KIRILLOV THEORY AND ITS APPLICATIONS

LECTURE BY JU-LEE KIM,
NOTES BY TONY FENG

Contents

1. Motivation (Akshay Venkatesh)

Let X be a spherical variety for the (reductive) group G. Then we have a moment map

$$T^*X \to \mathfrak{g}^*.$$

Let F be a p-adic field. We would like to know that for π an irreducible representation of $G(F)$, we have

$$\dim \operatorname{Hom}(\pi, C^\infty(X)) < \infty.$$

This is proved in [SV] for the "wavefront spherical varieties". It would be nice to have a complete proof in the class of p-adic fields, using the geometry of the moment map.

Here is the shred of an idea. Choose a nice open compact K of $G(F)$ and an irreducible K-representation τ. By Howe's Kirillov theory, τ corresponds to a K-stable subset $\mathcal{O}_\tau$ of $\mathfrak{g}^*$. (This is analogous to the parametrization of representations of nilpotent groups by coadjoint orbits, due to Kirillov.) Choose τ so that $\tau \subset \pi|_K$. We should have an analogous "correspondence"

$$\operatorname{Hom}(\tau, C^\infty(X)) \leftrightarrow \Phi^{-1}(\mathcal{O}_\tau) \subset T^*X.$$

The idea is that the arrow is "microlocal support".

For a generic point $\lambda \in \mathfrak{g}^*$, the property of being spherical forces $\Phi^{-1}(\lambda)$ to be a single (geometric) orbit of G_λ on T^*X (this is the only place so far where the spherical property is being used). This means that we should think of $\Phi^{-1}(\mathcal{O}_\tau)$ as a thickening of $\phi^{-1}(\lambda) = G_\lambda$-orbit.

On $\operatorname{Hom}(\tau, C^\infty(X))$ you have a Hecke action of $\operatorname{End}(i_K^G \tau)$. We want to show that it's finitely generated over the Hecke algebra. On some commutative subalgebra, this action should correspond to translation along the direction of the orbit. The vague idea is that the finiteness eventually comes from the fact that you had a single orbit.

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2. Howe's Kirillov theory

2.1. **Setup.** Let k be a p-adic field, with valuation ν_k.[1] Let $\Omega\colon k \to \mathbf{C}^\times$ be an additive character with conductor $\mathfrak{p}_k$.

Consider a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(k)$. Fix a nondegenerate bilinear form B on $\mathfrak{g}$. This gives an identification

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\sim} \mathfrak{g}^* \\ x &\mapsto B_x := (Y \mapsto B(X,Y)) \end{aligned}$$

Letting $\widehat{\mathfrak{g}} = \mathrm{Hom}(\mathfrak{g}, \mathbf{C})$, we also have an identification

$$\begin{aligned} \mathfrak{g}^* &\xrightarrow{\sim} \widehat{\mathfrak{g}} \\ B_x &\mapsto \Omega_x := (Y \mapsto \Omega(B(X,Y))). \end{aligned}$$

2.2. **Elementary exponentiable lattices.**

Definition 2.1. Let $\mathscr{L} \subset \mathfrak{g}$ be an open compact $\mathcal{O}_k$-module, closed under $[,]$. We say that $\mathscr{L}$ is *elementary exponentiable* (e.e.) if exp is defined on $\mathscr{L}$ and $\exp(\mathscr{L})$ is a group, which we denote by C.

Example 2.2. For $\mathfrak{g} = \mathfrak{gl}_n$, an e.e. lattice is a $\mathscr{L} \subset \mathfrak{gl}_n(\mathfrak{p}_k^\alpha) \cap \mathfrak{g}$ and $[\mathscr{L}, \mathscr{L}] \subset \varpi_k^\beta \mathscr{L}$, such that α is the in the range of convergence of exp and β makes the exponential of $\mathscr{L}$ into a group. To determine the meaning of the latter, we need

$$\log(\exp(X)\exp(Y)) \in \mathscr{L}.$$

We can calculate this explicitly using the Campbell-Baker-Hausdorff formula

$$\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \dots.$$

Let $\mathscr{L} \subset \mathfrak{g}$ be an e.e. lattice. So we have $\mathscr{L} \subset \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \xrightarrow{\sim} \widehat{\mathfrak{g}}$. Define

$$\mathscr{L}^\# := \{\lambda \in \mathfrak{g}^* \colon \lambda(\mathscr{L}) \subset \mathfrak{p}_k\} = \{x \in \mathfrak{g} \colon B(x, \mathscr{L}) \subset \mathfrak{p}_k\} \subset \mathfrak{g}^*$$

and

$$\mathscr{L}^\perp = \{\psi \in \widehat{\mathfrak{g}} \colon \psi|_{\mathscr{L}} = 1\} \subset \widehat{\mathfrak{g}}.$$

Then the diagram

$$\begin{array}{ccc} \mathscr{L}^* := \mathfrak{g}^*/\mathscr{L}^\# & \longrightarrow & \widehat{\mathfrak{g}}/\mathscr{L}^\perp =: \widehat{\mathscr{L}} \\ \uparrow & & \uparrow \\ \mathfrak{g}^* & \longrightarrow & \widehat{\mathfrak{g}} \end{array} \tag{2.1}$$

is commutative and $\mathrm{Ad}(\exp(\mathscr{L}))$-invariant.

Example 2.3. Take $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathscr{L} = \mathfrak{gl}_n(\mathfrak{p}_k^m)$. Then $\mathscr{L}^\# = \mathfrak{gl}_n(\mathfrak{p}_k^{-m+1})$ and $C = \exp \mathscr{L} = 1 + \mathfrak{gl}_n(\mathfrak{p}_k^m) = K_m$ (a standard congruence subgroup).

Definition 2.4. Let $\psi \in \widehat{\mathscr{L}}$. By the diagram (2.1) we get a corresponding $x_\psi + \mathscr{L}^\#$. We say that $x_\psi + \mathscr{L}^\#$ *represents* ψ, and is called the *dual blob* of ψ.

Definition 2.5. Given $\psi \in \widehat{\mathscr{L}}$, we define $f_\psi \colon \mathscr{L} \times \mathscr{L} \to S^1$ by $(X,Y) \mapsto \psi([X,Y])$.

Let $A = \{a \in \exp \mathscr{L} =: C \colon \mathrm{Ad}\, a(\psi) = \psi\}$. The *radical* of f_ψ is $\{x \in \mathscr{L} \mid f_\psi(x, \mathscr{L}) = 1\}$.

[1] We use the exponential map, so the proofs would break down in characteristic p. For large p, there may be a way to rectify this.

Remark 2.6. We can also calculate A as $A = \mathrm{Stab}_C(x_\psi + \mathscr{L}^\#)$.

Lemma 2.7. *In the situation of Definition 2.5,*

1. $\mathscr{I} = \log(A)$ *is an e.e. lattice and is a radical of* f_ψ.
2. *If* $A = C$, ψ *defines a character of* C *via* exp.

Proof. (1) Let $x \in \mathscr{L}$. We use $\mathrm{Ad}(\exp(x))$. Write

$$\mathrm{Ad}(\exp(x)) - \mathrm{Id} = \exp(\mathrm{ad}(x)) - 1 = \tau(x) \circ \mathrm{ad}\, x$$

where $\tau(x)$ is invertible. Hence

$$\begin{aligned}
\exp(x) \in A &\iff \mathrm{Ad}(\exp x)(\psi) = \psi \\
&\iff \mathrm{Ad}(\exp(x) - 1)(\mathscr{L}) \subset \ker \psi \\
&\iff \tau(x)\,\mathrm{ad}\, x(\mathscr{L}) \subset \ker \psi \\
&\iff \psi([x, \tau(x) y]) = 1 \text{ for all } y \in \mathscr{L} \\
&\iff \psi([x, \mathscr{L}]) = 1 \\
&\iff f_\psi(x, \mathscr{L}) = 1
\end{aligned}$$

For (2), we need to check that the formula $\exp(x) \mapsto \psi(x)$ defines a homomorphism of C. If $\mathrm{Stab}_C(\psi) = A = C$ then $\mathrm{rad}\, f_\psi = \mathscr{L}$, hence $\psi([\mathscr{L}, \mathscr{L}]) = 1$. By the Campbell-Baker-Hausdorff formula, we have

$$\psi(\log(\exp(x) + \exp(y)) = \psi(x + y + [\mathscr{L}, \mathscr{L}]) = \psi(x)\psi(y).$$

□

2.3. **Howe's Kirillov correspondence.**

Example 2.8. Let $k = \mathbf{Q}_p$, for $p > 3$. Then we can take $\mathscr{L} = \mathfrak{gl}_n(\mathfrak{p}_k)$, i.e. it is e.e. with $C = K_1$. Let

$$X = p^{-2d} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

where $t_i \in \mathcal{O}_k^\times$, and the the t_i are distinct modulo p. Then X defines a character ψ_x of $\mathscr{L}$, and $A = \mathrm{Stab}_c(\psi_X) = K_{2d+1} T_1$, where

$$T_1 = \begin{pmatrix} 1 + \mathfrak{p}_k & & \\ & \ddots & \\ & & 1 + \mathfrak{p}_k \end{pmatrix}.$$

We're going to build a representation from this data.

Let $\mathscr{J} \subset \mathscr{L}$ be an inclusion of e.e. lattices. Then $\mathscr{J}^\# \supset \mathscr{L}^\#$, and we have a restriction map

$$\begin{array}{ccc}
\widehat{\mathscr{L}} & \xrightarrow{\ r\ } & \widehat{\mathscr{J}} \\
\sim\uparrow & & \sim\uparrow \\
\mathfrak{g}/\mathscr{L}^\# & \longrightarrow & \mathfrak{g}/\mathscr{J}^\#
\end{array}$$

If $\exp(\mathscr{J}) \triangleleft C$, then the diagram is $\mathrm{Ad}\, C$-invariant. Set $B = \exp(\mathscr{J})$.

Definition 2.9. Let $\psi \in \mathscr{L}$ with $A = \mathrm{Stab}_C(\psi)$. In the notation above, we say that B *polarizes* ψ if $B \supset A$, $\#B/A = \#C/B$, and $r(\psi) \in \widehat{\mathscr{J}}$ is $\mathrm{Ad}\, B$-invariant.

Remark 2.10. If B polarizes ψ, then

(1) $r(\psi)$ defines a character of B.
(2) By Lemma 2.7, $f_\psi|_{\mathscr{J}\times\mathscr{J}} = f_{r(\psi)} = 1$, and $\mathscr{J}/\mathscr{I}$ forms a maximal isotropic subspace ("half of a polarization").
(3) $r^{-1}(r(\psi)) = \mathrm{Ad}(B)\psi$, and

$$\#(\mathrm{Ad}(B)\psi) = \#B/A = \#C/B = (\#\mathscr{O}_\psi)^{1/2},$$

where $\mathcal{O}_\psi \subset \mathscr{L}$ is the C-orbit of ψ.

Now, any dual blob $X \in \mathfrak{g}/\mathscr{L}^\#$ defines a character ψ of $\mathscr{L}$. In the setting of Example 2.8, we have $A = \mathrm{Stab}_C(\psi_x) = K_{2d+1}T_1$ and $B = K_{d+1}T_1$.

Lemma 2.11. *If B polarizes $\psi \in \widehat{\mathscr{L}}$, then $U(\mathcal{O}_\psi) := \mathrm{Ind}_B^C(\psi)$ is irreducible. Furthermore, we have*

(1) $\deg U(\mathcal{O}_\psi) = \#(C/B) = (\#\mathcal{O}_\psi)^{1/2}$ and
(2) the character of $U(\mathcal{O}_\psi)$ is

$$\frac{1}{(\#\mathcal{O}_\psi)^{1/2}} \sum_{\phi \in \mathcal{O}_\psi} \phi \circ \log .$$

In particular, $U(\mathcal{O}_\psi)$ depends only on the orbit $\mathcal{O}_\psi$.

Remark 2.12. If $\mathscr{L}$ is small enough, then for any $\psi \in \widehat{\mathscr{L}}$ there exists a polarizing group $A \subset B_\psi \subset C$. In this case, we say that $\mathscr{L}$ is *polarizable*.

Theorem 2.13. *Let $\mathscr{L} \subset \mathfrak{g}$ be e.e. and polarizable, and let $C = \exp(\mathscr{L})$. Define $\mathcal{E}(C)$ to be the set of irreducible smooth representation of C, and $\mathcal{O}(C)$ the set of C-orbits in $\widehat{\mathscr{L}} = \mathfrak{g}/\mathscr{L}^\#$. Then the association $U(\mathcal{O}_\psi) \leftrightarrow \mathcal{O}_\psi$ induces a bijection $\mathcal{E}(C) \leftrightarrow \mathcal{O}(C)$.*

2.4. **Representations of G.** We now aim to say something about representations of G itself.

Definition 2.14. Let $\mathscr{L}_1, \mathscr{L}_2 \subset \mathfrak{g}$ be e.e. polarizable subgroups. Let $C_i = \exp \mathscr{L}_i$, and $\rho_i \in \mathcal{E}(C_i)$. We say that $g \in G$ intertwines ρ_1 with ρ_2 if $\mathrm{Hom}({}^g\rho_1, \rho_2) \neq 0$ on ${}^gC_1 \cap C_2$. This is equivalent to following alternate formulations:

- the existence of $\phi_i \in \mathcal{O}_{\rho_i}$ such that ${}^g\phi_1 = \phi_2$ on ${}^g\mathscr{L}_1 \cap \mathscr{L}_2$,
- ${}^g\mathcal{O}_{\rho_1} \cap \mathcal{O}_{\rho_2} \neq \emptyset$ in ${}^g\widehat{\mathcal{L}_1 \cap \mathcal{L}_2}$,
- ${}^g(X_{\phi_1} + \mathscr{L}_1^\#) \cap (X_{\phi_2} + \mathscr{L}_2^\#) \neq \emptyset$.

Let $G \supset K = \exp(\mathscr{L})$. For $\pi \in \mathcal{E}(G)$, we have

$$\pi|_K = \bigoplus_{\rho \in \mathcal{E}(K)} m_\rho \rho.$$

If K is small enough, then every ρ appearing will intertwine with 1. This is equivalent to saying that the dual blobs are represented by nilpotent elements. There is a character expansion

$$\theta_\pi(f) = \sum_{\theta \in \mathcal{O}(0)} c_\theta \widehat{\mu}_\theta(f).$$

Here $\mathcal{O}(\Gamma)$ is the set of G-orbits θ in $\mathfrak{g}$ such that Γ is in the closure, so $\Theta(0)$ is the set of nilpotent orbits.

We have $K \subset G_{x,d(\pi)^+}$. The size of K dictates the convergence of the character expansion. If K is not too small, then new orbits start appearing. The first nontrivial orbits are from the "unrefined minimal K-types", introduced by Moy and Prasad. In order to explain this, we need to review the theory of Moy and Prasad.

3. Moy-Prasad theory

3.1. **The Moy-Prasad filtration.** For $x \in B(G)$, the affine building of G, given a sequence of real numbers

$$0 \leq r_1 \leq r_2 \leq \ldots \subset \mathbf{R}$$

Moy-Prasad defined filtrations

$$\mathfrak{g}_{x,r_1} \supset \mathfrak{g}_{x,r_2} \supset \ldots,$$
$$G_{x,r_1} \supset G_{x,r_2} \supset \ldots$$
$$\mathfrak{g}^*_{x,r_1} \supset \mathfrak{g}^*_{x,r_2} \supset \ldots$$

These filtrations are compatible with taking commutators:

$$(G_{x,s}, G_{x,t}) \subset G_{x,s+t} \quad \text{and} \quad [\mathfrak{g}_{x,s}, \mathfrak{g}_{x,t}] \subset \mathfrak{g}_{x,s+t}.$$

They also satisfy $gG_{x,r}g^{-1} = G_{gx,r}$.

We then define

$$G_{x,r^+} = \bigcup_{s>r} G_{x,s}$$

and

$$\mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s}.$$

Given $H \hookrightarrow G$, there is sometimes $B(H) \hookrightarrow B(G)$. It exists when we need it, namely when H is a twisted Levi subgroup of G.

Definition 3.1. Let $\mathfrak{g}_r = \bigcup_{x\in B(G)} \mathfrak{g}_{x,r}$ and $G_{r^+} = \bigcup_{s>r} \mathfrak{g}_s$.

Note that $\mathfrak{g}_r$ is open and closed, and $\mathrm{Ad}(G)$-invariant. These are called G-domains. They are thickenings of the nilpotent cone: note that $\mathfrak{n} = \bigcap \mathfrak{g}_r$.

3.2. **Depth.**

Definition 3.2. For $(\pi, V_\pi) \in \mathcal{E}(G)$, define the *depth* of π to be

$$\rho(\pi) := \min\{r \colon V_\pi^{G_{x,r^+}} \neq 0 \text{ for some } x \in B(G)\} \in \mathbf{Q}.$$

This implies

$$\pi \mid G_{x,\rho(\pi)^+} = \sum_{\tau\in\mathcal{E}(G_{x,\rho(\pi)^+})} m_\tau \tau$$

with $\tau \sim 1$ for all τ.

Theorem 3.3 (DeBacker-Waldspurger)**.** *Let $p \gg 0$. Then the Harish-Chandra–Howe local character expansion*

$$\theta_\pi = \sum_{\theta\in\mathcal{O}(0)} c_\theta \widehat{\mu}_\theta$$

is valid on $\mathfrak{g}_{\rho(\pi)^+}$.

We have an action of $G_{x,\rho(\pi)}$ on $V_\pi^{G_{x,\rho(\pi)+}}$. Moreover, if $\rho(\pi) > 0$ then $G_{x,\rho(\pi)}/G_{x,\rho(\pi)^+}$ is abelian, and

$$\left(\frac{G_{x,\rho(\pi)}}{G_{x,\rho(\pi)^+}}\right)^\vee \cong \frac{\mathfrak{g}_{x,-\rho(\pi)}}{\mathfrak{g}_{x,(-\rho(\pi))^+}}.$$

3.3. K-types.

Definition 3.4. For $y \in B(G)$ and $r \geq 0$,

(1) $(G_{y,0}, \tau)$ is an *unrefined minimal K-type of depth 0* if τ is inflated from a cuspidal representation of $G_{y,0}/G_{y,0^+}$.
(2) $(G_{y,r}, \chi)$ for $r > 0$ is an *unrefined minimal K-type of depth r* if $\chi|_{G_{y,r^+}}$ is trivial and the dual blob of χ doesn't contain nilpotent elements.

Theorem 3.5 (Moy-Prasad)**.**

(1) For any $\pi \in \mathcal{E}(G)$ of depth r, there exists an unrefined minimal K-type $(G_{x,\rho(\pi)}, \chi)$ such that $\pi|_{G_{\chi,\rho(\pi)}} > \chi$.
(2) Any two minimal K-types are "associates" (i.e. intertwined).

Example 3.6. Spherical principal series have minimal K-type $(I, 1)$.

Note that if $\rho(\pi) > 0$, then χ corresponds to a dual blob $\Gamma \in \mathfrak{g}_{x,-\rho(\pi)}/\mathfrak{g}_{x,(-\rho(\pi))^+}$.

Definition 3.7 (Adler-Roche)**.** Let $T \subset G$ be a maximal torus. We say that $\Gamma \in \operatorname{Lie} T =: \mathfrak{t}$ is *good of depth r* if

(1) $\Gamma \in \mathfrak{g}_r/\mathfrak{g}_{r^+}$,
(2) $\mathfrak{t}$ splits over a tamely ramified extension E,
(3) for any $\alpha \in \Phi(T, E)$,

$$\nu_E(d\alpha(\Gamma)) = r \text{ or } d\alpha(\Gamma) = 0.$$

Remark: this implies that $\Gamma \in \mathfrak{t}_r/\mathfrak{t}_{r^+}$.

We also declare that $0 \in \mathfrak{g}$ is good by definition.

Remark 3.8. Fintzen has recently proved that if G splits over a tame extension and $p \nmid |W|$, then any coset $\mathfrak{t}_r/\mathfrak{t}_{r^+}$ contains a good element.

Example 3.9. Let $\pi \in \mathcal{E}(\mathrm{GL}_n)$ be a principal series representation. Let $(\chi_1, \ldots, \chi_n)$ be a tuple of characters such that

- χ_i has conductor p^{2d},
- $\chi_i\chi_j^{-1}$ is nontrivial on p^{2d+1},

Then π contains the unrefined minimal K-type (K_{2d+1}, χ) corresponding to the dual blob

$$\varpi^{-2d-1}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} \tag{3.1}$$

where $\lambda_i \in \mathcal{O}_K^\times$, so (3.1) is a good element of depth $-2d-1$.

Example 3.10. If Γ is good and $C_G(\Gamma)$ is compact mod center, then π is supercuspidal.

Example 3.11 (Adler-Roche)**.** Let $\mathcal{H} = \mathcal{H}(G//G_{x,\rho(\pi)}, \chi)$, with the dual blob Γ corresponding to χ being good. We have $\mathcal{H} \cong \operatorname{End}_G(c-\operatorname{Ind}_{G_{x,\rho(\pi)}}^G \chi)$, and it is supported on $G_{x,0^+}G'G_{x,0^+}$ where $G' = C_G(\Gamma)$. (The support of $\mathcal{H}$ is defined to be $\{g \in G \colon \exists f \in \mathcal{H}$ such that $f(g) \neq 0\}$.)

Definition 3.12. An unrefined minimal K-type $(G_{x,\rho(\pi)}, \chi)$ is *weakly good* if it intertwines with a good type in G.

Depth zero unrefined K-types are also declared to be *good*.

4. Applications

Theorem 4.1 (Kim-Murnaghan)**.** *If $\rho(\pi) > 0$ and π contains a weakly good type represented by Γ, then we have the character expansion*

$$\theta_\pi = \sum_{\theta \in \mathcal{O}(\Gamma)} c_\theta \widehat{\mu_\theta}$$

valid on $\mathfrak{g}_{\rho(\pi)} \supsetneq \mathfrak{g}_{\rho(\pi)^+}$, where $\mathcal{O}(\Gamma)$ is the set of orbits whose closure contains Γ.

Remark 4.2.

(1) If Γ is regular, then $\theta_\pi = c_\theta \widehat{\mu}_{\theta_\Gamma}$.
(2) c_θ is "computable".

Theorem 4.3. *Let G be semisimple. Almost all tempered representations contain a weakly good type i.e. the Plancherel measure of the complement is 0.*

Proof.

(1) Let $G_{ss} \subset \mathfrak{g}$ be the subset of all good semisimple elements of depth < 0, union with $\{0\}$.

We define

$$\mathfrak{g}_0 = \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,0}.$$

Given a dual blob Γ, we define

$$\mathfrak{g}_\Gamma = G \cdot (\Gamma + C_{\mathfrak{g}}(\Gamma)_{r^+}),$$

which is an open and closed G-invariant set.

We let $\mathfrak{g}_\Gamma$ be the union of all dual blobs of unrefined minimal K-types intertwining with the good types of the form $(G_{\chi,\rho}, \chi_\Gamma)$. Given $\Gamma, \Gamma' \in G_{ss}$, we say $\Gamma \sim \Gamma'$ iff $\mathfrak{g}_\Gamma \cap \mathfrak{g}_{\Gamma'} \neq \emptyset$, iff $\mathfrak{g}_\Gamma = \mathfrak{g}_{\Gamma'}$.

We have

$$\mathfrak{g} = \overset{\circ}{\bigcup}_{\Gamma \in G_{ss}/\sim} \mathfrak{g}_\Gamma.$$

(2) We introduce the notation $\widetilde{G} := \mathcal{E}(G)$. For a dual blob Γ we let $\widetilde{G}_\Gamma$ be the set of irreducible smooth representations containing a minimal K-type whose dual blob is in $\mathfrak{g}_\Gamma$, and we set $\widetilde{G}_0$ to be the set of depth zero irreducible smooth representations.

We have

$$\widetilde{G} = \overset{\circ}{\bigcup}_{\Gamma \in G_{ss}/\sim} \widetilde{G}_\Gamma.$$

Now we're going to look at the tempered representations.

(3) The Plancherel formula says that for $f \in C_c^\infty(\mathfrak{g}_{0^+})$,

$$\int_{\mathfrak{g}} \widehat{f}(x)\, dx = f(0).$$

We want to show that

$$\int_{\mathfrak{g}} \widehat{f}(x)\, dx = \int_{\widehat{G}^{\text{temp}}} \Theta_\pi(f \circ \log) d\pi.$$

Decomposing,

$$\int_{\mathfrak{g}} \widehat{f}(x)\,dx = \bigoplus \int_{\mathfrak{g}_\Gamma} \widehat{f}(x)\,dx$$

and

$$\int_{\widehat{G}^{\text{temp}}} \mathcal{O}_\pi(f \circ \log) d\pi. = \bigoplus \int_{\widehat{G}_\Gamma^{\text{temp}}} \Theta_\pi(f \circ \log) d\pi + \int_{\widehat{G}^{\text{temp}} \setminus \mathring{\bigcup} \widehat{G}_\Gamma^{\text{temp}}} \Theta_\pi(f \circ \log) d\pi$$

So we want to match

$$\int_{\mathfrak{g}_\Gamma} \widehat{f}(x)\,dx = \int_{\widehat{G}_\Gamma^{\text{temp}}} \Theta_\pi(f \circ \log) d\pi.$$

Let $f_{y,\rho}$ be the characteristic function of $G_{y,\rho+1}$. If $0 \leq \rho < -\text{depth}(\Gamma)$, the depth of representations in $\widehat{G}_\Gamma^{\text{temp}}$ then both sides vanish.

If $\rho > -\text{depth}(\Gamma)$, then we match the two sides using the character expansion

$$\sum_{\mathcal{O} \in \mathcal{O}(\Gamma)} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}}.$$

The $c_{\mathcal{O}}$ are determined using the test functions $\text{ch}_{G_{x,r},\chi_\Gamma}$, i.e. the inflation of the character χ_Γ as a function supported on $G_{x,r}$.

The conclusion is that

$$\int_{\widehat{G}^{\text{temp}} \setminus \mathring{\bigcup} \widehat{G}_\Gamma^{\text{temp}}} \Theta_\pi(f \circ \log) d\pi = 0$$

for all such $f_{y,\rho}$. Since $\Theta_\pi(f)$ is always non-negative, this shows the vanishing of the Plancherel measure of non-tempered representations.

□

Remark 4.4. All smooth irreducible representations contain a good type, under the assumption that the group is tamely ramified. This can be proved using the geometry of the of Bruhat-Tits building.

Remark 4.5. The exhaustion theorem for supercuspidal reprsentations used a more refined notion of K-type and a more refined notion of character expansion. The refined notion extracts information on supercuspidal representations on $G_{x,0^+}$ (a topologically unipotent group).

To extract depth zero information, you need to analyze the Jacquet module.

References

[SV] Sakellaridis, Yiannis and Venkatesh, Akshay. *Periods and harmonic analysis on spherical varieties.*