DERIVED DEFORMATION RINGS FOR GROUP REPRESENTATIONS

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1. Homotopy theory of representations

Let me start by reviewing a few basic facts on homotopy theory. Let X be a "nice" connected space (e.g. a CW complex). Let G be a discrete group. We write BG for the classifying space of G, which is also known as a "$K(G,1)$". We write $\mathrm{Map}(X, BG)$ for the space of continuous maps from X to BG. The homotopy type of $\mathrm{Map}(X, BG)$ can be studied via its homotopy groups. Fixing a basepoint $x \in X$, the components are parametrized by $\mathrm{Hom}(\pi_1(X, x), G)$, but then we need to mod out by conjugation to account for the choice of basepoint. Therefore, we have

$$\pi_0 \mathrm{Map}(X, BG) = \mathrm{Hom}(\pi_1(X, x), G)/\mathrm{conj}.$$

Futhermore, we have

- $\pi_1(\mathrm{Map}(X, BG), \rho \colon \pi_1 \to G) \cong C_G(\mathrm{Im}\ \rho)$ (centralizer).
- $\pi_i(\mathrm{Map}(X, BG)) = 0$ for $i \geq 2$.

Thus we have a disjoint union of conjugacy classes

$$\mathrm{Map}(X, BG) = \coprod_{\rho \colon \pi_1(X) \to G} BC_G(\rho)$$

This presentation is good because it doesn't depend on basepoints of choices of representatives, and makes sense for non-discrete groups.

Remark 1.1. The components with trivial $C_G(\rho)$ are contractible. You can think of this as corresponding to ρ with "large image".

Remark 1.2. Framed deformations have to do with basepoints, where you get actual homomorphisms rather than up to conjugacy.

Date: December 12, 2017.

2. The classical theory

2.1. **The deformation functor.** We now introduce the setup for derived deformation rings. Let k be a finite field. Let W be the ring of Witt vectors of k. Let G be an algebraic group over W, e.g. $G = \mathrm{GL}_n$ or PGL_n.

Let A be a local Artinian ring. For every factorization

$$\begin{array}{ccccc} & & A & & \\ & \nearrow & & \searrow & \\ W & & \longrightarrow & & k \end{array}$$

we get a map $BG(A) \to BG(k)$, where for the moment the groups $G(A)$ and $G(k)$ are regarded as discrete. (In a moment we will take simplicial rings, and we will no longer be considering discrete groups.)

We want to think of the mapping spaces as a *substitute* for conjugacy classes of homomorphisms. Fix $\overline{\rho} \in \mathrm{Map}(X, BG(k))$, and consider the lifts to $\mathrm{Map}(X, BG(A))$.

$$\begin{array}{c} \mathrm{Map}(X, BG(A)) \\ \downarrow \\ \mathrm{Map}(X, BG(k)) \end{array}$$

Definition 2.1. The *homotopy fiber* $F^G_{X,\overline{\rho}}(A \xrightarrow{\epsilon} k)$ is the space of lifts

$$\begin{array}{ccc} & & BG(A) \\ & \overset{\rho}{\nearrow} & \downarrow \\ X & \xrightarrow{\overline{\rho}} & BG(k) \end{array}$$

together with a choice of homotopy h between the two compositions. This is a homotopy theoretic notion of lift, which is better than the naïve notion.

Remark 2.2. Often when we apply this, we make assumptions that imply $C_{G(k)}(\overline{\rho}) = 0$. In that case we can instead just require the *existence* of h. But in general it is better to include h in the notion of a lift.

Then $F^G_{X,\overline{\rho}}(A \xrightarrow{\epsilon} k)$ is a space which has no higher homotopy groups, and $\pi_0 F_{X,S}(A \to k)$ is the set of lifts

$$\begin{array}{ccc} & & G(A) \\ & \overset{\rho}{\nearrow} & \downarrow \\ \pi_1(X) & \xrightarrow{\overline{\rho}} & G(k)/\mathrm{conj} \end{array}$$

and $\pi_i F^G_{X,\overline{\rho}}(A \xrightarrow{\epsilon} k) = 0$ for all $i > 0$.

This is just a convoluted way of saying the following. Up to homotopy,

$$F^G_{X,\overline{\rho}}(A \xrightarrow{\epsilon} k) \cong \pi_0 F^G_{X,\overline{\rho}}(A \xrightarrow{\epsilon} k),$$

which is the "usual" deformation functor.

2.2. **The deformation ring.** The classical deformation ring $\mathcal{R}$ exists under mild assumptions, and represents this functor $F^G_{X,\overline{\rho}}$ in the sense that $F^G_{X,\overline{\rho}}(A \xrightarrow{\epsilon} k)$ is isomorphic to space of homomorphisms

$$\begin{array}{ccc} \mathcal{R} & \dashrightarrow & A \\ \downarrow & & \downarrow \\ k & = & k \end{array}$$

The ring $\mathcal{R}$ is complete local noetherian, or a pro-object in the category of local Artinian rings $A \to k$. We want advocate that the second point of view is better.

Remark 2.3. To obtain a theory that encompasses Galois groups, we consider pro-spaces $X = \{j \mapsto X_j\}$, and take

$$F_{X,\overline{\rho}} := \varinjlim_j F_{X(j),\overline{\rho}}.$$

3. The derived theory

We still fix a field k, but now we want to extend this theory to simplicial commutative rings $A_\bullet, \mathcal{R}$.

3.1. **Simplicial commutative rings.** A *simplicial commutative ring* (SCR) is a simplicial object in commutative rings, meaning a diagram of the form

$$A_\bullet = A_0 \leftleftarrows A_1 \ldots.$$

We want to think of $\pi_0(A_\bullet)$ as the "underlying commutative ring". For any simplicial commutative ring the homotopy groups form a graded commutative ring

$$\pi_* A_\bullet = \bigoplus_{i=0}^{\infty} \pi_i(A_\bullet).$$

Remark 3.1. Here $\pi_i(A_\bullet)$ is canonically isomorphic to the H_i of the chain complex

$$\ldots \leftarrow A_{i-1} \leftarrow A_i \leftarrow A_{i+1} \leftarrow \ldots$$

with differential $\partial = \sum (-1)^i d_i$. However this is not as good a point of view; for example the ring structure is invisible. It's better not to try to think in terms of this chain complex.

3.2. **Derived deformation functor.** To begin, we need to know what it means for a simplicial ring to be "Artinian", which is something we learned from Lurie's thesis [L].

Definition 3.2 (Lurie)**.** A simplicial commutative ring $A_\bullet$ is *Artinian local* if the underlying ring $\pi_0(A_\bullet)$ is Artinian local and $\pi_*(A_\bullet)$ is finitely generated as a module over $\pi_0(A_\bullet)$. (In particular, all but infinitely many of the homotopy groups are zero, and the non-zero ones are finitely generated over $\pi_0(A_\bullet)$.)

The domain for the derived version of $F_{X,\overline{\rho}}$ should be Artinian local simplicial commutative rings $(A_\bullet, \epsilon\colon \pi_0(A_\bullet)/\mathfrak{m} \xrightarrow{\sim} k)$.

Remark 3.3. The structure of an algebra over $W(k)$ comes "for free" (up to a contractible choice), so it does not need to be explicitly inserted into the definition.

We now want to design criteria for $F_{X,\overline{\rho}}$ to be representable.

Example 3.4. There is a functor from commutative rings to simplicial commutative rings, sending A to the constant simplicial ring A (with all morphisms being the identity). This is right adjoint to π_0. In particular, there is a canonical (counit) map $A_\bullet \to \pi_0(A_\bullet)$. You think of this as analogous to the map given by modding out by the nilpotent ideal.

This allows us to think of ordinary commutative rings as a full subcategory of simplicial commutative rings.

Definition 3.5. We say that a simplicial commutative ring is *homotopy discrete* if the counit

$$A_\bullet \to \pi_0(A_\bullet)$$

is a homotopy equivalence, i.e. the higher homotopy groups of $A_\bullet$ vanish.

The codomain of the derived $F_{X,\overline{\rho}}$ should be spaces, but for technical reasons it's better to say simplicial sets. If we ever say "space" we mean simplicial sets.

Let us first lay out what we are looking for in the derived deformation functor $F_{X,\overline{\rho}}$.

(1) It should be an extension of the classical $F_{X,\overline{\rho}}$.
(2) It should be *homotopy invariant*, meaning that if you have a morphism of Artinian local simplicial commutative rings

$$\begin{array}{ccc} A_\bullet & \xrightarrow{\sim} & A'_\bullet \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ k & = & k \end{array}$$

which is a homotopy equivalence, then

$$F_{X,\overline{\rho}}(f) \colon F_{X,\overline{\rho}}(A_\bullet) \to F_{X,\overline{\rho}}(A'_\bullet)$$

should also be a homotopy equivalence.
(3) There should be a compatibility with the non-derived theory. Namely, we have an inclusion of the category of Artinian local rings A into the category of Artinian local simplicial commutative rings, and taking π_0 recovers the classical theory, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \left\{ \begin{matrix} \text{Artinian local} \\ \text{commutative rings } A \end{matrix} \right\} & \xrightarrow{\text{classical } F_{X,\overline{\rho}}} & \text{Set} \\ \downarrow & & \uparrow{\scriptstyle \pi_0} \\ \left\{ \begin{matrix} \text{Artinian local simplicial} \\ \text{commutative rings } A \end{matrix} \right\} & \xrightarrow{F_{X,\overline{\rho}}} & \text{sSet} \end{array}$$

In the paper [GV] we explain how to define $F_{X,\overline{\rho}}$, and also in what sense it is representable. The proof of representable is like in the classical case, using a derived version of Schlessinger's criterion [L].

To define $F_{X,\overline{\rho}}$, we need to define the space "$BG(A)$". The association $A \mapsto BG(A)$ should be a functor from A to spaces. Then we just repeat everything we said about mapping spaces:

$$F_{X,\overline{\rho}}(A_\bullet \xrightarrow{\epsilon} k) = \left\{ \begin{array}{ccc} & & BG(A_\bullet) \\ & \overset{\rho}{\nearrow} & \downarrow{\scriptstyle \epsilon} \\ X & \xrightarrow{\overline{\rho}} & BG(k) \end{array} \right\}.$$

3.3. **Classifying spaces.** What is $BG(A)$? If A is discrete, one model for the classifying space of a group is the simplicial set which is the nerve of $G(A)$:

$$* \leftleftarrows G(A) \Lleftarrow G(A) \times G(A)$$

If A is discrete, $G(A) = \mathrm{Hom}(\mathcal{O}_G, A)$ and $G(A) \times G(A) = \mathrm{Hom}(\mathcal{O}_G \otimes \mathcal{O}_G, A)$ etc. So we could take the cosimplicial ring

$$W \rightrightarrows \mathcal{O}_G \Rrightarrow \mathcal{O}_G \otimes \mathcal{O}_G \otimes \dots$$

and try to set $BG(A)$ to be the homomorphisms from it to A.

However, this is a bad idea because the result won't be homotopy invariant. If $A_\bullet$, $B_\bullet$ are simplicial commutative rings then there is a space of homomorphisms $\mathrm{SCR}(A_\bullet, B_\bullet)$ but the construction $SCR(-,-)$ is not homotopy invariant in either factor. This is like how taking Hom out of a chain complex is bad if the complex isn't projective. Here the fix is to take a "cofibrant replacement" $c(A_\bullet) \xrightarrow{\sim} A_\bullet$. Then $\mathrm{SCR}(c(A_\bullet), -)$ is homotopy invariant.

We need to explain how to make a "space" of maps between two simplicial commutative rings.

Definition 3.6. If $X_\bullet, Y_\bullet$ are simplicial sets then $\mathrm{sSet}(X_\bullet, Y_\bullet)$ is the set of natural transformations between X and Y as functors $\Delta^{\mathrm{op}} \to \mathrm{Sets}$. This is naturally the set of 0-simplices of a simplicial set $\underline{\mathrm{sSet}}(X_\bullet, Y_\bullet)$. Here

$$\underline{\mathrm{sSet}}_p(X_\bullet, Y_\bullet) = \mathrm{sSet}(X \times \Delta[p], Y)$$

where $\Delta[p] = \Delta(-, [p])$ is the functor represented by $[p] \in \Delta$. Thus the category of simplicial sets is naturally enriched over itself.

Now, $Y_\bullet^{\Delta[p]}$ is the simplicial set of maps $\Delta[p] \to Y_\bullet$. If Y has an algebraic structure which is defined in terms of cartesian products, e.g. an abelian group structure, then this construction inherits that algebraic structure. In particular, if Y is a simplicial commutative ring then so is $Y_\bullet^{\Delta[p]}$. This makes SCR enriched over sSets.

The category SCR can be equipped with the structure of a model category in the sense of Quillen. This means that is has a notion of cofibrations and fibrations, etc. and part of this structure is the existence of cofibrant replacements. Now $\mathrm{SCR}(A_\bullet, -)$ is homotopy invariant if $A_\bullet$ is cofibrant. Intuitively this is something like "projective" but we don't want to define it formally.

Example 3.7. A free commutative ring on a simplicial set (meaning levelwise) is cofibrant. More generally, a sufficient criterion for a simplicial commutative ring to be cofibrant is that in each degree it has to be polynomial on a set (finite or infinite), such that the sets are closed under degeneracies (but not under face maps). Not every cofibrant thing is of the form, but this criterion supplies everything that you need in practice.

Fact 3.8. *There exists a functorial cofibrant replacement*

$$c(A_\bullet) \xrightarrow{\sim} A_\bullet.$$

Thus we get a cosimplicial commutative ring

$$W \rightrightarrows c(\mathcal{O}_G) \Rrightarrow c(\mathcal{O}_G \otimes \mathcal{O}_G) \dots$$

Remark 3.9. In general we do not claim that $c(\mathcal{O}_G \otimes \mathcal{O}_G) \cong c(\mathcal{O}_G) \otimes c(\mathcal{O}_G)$, i.e. that there is an op lax monoidal cofibrant replacement functor. Thus our construction does not present $BG(A)$ as the bar construction of some group $G(A)$. In fact it is more convenient to define $BG(A)$ than $G(A)$. We could then go and define $G(A) = \Omega BG(A)$.

The point is that $\mathrm{SCR}(\mathcal{O}_G, A)$ is a bad model for $G(A)$ because it is not homotopy invariant, so we replace it by $\mathrm{SCR}(c(\mathcal{O}_G), A_\bullet)$. Similarly, we get a homotopy invariant version of $N_\bullet(G(A))$,

$$[p] \mapsto \mathrm{Hom}(c(\mathcal{O}_G^{\otimes p}), A_\bullet).$$

This nerve is a simplicial object in simplicial sets. To extract the desired simplicial set $BG(A)$ we take the geometric realization $BG(A_\bullet) = |N_\bullet G(A_\bullet)|$, which in this case is just the diagonal simplicial set:

$$BG(A_\bullet)_p = N_p G(A_p).$$

This gives a homotopy invariant construction $A_\bullet \to BG(A_\bullet)$, which recovers the classical construction in the sense that if $A_\bullet \xrightarrow{\sim} \pi_0(A_\bullet)$, then $BG(A_\bullet) \cong B(G(\pi_0(A)))$.

We now define a deformation functor as before.

$$F_{X,\overline{\rho}}(A_\bullet \xrightarrow{\epsilon} k) = \left\{ \begin{array}{ccc} & & BG(A_\bullet) \\ & \overset{\rho}{\nearrow} & \downarrow \epsilon \\ X & \xrightarrow{\overline{\rho}} & BG(k) \end{array} \right\}.$$

(We imagine these as simplicial sets; if X is a space we should take geometric realizations.) This defines a homotopy invariant functor

$$F_{X,\overline{\rho}} \colon \left\{ \begin{smallmatrix} \text{Artin local simplicial} \\ \text{commutative rings} \end{smallmatrix} \right\} \to \text{sSets}.$$

Now we turn our attention to the issue of representability. In his thesis [L] Lurie gives a "derived Schlessinger criterion", and we check that our functor does satisfy this criterion. Let's not say much about this; it's not one of the more difficult steps. The point is that we want F to be equivalent to the functor represented by a pro-object $\mathcal{R} = \{j \mapsto \mathcal{R}(j)\}$ in Artin local simpicial commutative rings, such that

$$A \mapsto \varinjlim_j \mathrm{SCR}_{\bullet/k}(\mathcal{R}(j), A_\bullet)$$

In order to get homotopy invariance, we need to demand that the $\mathcal{R}(j)$ are cofibrant.

Remark 3.10. In simplicial sets homotopy groups commute with filtered colimit. The colimit is morally a homotopy colimit, but in our model it is not necessary (which is technically convenient).

3.4. **The tangent complex.** Consider a functor

$$F \colon \left\{ \begin{smallmatrix} \text{Artin local simplicial} \\ \text{commutative rings} \end{smallmatrix} \right\} \to \text{sSets}.$$

We are interested in the question of when F is pro-representable by $\mathcal{R}$.

The representing object should have a cotangent complex, and the tangent complex should be "$\mathrm{Hom}_{\mathcal{R}}(\Omega^1_{\mathcal{R}}, k))$". There is a way to express this only in terms of the functor F, which is expressed in terms of evaluating F on objects "$k \oplus k[n]$" (which we also learned from Lurie's thesis). This is a square-zero extension.

When $n = 0$, $k \oplus k[0] = k[\epsilon]/\epsilon^2$ is the usual dual numbers viewed as a constant simplicial ring. Now we want a version where ϵ is in π_n instead. So we define $k[n]$ to be the simplicial k-module with

$$\pi_i k[n] = \begin{cases} k & i = n \\ 0 & \text{otherwise} \end{cases}$$

One way to construct this is to apply Dold-Kan to the chain complex $k[n]$.

We then define $k \oplus k[n]$ to be the levelwise square 0 extension. This is a simplicial commutative ring with

$$\pi_*(k \oplus k[n]) = k[\epsilon]/\epsilon^2$$

where $|\epsilon| = n$. One of the properties of F is that $F = F_{X,\overline{\rho}}$ satisfies

$$F(k = k) \cong *$$

is contractible, and there is a canonical homotopy equivalence

$$F(k \oplus k[n]) \cong \Omega F(k \oplus k[n+1]).$$

In particular $F(k \oplus k[0])$ (the first-order deformations) comes with deloopings

$$\Omega F(k \oplus k[1]) \cong \Omega^2 F(k \oplus k[2]) \cong \dots$$

so $F(k \oplus k[n])$ forms an "Ω-spectrum". The spectrum is what we define to be the tangent complex $t\mathcal{F}$. It then makes sense to talk about negative homotopy groups. For example,

$$\pi_{-i} tF = \pi_{n-i} F(k \oplus k[n]) \quad n \gg 0.$$

Remark 3.11. There is a way to produce a spectrum from a $\mathbf{Z}$-graded chain complex of k-vector spaces. The construction goes as follows. If C_* is a chain complex, $\Sigma^n C_* = C_*[n]$. Then we can form an object $|\Sigma^n C_*|$ by shifting, truncating the complex to be in non-positive degrees, and then applying Dold-Kan, to get a spectrum.

It is a fact that our tangent complex $\mathcal{F}$ does arise from a $\mathbf{Z}$-graded chain complex in this way. (It's mostly in negative degrees, so if you dualize you get something mostly in positive degrees, which might be psychologically comforting.)

Lurie's derived Schlessinger's criterion roughly asks for the following conditions on the functor F:

- it should be homotopy invariant,
- it should preserve pullbacks

$$\begin{array}{ccc} A_\bullet \times_{B_\bullet} C_\bullet & \longrightarrow & A_\bullet \\ \downarrow & & \downarrow \\ C_\bullet & \longrightarrow & B_\bullet \end{array}$$

- tF has no homotopy in positive degrees, i.e. $F(k \oplus k[0]) \xrightarrow{\sim} \pi_0 F(k \oplus k[0])$. This expresses the intuition that "the moduli problem has no automorphisms".

The upshot is that we get a pro-([cofibrant] Artin local simplicial commutative ring) over k),

$$\mathcal{R} = \{j \mapsto \mathcal{R}(j)\}.$$

It's better to think of $\mathcal{R}$ as a pro-object. You could try taking

$$\pi_i \mathcal{R} := \varprojlim_j \pi_i(\mathcal{R}(j))$$

but this is badly behaved in general because there's a $\varprojlim^1$. It well-behaved if k is finite because then $\varprojlim^1 = 0$. One could take a homotopy limit, but we don't do this.

We can then define

$$\pi_*(\mathcal{R}) = \bigoplus_{i=0}^{\infty} \pi_i(\mathcal{R}).$$

Remark 3.12. Because this is a pro-object, it could have homotopy in arbitrarily high degrees, just as taking a limit of Artinian objects need not produce something Artinian.

Now we consider what the tangent complex looks like for $F = F_{X,\overline{\rho}}$, the functor sending $(A \xrightarrow{\epsilon} k)$ to the space of lifts

$$\begin{array}{ccc} & & BG(A_\bullet) \\ & \nearrow & \downarrow \\ X & \xrightarrow{\overline{\rho}} & BG(k) \end{array}$$

In particular consider lifts to $A_\bullet = k \oplus k[n]$:

$$\begin{array}{ccc} & & BG(k \oplus k[n]) \\ & \nearrow & \downarrow \\ X & \xrightarrow{\overline{\rho}} & BG(k) \end{array}$$

Think of $BG(k)$ as a $K(\pi, 1)$. The $BG(k \oplus k[n])$ only has $\pi_1 = G(k)$ and $\pi_{n+1} = \mathfrak{g} \otimes_W k$. For any space you have an action of π_1 on higher homotopy groups, which in this case is the adjoint action.

From this we can get a description of the homotopy fiber:

$$\pi_{-n}(tF) = \pi_0(F(k \oplus k[n])) = H^{n+1}(X; \mathfrak{g})$$

where $\pi_1(X)$ acts on $\mathfrak{g}$ by

$$\pi_1(X) \xrightarrow{\overline{\rho}} G(k) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}).$$

This gives a formula for "$\pi_0 \operatorname{Hom}(\mathcal{R}, k \oplus k[n])$". This is like how it's easier to find formulas for cohomology than homotopy groups in topology.

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