A REMARK ON COHOMOLOGY OF LOCALLY SYMMETRIC SPACES

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1. Introduction

This will be an informal discussion of some ideas. There will be no proofs, but we will formulate some conjectures.

Let G be a reductive group and K a maximal compact subgroup of $G(\mathbf{R})$. Let $H = G/K$ be the associated symmetric space. This is contractible.

Suppose $\Gamma \subset G$ is an arithmetic subgroup. Consider the locally symmetric space

$$X_\Gamma := \Gamma \backslash H.$$

We may assume that Γ is sufficiently small, so that it has no torsion elements. Then it will act freely on H, so

$$H^*(\Gamma) = H^*(\Gamma \backslash X_\Gamma).$$

We will discuss how to think about these cohomology groups. First I want to explain why this is an important problem, from my point of view.

If you consider one particular space and compute its cohomology, this is maybe not so important. The important thing is that we have a *family* of spaces, and the family has a "Hecke structure". Namely, if $\Gamma \supset \Gamma'$ is an inclusion of finite index then we have maps

$$H^*(\Gamma) \to H^*(\Gamma') \quad \text{and} \quad H^*(\Gamma) \leftarrow H^*(\Gamma').$$

Indeed, $\Gamma \backslash X \leftarrow \Gamma' \backslash X$ is a finite covering, so we have pushforward and pullback maps. These are used to define an action of the Hecke algebra.

Second, the spaces $\Gamma \backslash X$ are have a "motivic structure". I will not try to make this precise. They have Betti and de Rham realizations. In the case where X is Hermitian symmetric, they are even algebraic varieties so they also have p-adic cohomology theories, equipped with an action of Galois groups.

We will focus on the de Rham realization, i.e. the de Rham cohomology $H^*_{\mathrm{dR}}(X_\Gamma)$. This is computed by the de Rham complex,

$$H^*_{\mathrm{dR}}(X_\Gamma) = H^*(\Omega^*_{\mathrm{dR}}(X_\Gamma)) = H^*(\mathfrak{g}, K; \mathcal{F})$$

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where $\mathcal{F}$ is the space of functions on $Z_\Gamma := \Gamma\backslash G$. (The space Z_Γ is somehow the more correct object, as opposed to X_Γ. It has an action of G, and fits into the automorphic picture.)

In the case where X_Γ is compact, we have

$$H^*_{\mathrm{dR}}(X_\Gamma) = H^*(\mathfrak{g}, K; C^\infty(Z_\Gamma)).$$

We can think of $C^\infty(Z_\Gamma)$ as the space of smooth vectors in $L^2(Z_\Gamma)$. Then we can decompose $L^2(Z_\Gamma)$ spectrally,

$$L^2(Z_\Gamma) = \bigoplus L_\pi$$

which induces

$$C^\infty(Z_\Gamma) = \bigoplus L_\pi^\infty$$

hence

$$H^i(\Gamma) = \bigoplus_\pi H^i(\mathfrak{g}, K; L_\pi^\infty).$$

Thus in the compact case we get a spectral decomposition of cohomology. Furthermore, we have a Poincaré duality.

Now suppose that X_Γ is *not* compact. Then there are two kinds of possibilities for $\mathcal{F}$.

(1) For $\mathcal{F} = C^\infty(Z_\Gamma)$, which yields the ordinary cohomology

$$H^i(\mathfrak{g}, K; \mathcal{F}) = H^*(X_\Gamma).$$

(2) For $\mathcal{F} = C_c^\infty(Z_\Gamma)$, which yields the compactly supported

$$H^i(\mathfrak{g}, K; \mathcal{F}) = H_c^*(X_\Gamma).$$

There is a duality between these two cohomologies.

It turns out that neither of these function spaces is good: we don't have a good spectral decomposition. The basic problem which we want to consider is that replacing these by a better cohomology theory.

2. Schwartz functions

We give some background on function spaces.

Example 2.1. On $\mathbf{R}$, with the translation action of $\mathbf{R}$, we have the function space $C^\infty(\mathbf{R})$ and $C_c^\infty(\mathbf{R})$. Neither space has a good spectral description. We should instead consider the Schwartz space $\mathcal{S}(\mathbf{R})$, which has a good spectral description; for instance it has a notion of Fourier transform.

The general philosophy is that we want to replace the function spaces by ones which are meaningful from the point of view of representation theory. There is a standard way to do this.

Let G be a reductive group. Pick an embedding $\alpha\colon G \hookrightarrow \mathrm{SL}_n$. Define $\rho(g) = ||\alpha(g)||$. This has the property that

$$\rho(g_1 g_2) \leq \rho(g_1)\rho(g_2).$$

If you normalize correctly, then (for good α) you will have $\rho(g^{-1}) = \rho(g)$. This defines a "size function" on G, which allows us to define a notion of Schwarz function.

Definition 2.2. We say that a function f on G has *moderate growth* if $f \cdot \rho^{-N}$ is bounded for some N.

This is the standard definition, but the following equivalent formulation is better:

We say a function f on G has *moderate growth* if $f \cdot \rho^{-N} \in L^2(G)$ is bounded for some N.

Example 2.3. For functions on $\mathbf{R}$, the conditions $f||x||^N < C$ for every N and $f||x||^N \in L^2$ are the same.

The point is that there are two ways to measure the size. One is pointwise, and another is L^2. The second is the correct way.

Definition 2.4. The *space of Schwartz functions* $S(G)$ consists of functions which have uniform moderate growth (umg), meaning "all derivatives are of moderate growth."

Formally, $S(G)$ consists of f such that for every N, for every $X, Y \in U(\mathfrak{g})$ we have

$$[\ell(X)r(Y)f] \cdot \rho^N \in L^2(G).$$

Evidently $\mathcal{S}(G)$ has a left and right action of G.

We will also refer to the dual space $S^*(G)$, or interchangeably its smooth subspace.

Remark 2.5. The notion of smooth subspace requires some care. We can write

$$S(G) = \bigcap_N S^N(G),$$

and then we should define

$$S^*(G) = \bigcup_N (S^N(G)^*).$$

This means that we are really considering "uniformly smooth" distributions.

We now move towards defining a notion of Schwartz functions on more general G-homogeneous spaces.

Definition 2.6. If $X = G/H$ is a homogeneous space, define

$$\rho(x) := \min_{g \mapsto x} \rho(g).$$

Then the *Schwartz functions* $\mathcal{S}(X)$ to be the space of functions f on X such that for all N, for all $X \in U(\mathfrak{g})$ (note the order of the quantifiers)

$$Xf \cdot \rho^N \in L^2(X).$$

The main example is $X = Z_\Gamma := G/\Gamma$. The idea is that you can now replace $C_c^\infty(Z_\Gamma)$ by $\mathcal{S}(Z_\Gamma)$. This latter space somehow has a much better description in spectral terms. You replace $C_\infty(Z_\Gamma)$ by $\mathcal{S}^*(Z_\Gamma)^\infty$. (Taking the smooth part doesn't change the $(\mathfrak{g}, K)$-cohomology, so we'll drop i.) Thus we replace

$$H^*(\mathfrak{g}, K; C_c^\infty(Z_\Gamma)) \rightsquigarrow H^*(\mathfrak{g}, K; \mathcal{S}(Z_\Gamma)).$$

We are interested in the case where Z_Γ is not compact. Then we want to decompose $H^*(\mathfrak{g}, K; S^*(Z_\Gamma)^\infty)$. This is "not computable". Most of it (the cuspidal stuff) is computable, but the boundary is very complicated.

3. A conjectural cohomology theory

There is a general idea, not mine, that $H^*(X_\Gamma)$ is not the correct object to study. Instead, there should be some slightly different object $H_I^*(X_\Gamma)$ with better properties.

For example, we had the two objects $H_c^i(X_\Gamma)$ and $H^i(X_\Gamma)$, which are the same in the compact case but not in general. We would like to introduce something which is morally the image of $H_c^i(X_\Gamma) \to H^i(X_\Gamma)$. The thing would then be *self*-dual.

We want this cohomology theory $H_I^*(X_\Gamma)$ to satisfy:

(1) It is automorphic, i.e. forms a Hecke system.

(2) It is "motivic" (and moreover "pure").

This comes from an analogy with the Goresky-MacPherson intersection cohomology. Let's take a digression to explain it.

3.1. **Analogy to intersection cohomology.** Let X be a projective variety over $\mathbf{C}$. Then we can consider $H^i(X)$ and $H_i(X)$. If X is smooth then these two are isomorphic. If not, then they are usually "very close" but not quite the same. Goresky-MacPherson introduced a notion of intersection cohomology which in some sense lies between the two: $H_{IC}(X)$. In terms of motives, this is like considering the pure part of a mixed motive. ♠♠♠ TONY: [???]

By now it is clear that intersection cohomology is the "correct" object. If there is time, I want to explain why intersection cohomology is "computable" while H^i and H_i are "uncomputable".

3.2. **L^2-cohomology.** Here is an attempt to define H_I^*. We can try to define it to be the L^2-cohomology of X_Γ. I won't give the precise definition of L^2-cohomology, because I don't like it, and I don't actually think that it's right. Zucker's Conjecture, which was proved, says that if X is hermitian, so that X_Γ is an algebraic variety, then

$$H^*_{L^2} \cong H^*_{IC}((\overline{X_\Gamma})_{BB})$$

where BB is the Baily-Borel compactification. This is some kind of minimal compactification, which is highly singular. This shows that in the hermitian case $H^*_{L^2}$ is a motivic construction.

I don't like the definition of L^2 cohomology. I don't think it's "correct". Outside the hermitian case, you get a pathological answer - it's infinite dimensional. The simplest case is $G = \mathbf{R}^*$, then already $H_{L^2}(\mathbf{R}^*)$ is infinite-dimensional. I don't understand why it works in the hermitian case, even though I've read the proof.

4. Harish-Chandra Schwartz space

We want to formulate a new definition and conjecture, concerning a "tempered" cohomology. For this we make a digression to remind you about the Harish-Chandra-Schwartz space. Harish-Chandra studied representations of real reductive groups, and one of his main goals was to describe the Plancherel measure governing the decomposition of $L^2(G)$. He realized that the Schwartz space is not good for spectral analysis, so he defined a larger space $\mathcal{C}(G) \subset L^2(G)$ containing $\mathcal{S}(G)$. This is the "Harish-Chandra-Schwartz space". This space has a much nice spectral description. One problem is that the original definition is very technical, which was useful but also hard to generalize.

Here is the definition that I prefer. We have the size function $\rho(g)$. Previously we defined the (smooth) Schwartz distributions to consist of functions such that multiplication by powers of ρ lie in L^2, and then we took the (uniformly) smooth part. But instead consider the size function

$$\nu(g) := \log \rho(g).$$

Define $\mathcal{C}(G)$ to consist of f such that for any N, for any $X, Y \in U(\mathfrak{G})$,

$$\ell(X) r(Y) f \cdot \nu^N \in L^2(G).$$

Here it is really important that we asked for $\ell(X)r(Y)f \cdot \nu^N \in L^2$ instead of bounded functions, because they are no longer equivalent. (Before there were the two formulations, which gave the same answer. Here we get two different answers, and this one is the correct one.)

Harish-Chandra showed that this space has a very nice decomposition with respect to discrete series represesentations of Levi subgroups.

Remark 4.1. Deligne comments that for $G = \mathbf{R}^*$, this recovers the usual Schwartz space on $\mathbf{R}$ via the exponential function.

5. Tempered cohomology

For Z a quotient space of G, say $Z = \Gamma\backslash G$, we can define functions $\rho(z)$ and $\nu(z)$. Define $C(Z)$ to be the space of functions f on Z such that for all N, for all $X \in U(\mathfrak{g})$ we have

$$Xf\nu(z)^N \in L^2(Z).$$

Our suggestion is that we should define

$$H_I^*(X_\Gamma) := H^*(\mathfrak{g}, K; C(Z_\Gamma)).$$

We conjecture is that this is the "right" cohomology theory. The first concrete test is that this should be self-dual. The second is that if X is a hermitian symmetric space then this should coincide with L^2 cohomology, since that gives the correct answer. Using Zucker's conjecture, this shouldn't be difficult to check. The cohomology should be also finite-dimensional.

The more subtle question is whether it has a canonical rational structure. (In the hermitian case this comes from Zucker's conjecture.) This would reflect a "motivic nature". The point is that this should be true even for non-hermitian spaces.

What brought me to this train of thought was reflecting that the computation of cohomology is difficult, in the same way as the cohomology of the singular varieties. We want to come back and explain what we mean by this. The idea is that there is a class of varieties which we can understand very explicitly, namely the toric varieties. Namely, suppose $\Lambda \cong \mathbf{Z}^n$ is a lattice in $V \cong \mathbf{R}^n$. Suppose now that P is a convex polyhedron in V with vertices in Λ, of dimension n. Then there is a standard procedure to construct an algebraic variety over $\mathbf{C}$ with an action of a torus (whose character group is Λ). The action has finitely many orbits, corresponding to the orbits of the polyhedron. If X_P is smooth, then you can compute the cohomology $H^i(X_P)$. If not, then there is a standard "algorithm" you can compute $H_{IC}^*(X)$, but there is no algorithm to compute the ordinary cohomology. Indeed, Mark McConnell gave an example of two polyhedra with the same simplicial structure, but which have different ordinary cohomologies.

What is the connection to automorphic forms? This is a suggestion of Kazhdan. If X is a projective variety, $\widetilde{X} \to X$ a smooth resolution. Then you have $H^*(\widetilde{X}) = H_{IC}^*(\widetilde{X})$. It contains $H_I^*(X)$. Although the resolution is not canonical, the subspace $H_I^*(X)$ is the thing which is "common to all resolutions".

Suppose now X is quasiprojective and smooth. Then you can consider a smooth compactification $X \subset \overline{X}$. Then $H^i(\overline{X})$. Kazhdan's suggestion is that there is a cohomology theory which is "common to all compactifications".

The construction of Baily-Borel compactification is very intricate (unlike the toroidal compactifications). It is basically just an existence theorem.

Tcategory of smooth projective varieties is contained in the category of smooth varieties and the projective varieties. From Deligne's works it is clear that the there should be a duality that exchanges the two bigger categories.