TRANSFER OPERATORS FOR RELATIVE FUNCTORIALITY

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Contents

I will talk about Langlands' "Beyond Endoscopy" proposal, in a broader scope that also encompasses the relative Langlands program. It aims to compare stable trace formulas in a new way. However, the nature of these comparisons is completely unclear. Even the case where the L-groups are *equal* (classically this is the setting for the Jacquet-Langlands correspondence) the comparison is very interesting, and encompasses for instance the Gan-Gross-Prasad Conjecture. I am trying to enlarge the program to have more tractable cases available.

1. Notation and basic setup

Let k be a global field, $\mathbb{A}_k$ the adeles. Let F be a local field and $[G] := G(k)\backslash G(\mathbb{A}_k)$.

For a smooth variety X, we let $\mathcal{S}(X) = \mathcal{S}(X(F))$ be the space of $\mathbf{C}$-valued Schwartz measures. This means that there is rapid decay towards all boundaries; for example on $\mathbf{G}_m$ you have rapid decay near 0 and ∞. Roughly you can think of a Schwartz measure as a Schwartz function times a polynomial volume form. Since in the global case you have canonical Tamagawa measures, you can confuse Schwartz measures with Schwartz functions.

When we have a group G acting on (quasi-affine) variety X, we denote by $X//G :=$ Spec $k[X]^G$ the coarse quotient. We want to discuss the notion of "Schwartz measures on the quotient". There are three versions of this notion:

(1) A sophisticated and correct one: Schwartz measures on the quotient stack $[X/G]$.
(2) An intermediate notion: the coinvariants $\mathcal{S}(X)_G$.
(3) A coarse notion: the pushforward of a measure from X to $X//G$. You should think of this as stable orbital integrals: the pushforward measure is given by the integral along the fiber, which is typically a stable conjugacy class, so this is a stable orbital integral. We will denote this by $\mathcal{S}^{\mathrm{st}}(X/G)$.

Let $\psi\colon F \to \mathbf{C}^\times$ or $\mathbb{A}_k \to \mathbf{C}^\times$ be an additive character. We let $N \subset B$ be a maximal unipotent subgroup of a Borel. If $X = (N\backslash G, \psi)$ then we can take a Whittaker model $\mathcal{S}(X) := \mathcal{S}(N, \psi)$ for the definition of $\mathcal{S}(X)$.

Notation: the quotient H/H means H modulo H-conjugacy.

Remark 1.1. This is an elementary but useful remark. When X is a group H (the "group case"), we think of X as a symmetric space for $G := H \times H$, with G acting via $(h_1, h_2) \cdot h = h_1^{-1} h h_2$. Then we can view $H/H = (X \times X)/G$ where G acts diagonally on $X \times X$, as we can use the G action to trivialize the first component of X, leaving the conjugation action on the second factor.

2. Langlands functoriality and "beyond endoscopy"

2.1. **The idea of "Beyond Endoscopy".** The Langlands functoriality conjecture says that a map ${}^L H_1 \to {}^L H_2$ induces a map

$$\{\text{packets of irreps for } H_1\} \mapsto \{\text{packets of irreps of } H_2\}.$$

One way to think of this as follows: you have a stable character for a packet Π, given by

$$\Theta_\Pi := \sum_{\pi \in \Pi} \Theta_\pi.$$

(This is all local right now.)

So functoriality gives a map from stable characters of H_1 to stable characters for H_2.

Remark 2.1. This is not quite what one does in endoscopy - there one takes a different linear combination of the characters within the L-packet.

Langlands suggested considering the dual map of stable test measures:

$$\mathcal{S}^{\mathrm{st}}(H_1/H_1) \xleftarrow{\mathcal{T}} \mathcal{S}^{\mathrm{st}}(H_2/H_2).$$

We want to describe this map - it should satisfy the condition that $\mathcal{T}^*(\Theta_{\Pi_1}) = \Theta_{\Pi_2}$ is a stable character for all tempered Π_1.

Remark 2.2. Langlands studied this for $H_1 = T$, $H_2 = \mathrm{GL}_2$. Daniel Johnstone studied this for $H_1 = T$ and $H_2 = \mathrm{GL}_n$.

We would like to use this to prove a comparison of stable trace formulas.

$$\mathrm{STF}_{H_2}(f) \rightsquigarrow \mathrm{STF}_{H_1}(\mathcal{T} f).$$

Here STF stands for the stable trace formula, viewed as a functional on measures. Why have we written $\rightsquigarrow$ instead of equality? There should not be equality on the nose: we need to extract a part of the trace formula for H_2 corresponding to the contribution from the spectrum of H_1 to the spectrum of H_2, which Langlands proposed doing using L-functions.

2.2. **A baby case.** Let T_H be the *universal Cartan* of H (the quotient B/N for any Borel B, which is well-defined because all Borels are conjugate). We assume that T_H is split for simplicity.

We have a map ${}^L T_H = \widehat{T}_H \to {}^L H$, which should induce a transfer from representations of H to representations of T. We want to understand the dual map

$$\mathcal{T}\colon \mathcal{S}(H) \to \mathcal{S}(H/H) \to \mathcal{S}(T_H).$$

In terms of representations, it is easy to say what is going on. Let $H_1 = T_H$ and $H_2 = H$. For $\chi \in \widehat{T}_H$ the local Langlands correspondence assigns

$$\Pi_\chi = \{I(\chi) := \mathrm{Ind}_B^H(\chi \cdot \delta^{1/2})\}.$$

The character of this representation is known:

$$\Theta_{\Pi_\chi}(t) = D_H^{-1/2}(t) \sum_{w \in W} {}^w\chi(t)$$

where D_H is the *Weyl denominator* (the thing that comes up in the Weyl character formula). The condition we need is that

$$\int_{T_H} (\mathcal{T}f)(t)\chi(t)\,dt = \int_H f(h)\Theta_\chi(h)\,dh.$$

Using this we can compute $\mathcal{T}f$ explicitly.

This is opposite to what we want to do, which is to start with the transfer $\mathcal{T}$ and use it to pull back representations.

To globalize this, we want a map

$$\mathcal{S}(H/H(\mathbb{A})) \to \mathcal{S}(T_H(\mathbb{A})).$$

The trace formula for H gives a distribution on the left side. Similarly the trace formula for T_H gives a distribution on the right side. This fits into a diagram

$$\begin{array}{ccc} \mathcal{S}(H/H(\mathbb{A})) & \longrightarrow & \mathcal{S}(T_H(\mathbb{A})) \\ \big\downarrow{\scriptstyle \mathrm{TF}} & & \big\downarrow{\scriptstyle \mathrm{TF}} \\ \mathbf{C} & \longrightarrow & \mathbf{C} \end{array}$$

This diagram does not commute using the usual (non-invariant) trace formula. We can view the trace formula as a "Laurent series"

$$\mathrm{TF}_H(f) = \frac{1}{s^r}\mathrm{TF}_{H,-r}(f) + \ldots + \mathrm{TF}_{H,0}(f).$$

Here $\mathrm{TF}_{H,0}(f)$ is the usual trace formula, and it is not invariant. The leading term is invariant, and is what will be compared.

3. The spectrum of a spherical variety

3.1. **Spherical varieties.** We said earlier that one can think of a group $X = H$ as being a symmetric space for $G = H \times H$. There is a broader context for this, mamely that of *spherical varieties*. This means (in characteristic 0) that X is an affine normal variety with G-action, such that $\overline{k}[X]$ is a multiplicity-free direct sum of highest weight modules. This is equivalent to saying that the Borel B has an open orbit.

This is a convenient class that gives Euler products for L-functions.

Example 3.1. Symmetric spaces are spherical varieties: $X = O_n \backslash \mathrm{GL}_n$ (with $G = \mathrm{GL}_n$), or $X = \mathrm{Sp}_{2n} \backslash \mathrm{GL}_{2n}$ (with $G = \mathrm{GL}_{2n}$).

Example 3.2. $X = \mathrm{GL}_n \backslash \mathrm{GL}_n \times \mathrm{GL}_{n+1}$ (with $G = \mathrm{GL}_n \times \mathrm{GL}_{n+1}$), or the Gan-Gross-Prasad settings with GL_n replaced by SO_n or U_n.

Example 3.3. The Whittaker situation: $X = N \backslash G$ (with $G = G$).

3.2. **The local spectrum.** We will define the *spectrum* of a spherical variety X. We can decompose

$$L^2(X) = \int_{\widehat{G}} \pi \mu(\pi)$$

where $\mu(\pi)$ is the Plancherel measure. The π appearing above form the support of the *local spectrum* of X.

Example 3.4. For $X = N\backslash G$, we have

$$L^2(X) = \int_{\chi \in \widehat{T}_G} I(\chi)\, d\chi.$$

3.3. **Relative characters.** Let $\Phi_1, \Phi_2 \in L^2(X)$. Then the Plancherel formula gives

$$\int_X \Phi_1 \Phi_2 \, dx = \int_{\widehat{G}} J_\pi(\Phi_1 \otimes \Phi_2)\, \mu(\pi).$$

Here J_π is a "relative character" (terminology by analogy to the relative trace formula). A relative character (for irreducible π) is a composition

$$J_\pi \colon \mathcal{S}(X \times X) \xrightarrow{G \times G \text{ equiv}} \pi \otimes \widetilde{\pi} \xrightarrow{\langle , \rangle} \mathbf{C}$$

where $\widetilde{\pi}$ is the contragredient of π, and $\langle , \rangle$ is the canonical pairing. We will assume that the map $\mathcal{S}(X \times X) \xrightarrow{G \times G \text{ equiv}} \pi \otimes \widetilde{\pi}$ is unique up to scalar, so J_π is unique to scalar.

Example 3.5. In the group case $X = H$,

$$J_\pi(\Phi_1 \otimes \Phi_2) \propto \operatorname{Tr}(\tau(\Phi_1^\vee * \Phi_2))$$

where the representation π of $H \times H$ necessarily factors as $\pi = \tau \otimes \widetilde{\tau}$, with τ a representation of H and $\widetilde{\tau}$ its contragredient. We can take the right side as a canonical normalization of J_π. This means that the Plancherel measure μ is canonical, in this case.

For general X, once we fix $\psi = \psi_X$ we should normalize J_π in some way (there is no intrinsically canonical choice).

3.4. **The relative trace formula.** We need a $G(\mathbb{A})^{\text{diag}}$-equivariant map

$$\mathcal{S}(X \times X(\mathbb{A})) \xrightarrow{\text{RTF}_{X \times X/G}} \mathbf{C}.$$

We specify this on functions of the form $\Phi_1 \otimes \Phi_2$ for $\Phi_1, \Phi_2 \in \mathcal{S}(X(\mathbb{A}))$. The recipe is as follow. For $\Phi_1, \Phi_2 \in \mathcal{S}(X(\mathbb{A}))$ we set

$$\Sigma\Phi_1(g) := \sum_{\gamma \in X(k)} \Phi_1(\gamma g)$$
$$\Sigma\Phi_2(g) := \sum_{\gamma \in X(k)} \Phi_2(\gamma g).$$

These are smooth functions on $[G]$, and then we take their inner product to get something in $\mathbf{C}$.

Example 3.6. In the group case $X = H$, $G = H \times H$,

$$\Sigma\Phi(h_1, h_2) = K_\Phi(h_1, h_2).$$

is the usual kernel function, so

$$\text{RTF}(\Phi_1 \otimes \Phi_2) = \langle K_{\Phi_1}, K_{\Phi_2} \rangle_{[H \times H]} = \langle R(\Phi_1), R(\Phi_2) \rangle_{HS} = \operatorname{Tr}(R(\Phi_1^\vee * \Phi_2)).$$

Example 3.7. If $X = (N, \psi)\backslash G$ then $\Sigma\Phi$ is a Poincaré series, and $\mathrm{RTF}_{(N,\psi)\backslash G/(N,\psi)}$ is the Kuznetsov trace formula for G.

The relative trace formula says that a geometric expansion of RTF equals a spectral expansion of RTF.

3.4.1. *The geometric side.* The geometric expansion is a sum of orbital integrals for $X \times X(k)/G(k)$. Formally, if $\mathcal{X} = [X \times X/G]$ then

$$\mathrm{RTF}(f) = \sum_{\xi \in \mathcal{X}(k)} f(\xi).$$

(Globally we have canonical Tamagawa measures which we use to identify measures with functions.)

3.4.2. *The spectral side.* On the spectral side, $\mathrm{RTF}(\Phi_1, \Phi_2)$ is a sum over automorphic representations π of a trace,

$$\mathrm{RTF}(\Phi_1, \Phi_2) = \sum_{\pi \in \widehat{G}_{\mathrm{Aut}}} J_\pi^{\mathrm{Aut}}(\Phi_1 \otimes \Phi_2)$$

where

$$J_\pi^{\mathrm{Aut}}(\Phi_1 \otimes \Phi_2) = \langle (\Sigma\Phi_1)_\pi, (\Sigma\Phi_2)_{\widetilde{\pi}} \rangle$$

where we are naively pretending that π embeds in $L^2([G])$. This $J_\pi^{\mathrm{Aut}}(\Phi_1 \otimes \Phi_2)$ is called a *period relative character*. The π for which $J_\pi^{\mathrm{Aut}} \neq 0$ are called X*-distinguished representations.*

Remark 3.8. If $X = H\backslash G$, you can write J_π^{Aut} as

$$J_\pi^{\mathrm{Aut}} = \sum_{(\varphi, \widetilde{\varphi})} (\dots) \int_{[H]} \varphi \int_{[H]} \widetilde{\varphi}$$

where the sum is over a dual basis $(\varphi, \widetilde{\varphi})$ of $(\pi, \widetilde{\pi})$. The mysterious factors are powers of 2 that measure the size of an Arthur packet, but they are present because we normalize our conventions for orbital integrals rather than stable orbital integrals.

3.5. **The generalized Ichino-Ikeda Conjecture.** In [SV] we formulate:

Conjecture 3.9 (Generalized Ichino-Ikeda conjecture)**.** *Under certain assumptions on X, we have*

$$J_\pi^{\mathrm{Aut}} = \prod_v J_{\pi_v}^{\mathrm{Planch}}$$

where $J_{\pi_v}^{\mathrm{Planch}}$ is a local relative character normalized with a distinguished Plancherel measure.

We emphasize that we have not explained how to define $J_{\pi_v}^{\mathrm{Planch}}$ here, but that there is a reasonable way to normalize it in general.

Let's try to say something to demystify the global periods. If $X = H\backslash G$ then we can think of $X \times X/G^{\mathrm{diag}} = H\backslash G/H$. Thus we can think of $f \in \mathcal{S}((X \times X)/G)$ as the pushforward of $F \in \mathcal{S}(G)$. Then

$$J_\pi(f) = \sum_{(\varphi, \widetilde{\varphi}) \text{ of } (\pi, \widetilde{\pi})} \int_{[H]} \varphi(h)\, dh \int \pi(F)\widetilde{\varphi}\, dh$$

Example 3.10. The original Ichino-Ikeda Conjecture concerned $X = \mathrm{SO}_n \setminus \mathrm{SO}_n \times \mathrm{SO}_{n+1}$. The conjecture predict that

$$J_\pi = \prod_v J_{\pi_v}^{\mathrm{Planch}}$$

but instead of phrasing things in terms of $J_{\pi_v}^{\mathrm{Planch}}$, they use an explicit expression which is related via

$$J_{\pi_v}^{\mathrm{Planch}}(f) = \sum_{(v,\widetilde{v}) \text{ of } (\pi_v,\widetilde{\pi}_v)} \int_{H(k_v)} \langle \pi_v(h), \widetilde{\pi}_v(F)\widetilde{v}\rangle \, dh = \int_{H(k_v)} \Theta_{\pi_v}(h\cdot F)\, dh.$$

For the case $\mathrm{SO}_n \setminus \mathrm{SO}_n \times \mathrm{SO}_{n+1}$, when $F = \mathbb{I}_{G(\mathcal{O}_v)}$ and π_v is unramified, then

$$J_{\pi_v}^{\mathrm{Planch}}(f) = \frac{L(\pi_{v1} \times \pi_{v2}, 1/2)}{L(\pi_v, \mathrm{Ad}, 1)}$$

where $\pi_v = \pi_{v1} \otimes \pi_{v2}$.

Example 3.11. Let $X = (N, \psi)\backslash G$. Then the conjecture predicts

$$J_\pi = \prod_v J_{\pi_v}^{\mathrm{Planch}}$$

with almost every factor being

$$J_{\pi_v}^{\mathrm{Planch}}(f) = \frac{1}{L(\pi_v, \mathrm{Ad}, 1)}$$

where f is the pushforward of $F = \mathbb{I}_{G(\mathcal{O}_v)}$. This is conjectural except for GL_n (Jacquet) and $\widetilde{\mathrm{Sp}}_{2n}$ (Lapid-Mao).

Note that we are seeing a ratio of L-functions, with the denominator being the adjoint L-function. The numerator is called L_X.

Example 3.12 (Ichino-Ikeda case)**.** For $X = \mathrm{SO}_n \setminus \mathrm{SO}_n \times \mathrm{SO}_{n+1}$ (with $G = \mathrm{SO}_n \times \mathrm{SO}_{n+1}$), we have

$$L_X = L(\pi_{v1} \times \pi_{v2}, 1/2).$$

Example 3.13 (Whittaker case)**.** For $X = (N, \psi)\backslash G$ we have $L_X = 1$.

Example 3.14 (Group case)**.** For $X = H$ and $G = H \times H$, the representation π must be of the form $\pi = \tau \otimes \tau^\vee$. In this case $J_{\pi_v}^{\mathrm{Planch}} = 1$, as is tautological from our normalization, so we have $L_X = L(\tau, \mathrm{Ad}, 1)$.

Thus the spectral side of the RTF is a sum over automorphic representations of these (ratios of) L-values.

3.6. **The L-group of a spherical variety.** It turns out that "most" spherical varieties X have an L-group ${}^L X \to {}^L G$ which controls the spectrum. (We are sweeping an Arthur SL_2 under the rug.)

Example 3.15. For $X = (N, \psi)\backslash G$, we should have ${}^L X = {}^L G$ because every tempered L-packet is expected to have a generic (i.e. X-distinguished) element, so every automorphic representation of G should contribute to $L^2(X)$. Since we haven't explained the definition of ${}^L X$, this is only a heuristic.

Example 3.16. For $X = \mathrm{SO}_n \setminus \mathrm{SO}_n \times \mathrm{SO}_{n+1}$, we should have ${}^L X = {}^L G$ because every tempered L-packet is expected to have an X-distinguished element.

Example 3.17. For the group case $X = H\backslash H \times H$, we should have ${}^L X = {}^L H \xrightarrow{\mathrm{Id},c} {}^L(H \times H)$, where c is the Chevalley involution, because the $G = H \times H$-representations appearing in $L^2(X)$ are only those of the form $\tau \otimes \widetilde{\tau}$.

3.7. **The L-function of a spherical variety.** We also have a global L-function L_X attached to a spherical variety X. This has the form (a product of) $L(\pi, r, s)$ where $r\colon {}^L X \to \mathrm{GL}(V)$ and $s \in \mathbf{C}$.

Example 3.18. For $X = \mathrm{GL}_n \backslash \mathrm{PGL}_{n+1}$, we should have ${}^L X = \mathrm{SL}_2$ because

$$L^2(X) = \mathrm{Ind}_{P_{2,n-1}}^{\mathrm{PGL}_{n+1}}(L^2(N, \psi\backslash \mathrm{PGL}_2))$$

where $P_{2,n-1}$ is the standard parabolic subgroup of partition type $(2, n-1)$ and acts on $L^2(N, \psi\backslash \mathrm{PGL}_2)$ through projection to PGL_2.

For $n = 1$, we are looking at $X = \mathbf{G}_m\backslash \mathrm{PGL}_2$, so $L_X = L(\pi, 1/2)^2$ because you should have square of the period for $\mathbf{G}_m\backslash \mathrm{PGL}_2$ (the adjoint L-value is coming from our different normalization than the usual one), which would give $L(\pi, 1/2)$.

4. Relative functorality

4.1. **"Beyond Endoscopy" for spherical varieties.** Again we discuss the local setting. Suppose we have spherical varieties (X_1, G_1) and (X_2, G_2). A map

$${}^L X_1 \to {}^L X_2$$

should induce a map from X_1-distinguished packets to X_2-distinguished packets, hence a map from stable relative characters for X_1 to stable relative characters for X_2, which can be interpreted dually as a transfer operator

$$\mathcal{S}(X_1 \times X_1/G_1) \xleftarrow{\mathcal{T}} \mathcal{S}(X_2 \times X_2/G_2)$$

so that there is some sort of comparison of stable relative trace formulas

$$\mathrm{RTF}_{X_2 \times X_2/G_2}(f) \rightsquigarrow \mathrm{RTF}_{X_1 \times X_1/G_1}(\mathcal{T} f).$$

The problem is nontrivial already when ${}^L X_1 \cong {}^L X_2$. (This case was used to reprove Waldspurger's form. It would be enough by itself to give Gross-Prasad.) In this case you can formulate precise desiderata. We will begin by describing a more naïve version, which we will then correct.

(1) Locally, there should be a transfer operator which is a linear bijection

$$\mathcal{S}(X_1 \times X_1/G_1) \xleftarrow{\mathcal{T}} \mathcal{S}(X_2 \times X_2/G_2)$$

Secretly this should realize functoriality, so $\mathcal{T}^*$ should take stable relative characters to stable relative characters for the same L-parameter. However this would be the *outcome* of having the theory of the transfer operator, so we cannot use it to construct $\mathcal{T}$.

The $\mathcal{T}$ should satisfy a fundamental lemma for the Hecke algebra.

(2) Globally, this should fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{S}(X_2 \times X_2/G_2(\mathbb{A})) & \xrightarrow{\mathcal{T}} & \mathcal{S}(X_1 \times X_1/G_1(\mathbb{A})) \\ & {\scriptstyle \mathrm{RTF}_{X_2 \times X_2/G_2}} \searrow \quad \swarrow {\scriptstyle \mathrm{RTF}_{X_1 \times X_1/G_1}} & \\ & \mathbf{C} & \end{array} \tag{4.1.1}$$

This would let you transfer questions about periods.

However, these two desiderata are already incompatible. Indeed, we have already seen that spherical varieties with the same L-group can have different L-functions. (For example, for any spherical variety we can find a Whittaker spherical variety with the same L-group, which automatically has L-function equal to 1.) The L-functions arise as special cases of periods, so we cannot have this sort of RTF (4.1.1) as is.

4.2. **Non-standard test measures.** Since in the Whittaker case the L-function was trivial, we might as well take it for one side. From now on, we take X_2 to be the Whittaker period for the quasisplit group G^* such that ${}^LG^* \cong {}^LX_1$. Then

$$X_2 \times X_2/G_2 \cong (N,\psi)\backslash G^*/(N,\psi).$$

We want to define a transfer operator

$$\mathcal{T} \colon \mathcal{S}((N,\psi)\backslash G^*/(N,\psi)) \xrightarrow{\sim} \mathcal{S}(H\backslash G/H)$$

with a RTF as in (4.1.1). However, we have already noted that RTF for a basic function in the Whittaker case (Example 3.11) is $\frac{1}{L(\mathrm{Ad},1)}$ on the left side and $\frac{L_X}{L(\mathrm{Ad},1)}$ on the right side. So we need to use a new basic function for this to map. Thus we consider a transfer operator

$$\mathcal{T} \colon \mathcal{S}^-_{L_X}((N,\psi)\backslash G^*/(N,\psi)) \xrightarrow{\sim} \mathcal{S}(H\backslash G/H).$$

where $\mathcal{S}^-_{L_X}((N,\psi)\backslash G^*/(N,\psi))$ is a larger space of "nonstandard" test measures for the Kuznetsov formula. We'll explain how one can cook up a function in $\mathcal{S}^-_{L_X}((N,\psi)\backslash G^*/(N,\psi))$ for which the RTF outputs $\frac{L_X}{L(\mathrm{Ad},1)}$.

Classically the KTF computes the inner product of two Poincaré series, i.e the Whittaker coefficients of Poincaré series. That is, if Φ_n is the nth Poincaré series then

$$\mathrm{KTF}(\Phi_n \otimes \Phi_1) = \sum_{\Pi} \sum_{\varphi \in \pi} a_n(\varphi)\overline{a_1(\varphi)}.$$

We want to add in a new test function f^0 such that $L_X = \mathrm{KTF}(f^0)$. To illustrate how this works, imagine expanding out a local factor of L_X:

$$L_{X,v} = \frac{1}{\det(\mathrm{Id} - q^{-s} r(-))} = \sum_{n \geq 0} q^{-ns} \operatorname{Tr}(\operatorname{Sym}^n r(-)).$$

Let $\varphi \in \mathcal{S}((N,\psi)\backslash G^*/(N,\psi))$ be the pushforward of the function $\mathbb{I}_{G(\mathcal{O}_v)}$ on G. We know that the local period of f is $\frac{1}{L(\mathrm{Ad},1)}$, so we replace

$$f \rightsquigarrow f^0 := \sum_n q^{-ns} h_n * f \tag{4.2.1}$$

where h_n is the Hecke operator corresponding to $\operatorname{Tr}(\operatorname{Sym}^n r(-))$.

Example 4.1. We'll give an example of how to write down f^0. For $G^* = \mathrm{SL}_2$, the usual space of test measures $\mathcal{S}((N,\psi)\backslash G^*/(N,\psi))$ are certain measures on $N\backslash G^*//N = \mathbf{G}_a$, where the identification of the quotient is via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c.$$

To make a measure on this quotient, we first choose a section, e.g.

$$c \mapsto \begin{pmatrix} & c \\ -c^{-1} & \end{pmatrix}$$

For $\phi \in \Phi(g)\,dg \in \mathcal{S}(\mathrm{SL}_2)$, $\pi_! f$ is a measure on $\mathbf{G}_a(F) = F$ given by

$$\pi_!\phi(t) = |c|^{1/2}\left(\int \Phi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix} & c\\ -c^{-1} & \end{pmatrix}\begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi^{-1}(x+y)\,dxdy\right)d^*c$$

Then we form f^0 as a formal combination of $\pi_!\phi_n$ for appropriate $\phi_n = h_n * \mathbb{I}_{G(\mathcal{O}_v)}$, as in (4.2.1).

5. Examples

Next time we'll start by explicitly describing

$$\mathcal{T}\colon \mathcal{S}^-(N,\psi\backslash\,\mathrm{SL}_2\,/N,\psi) \to \mathcal{S}^{st}(\mathrm{SL}_2\,/\,\mathrm{SL}_2)$$

and

$$\mathcal{T}\colon \mathcal{S}^-_{L(\mathrm{Std},1/2)^2}(N,\psi\backslash\,\mathrm{PGL}_2\,/N,\psi) \to \mathcal{S}(\mathbf{G}_m\backslash\,\mathrm{PGL}_2\,/\mathbf{G}_m)$$

6. Global applications

References

[SV] Sakellaridis, Yiannis and Venkatesh, Akshay. *Periods and harmonic analysis on spherical varieties.*