COHOMOLOGY OF ARITHMETIC GROUPS AND AUTOMORPHIC FORMS: AN INTRODUCTION, II

LECTURES BY LAURENT CLOZEL,
NOTES BY TONY FENG

Contents

We'll start off with some complements to and applications of the theory discussed last time. As before we will mostly stick to the cocompact case, but at the end we'll give a preview of what happens in the noncompact case.

1. Classification of cohomological $(\mathfrak{g}, K)$-modules

Last time we saw that for (cocompact) $\Gamma \subset G$, the cohomology $H^*(\Gamma \backslash X)$ can be described in terms of the $(\mathfrak{g}, K)$-cohomology of $(\pi, V) \subset L^2(\Gamma \backslash G)$.

Let (π, V) be a unitary $(\mathfrak{g}, K)$-module. If $H^*(\mathfrak{g}, K; V) \neq 0$ then we know that:

- The Casimir acts by 0, but moreover by Wigner's theorem we have $\omega_\pi = \omega_{\mathbf{C}}$.
- We have $\mathrm{Hom}_K(\Lambda^k \mathfrak{p}, V) \neq 0$ for some k.

By a theorem of Harish-Chandra, there are only finitely many π satisfying these conditions. They have been classified by work of Parthasarathy, Kumaresan, and finally Vogan-Zuckerman. Let's try to state the result. For our formulation, it is important that G is semisimple with no real compact factors.

1.1. θ-stable parabolics.

Consider the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

There is a notion of *θ-stable parabolic subalgebra* $\mathfrak{q} \subset \mathfrak{g}$. They are all obtained by the following construction.

Pick a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$ and let $x \in i\mathfrak{t}_0$. (By compactness the roots of G have purely imaginary values on $\mathfrak{t}_0$, which is why we put the factor of i). We will define

$$\mathfrak{q} := \mathfrak{l} \oplus \mathfrak{u},$$

a decomposition into a Levi plus a nilpotent. Here

- $\mathfrak{l} := \mathfrak{g}_x$ (the centralizer of x) and

Date: October 17, 2017.

- $\mathfrak{u} := \sum_\alpha \mathfrak{g}_\alpha$ with $\langle \alpha, x\rangle > 0$, where $\mathfrak{g}_\alpha$ is the root space for the root α of $\mathfrak{t}$ in $\mathfrak{g}$.

Warning 1.1. We do not assume that $\mathfrak{t}_0$ is a Cartan subalgebra in $\mathfrak{g}_0$.

1.2. The Vogan-Zuckerman classification.

Theorem 1.2 (Vogan-Zuckerman, '84)**.** *We have the following facts.*

1. *There is a unique irreducible unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}$, with (trivial) infinitesimal character $\omega_{\mathbf{C}}$, containing as its minimal K-type. the representation of K with highest weight $2\rho(\mathfrak{u}\cap\mathfrak{p})$.*
2. *We'll describe the degrees where cohomology occurs. Let $R = \dim_{\mathbf{C}}(\mathfrak{u}\cap\mathfrak{p})$. Then*
$$H^R(\mathfrak{g}, K; A_{\mathfrak{q}}) \cong \mathbf{C}$$
and furthermore $H^i(\mathfrak{g}, K; A_{\mathfrak{q}}) \implies i \geq R$.

Remark 1.3. Let us try to explain any mysterious parts of the statement:

- Here $2\rho(\mathfrak{u}\cap\mathfrak{p})$ is the sum of the roots of $\mathfrak{t}$ occurring in $\mathfrak{u}\cap\mathfrak{p}$.
- We equate the highest weight representation of K with the corresponding element of $\mathfrak{t}^*$. The minimal K-type is essentially the K-type in $A_{\mathfrak{q}}$ whose highest weight is the "shortest" in a certain (non-obvious) sense.

1.3. Trivial cohomology and the compact dual.

Example 1.4. (Matsushima) For $A_{\mathfrak{q}} = \mathbf{C}$, we have an isomorphism
$$H^*(\mathfrak{g}, K; \mathbf{C}) \cong H^*(\widetilde{X})$$
where $\widetilde{X}$ is the *compact dual* of the symmetric space $X = G/K$, defined as follows. Consider the real form
$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \subset \mathfrak{g}.$$
We define
$$\mathfrak{u}_0 := \mathfrak{k}_0 + i\mathfrak{p}_0 \subset \mathfrak{g}.$$
This is a real Lie subalgebra, and induces a compact group U still containing K. Then $\widetilde{X} = U/K$ (this is a symmetric space). Since this is compact, we can apply Matsushima's formula, plus the fact that the cohomology of compact manifolds is represented by *invariant* forms (by an averaging argument), to get
$$H^*(\widetilde{X}, \mathbf{C}) \cong H^*(\mathfrak{u}, K; \mathbf{C}).$$
But then we see that
$$H^*(\mathfrak{u}, K; \mathbf{C}) = H^*(\mathfrak{g}, K; \mathbf{C})$$
by direct comparison of the defining complexes: the K is the same and the terms in the complex are $\Lambda^k\mathfrak{p}$ for both.

Warning 1.5. There are differences in the signs for the complexes defining $H^*(\mathfrak{u}, K; \mathbf{C})$ and $H^*(\mathfrak{g}, K; \mathbf{C})$.

Remark 1.6. Now $H^*(\mathfrak{g}, K; \mathbf{C})$ acts on the whole cohomology
$$H^*(\Gamma\backslash X; \mathbf{C}) \cong \bigoplus_V H^*(\mathfrak{g}, K; V)$$
because we have an action of $H^*(\mathfrak{g}, K; \mathbf{C})$ on $H^*(\mathfrak{g}, K; V)$ by $\mathbf{C}\otimes V \to V$. In particular $H^*(\mathfrak{g}, K; \mathbf{C})$ is an algebra, and endows $H^*(\Gamma\backslash X; \mathbf{C})$ with the structure of an algebra over it.

Fact 1.7. *If G is simple, then for $A_{\mathfrak{q}} \ncong \mathbf{C}$ we have $R \geq \operatorname{rank}_{\mathbb{R}} G$.*

Together with Theorem 1.2 this fact implies the following theorem, because in the decomposition

$$H^*(\Gamma\backslash X) \cong \bigoplus H^*(\mathfrak{g}, K; V)$$

the nontrivial $(\mathfrak{g}, K)$-modules contribute only in degrees $R \geq \operatorname{rank}_{\mathbb{R}} G$.

Theorem 1.8 (Zuckerman, Borel-Wallach)**.** *For G simple, and for $k < \operatorname{rank}_{\mathbb{R}} G$, the map*

$$j\colon H^k(\mathfrak{g}, K; \mathbf{C}) \to H^k(\Gamma\backslash X; \mathbf{C})$$

is an isomorphism. Here j is the map induced by the inclusion of the constant functions $\mathbf{C} \subset C^\infty(\Gamma\backslash G)$.

2. The Hermitian symmetric case

2.1. **Hodge theory.** In this section we discuss some results for the case where X is *hermitian symmetric.* This means that the real tangent space

$$\mathfrak{p}_0 = T_o(X) \subset \mathfrak{p} = T_o(X) \otimes \mathbf{C}$$

has a complex structure (i.e. a J with $J^2 = -1$), which is K-invariant. (This is equivalent to the condition that the complex structure on X be G-invariant.) Then we can decompose

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \tag{2.1.1}$$

and $X, \Gamma\backslash X$ are Kähler varieties. We can thus apply Hodge theory.

By (2.1.1) we have the decomposition

$$C^k(\mathfrak{g}, K; V) = \operatorname{Hom}_K(\Lambda^k \mathfrak{p}, V) = \bigoplus_{p+q=k} \operatorname{Hom}_K(\Lambda^p \mathfrak{p}^+ \otimes \Lambda^q \mathfrak{p}^-, V). \tag{2.1.2}$$

Not surprisingly this coincides with the Hodge decomposition.

Theorem 2.1 (Borel-Wallach)**.** *We have a decomposition*

$$H^k(\Gamma, E) \cong \bigoplus_{p,q} H^{p,q}(\Gamma, E)$$

where p, q are obtained (in Matsushima's formula) by (2.1.2).

Warning 2.2. Even with a coefficient system (local system) E, the degrees (p, q) are the ones occurring for the cohomology of $\Gamma\backslash X$, i.e. $p, q \leq \dim_{\mathbf{C}}(\Gamma\backslash X)$. This does not coincide with what you might expect, as we will explain next.

2.2. **Hodge theory corrected.** In the applications, e.g. to Shimura varieties, we have a local system $\mathcal{E}$ (coming from the geometry) which is not constant. Then you have to correct the Hodge theory.

We'll explain in the case of modular curves. (This is sort of cheating because it's noncompact, but the same phenomenon appears.) Let $S := \Gamma\backslash X = \Gamma\backslash\mathfrak{h}$ where $\mathfrak{h}$ is the Poincaré upper half plane and $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup, e.g. $\Gamma = \Gamma_0(N)$. In this situation we have the universal elliptic curve $\mathcal{E} \to S$, with a level structure.

For $s \in S$, we denote the fiber elliptic curve by $\mathcal{E}_s$. We have the Leray spectral sequence

$$H^p(S, H^q(\mathcal{E}_s)) \implies H^{p+q}(\mathcal{E}). \tag{2.2.1}$$

(All coefficients are over $\mathbf{C}$.) Now, $H^q(\mathcal{E}_s)$ is a local system on S.

In the compact Kähler case one would have degeneration of this spectral sequence. That does not hold here because S is noncompact, but it does hold after passing to the "cuspidal" part, so we have an inclusion

$$H^1_{\mathrm{cusp}}(S, \mathcal{H}^1(\mathcal{E}_s)) \subset H^2(\mathcal{E}).$$

Here the local system is simply $H^1(\mathcal{E}_s, \mathbf{Z}) \cong \mathbf{Z}^2$ which is the standard representation of Γ. Consequently, (by Eichler-Shimura)

$$H^1_{\mathrm{cusp}}(S, \mathrm{Std})$$

is the space of (holomorphic plus antiholomorphic) cusp forms of classical weight 3.

The conclusion is then that (recalling that cusp forms of classical weight 2 have weight 1, in the usual geometric sense) forms of classical weight 3 have geometric weight 2. The Hodge structure on them *should be* $(2,0) \oplus (0,2)$. But (2.2.1) assigns the weight $(1,1)$. Deligne explained how to correct this (unpublished); for references see [Zu81] and [Oda].

3. Applications and questions

3.1. Stable cohomology and algebraic K-theory. We return to the case of general locally symmetric spaces (not just hermitian symmetric).

We define the "trivial cohomology" $H^k_{\mathrm{triv}}(\Gamma\backslash X; \mathbf{C})$ by

$$H^k(\Gamma\backslash X; \mathbf{C}) \supset H^k_{\mathrm{triv}}(\Gamma\backslash X; \mathbf{C}) := H^k(\widetilde{X}; \mathbf{C}).$$

As we saw in Theorem 1.8, for G simple and $\Gamma\backslash G$ cocompact this is an isomorphism for $k < \operatorname{rank} G$, . In fact this is sort of phenomenon is true more generally.

Theorem 3.1 (Borel '75)**.** *This remains true in general for $k < \rho(G)$.*

We don't define $\rho(G)$, but it tends to ∞ in most important families of arithmetic groups, such as Example 3.2. In particular if we fix k, we can compute the projective limits $\varprojlim H^k$ for natural embeddings of spaces $\Gamma\backslash X$.

Example 3.2. Consider the inclusion $\mathrm{GL}(n, \mathbf{Z}) \hookrightarrow \mathrm{GL}(n+1, \mathbf{Z})$ by

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

Hence $H^*(\mathrm{GL}(n, \mathbf{Z}); \mathbf{C}) = H^k(\Gamma_n\backslash X_n, \mathbf{C})$ has

$$\varprojlim_{n\to\infty} H^k(\mathrm{GL}(n, \mathbf{Z}), \mathbf{C}) \cong \varprojlim_{n\to\infty} H^k(\widetilde{X}_n; \mathbf{C}).$$

This is called the *stable cohomology* of GL_n. It is known that this computes the K-theory of $\mathbf{Z}$ with rational coefficients. By exactly the same argument, we can compute the rational K-theory of $\mathcal{O}_F$ for a number field F.

3.2. Representations contributing to L^2. Let G be a reductive group over $\mathbf{Q}$, and $\Gamma \subset G(\mathbf{Q})$ an arithmetic subgroup. Then we can describe $H^*(\Gamma, \mathbf{C})$ in terms of the occurrences of the representations $A_{\mathfrak{q}} \subset L^2(\Gamma\backslash G)$. But which $A_{\mathfrak{q}}$ actually occur?

There has been a lot of progress thanks to work of Arthur, with the consequence that $L^2(\Gamma\backslash G)$ is known for a lot of classical groups. An analysis of this problem using Arthur's work is expected to appear in [AMR].

An informal conjecture predicts that if G is split over $\mathbf{Q}$, then all $A_{\mathfrak{q}}$'s occur for some Γ.

This is known for GL_n, thanks to a result of Birgit Speh. Using the work of Arthur, this probably can be proven for all G.

Conjecture 3.3 (Bergeron)**.** *Let G be a simple group over $\mathbf{Q}$. Then $A_{\mathfrak{q}}$ occurs if and only if the Levi subalgebra $\mathfrak{l}_0$ is conjugate to $\underline{\mathfrak{l}}_0$ defined over $\mathbf{Q}$ under $G(\mathbb{R})$.*

3.3. **The Hodge Conjecture.** In the Hermitian symmetric spaces, this description of the cohomology can be used in relation to geometric problems, in particular the Hodge Conjecture and the Tate Conjecture.

As we have seen, the trivial cohomology embeds as

$$j\colon H^*(\widetilde{X};\mathbf{C}) \hookrightarrow H^*(\Gamma\backslash X;\mathbf{C}).$$

Assume that Γ is a congruence subgoup. (Since we have not defined this yet, here is a definition. If $G/\mathbf{Q}$ is simple and simply connected, then a congruence subgroup of $G(\mathbf{Q})$ is one of the form $\Gamma = G(\mathbf{Q})\cap K_f$ with $K_f \subset G(\mathbb{A}_f) = \prod' G(\mathbf{Q}_p)$ is a compact open subgroup.)

Then the Hecke algebra $\mathcal{H} := C_c^\infty(K_f\backslash G(\mathbb{A}_f)/K_f,\mathbf{Z})$ acts naturally on $C^\infty(\Gamma\backslash G)$, and thus on $H^*(\Gamma\backslash X;\mathbf{C})$. There is a natural homomorphism $\deg\colon \mathcal{H}\to\mathbf{Z}$, given by $h\mapsto \deg h$, which assigns to a Hecke operator the degree of the correspondence. Then $\mathcal{H}$ acts on $\mathbf{C}$ (the trivial representation of $G(\mathbb{A}_f)$) by deg.

It is then easy to check that

$$H^k_{\text{triv}}(\Gamma\backslash X;\mathbf{C}) = \{\omega\in H^k(\Gamma\backslash X;\mathbf{C})\colon \omega * h = (\deg h)\omega\}.$$

A consequence of this is that the trivial cohomology (seen as a subspace of $H^k(\Gamma\backslash X;\mathbf{C})$) is a subspace defined over $\mathbf{Q}$.

But H_{triv} is composed entirely of (p,p) classes. Indeed

$$H^k(\mathfrak{g};K;\mathbf{C}) \cong H^k(\mathfrak{u},K;\mathbf{C}) \cong H^k(\widetilde{X};\mathbf{C}).$$

Now, the varieties $\widetilde{X}$ are generalized Grassmannians, for which it's well-known that all cohomology is of Hodge type. Since the only difference between $H^k(\mathfrak{g};K;\mathbf{C})$ and $H^k(\mathfrak{u},K;\mathbf{C})$ cohomology has to do with signs, the same is true for $H^*(\mathfrak{g},K;\mathbf{C})$.

Theorem 3.4. $H^*(\widetilde{X})\subset H^*(\Gamma\backslash X)$ *is generated by cycle classes.*

This is "well-known" (to the experts) but not quite obvious. The proof is too long to explain right now.

3.4. **The Tate conjecture.** Let V be a smooth, projective variety over a number field F. Then we can consider the étale cohomology $H^{2k}_{\text{ét}}(V_{\overline{F}};\mathbf{Q}_\ell(k))$. The Tate conjecture predicts that $(H^{2k}_{\text{ét}}(V_{\overline{F}};\mathbf{Q}_\ell(k))^{\text{Gal}(\overline{F}/F)}$ is generated by cycle classes of codimension k.

In case where G is a $\mathbf{Q}$-group yielding a Hermitian quotient, the quotients are defined over number fields (Shimura varieties) and we can use of the cohomology to attack the Tate conjecture.

There are two ways to proceed in very particular cases.

- You can try to understand which classes should exist from the description of the Galois action, and then you construct the cycles from embeddings of smaller Shimura varieties coming from $H\subset G$.
- You can start with the Lefschetz theorem on hyperplane sections, and use the richer Lefschetz theory coming from the algebra structure over $H^*(\widetilde{X})$, cf. Remark 1.6, and the fact that the latter is all generated by cycle classes.

This was started by Harder-Langlands-Rapoport (1986), and culminated in a recent paper Bergeron-Millson-Moeglin (2016), using the full strength of Arthur's results.

4. The noncompact case

4.1. **Cohomology and automorphic forms.** First we say what replaces Matsushima's formula in the noncompact case.

Let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic subgroup. Then we have as before

$$H^*(\Gamma, E) \cong H^*(\Gamma \backslash X, \mathcal{E}).$$

Secondly it remains true that

$$H^*(\Gamma \backslash X) \cong H^*(\mathfrak{g}, K; C^\infty(\Gamma \backslash G)).$$

But C^∞ is too large and too soft; we need to replace it by automorphic forms.

Definition 4.1. We say a smooth function $f \colon \Gamma \backslash G \to \mathbf{C}$ is an *automorphic form* if

(1) f is K-finite,
(2) f is $\mathfrak{z}$-finite,
(3) f is of moderate growth on $\Gamma \backslash G$.

This is sort of cheating because we haven't defined what "moderate growth" means in (3). Let's just explain what it means in the classical case. For subgroups of $\mathrm{SL}(2, \mathbf{Z})$, considering the standard fundamental domain, it is equivalent to $f = O(y^N)$.

Theorem 4.2 (Franke '98)**.** *Let $\mathcal{A}(\Gamma \backslash G)$ be the space of automorphic forms. Then the map*

$$\mathcal{A}(\Gamma \backslash G) \hookrightarrow C^\infty(\Gamma \backslash G)$$

induces an isomorphism on $(\mathfrak{g}, K)$-cohomology.

Although the statement is simple, this is a very hard result!

4.2. **The L^2-theory.** We want to make a connection with $L^2(\Gamma \backslash G)$. For $\Gamma \backslash G$ not compact, this does not decompose discretely under G. Consider the discrete part

$$L^2_{\mathrm{disc}}(\Gamma \backslash G) \subset L^2(\Gamma \backslash G)$$

where $L^2_{\mathrm{disc}}(\Gamma \backslash G)$ is defined as the direct sum of all irreducible representations (π, H) with $H \hookrightarrow L^2(\Gamma \backslash G)$ as a G-module. Thus we can write

$$L^2_{\mathrm{disc}}(\Gamma \backslash G) = \widehat{\bigoplus} H_\pi, \qquad (H_\pi \text{ irreducible}).$$

We can then try to imitate the compact theory. We can look at the smooth vectors $L^2_{\mathrm{disc},\infty} \subset L^2_{\mathrm{disc}}$, which composed of smooth L^2-functions on $\Gamma \backslash G$ (but not necessarily all of them).

We can then look at

$$H^k(\mathfrak{g}, K; L^2_{\mathrm{disc},\infty}) = \bigoplus_{\text{finite}} H^k(\mathfrak{g}, K; V_\pi).$$

(This picks out the cohomological π.) So we have a finite dimensional space, computed by the representation theory. This is called $H^k_{(2)}(\Gamma \backslash X, \mathbf{C})$; although one should be careful that there are other notions with similar notation.

What is this space intrinsically in terms of $\Gamma \backslash X$?

Theorem 4.3. *The space*

$$H^*(\mathfrak{g}, K; L^2_{\mathrm{disc},\infty})$$

is naturally isomorphic to the space of L^2, harmonic forms on $\Gamma \backslash X$.

(This is a difficult theorem.) As a consequence, we get a natural map

$$H^*_{(2)}(\Gamma\backslash X, \mathbf{C}) \to H^*(\Gamma\backslash X).$$

This is never surjective for noncompact X, and generally not injective.

The space $H^k_{(2)}$ is given by L^2 harmonic forms, hence by automorphic forms. (This implication is not obvious, but not as hard at the other results we have been discussing.)

Fact 4.4. *We have an inclusion*

$$L^2_{\text{cusp}}(\Gamma\backslash G) \subset L^2_{\text{disc}}(\Gamma\backslash G).$$

We can take the smooth vectors and then forming the corresponding cohomology:

$$H^*_{\text{cusp}}(\Gamma\backslash X) := H^*(\mathfrak{g}, K; L^2_{\text{cusp},\infty}(\Gamma\backslash G)).$$

Theorem 4.5 (Borel-Langlands)**.** *The map*

$$H^*_{\text{cusp}}(\Gamma\backslash X) \to H^*(\Gamma\backslash X)$$

is injective.

Remark 4.6. We can compute cohomology of a Hilbert space, like $L^2(\Gamma\backslash G)$, in two different ways. One would be to pass to smooth vectors, and then take $(\mathfrak{g}, K)$-cohomology in the ordinary sense. The other would be to work with a complex of Hilbert spaces, and treat the differentials as maps in a functional-analytic sense. It is a non-trivial fact that these two methods ultimately give the same answer.

Warning 4.7. We have a decomposition

$$L^2(\Gamma\backslash G) = L^2_{\text{disc}} \oplus L^2_{\text{cont}}.$$

Taking smooth vectors, we get

$$L^2(\Gamma\backslash G)_\infty = L^2_{\text{disc},\infty} \oplus L^2_{\text{cont},\infty}$$

We can compute $H^*(\mathfrak{g}, K; L^2_{\text{cont},\infty})$. What does its image look like in $H^*(\mathfrak{g}, K; L^2(\Gamma\backslash G)_\infty)$? This is often pathological, and infinite-dimensional in general. This was understood by Borel-Casselman.

References

[AMR] N. Arancibia, C.Moeglin, D. Renard, Paquet dâĂŹArthur des groupes classiques et unitaires (le cas des representation cohomologiques), preprint.

[Oda] Oda, Takayuki. Hodge structures attached to geometric automorphic forms. Automorphic forms and number theory (Sendai, 1983), 223–276, Adv. Stud. Pure Math., 7, North-Holland, Amsterdam, 1985.

[Zu81] Zucker, Steven. Locally homogeneous variations of Hodge structure. Enseign. Math. (2) 27 (1981), no. 3-4, 243–276 (1982).