COHOMOLOGY OF ARITHMETIC GROUPS AND EISENSTEIN SERIES: AN INTRODUCTION, I

LECTURES BY JOACHIM SCHWERMER,
NOTES BY TONY FENG

Contents

1. Setup

[1]

Let k be a number field with ring of integers $\mathcal{O}_k$. Let $V = V_\infty \cup V_f$ be the set of places, where V_∞ consists of the subset of archimedean places. For each $v \in V_\infty$, we denote by σ_v the corresponding embedding

$$k \hookrightarrow k_v = \mathbf{C} \text{ or } \mathbf{R}.$$

Let G be a connected reductive/semisimple algebraic group over k. Choose an embedding $\rho \colon G \hookrightarrow \mathrm{GL}_n$. Then we can define

$$G(\mathcal{O}_k) := G(k) \cap \mathrm{GL}_n(\mathcal{O}_k).$$

Let

$$G_\infty = \mathrm{Res}_{k/\mathbf{Q}} G(\mathbf{R}) = \prod_{v \in V_\infty} G(k_v)$$

which is a real Lie group. There is a diagonal embedding

$$G(k) \hookrightarrow G_\infty.$$

For $\Gamma \subset G(\mathcal{O}_k)$ a finite index subgroup, we can view Γ as a discrete subgroup in G_∞. It is an *arithmetic group.*

For $v \in V_\infty$, let $K_v \subset G_v$ be a maximal compact subgroup. Let $X_v = K_v \backslash G_v$ for $v \in V_\infty$. Then we have $X_v \cong \mathbf{R}^{d(G_v)}$ where $d(G_v) = \dim G_v - \dim K_v$.

Let $X = \prod_{v \in V_\infty} X_v$, which has dimension $d(G) = \sum d(G_v)$.

The arithmetic subgroup Γ acts properly discontinuously on X. If Γ is torsion-free then it moreover acts freely, and X/Γ is a manifold. (For this statement it is necessary to assume that G is semisimple, although it is not hard to salvage for reductive G.)

Let me remind you of the following important *compactness criterion.*

Date: October 24, 2017.
[1]Thanks to Joachim Schwermer for corrections.

Theorem 1.1. *(1) The quotient $G_\infty/G(\mathcal{O}_k)$ has finite Haar measure if and only if $X_k^*(G) = 1$. (Here $X_k^*(G)$ is the group of characters of G defined over k.) Note that the condition is the same for any Γ.*

(2) Furthermore, X is compact if and only if $X_k^*(G) = 1$ and every unipotent element in $G(k)$ belongs to the radical RG.

Corollary 1.2. *Suppose k has at least one place $v \in V_\infty$ such that G_v is compact, and $|V_\infty| \geq 2$. Then $G_\infty/G(\mathcal{O}_k)$ is compact if and only if $X_k(G) = 1$.*

Example 1.3. If G is the isometry group of a quadratic form which is definite at some $v \in V_\infty$, then the compactness assumption of Corollary 1.2 is satisfied.

2. Cohomology at infinity of locally symmetric spaces

We want to consider $H^*(X/\Gamma, E)$ where (η, E) is a rational representation of G. This coincides with $H^*_{\mathrm{dR}}(X/\Gamma, E)$, and as was explained in Laurent's talks, this can be interpreted in terms of $(\mathfrak{g}, K)$-cohomology:

$$H^*_{\mathrm{dR}}(X/\Gamma, E) \cong H^*(\mathfrak{g}, K; C^\infty(G_\infty/\Gamma) \otimes E).$$

If Γ is a *congruence* subgroup, then by a very deep result of Franke, we may further identify

$$H^*(\mathfrak{g}, K; C^\infty(G_\infty/\Gamma) \otimes E) \cong H^*(\mathfrak{g}, K; \mathcal{A}(G_\infty/\Gamma) \otimes E)$$

where $\mathcal{A}(G_\infty/\Gamma)$ is the space of automorphic forms.

Suppose $\operatorname{rank}_k G > 0$. Then we are in the situation where X/Γ is not compact, but it is of finite volume. We have various compactifications of X/Γ, and we are interested in the cohomology that relates to the boundary of these compactifications.

2.1. **Example: modular curves.** Think of the fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$ acting on $\mathbb{H} = \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbf{R})$. It has an infinite direction $y \to \infty$, which looks like a cusp because the metric decays like $\frac{dxdy}{y^2}$. One way to compactify is to add a point at ∞. Another option is to instead add a circle at ∞.

What are the relative advantages of these compactifications? In the first case you get a Riemann surface. On the other hand, the compactification in the second case is homotopy equivalent to the space obtained by cutting off the cusp at some finite length. More precisely, you can define a distance function to the cusp, and this is is a Morse function with no critical points far out.

We'll denote by $\overline{\mathbb{H}}/\Gamma$ the second compactification. Then as we have discussed, the inclusion $\mathbb{H}/\Gamma \hookrightarrow \overline{\mathbb{H}}/\Gamma$ induces an isomorphism:

$$H^*(\overline{\mathbb{H}}/\Gamma, \mathbf{C}) \xrightarrow{\sim} H^*(\mathbb{H}/\Gamma, \mathbf{C}).$$

On the other hand, we have a map

$$H^*(\overline{\mathbb{H}}/\Gamma, \mathbf{C}) \to H^*(\partial(\overline{\mathbb{H}}/\Gamma), \mathbf{C}). \tag{2.1.1}$$

Now, $\partial(\overline{\mathbb{H}}/\Gamma)$ is a disjoint union of S^1, so we know its cohomology.

Consider the map (2.1.1) in degree 0. The left hand side is $\mathbf{C}$, and the right side is $\mathbf{C}^{\#\text{cusps}}$. The map is injective. (The kernel comes from $H^0_c(\mathbb{H}/\Gamma, \mathbf{C}) = 0$.)

Now consider the map (2.1.1) in degree 1, which is the more interesting case. We have the long exact sequence

$$H^1(\overline{\mathbb{H}}/\Gamma, \mathbf{C}) \xrightarrow{\gamma^1} H^1(\partial(\overline{\mathbb{H}}/\Gamma), \mathbf{C}) \to H^2(\overline{\mathbb{H}}/\Gamma, \partial(\overline{\mathbb{H}}/\Gamma), \mathbf{C}) \cong H_0(\overline{\mathbb{H}}/\Gamma, \mathbf{C}) \cong \mathbf{C}$$

so we see that Im γ^1 is of codimension 1. This raises the question: how we can describe this codimension 1 defect?

2.2. **Example: arithmetic hyperbolic 3-manifolds.** Let $k = \mathbf{Q}(\sqrt{d})$ be a quadratic imaginary field. Then the locally symmetric space attached to SL_2/k is $X = \mathbb{H}^3/\Gamma$ where $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_k)$. This is a 3-dimensional hyperbolic manifold.

Consider the restriction map

$$\gamma^*\colon H^*(\overline{X}/\Gamma, \mathbf{C}) \to H^*(\partial(\overline{X}/\Gamma), \mathbf{C}).$$

Now the boundary components are not circles but tori T^2, if $d \neq -1, -3$.

The map $\gamma^0\colon H^0(\overline{X}/\Gamma, \mathbf{C}) \to H^0(\partial\overline{X}/\Gamma, \mathbf{C})$ is as before. What about γ^1?

$$\gamma^1\colon H^1(\overline{X}/\Gamma, \mathbf{C}) \xrightarrow{\gamma^1} H^1(\partial(\overline{X}/\Gamma), \mathbf{C})$$

It is a fact that $\dim \mathrm{Im}\ \gamma^1 = \frac{1}{2}\dim H^1(\partial(\overline{X}/\Gamma), \mathbf{C}) = \#\text{cusps}$. This is a general phenomenon of algebraic topology; in fact by using Poincaré duality one shows that the image of γ^1 is maximal isotropic under Poincaré duality.

3. Borel-Serre compactification

We will now discuss the Borel-Serre compactification in more detail. References are [BS73], and [Har71] for a different approach. We are going to elide some technical subtleties.

If H is a connected algebraic group over $k \subset \mathbf{R}$, we define

$$^0H := \cap_{\chi \in X_k^*(H)} \ker(\chi^2)$$

This is evidently a normal subgroup of H, defined over k.

Let S be a maximal k-split torus in the center of H. Setting $A = S(\mathbf{R})^0$, we have

$$H(\mathbf{R}) = A \ltimes {}^0H(\mathbf{R})$$

Fact 3.1. *We have the following two facts:*

1. *$^0H(\mathbf{R})$ contains all maximal compact subgroups of $H(\mathbf{R})$.*
2. *If $k = \mathbf{Q}$, $^0H(\mathbf{R})$ contains all arithmetic subgroups of H.*

From now on we assume that G is defined over $\mathbf{Q}$; we can always arrange this by taking restriction of scalars. Let $P \subset G$ be a parabolic $\mathbf{Q}$-subgroup. Write the Langlands decomposition

$$P(\mathbf{R}) = A_P M_P N_P$$

with $M_P = {}^0L_P(\mathbf{R})$, and $L_P = P/N_P$. (Implicitly we've chosen a good lifting of L_P to P. This must be done with some care, which is tied in to the notion of "θ-stable parabolic", but the lift might not be defined over $\mathbf{Q}$. We ignore this point.)

Let $\overline{A}_P$ be the closure of A_P under the open embedding $A_P \to \mathbf{R}^{\Delta - J(P)}$, via the identification

$$A_P \cong (\mathbf{R}_+^\times)^{\Delta - J(P)}.$$

(In our conventions, if P is a maximal parabolic then $J(P)$ has one element.) We have the Cartan decomposition

$$X = K\backslash G = K\backslash K \cdot P(\mathbf{R}) = K \cap P(\mathbf{R})\backslash P(\mathbf{R}).$$

Then we have a *geodesic action*

$$A_P \times X \to X$$

given by (writing $K_P = K \cap P(\mathbf{R})$)

$$(a, K_P g) \mapsto a K_P g$$

which is well-defined because A_P commutes with $K_P \subset M_P$, and is independent of the choice of basepoint. Furthermore, each orbit under this action is a totally geodesic submanifold with respect to any invariant Riemannian metric.

Example 3.2. If $X = \mathbb{H}$, so $G = \mathrm{SL}_2 / \mathbf{Q}$, then the action is

$$(t, x+iy) \mapsto x + it^2 y.$$

Thus the orbits are vertical half-lines.

Let $\mathcal{P}$ be the set of $\mathbf{Q}$-parabolic subgroups. For any $P \in \mathcal{P}$, we define $e(P) := A_P \backslash X$. Then we set

$$\overline{X} := \coprod_{P \in \mathcal{P}} e(P).$$

For each $P \in \mathcal{P}$, define

$$X(P) = \coprod_{\substack{Q \in \mathcal{P} \\ Q \supset P}} e(Q).$$

If $P, Q \in \mathcal{P}$ then $X(P) \cap X(Q) = X(R)$ where R is the smallest parabolic over $\mathbf{Q}$ containing P and Q.

Note that $e(G) = A_G \backslash X$, which is just X because A_G is trivial. Thus X is included as the open subset $e(G)$.

Theorem 3.3 (Borel-Serre)**.** *We have the following facts.*

1. *There exists a unique topology on $\overline{X}$ such that the action of $G(k)$ on X extends continuously to an action of homeomorphisms on $\overline{X}$, and such that $X(P)$ is open in $\overline{X}$ for $P \in \mathcal{P}$. The normalizer of $e(P)$ is the parabolic P.*
2. *The e_P are permuted under this action.*
3. *If $\Gamma \subset G(k)$ is an arithmetic subgroup, then it acts properly on $\overline{X}$ and $\overline{X}/\Gamma$ is compact. The map $X/\Gamma \to \overline{X}/\Gamma$ is a homotopy equivalence.*

Let $\Gamma \subset G(\mathbf{Q})$, and $\pi \colon \overline{X} \to \overline{X}/\Gamma$. The action of Γ will identify some boundary components in $\overline{X}$. What about the "local picture" of $X(P)$?

There exists a neighborhood of $e(P)$ in $\overline{X}$ such that given $x, y \in U$ we have $x \sim_\Gamma y$ if and only if $x \sim_{\Gamma \cap P} y$. So we can define $e'(P) = \pi(e(P))$.

Theorem 3.4. *We have*

$$\overline{X}/\Gamma = \coprod_{P \in \mathcal{P}/\Gamma} e'(P)$$

where $\mathcal{P}/\Gamma$ is the set of Γ-conjugacy classes of parabolic subgroups of G. (It is a finite set.)

Consider an open neighborhood of $e'(P)$ on X/Γ. We have $e'(P)$ at ∞. The A_P "paves the way" to $e'(P)$.

Remark 3.5. In Harder's approach you cut off the cusp at a finite part rather than add something at ∞. This idea will be useful to us later.

4. Eisenstein cohomology

4.1. **Restriction to the boundary.** We have the map

$$\gamma^* \colon H^*(\overline{X}/\Gamma, E) \to H^*(\partial(\overline{X}/\Gamma), E).$$

We have just described the boundary as

$$\partial(\overline{X}/\Gamma) = \coprod_{\substack{P \in \mathcal{P}/\Gamma \\ P \neq G}} e'(P).$$

For the maximal parabolic, $\operatorname{codim} e'(P) = 1$. In general, the *parabolic rank* is the dimension of the maximal split torus, and is the codimension of the boundary component corresponding to P.

For a fixed P consider in particular the map

$$\gamma^* \colon H^*(\overline{X}/\Gamma, E) \to H^*(e'(P), E).$$

We can think of restricting a differential form to the subset by slicing off this cusp. We're going to explain this now. Let

$$A_{P,t} = \{a \in A_P \mid \alpha(a) \geq t \text{ for all } \alpha \in \Delta(P, A_P)\}.$$

Define

$$\overline{A}_{P,t} = \{a \in \overline{A}_P \mid \alpha(a) \geq t \text{ for all } \alpha \in \Delta(P, A_P)\}.$$

Recall the geodesic action furnishes an isomorphism

$$\mu_0 \colon A_P \times e(P) \xrightarrow{\sim} X$$

compatible with the action of $\Gamma \cap P$ on each side. Then there exists a $t_0 > 0$ such that Γ-equivalence and $(\Gamma \cap P)$-equivalence on $U_{P,t} := \mu_0(A_{P,t} \times e(P))$ coincide for any $t > t_0$.

Hence we have

$$A_{P,t} \times e(P)/\Gamma \cap P = A_{P,t} \times e'(P) \xrightarrow{\sim} \pi(U_{P,t}).$$

Define $\overline{U}_{P,t} = \mu_0(\overline{A}_{P,t} \times e(P))$, so this tells us that we have an isomorphism

$$\overline{A}_{P,t} \times e'(P) \xrightarrow{\sim} \pi(\overline{U}_{P,t}).$$

such that

$$\pi(\overline{U}_{P,t}) \cap X/\Gamma \xrightarrow{\sim} A_{P,t} \times e'(P).$$

So for $t > t_0$ and $a \in A_{P,t}$, any cohomology class $[x] \in H^*(X/\Gamma, E)$ restricts to $[x]_{a \cdot e'(P)}$ in a way that is *independent of* a (this is a crucial point!). Therefore we have a well-defined restriction $\gamma_P^*([x]) = [x]|_{e'(P)}$.

4.2. **Cohomology of the boundary faces.** Next we have to analyze the cohomology of the face $e'(P)$. Denote

$$\kappa \colon P \to P/N = L_P.$$

The restriction map uses the fibration of $e'(P)$:

$$\underbrace{N(\mathbf{R})/N(\mathbf{R}) \cap \Gamma}_{=:F_N} \to e'(P) \to Z_P/\Gamma_{(P)}$$

where $\Gamma_{(P)} = \kappa(\Gamma \cap P)$, and $Z_P = \kappa(K \cap P(\mathbf{R}))\backslash {}^0L_P(\mathbf{R})$.

Proposition 4.1. *The spectral sequence in cohomology associated to this fibration degenerates at E_2, so we have*

$$H^*(e'(P)) \cong H^*(Z_P/\Gamma_{(P)}, H^*(F_N, E)). \tag{4.2.1}$$

This reduces the question to one about groups of lower rank. So we need to understand the cuspidal spectrum of $Z_P/\Gamma_{(P)}$. For example, if $G = \mathrm{SL}_n$ then this could be the locally symmetric space of SL_{n-1}. There are a bunch of constraints on the cuspidal automorphic forms here to lift to Eisenstein series. There are topological constraints having to do with the coefficient system, but there are also global constraints coming from analysis.

Proposition 4.2 (van Est, Nomizu)**.** *The restriction maps*

$$\Omega^*(N(\mathbf{R}), E)^{N(\mathbf{R})} \to \Omega^*(N(\mathbf{R}), E)^{\Gamma \cap N(\mathbf{R})}$$

induces an isomorphism in cohomology.

The cohomology of the left side is Lie algebra cohomology:

$$\Omega^*(N(\mathbf{R}), E)^{N(\mathbf{R})} \cong H^*(\mathfrak{n}, E).$$

Thus we obtain

$$H^*(\mathfrak{n}, E) \cong \Omega^*(N(\mathbf{R}), E)^{\Gamma \cap N(\mathbf{R})}.$$

Note that the Levi quotient acts on the unipotent radical, so $H^*(\mathfrak{n}, E)$ is an $L_p(\mathbf{R})$-module. We also have an action of $\Gamma_{(P)}$ on the fiber F_N, since this is a general fact about any fibration, and these are compatible. In other words, the restriction of the first action to $\Gamma_{(P)}$ coincides with the second.

4.3. **A result of Kostant.** We now investigate how the cohomology of the fiber decomposes into irreducibles under the Levi action. The description will come from a result of Kostant.

Let $P_0 \subset G$ be a minimal parabolic subgroup over $\mathbf{Q}$. Let M_{P_0} be a Levi component. Take a standard parabolic $P \supset P_0$ over $\mathbf{Q}$. We have $A_P \subset A_0 := A_{P_0}$, and we can also arrange that $M_P \supset M_{P_0}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, containing $\mathfrak{a}_0$.

Let $\Phi = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and $\Phi_0 = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{a}_{0,\mathbf{C}})$. Choose compatible orderings of positive roots, i.e. $\Phi^+ \cap \Phi_0 = \Phi_0^+$.

Let $\mathfrak{b} = \mathfrak{h} \cap {}^0\mathfrak{m}_P$ be a Cartan of ${}^0\mathfrak{m}_P$, and choose a complement $\mathfrak{a}_P$ so that $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}_P$. Then we get dually $\mathfrak{h}_{\mathbf{C}}^\vee = \mathfrak{b}_{\mathbf{C}}^\vee \oplus \mathfrak{a}_{P,\mathbf{C}}^\vee$. Write $\Phi_P = \Phi({}^0\mathfrak{m}_{P,\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ and $\Delta_P = \Delta \cap \Phi_P$.

Write $W = W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and $W_P = W({}^0\mathfrak{m}_{P,\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$.

Lemma 4.3. *In any right coset of W_P in W, there exists a unique element with the following properties:*

(1) w is the unique element of minimal length in the coset,
(2) $w^{-1}(\alpha) > 0$ for every $\alpha \in \Delta_P$.

In fact each of these properties individually characterizes w. Hence the map $W \to W_P \backslash W$ has a canonical section, whose image W^P is the set of minimal coset representatives, and can also be expressed as

$$\{w \in W \mid w^{-1}(\Delta_P) \subset \Phi^+\}.$$

Theorem 4.4 (Kostant)**.** *Let (η, E) be a representation of $G(\mathbf{C})$ with highest weight $\lambda \in \mathfrak{h}_{\mathbf{C}}$. Then $H^*(\mathfrak{n}, E)$ decomposes as $L_p(\mathbf{C})$-module in any degree q:*

$$H^q(\mathfrak{n}, E) = \bigoplus_{\substack{w \in W^P \\ \ell(w) = q}} F_{\mu_w}$$

where F_{μ_w} is an irreducible $L_p(\mathbf{C})$-module with highest weight $\mu_w = w(\lambda + \rho) - \rho$. The μ_w are all distinct as w varies through W^P.

Let's apply this to (4.2.1):

$$H^*(e'(\Gamma), E) = H^*(Z_\Gamma/\Gamma_{(p)}, H^*(\mathfrak{n}, E)).$$

By Theorem 4.4, this can be written as

$$= H^*({}^0\mathfrak{m}_P, K_{M_P}; C^\infty({}^0M_P(\mathbf{R})/\Gamma_{(P)}) \otimes \bigoplus_{w\in W^P} F_{\mu_w}).$$

Inside we have the cuspidal cohomology, which comes from the space of cuspidal automorphic forms:

$$H^*_{\text{cusp}}({}^0\mathfrak{m}_P, K_{M_P}; \bigoplus_{w\in W^P} F_{\mu_w}).$$

Consider cuspidal automorphic representations (π, H_π) in $L^2_0({}^0M_P(\mathbf{R})/\Gamma_{(P)})$. The idea is to take $[\varphi] \in H^*_{\text{cusp}}({}^0\mathfrak{m}_P, K_{M_P}; \bigoplus_{w\in W^P} F_{\mu_w})$, and say that it's of type (π, w) for $w \in W^P$ if $0 \neq [\varphi] \in H^*({}^0\mathfrak{m}_P, K_{M_P}; H_\pi \otimes F_{\mu_w})$.

This is the starting point for the construction of Eisenstein series. It gets "easier" if (η, E) has regular highest weight. Why? This forces the representation π to be tempered, as we shall state in a proposition below. So the non-tempered stuff cannot contribute if the regularity of E transfers to the regularity of F_{μ_w}.

Proposition 4.5. *If the highest weight of (η, E) is regular, then the highest weight of F_{μ_w} as an $L_p(\mathbf{R})$-module is regular.*

References

[BS73] Borel, A.; Serre, J.-P. Corners and arithmetic groups. Avec un appendice: Arrondissement des variétés á coins, par A. Douady et L. Hérault. Comment. Math. Helv. 48 (1973), 436–491.

[Har71] Harder, G. A Gauss-Bonnet formula for discrete arithmetically defined groups. Ann. Sci. École Norm. Sup. (4) 4 (1971), 409–455.