(\omega^2\chi^{-1}, -1).$$

We want to sketch the proof. The Eisenstein series induces an isomorphism $\mathfrak{H}_{\mathfrak{m}}/I \xrightarrow{\sim} \Lambda$. Since $\widetilde{\chi}(\xi)/2$ is the constant coefficient of $E_{2,\chi^{-1}}$, modding out by this should make the Eisenstein series "look like a cusp form", i.e. induce a map

$$\mathfrak{h}_{\mathfrak{m}}/I \twoheadrightarrow \Lambda/\xi.$$

The hard part is injectivity.

Mazur-Wiles proved injectivity as a consequence of the proof of the Iwasawa main conjecture, but this seems a little backward. Emerton observed that there is a direct proof, which now present.

Let $\mathfrak{S} := S_2(N, \mathbf{Z}_p) \hookrightarrow M$. There is a perfect pairing

$$\mathfrak{h} \times \mathfrak{S} \to \mathbf{Z}_p$$

given by $(T, f) \mapsto a_1(Tf)$. This extends to a perfect pairing

$$\mathfrak{H} \times \mathfrak{M}^0 \to \mathbf{Z}_p$$

where $\mathfrak{M}^0$ consists of modular forms with q^n-coefficient in $\mathbf{Z}_p$ for $n \geq 1$ and constant coefficient in $\mathbf{Q}_p$.

If $\theta \neq \omega^2$ then $\mathfrak{M}_\theta = \mathfrak{M}^0_\theta$. We have an exact sequence

$$0 \to \mathfrak{S}_{\mathfrak{m}} \to \mathfrak{M}_{\mathfrak{m}} \xrightarrow{T_*} \Lambda \to 0.$$

Think of Λ as being generated by Eisenstein series and T_* as the constant term. This sequence splits over $\mathbf{Q}_p$, but not over $\mathbf{Z}_p$. The rational splitting

$$\mathfrak{M} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \leftarrow \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}_p$$

is an equivariant version of $1 \mapsto \frac{2}{\widetilde{\chi}(\xi)} E_{2,\chi^{-1}}$. It induces a splitting $s \colon \mathfrak{M}_{\mathfrak{m}} \otimes \mathbf{Q}_p \to \mathfrak{S}_{\mathfrak{m}} \otimes \mathbf{Q}_p$. The *congruence module* is $s(\mathfrak{M}_{\mathfrak{m}})/\mathfrak{S}_{\mathfrak{m}}$ which by what we said is $\Lambda/(\xi)$. But we want the statement for $\mathfrak{h}_{\mathfrak{m}}/I$, so we take the dual sequence.

$$\begin{array}{ccccccccc}
 & & & & I & & I & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda & \xrightarrow{1 \mapsto T_0} & \mathfrak{H}_{\mathfrak{m}} & \longrightarrow & \mathfrak{h}_{\mathfrak{m}} & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda & \xrightarrow{1 \mapsto \xi} & \mathfrak{H}_{\mathfrak{m}}/I = \Lambda & \longrightarrow & \mathfrak{h}_{\mathfrak{m}}/I & \longrightarrow & 0
\end{array}$$

4. Construction of Υ

The map Υ will be a canonical version of the cocycles which appear in the proof of Mazur-Wiles.

First we recall some facts about ordinary Hecke algebras. Let $\mathcal{T} = H^1_{\text{ét}}(X_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))^{\text{ord}}$. We have $U_p \in (\mathfrak{h}^{\text{ord}})^*$. What properties does $\mathcal{T}$ have?

(1) It is ordinary, so we have an exact sequence

$$0 \to \mathcal{T}_{\text{sub}} \to \mathcal{T} \to \mathcal{T}_{\text{quo}} \to 0$$

where $\mathcal{T}_{\text{sub}}, \mathcal{T}_{\text{quo}}$ have rank 1 over $\mathfrak{h}^{\text{ord}}$. In fact $\mathcal{T}_{\text{sub}}$ is free, and $\mathcal{T}_{\text{quo}}$ is unramified as a $G_{\mathbf{Q}}$-module.

We have a similar story for the open modular curve. Let $\widetilde{\mathcal{T}} = H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))^{\text{ord}}$. Then there is an exact sequence

$$0 \to \widetilde{\mathcal{T}}_{\text{sub}} \to \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}_{\text{quo}} \to 0$$

The natural map induces an equality on the subs.

(2) There is a p-adic Eichler-Shimura Theorem (due to Ohta). There is a functor D from unramified $\mathfrak{H}[G_{\mathbf{Q}_p}]$-modules to compact $\mathfrak{H}$-modules, given by

$$D(M) = (M \widehat{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}_p}))^{\varphi_p \otimes \varphi_p = 1}.$$

The functor D is naturally but not canonically isomorphic to the forgetul functor. However it is canonical if we restrict to trivial $G_{\mathbf{Q}_p}$-modules.

The upshot is that $D(\mathcal{T}_{\text{quo}}) \cong \mathfrak{S}^{\text{ord}}$ and $D(\widetilde{\mathcal{T}}_{\text{quo}}) \cong \mathfrak{M}^{\text{ord}}$.

(3) There is a twisted Poincaré duality

$$\mathcal{T} \times \mathcal{T} \to \mathbf{Z}_p[\Delta]^{\iota}(1).$$

The twisting ι means that if $\sigma \in G_{\mathbf{Q}}$ has $\sigma(\zeta_N) = \zeta_N^j$, then it acts as $[j]^{-1}$ on $\mathbf{Z}_p[\Delta]$. The pairing is

$$(x, y) = \sum_{j \in (\mathbf{Z}/N\mathbf{Z})^{\times}} (x \smile \langle j \rangle^{-1} w_N y)[j].$$

With this definition, we have

$$(Tx, y) = (x, Ty) \text{ for } T \in \mathfrak{h}$$

and the pairing is $G_{\mathbf{Q}}$-equivariant.

Theorem 4.1 (S, Fukaya-Kato)**.** *Let $T = \mathcal{T}_{\mathfrak{m}}/I\mathcal{T}_{\mathfrak{m}}$. There is an exact sequence*

$$0 \to T^+ \to T \to T^- \to 0$$

of $\mathfrak{h}[G_{\mathbf{Q}}]$-modules such that $T^+ \cong S_{\mathfrak{m}}/IS_{\mathfrak{m}}$ has trivial $G_{\mathbf{Q}}$-action, and $T^- \cong (\Lambda/\xi)^{\iota}(1)$ canonically (i.e. has a canonical generator). Moreover, the sequence is locally split at all $\ell \mid N$.

How does this give what we want? When we have an exact sequence like this, we get a 1-cocycle $G_{\mathbf{Q}} \to \operatorname{Hom}(T^-, T^+)$. Composing this with the map $\operatorname{Hom}(T^-, T^+) \to T^+$ given by evaluation on the canonical generator, we get a cocycle $G_{\mathbf{Q}} \to T^+$. Now restrict this cocycle to $G_{\mathbf{Q}(\mu_{Np^\infty})}$. It factors through X_∞, the Galois group of the maximal unramified abelian pro-p extension (by local splitness).

The map is not Λ-equivariant, but becomes equivariant after twisting by 1. That is, we get a map

$$\Upsilon' \colon X_\infty(1) \to T^+$$

with the equivariance property $\sigma_j x \mapsto \langle j \rangle^{-1} \Upsilon'(x)$ (this was the reason for the twist) where $\sigma_j(\zeta_{Np^r}) = \zeta_{Np^r}^j$. Then we make a choice of of compatible sequence of roots of unity to identify X_∞ with $X_\infty(1)$, transferring the map to $X_\infty \to T^+$.

Now we have to descend back down to Y. The key point is that we have an isomorphism

$$X_\infty(1)' \cong \varprojlim H^2(\mathbf{Z}[1/p, \mu_{Np^r}]; \mathbf{Z}_p(2))',$$

the transition maps being corestrictions. This can be rephrased as $H^2_{\text{Iw}}(\mathbf{Z}[1/p, \mu_{Np^\infty}]; \mathbf{Z}_p(2))$. These are cohomological dimension 2, so corestriction gives an isomorphism on coinvariants.

So taking coinvariants for $\mathrm{Gal}(\mathbf{Q}(\mu_{Np^\infty})/\mathbf{Q}(\mu_N))$ we get

$$\begin{array}{ccc} X_\infty(1)' & \xrightarrow{\sim} & \varprojlim H^2(\mathbf{Z}[1/p,\mu_{Np^r}];\mathbf{Z}_p(2))' \\ & & \Big\downarrow{\scriptstyle \mathrm{Gal}(\mathbf{Q}(\mu_{Np^\infty})/\mathbf{Q}(\mu_N))-\mathrm{coinv}} \\ \Big\downarrow{\scriptstyle \Upsilon'} & & H^2(\mathbf{Z}[1/p,\mu_N];\mathbf{Z}_p(2))' \\ & \swarrow{\scriptstyle \Upsilon} & \\ S_{\mathfrak{m}}/IS_{\mathfrak{m}} & & \end{array}$$

The map Υ' factors through the $\mathrm{Gal}(\mathbf{Q}(\mu_{Np^\infty})/\mathbf{Q}(\mu_N))$-coinvariants by inspection of the equivariance condition.

5. Sharifi's Conjecture

Conjecture 5.1 (Sharifi)**.** *The maps Υ and ϖ are inverse isomorphisms*

$$\Upsilon\colon Y' \to S'/IS'$$

and

$$\varpi\colon S'/IS' \to Y'.$$

There's an issue of choosing lattices. Wiles chooses the smallest possible T_{sub}. He shows it is surjective, hence the characteristic ideal of the image divides the characteristic ideal of the domain. Here we are choosing a "natural" lattice. But for all we know the map could be 0. The conjecture implies that the lattice must be extremal.

Theorem 5.2 (Fukaya-Kato, FKS)**.** *We have $\xi'\Upsilon\circ\varpi = \xi'\colon S/IS \to S/IS$ where $\widetilde{\chi}(\xi') = L_p'(\omega^2\chi^{-1},-1)$ for all χ.*

Some progress has been made recently by Ohta:

Theorem 5.3 (Ohta)**.** *If $p \nmid \varphi(N)$ and $\theta|_{(\mathbf{Z}/p\mathbf{Z})^\times}$ has nontrivial kernel, then Υ_θ is an isomorphism.*

Fukaya and Kato additionally showed that if $\mathfrak{H}_{\mathfrak{m}}$ and $\mathfrak{h}_{\mathfrak{m}}$ are both Gorenstein and the p-adic power series interpolating L-functions $L_p(\omega^2\chi^{-1},s-1)$ has no square factors, then the Conjecture holds.

Wake and Wang-Erickson prove that if $p \nmid h^+_{\mathbf{Q}(\mu_N)}$ then $\mathfrak{H}_{\mathfrak{m}}$ and $\mathfrak{h}_{\mathfrak{m}}$ are Gorenstein.

6. Another construction of Υ

Here's another construction of Υ which is sometimes useful. The sequence

$$0 \to T^+ \to T \to T^- \to 0$$

is locally split, as we have discussed (Theorem 4.1). There is connecting homomorphism

$$H^2(\mathbf{Z}[1/N], T^-(1)) \xrightarrow{\partial} H^3_c(\mathbf{Z}[1/N], T^+(1))$$

Now, we have

$$H^2(\mathbf{Z}[1/N], T^-(1)) \cong H^2(\mathbf{Z}[1/N], (\Lambda/\xi)^\iota(2)) \cong H^2(\mathbf{Z}[1/N], \Lambda^\iota(2))/\xi$$

where the last isomorphism follows from the vanishing of $H^3(\mathbf{Z}[1/N],-)$. By Shapiro's Lemma, $H^2(\mathbf{Z}[1/N],\Lambda^\iota(2))/\xi \cong H^2(\mathbf{Z}[1/N,\mu_N],\mathbf{Z}_p(2))'/\xi$. But ξ kills $H^2(\mathbf{Z}[1/N,\mu_N],\mathbf{Z}_p(2))'$ already by a Stickelberger-type theorem, so in the end we just get that the domain of this boundary map is $H^2(\mathbf{Z}[1/N],\Lambda^\iota(2))/\xi \cong Y'$.

Now you might guess that the target is S/IS. Let's see: by Poitou-Tate duality we have

$$H^3_c(\mathbf{Z}[1/N], T^+(1)) \cong H^0(\mathbf{Z}[1/N], (T^+)^\vee)^\vee$$

where $\vee$ is the Pontrjagin dual, so this is just $(((T^+)^\vee)^{\mathbf{G}_\mathbf{Q}})^\vee = T^+$ by the triviality of Galois action on T^+, which by Theorem 4.1 is S'/IS'.

Remark 6.1. What is compactly supported cohomology? We define the *compactly supported cochains*

$$C_c(\mathbf{Z}[1/N]; A) := \mathrm{Cone}\left(C(\mathbf{Z}[1/N]; A) \to \bigoplus_{\ell \mid N} C(\mathbf{Q}_\ell, A)\right)[-1].$$

This gives a long exact sequence in cohomology by construction.

Next we define

$$C_f(\mathbf{Z}[1/N], T(1)) := \mathrm{Cone}\left(C(\mathbf{Z}[1/N], T(1)) \to \bigoplus_{\ell \mid N} C(\mathbf{Q}_\ell, T^+(1))\right)[1]$$

using the local splittings. By construction, there is an exact sequence of complexes

$$0 \to C_c(\mathbf{Z}[1/N], T^+(1)) \to C_f(\mathbf{Z}[1/N], T(1)) \to C(\mathbf{Z}[1/N], T^-(1)) \to 0.$$

The associated long exact sequence then induces the boundary map used above.

7. Proof of Theorem 4.1

Consider $\widetilde{\mathcal{T}}/\mathcal{T} \cong \widetilde{\mathcal{T}}^+/\mathcal{T}^+ \cong M/S$, where $M = H_1(X_1(N), C_1(N); \mathbf{Z}_p)^+$. In turn, M/S is isomorphic to Λ via the generator $\{0 \mapsto \infty\}$, essentially by a result of Ohta.

On the other hand we have the Manin-Drinfeld style splitting

$$s \colon \widetilde{\mathcal{T}} \otimes \mathbf{Q}_p \cong \mathcal{T} \otimes \mathbf{Q}_p.$$

We can again consider the congruence module $s(\widetilde{\mathcal{T}})/\mathcal{T}$, which again is $\Lambda/(\xi)$. Galois acts trivially on this quotient.

Recall Ohta's pairing

$$\mathcal{T} \times \mathcal{T} \to \Lambda^\iota(1).$$

We can extend this to a map

$$s(\widetilde{\mathcal{T}}) \times \mathcal{T} \to \frac{1}{\xi}\Lambda^\iota(1). \tag{7.1}$$

Let $T = \mathcal{T}/I\mathcal{T}$. Then (7.1) descends to

$$s(\widetilde{\mathcal{T}})/\mathcal{T} \times T \to (\frac{1}{\xi}\Lambda/\Lambda)^\iota(1) \xrightarrow{\xi} (\Lambda/\xi)^\iota(1).$$

Pairing with the generator $\{0 \to \infty\}$ of $s(\widetilde{\mathcal{T}})/\mathcal{T}$ gives a map $T \twoheadrightarrow (\Lambda/\xi)^\iota(1) =: Q$ of $\mathfrak{h}/I[G_\mathbf{Q}]$-modules. We have an extension of the form

$$0 \to P \to T \to Q \to 0 \tag{7.2}$$

Consider the sequence

$$0 \to \mathcal{T}_{\mathrm{sub}} \to \mathcal{T} \to \mathcal{T}_{\mathrm{quo}} \to 0.$$

The point is that this has a splitting when restricted to $G_{\mathbf{Q}_p}$. On θ-parts, if $(\theta\omega^{-1})|_{(\mathbf{Z}/p\mathbf{Z})^\times} \neq 1$ then we get a splitting by looking at the action of I_p. If it's $(\theta\omega^{-1})|_{(\mathbf{Z}/p\mathbf{Z})^\times} = 1$ but $(\theta\omega^{-1})(p) \neq 1$, we get a splitting by looking at Frobenius.

Using this we deduce that $0 \to P \to T \to Q \to 0$ is locally split. With respect to the splitting $T = T_{\mathrm{sub}} \oplus T_{\mathrm{quo}}$, we can write

$$\overline{\rho} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}.$$

Since $\overline{b} = 0$ and $\overline{c}|_{G_{\mathbf{Q}_p}} = 0$, and $\overline{a}\overline{d} = \det \overline{\rho}(\sigma) = \chi_p(\sigma)\langle\sigma\rangle$. On the other hand, $\overline{a}(\sigma)$ describes the action of σ on $T_{\mathrm{sub}} \cong (\Lambda/\xi)^{\iota}(1)$, which is $\chi_p(\sigma)\langle\sigma\rangle$. Hence we deduce $\overline{d}(\sigma) = 1$.

Now, consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & T_{\mathrm{sub}} & & & & \\
 & & & & \downarrow & \searrow & & & \\
0 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & \searrow & \downarrow & & & & \\
 & & & & T_{\mathrm{quo}} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Now, T_{sub} and Q are both abstractly isomorphic to Λ/I. Also, Q can't map to T_{quo} because the actions are incompatible. So this forces $T_{\mathrm{sub}} \cong T^-$ and then $T^+ \cong T_{\mathrm{quo}}$. This also gives the local splitting of (7.2). □

8. A loose end

We showed earlier that $H^2(\mathbf{Z}[1/N], T^-(1)) \cong Y$. We'd like to explain why we also have $H^1(\mathbf{Z}[1/N], T^-(1)) \cong Y$. There is a long exact sequence associated to

$$0 \to \Lambda \to \Lambda \to \Lambda/(\xi) \to 0$$

which looks like

$$0 \to H^1(\mathbf{Z}[1/N], \Lambda^{\iota}(2))/(\xi) \to H^1(\mathbf{Z}[1/N], T^-(1)) \to H^2(\mathbf{Z}[1/N], \Lambda^{\iota}(2))[\xi] \to 0$$

Now as we said earlier, $H^2(\mathbf{Z}[1/N], \Lambda^{\iota}(2)) \cong Y$, and it's already killed by ξ. So we just need just that $H^1(\mathbf{Z}[1/N], \Lambda^{\iota}(2))/(\xi) = 0$. This group comes from units. Going up to $\mathbf{Z}[1/N, \mu_N]$ by Shapiro's lemma as before: $H^1(\mathbf{Z}[1/N], \Lambda^{\iota}(2))/(\xi) = 0 \cong H^1(\mathbf{Z}[1/N, \mu_N], \mathbf{Z}_p(2))/(\xi) = 0$. Then thanks to the (2) twist, you get "odd" units instead of N-units (which would have been (1)). Anyway, the point is that, thanks to the assumption $\theta \neq \varpi^2$, we win because $(\mathbf{Z}_p(2)_{G_{\mathbf{Q}(\mu_N)}})' = 0$.

The upshot is that $H^1(\mathbf{Z}[1/N], T^-(1))$ and $H^2(\mathbf{Z}[1/N], T^-(1))$ are both identified with Y. A natural map from H^1 to H^2 is cupping with the (logarithm of the) cyclotomic character $\chi_p \in H^1(\mathbf{Z}[1/N], \mathbf{Z}_p)$. What does this correspond to on Y? It turns out to be multiplication by the derivative of the p-adic L-function, ξ'.

$$\begin{array}{ccc}
H^1(\mathbf{Z}[1/N], T^-(1)) & \longrightarrow & H^2(\mathbf{Z}[1/N], T^-(1)) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\quad \xi' \quad} & Y
\end{array}$$