RELATION OF SPHERICAL VARIETIES TO REPRESENTATION THEORY

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1. Ideology

For a spherical variety X under an action of G, we want to be able to attach

(1) an L-group ${}^L G_X$[1], which should be equipped with a map ${}^L X \times \mathrm{SL}_2 \to {}^L G$,
(2) an L-value, which is something of the form $L(\bullet, r, s)$ where r is a representation ${}^L X \to \mathrm{GL}(V)$ and s_0 is a point of evaluation. (The point of evaluation is somehow superfluous; we can replace ${}^L X$ by ${}^L X \times \mathbf{G}_m$.)

Example 1.1. The L-group depends on G. For $X = \mathrm{PGL}_2$ acted on by $\mathrm{PGL}_2 \times \mathrm{PGL}_2$, the representations appearing are $\widetilde{\tau} \times \tau$.

For $X = \mathrm{PGL}_2$ acted on by $\mathbf{G}_m \times \mathrm{PGL}_2$, the representations appearing are $\chi \otimes \tau$. The represents appear in the second case are richer, so the L-group should be correspondingly richer.

Example 1.2. If $X = H\backslash G$, $\pi \in L^2([G] = G(k)\backslash G(\mathbf{A}))$, then for $\varphi \in \pi$ we can consider the period

$$\varphi \xrightarrow{\mathcal{P}_H} \left(g \mapsto \int_{[H]} \varphi(h \cdot g) dh \right) \in C^\infty(H\backslash G(\mathbf{A})).$$

So (1) should answer the question of which π for G embed in $C^\infty(H\backslash G)$.

For $X = G\backslash G = \mathrm{pt}$, we have ${}^L X = 1$ and the only such π is the trivial representation. We have a map ${}^L X \times \mathrm{SL}_2 \to {}^L G$ induced by the principal nilpotent.

The L-values in (2) should be

$$|\mathcal{P}_H|^2 \sim \frac{L_X(\pi)}{L(\pi, \mathrm{Ad}_{\widehat{G}_X}, 1)}.$$

Example 1.3. For $X = \mathbf{G}_m \backslash \mathrm{PGL}_2$, we have ${}^L G_X = {}^L G$, and $L_X = L(\mathrm{std}, 1/2)^2$.

For $X = (N, \psi)\backslash G$, we have ${}^L G_X = {}^L G$, and $L_X = 1$.

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[1] At this point we only have a definition for $\widehat{G}_X$.

2. History

Brion, Luna, Vust, Knop do invariant theory on X and $T^*X \to \mathfrak{g}^*$. This gives rise to data for $\widehat{G}_X$, encoding e.g. $\mathfrak{a}_X^*, W_X$, "spherical roots". However the root system doesn't quite define the dual group.

Gaitsgory and Nadler constructed the dual group $\widehat{G}_X$ using the Tannakian formalism. The expectation is that it is attached to the data above. However, this has not been completely checked.

Sakellaridis-Venkatesh gives a combinatorial description of the dual group.

Knop-Schalke have now defined $\widehat{G}_X$ for *any* G-variety X.

There are some issues here:

(1) Finish the work of Gaitsgory-Nadler, showing that their $\widehat{G}_X$ is as expected and that we have a map $\widehat{G}_X \times \mathrm{SL}_2 \to \widehat{G}$.
(2) Define ${}^L G_X$.

There are two aspects of invariant theory: the theory of compactifications, and the theory of the moment map $T^*X \to \mathfrak{g}^*$. By connecting them, Knop discovered the cone of G-invariant valuations $V(X) \subset \mathfrak{a}_X$.

Theorem 2.1 (Brion-Knop)**.** *$V(X)$ is a fundamental domain for W_X, which is a finite reflection group.*

All this theory misses the data that gives rise to L_X! The reason is that Knop uses invariant theory, whereas it's better to work with stacks. We have a map from the stack to the coarse quotient:

$$[T^*X/G] \to T^*X//G.$$

Example 2.2. The ratio

$$\frac{L_X(\pi)}{L(\pi, \mathrm{Ad}, 1)}$$

should be the value of certain *local zeta integrals*, e.g. in the non-archimedean case and π is unramified. In my paper "Spherical functions on spherical varieties", I found that the data entering into L_X depends on the divisor Δ_X (the sum of the B-stable but not G-stable divisors) that was ignored in the invariant theory.

If G is split, then $\widehat{G}_X$ should encode the harmonic analysis of $X = X(F)$ for F a local field.

Already using the theory of compactifications, one can get valuable information. For example, let $I(\chi)$ be the normalized induced representation

$$I(\chi) = \mathrm{Ind}_{P(X)}^G(\chi \cdot \delta_{P(X)}^{1/2}).$$

We ask which $I(\chi) \hookrightarrow C^\infty(X)$? Dually we want a map

$$\mathcal{E}(X) \twoheadrightarrow I(\chi^{-1}).$$

The obvious thing to try is integration. Michal has shown that $X^\circ = A_X \times U_{P(X)}$. So for $\varphi \in \mathcal{S}(X)$, we define

$$\varphi \mapsto \int_{X^\circ} \varphi(au)\chi^{-1}(a) da du \in I(\chi^{-1}).$$

For this to make sense we need χ to be a character of A_X. We would then have a Langlands parameter $W_F \to \widehat{A}_X$, which is a Cartan in $\widehat{G}_X$. Thus we have an unramified Langlands parameter in $\widehat{G}_X$.

What does the Weyl group tell us? We have $I(\chi^{-1}) \cong I({}^w\chi^{-1})$. Applying the intertwining operator $I(\chi^{-1}) \to I({}^w\chi^{-1})$, do we get the same integral

$$\varphi \mapsto \int_{X^\circ} \varphi(au)^w\chi^{-1}(a)dadu?$$

Theorem 2.3. *The diagram commutes up to scalar if and only if* $w \in W_X \subset N_W(\mathfrak{a}_X^*)/Z(\mathfrak{a}_X^*)$.

In other words, applying $w \in W_X$ gives the same Langlands parameter.

Example 2.4. For $X = N\backslash G$, $I(\chi) \subset C^\infty(N\backslash G)$ and also $I({}^w\chi) \subset C^\infty(N\backslash G)$; they are abstractly isomorphic but not the same. Thus we get that the multiplicity of $I(\chi)$ in $C^\infty(N\backslash G)$ is generically $\#W_X$.