The Arbeitsgemeinschaft 2017:
Higher Gross-Zagier Formulas

Notes by Tony Feng[1]

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Contents

Note to the reader

This document consists of notes I live-TEXed during the Arbeitsgemeinschaft at Oberwolfach in April 2017. They should not be taken as a faithful transcription of the actual lectures; they represent only my personal perception of the talks. Moreover, they have been edited from their original form.

No doubt many typors and errors remain, for which I take full responsibility. I have also not attempted to sync the notation used across different talks. The reader is warned that there are some fairly significant inconsistencies in notation! Despite these flaws, I hope the notes will be useful to some readers.

Almost all of the mathematics here is contained in the paper "Shtukas and the Taylor Expansion of L-functions" by Zhiwei Yun and Wei Zhang. These notes are but an expository first glimpse of their incredible work. Special thanks to Zhiwei and Wei for also organizing the workshop, and to the speakers.

Comments and corrections can be sent to me at tonyfeng@stanford.edu. I thank Arthur Cesar le Bras for corrections.

Part 1

Day One

1. An overview of the Gross-Zagier and Waldspurger formulas (Yunqing Tang)

1.1. The modular curve $X_0(N)$.

1.1.1. *The open modular curve.* To state the Gross-Zagier formula, we need to introduce modular curves. We begin by defining the *open modular curve* $Y_0(N)$. Over a field of characteristic 0, it is the moduli space of pairs (E', C) where E' is an elliptic curve and C is a subgroup of E' isomorphic to $\mathbf{Z}/N\mathbf{Z}$.

The complex points $Y_0(N)(\mathbf{C})$ have the structure of the locally symmetric space $\Gamma_0(N)\backslash\mathbf{H}$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \colon c \equiv 0 \pmod N \right\}.$$

The point $\tau \in \mathbf{H}$ parametrizes the curve $\mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$, with N-torsion point being $\frac{\tau}{N}$.

1.1.2. *Cusps.* The *cusps* of $\Gamma_0(N)$ are in bijection with the set

$$\Gamma_0(N)\backslash\mathbf{P}^1(\mathbf{Q}) = \bigsqcup_{d \mid N} (\mathbf{Z}/f_d\mathbf{Z})^\times \quad f_d = \gcd(d, N/d).$$

We define $X_0(N)$ as the compactification of $Y_0(N)$ obtained by adjoining a point for each cusp. There is a moduli interpretation of $X_0(N)$ as parametrizing isogenies of *generalized elliptic curves*

$$\phi \colon E' \to E''$$

such that $\ker\phi \cong \mathbf{Z}/N\mathbf{Z}$ and $\ker\phi$ meets every component of E'. A generalized elliptic curve is a family whose geometric fibers are either an elliptic curve or a "Néron n-gon" of $\mathbf{P}^1$'s.

There are two special cusps on $X_0(N)$:

- The cusp ∞ corresponds to the n-gon for $n = 1$, which is the nodal cubic.
- The cusp 0 corresponds to the N-gon.

1.1.3. *CM points.* In terms of the uniformization of $X_0(N)$ by $\mathbf{H}$, CM points correspond to $\tau \in \mathbf{H}$ such that there exist $a, b, c \in \mathbf{Z}$ such that

$$a\tau^2 + b\tau + c = 0.$$

We can assume that $\gcd(a, b, c) = 1$. With this assumption, the discriminant $D = b^2 - 4ac$ is the discriminant of $\mathrm{End}_{\mathbf{C}}(E_\tau) \cong \mathbf{Z} + \mathbf{Z}[\frac{D+\sqrt{D}}{2}]$.

1.1.4. *Heegner points.* Heegner points are a special type of CM points. Fix K to be an imaginary quadratic field of discriminant D over $\mathbf{Q}$. Assume D is odd. The *Heegner condition* says that for all $p \mid N$,

(1) p is split or ramified in K, and
(2) $p^2 \nmid N$.

Remark 1.1. These conditions are equivalent to saying that D is a square $\pmod{4N}$.

The Heegner condition is equivalent to the existence of a point $x := (\phi \colon E' \to E) \in X_0(N)(\overline{\mathbf{Q}})$ satisfying

$$\mathrm{End}_{\overline{\mathbf{Q}}}(E') = \mathrm{End}_{\overline{\mathbf{Q}}}(E'') = \mathcal{O}_K.$$

The theory of complex multiplication implies that Heegner points are defined over the Hilbert class field of K, which we denote by H. In terms of the complex uniformization, the Heegner point x corresponds to

$$x = [\mathbf{C}/\mathcal{O}_K \to \mathbf{C}/\mathcal{N}^{-1}\mathcal{O}_K]$$

where $\mathcal{N} \subset \mathcal{O}_K$ is an ideal of norm N. Its existence is guaranteed by the Heegner condition as follows. For every $p \mid N$ we can choose $\mathfrak{p} \subset \mathcal{O}_K$ such that $\operatorname{Nm}\mathfrak{p} = p$, and then set $\mathcal{N} = \prod_p \mathfrak{p}^{v_p(N)}$.

Finally, we can form a degree 0 divisor on $X_0(N)$ from the Heegner point, which will actually be defined over K, as follows: let

$$P := \sum_{\sigma \in \operatorname{Gal}(H/K)} (\sigma(x) - \infty).$$

1.2. Néron-Tate height. We now define the "Néron-Tate height". This construction can be done for any abelian variety, but we will only do it for Jacobians; this is all we need to state Gross-Zagier.

Suppose we have a line bundle $\mathcal{L}$ on $J_0(N)$, corresponding to twice a theta divisor Θ. (More This is ample, so we can use it to define a height. Namely, we can pick a large power of n and use $\mathcal{L}^{\otimes n}$ to embed

$$\mathcal{L}^{\otimes n} \colon J_0(N) \hookrightarrow \mathbf{P}^m.$$

On projective space we have the standard height function due to Weil, which we can restrict to $J_0(N)$ to obtain a height function $\frac{1}{n} h^K_{\mathcal{L}^{\otimes n}}$. To make this well defined, we normalize: define $h^K_{\mathcal{L}}$ on $J_0(N)(K)$ by $\frac{1}{n} h^K_{\mathcal{L}^{\otimes n}}$.

Definition 1.2. The *Néron-Tate height* for $J_0(N)$ is defined to be

$$\hat{h} := \lim_{n \to \infty} \frac{h^K_{\mathcal{L}}(2^n x)}{4^n}.$$

This satisfies

$$\hat{h}(2x) = 4\hat{h}(x).$$

Remark 1.3. The Néron-Tate height can be decomposed into a sum of local terms, which is used in the original proof of the Gross-Zagier formula.

1.3. L-functions. Let f be a weight 2 newform for $\Gamma_0(N)$. (This means that f is a cuspidal Hecke eigenform, orthogonal to modular forms coming from smaller level.) We have a Fourier expansion

$$f = \sum_{n \geq 1} a_n q^n.$$

If $a_n \in \mathbf{Z}$ for all n, then by Eichler-Shimura we have an elliptic curve $E/\mathbf{Q}$ with conductor N. Conversely, for an elliptic curve $E/\mathbf{Q}$ the modularity theorem (Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor) produces a modular form with the same L-function.

The modular form f can be viewed as an automorphic form for $\mathrm{GL}_2/\mathbf{Q}$. If f_k denotes its base change to K, then

$$L(f_K, s) = L(f, s)L(f \otimes \eta_{K/\mathbf{Q}}, s) \tag{1.1}$$

where $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to $K/\mathbf{Q}$ by class field theory. Explicitly, we can write

$$L(f, s) = \sum a_n q^n$$
$$L(f \otimes \eta_{K/\mathbf{Q}}, s) = \sum \eta(n) a_n q^n$$

Remark 1.4. The base change for automorphic forms can be understood concretely in terms of elliptic curve. If f corresponds to the elliptic curve E under Eichler-Shimura, then

$$L(f_K, s) = L(E_K, s).$$

Thus (1.1) becomes

$$L(E_K, s) = L(E, s)L(E^D, s)$$

where E^D is the quadratic twist of E by D. This has an Euler product

$$L(E_K, s) = \prod_{v \text{ finite place of } K} L_v$$

where for good reduction v,

$$L_v = (1 - a_v q_v^{-s} + q_v^{1-2s})^{-1}, \quad a_v = q_v + 1 - \#E(\mathbf{F}_v),$$

and in the bad reduction case,

$$L_v = (1 - a_v q_v^{-s})^{-1}$$

where $a_v = 1$ for split multiplicative reduction, $a_v = -1$ for a nonsplit multiplicative reduction, and $a_v = 0$ for additive reduction. (This can again be phrased in terms of a point count for the non-singular locus of the reduction.)

The Heegner condition implies that

$$\epsilon(L(f_K, s)) = -1 \implies L(f_K, 1) = 0.$$

1.4. Gross-Zagier.

1.4.1. *The elliptic curve case.* Let $\phi\colon X_0(N) \to E$ be the modular parametrization, sending $\infty \mapsto e$. Thanks to the modularity theorem of Wiles, this parametrization is induced by a modular form f. We define

$$P(\phi) := \sum_{\sigma \in \mathrm{Gal}(H/K)} \phi(\sigma(x)) \in E(K).$$

Theorem 1.5 (Gross-Zagier). *We have*

$$\hat{h}(P(\phi)) = \frac{\deg \phi \cdot u^2 \cdot |D|^{1/2}}{8\pi^2 ||f||_{\mathrm{Pet}}} L'(E_K, 1)$$

where $u = |\mathcal{O}_K^\times|$, *and*

$$||f||_{\mathrm{Pet}} := \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{f(z)}\, dxdy$$

We can rewrite this in terms of modular forms, which fits better with the generalization to automorphic forms.

Definition 1.6. The *Hecke algebra* is the algebra of correspondences on $X_0(N)$ generated by

$$T_m\colon [E \xrightarrow{\phi} E'] \mapsto \sum_{\substack{C \subset C: \\ \#C = m \\ C \cap \ker\phi = e}} [E/C \to E'/C].$$

It acts on $X_0(N)$, hence also on $J_0(N)$. Let $P(f)$ be the isotypic component of $J_0(N) \otimes \mathbf{Q}$, where we need to extend scalars because the idempotent has denominators. Then the reformulation of Gross-Zagier is:

$$\hat{h}(P(f)) = \frac{u^2 \cdot |D|^{1/2}}{8\pi^2} \frac{L'(F_K, 1)}{||f||_{Pet}}.$$

Remark 1.7. The proof considers the height pairing

$$\langle (x - \infty), T_M(\sigma(x) - \infty)\rangle_{NT}$$

for $X_0(N)$. This is the Fourier coefficient of a cusp form of weight 2 on $X_0(N)$. It is part of a general philosopy of Kudla that the generating series for special cycles is a modular form. The L-function is also associated to a modular form. The proof goes by arguing that these two forms coincide, up to an old form. The higher Gross-Zagier also has to do with this.

1.5. Generalized Heegner conditions. We now explain a generalization of of Heegner points, following work of Zhang and Yuan-Zhang-Zhang.

Let $(N, D) = 1$. Assume $N = N^+N^-$ where N^- is squarefree and its number of prime factors is even. In this case we can have a quaternion algebra B ramified at N^-, giving rise to a Shimura curve

$$X = B^\times(\mathbf{Q})\backslash \mathbf{H}^\pm \times B^\times(\mathbf{A}_f)/U.$$

From an elliptic curve $E/\mathbf{Q}$ we get a modular form f. By Jacquet-Langlands, we get a modular parametrization $X \to E$. For an embedding $K \to B(\mathbf{Q})$ of an imaginary quadratic field K, we get a Heegner point $x \in X(H)$, where H is the Hilbert class field of K. (The Shimura curve parametrizes abelian surfaces with real multiplication, while the CM point parametrizes things with endomorphism by $\mathcal{O}_K$. The Heegner condition forces endomorphisms by the maximal order. In particular, this implies that the CM point is defined over H.)

Definition 1.8. We define the generalized Heegner point

$$P(\phi) := \sum_{\sigma \in \mathrm{Gal}(H/K)} \phi(\sigma(x)) \in E(K).$$

THEOREM 1.9 (Zhang, YZZ). *We have*

$$\hat{h}(P(\phi)) = \frac{L'(E/K,1)}{||f||_{\mathrm{Pet}}}.$$

1.6. Waldspurger formula. We normalize so that the center of the L-function is $1/2$.

Let F be a number field and $\mathbf{A} = \mathbf{A}_F$. Let B be a quaternion algebra over F, and G the algebraic group associated to $B^\times$. Denote the center of G by $Z_G = F^\times$. Let K/F be a quadratic extension with a given embedding $K \hookrightarrow B$. Let $T = \mathrm{Res}_{K/F}\,\mathbf{G}_m$; note that we can naturally view $T \subset G$. Let η be the quadratic Hecke character associated to K/F.

Let π be an irreducible cuspidal automorphic representation of G, and ω_π the central character. Let π_K denote the base change of π to K. Let

$$\chi\colon T(F)\backslash T(\mathbf{A}) \to \mathbf{C}^\times$$

be a unitary character, such that $\omega_\pi \cdot \chi|_{\mathbf{A}^\times} = 1$. (The purpose of χ is to get a trivial central character.)

The Waldspurger formula concerns a *period integral.* We define

$$P_\chi\colon \pi \to \mathbf{C}$$

by

$$f \mapsto P_\chi(f) = \int_{T(F)\backslash T(\mathbf{A})/\mathbf{A}^\times} f(t)\chi(t)\,dt.$$

THEOREM 1.10 (Waldspurger). *For $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$ (the contragredient representation), we have*

$$P_\chi(f_1)P_\chi(f_2) \sim \frac{L(\pi_K \otimes \chi, 1/2)}{L(\pi, \mathrm{Ad}, 1)}\alpha(f_1 \otimes f_2)$$

where $\alpha = \prod_v \alpha_v$ is a product of local terms

$$\alpha_v \in \mathrm{Hom}_{K_v^\times}(\pi_v \otimes \chi_v, \mathbf{C}) \otimes \mathrm{Hom}_{K_v^\times}(\widetilde{\pi}_v \otimes \chi_v^{-1}, \mathbf{C}),$$

normalized by Waldspurger (so in particular, they are 1 in the spherical case).

2. The stacks Bun_n and Hecke (Timo Richarz)

2.1. Why stacks? In algebraic geometry one would like to have a classifying space BGL_n for vector bundles, such that

$$\mathrm{Hom}(S, \mathrm{BGL}_n) = \{\text{vector bundles of rank } n \text{ on } S\} / \sim .$$

Such an object can't be represented by a scheme, since a vector bundle is locally trivial, so any map $S \to B\,\mathrm{GL}_n$ would need to be locally constant, and for maps of schemes locally constant implies constant.

There are several possible ways to wriggle out of this situation.

(1) Add extra data (e.g. level structure) in order to eliminate automorphisms.
(2) Don't pass to isomorphism classes.

Stacks are the result of the second option.

2.2. Bun_n as a stack. Let k be a field.

Definition 2.1. A *stack* $\mathcal{M}$ is a sheaf of groupoids

$$\mathcal{M} \colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{Grp} \subset \mathbf{Cat}$$

i.e. an assignment

- for all S a groupoid $\mathcal{M}(S)$,
- for every $S \xrightarrow{f} S'$ a pullback functor $f^* \colon \mathcal{M}(S') \to \mathcal{M}(S)$,
- for all $S \xrightarrow{f} S' \xrightarrow{g} S''$ a transformation

  $$\varphi_{f,g} \colon f^* \circ g^* \implies (g \circ f)^*$$

  such that objects and morphisms glue (in the appropriate topology).

Example 2.2. The classifying stack

$$\mathrm{BGL}_n := [\mathrm{pt} \,/\, \mathrm{GL}_n]$$

takes S to the groupoid of vector bundles of rank n on S.

Example 2.3. Let X be a smooth, projective, connected curve over k. We define the stack Bun_n taking S to the groupoid of vector bundles of rank n on $X \times S$.

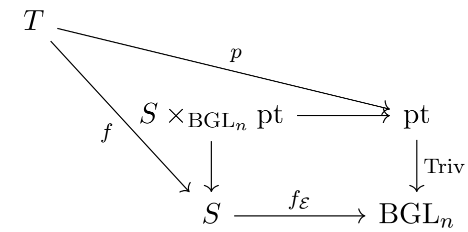
How do you make this geometric? We have a map $\mathrm{pt} \to \mathrm{BGL}_n$ corresponding to the trivial bundle. If $\mathcal{E}$ is a rank n vector bundle on S, then we get by definition a classifying map

$$f_{\mathcal{E}} \colon S \to \mathrm{BGL}_n.$$

Consider the fibered product

$$\begin{array}{ccc} S \times_{\mathrm{BGL}_n} \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ S & \xrightarrow{f_{\mathcal{E}}} & \mathrm{BGL}_n \end{array}$$

To understand what the fibered product is, let's compute its functor of points is.



Its T-valued points are

$$\{(f, \varphi \colon \mathrm{Triv} \circ p \xrightarrow{\sim} f_{\mathcal{E}} \circ f\} = \underline{\mathrm{Isom}}(\mathcal{O}_S^{\oplus n}, \mathcal{E})(T),$$

which is the frame bundle of $\mathcal{E}$. Let's think about what this means.

(1) We can recover $\mathcal{E} = \mathcal{O}_S^n \times_{\mathrm{GL}_n} \underline{\mathrm{Isom}}(\mathcal{O}_S^{\oplus n}, \mathcal{E})$, i.e. the map $\mathrm{pt} \to \mathrm{BGL}_n$ is the universal vector bundle.
(2) The map $\mathrm{pt} \to \mathrm{BGL}_n$ is a smooth surjection after every base change.

Inspired by these examples, we make a definition.

Definition 2.4. A stack $\mathcal{M}$ is called *algebraic* if

(1) For all maps $S \to \mathcal{M}$ and $S' \to \mathcal{M}$ from schemes S, S', the fibered product $\mathcal{S} \times_{\mathcal{M}} \mathcal{S}'$ is a scheme .
(2) There exists a scheme U together with a smooth surjection $U \to \mathcal{M}$ called an atlas.
(3) The map $U \times_{\mathcal{M}} U \to U \times U$ is qcqs.

An algebraic stack $\mathcal{M}$ is *smooth* (resp. locally of finite type, ...) if there is an atlas $U \twoheadrightarrow \mathcal{M}$ such that U is smooth (resp. locally of finite type, ...).

Example 2.5. (Picard stack) We define $\mathrm{Pic}_X = \mathrm{Bun}_{X,1}$. Let Jac_X be the Jacobian of X. This is the coarse moduli space of Pic_X, so we have a map

$$\mathrm{Pic}_X \to \mathrm{Jac}_X$$

which preserves the labelling of connected components by degree. Suppose you have $x \in X(k) \neq \emptyset$. Then we actually have an isomorphism

$$\mathrm{Pic}_X \xrightarrow{\sim} \mathrm{Jac}_X \times B\mathbf{G}_m$$

where the map $\mathrm{Pic}_X \to B\mathbf{G}_m$ corresponds to the restriction of the universal line bundle on $X \times \mathrm{Pic}_X$ to $\{x\} \times \mathrm{Pic}_X$.

This shows that Pic_X is a smooth algebraic stack locally of finite type of dimension $g(X) - 1$.

THEOREM 2.6. Bun_n *is a smooth algebaic stack locally of finite type over* k*, of dimension* $n^2(g(X) - 1)$*, and* $\pi_0(\mathrm{Bun}_n) = \mathbf{Z}$.

PROOF. Choose an ample line bundle $\mathcal{O}_X(1)$ on X. Define U to be the union over N of $(\mathcal{E}, \{s_i\})$ such that

- $\mathcal{E}(N)$ is globally generated,
- $H^1(X, \mathcal{E}(N)) = 0$, and
- the $\{s_i\}$ are a basis of $H^0(X, \mathcal{E}(N))$.

This U is represented by a smooth scheme, by the theory of Quot schemes, and $U \to \mathrm{Bun}_n$ is an atlas. (The obstruction to deforming a basis lies in $H^1(X, \mathcal{E}(N))$, which we have asked to vanish.) □

Example 2.7. Let $X = \mathbf{P}^1_k$. Then $[\mathrm{pt} / \mathrm{GL}_n]$, corresponding to the trivial bundle, is an open immersion in Bun_n because $H^1(\mathbf{P}^1_k, \mathfrak{g}) = 0$. For example, $\mathrm{Bun}^0_2(k) = \{\mathcal{O}^{\oplus 2}, \mathcal{O}(1) \oplus \mathcal{O}(-1), \ldots\}$ so the automorphism groups get bigger as the points get more special.

2.3. Adelic uniformization of Bun_n.

2.3.1. *Weil's uniformization.* Let $k = \mathbf{F}_q$. Let F be a the function field of X, and $|X|$ the set of closed points. For $x \in |X|$ denote by $\mathcal{O}_x$ the completed local ring at x. This is non-canonically isomorphic to $k_x[[\varpi_x]]$. We also set $F_x = \mathrm{Frac}(\mathcal{O}_x)$, which is non-canonically isomorphic to $k_x((\varpi_x))$. Recall the ring of adeles

$$\mathbf{A} = \prod_{x \in |X|}{}' (F_x, \mathcal{O}_x) = \{(a_x) \in \prod F_x \mid a_x \in \mathcal{O}_x \text{ for almost all } x \in |X|\}.$$

Theorem 2.8 (Weil). *There is a canonical isomorphism of groupoids*

$$\mathrm{GL}_n(F) \backslash \left(\mathrm{GL}_n(\mathbf{A}) / \prod_{x \in |X|} \mathrm{GL}_n(\mathcal{O}_x) \right) \xrightarrow{\sim} \mathrm{Bun}_n(k).$$

Here if S is a set with a group action of G, then S/G can be considered as a groupoid, whose objects are orbits and automorphisms are stabilizers.

Example 2.9. For $n = 1$, this gives

$$F^\times \backslash \mathbf{A}^\times / \prod \mathcal{O}_X^\times = F^\times \backslash \left(\prod_x F_x^\times / \mathcal{O}_x^\times \right) = F^\times \backslash \mathrm{Div}(X) = \mathrm{Pic}_X(k).$$

Proof. Consider the set

$$\Sigma := \left\{ (\mathcal{E}, \{\alpha_x\}, \tau) \colon \begin{array}{c} \mathrm{rank}\, \mathcal{E} = n \\ \alpha_x \colon \mathcal{E}|_{\mathrm{Spec}\ \mathcal{O}_x} \cong \mathcal{O}_x^{\oplus n} \\ \tau \colon \mathcal{E}|_{\mathrm{Spec}\ F} \cong F^{\oplus n} \end{array} \right\}.$$

We seek to define a $\mathrm{GL}_n(F) \times \prod \mathrm{GL}_n(\mathcal{O}_x)$-equivariant map

$$\Sigma \to \mathrm{GL}_n(\mathbf{A}). \tag{2.1}$$

Once we have this, we get a map of quotients

$$\begin{array}{ccc} \Sigma & \longrightarrow & \mathrm{GL}_n(\mathbf{A}) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_n(k) & \dashrightarrow & \mathrm{GL}_n(F) \backslash (\mathrm{GL}_n(\mathbf{A}) / \prod \mathrm{GL}_n(\mathcal{O}_x)). \end{array}$$

We'll just show you how to define the map (2.1). Given $(\mathcal{E}, \{\alpha_x\}, \tau) \in \Sigma$, we get $gx \in \mathrm{GL}_n(F_x)$ given by

$$F_x^n \xrightarrow{\alpha_x^{-1}} \mathcal{E}|_{\mathrm{Spec}\ F_x} \xrightarrow{\tau} F_x^{\oplus n}.$$

□

2.3.2. *Level structure.* Given $D = \sum d_x \cdot x$ an effective divisor, we can look at the double quotient

$$\mathrm{GL}_n(F)\backslash\left(\mathrm{GL}_n(\mathbf{A})/K_D\right) \cong \left\{(\mathcal{E}, \alpha) \mid \alpha\colon \mathcal{E}|_D \cong \mathcal{O}_D^{\oplus n}\right\}$$

where $K_D = \ker\left(\prod_{x\in|X|}\mathrm{GL}_n(\mathcal{O}_x) \to \prod_{x\in|X|}\mathrm{GL}_n(\mathcal{O}_x/\varpi_x^{d_x})\right)$.

2.3.3. *Split groups.* If G is any (not necessarily reductive) algebraic group split over k, then

$$G(F)\backslash\left(G(\mathbf{A})/\prod_{x\in|X|}G(\mathcal{O}_X)\right) \cong \mathrm{Bun}_G(k).$$

If G is not split, then we instead get an injection, with the right side having terms related to inner twists of G.

2.4. Hecke stacks. Let $r \geq 0$ and $\mu = (\mu_1, \ldots, \mu_r)$ a sequence of dominant coweights of GL_n such that μ_i is either $\mu_+ = (1, 0, \ldots, 0)$ or $\mu_- = (0, \ldots, 0, -1)$.

Definition 2.10. The *Hecke stack* Hk_n^μ is the stack defined by $\mathrm{Hk}_n^\mu(S)$ is the groupoid classifying the following data:

- a sequence $(\mathcal{E}_0, \ldots, \mathcal{E}_r)$ of rank n vector bundles on $X \times S$.
- a sequence $(x_1, \ldots, x_r)$ of morphisms $x_i\colon S \to X$, with graphs $\Gamma_{x_i} \subset X \times S$,
- maps $(f_1, \ldots, f_r)$ with
  $$f_i\colon \mathcal{E}_{i-1}|_{X\times S\backslash\Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_i|_{X\times S\backslash\Gamma_{x_i}}$$
  such that if $\mu_i = \mu_+$, then f_i extends to $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i$ whose cokernel is an invertible sheaf on Γ_{x_i}, and if $\mu_i = \mu_-$ then f_i^{-1} extends to $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1}$ whose cokernel is an invertible sheaf on Γ_{x_i}.

For $i = 0, \ldots, r$ we have a map

$$p_i\colon\ \mathrm{Hk}_n^\mu \to \mathrm{Bun}_n$$

sending $(\underline{\mathcal{E}}, \underline{x}, \underline{f}) \mapsto \mathcal{E}_i$ and

$$p_X\colon\ \mathrm{Hk}_n^\mu \to X^r$$

sending $(\underline{\mathcal{E}}, \underline{x}, \underline{f}) \mapsto \underline{x}$.

Lemma 2.11. *The morphism*

$$(p_0, p_X)\colon\ \mathrm{Hk}_n^\mu \to \mathrm{Bun}_n \times X^r$$

is representable by a proper smooth morphism of relative dimension $r(n-1)$, whose fibers are iterated $\mathbf{P}^{n-1}$-bundles.

Proof. Once we have fixed a reference bundle, the fibers are iterated modifications, which amounts to a choice of a hyperplane in an n-dimensional vector space. □

3. (Moduli of Shtukas I)
(Doug Ulmer)

3.1. Hecke stacks. Let X be a (smooth, projective, geometrically connected) curve over $\mathbf{F}_q$. Let $F = \mathbf{F}_q(X)$. Fix integers $n \geq 1$ and $r \geq 0$. Let $\mu = (\mu_1, \ldots, \mu_r)$ with each $\mu_i = \pm 1$. Usually we require that r is even, and moreover that $\sum \mu_i = 0$.

In the previous talk we met the Hecke stack Hk_n^μ, parametrizing the stack of modifications of type μ of rank n vector bundles. If S is an $\mathbf{F}_q$-scheme, then $\mathrm{Hk}_n^\mu(S)$ is the groupoid of

- vector bundles $(\mathcal{E}_0, \ldots, \mathcal{E}_r)$ on $X \times S$.
- If $\mu_i = +1$, a map $\phi_i \colon \mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i$ with cokernel an invertible sheaf supported on Γ_{x_i}. If $\mu_i = -1$, a map $\phi_i \colon \mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1}$ with cokernel an invertible sheaf supported on Γ_{x_i}.

Example 3.1. If $\mu_i = +1$, demanding a map

$$\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}_{i-1} \otimes \mathcal{O}(\Gamma_{x_i}).$$

amounts to specifying a line in an n-dimensional vector space.

We have a map

$$\mathrm{Hk}_n^\mu \to \mathrm{Bun}_n \times X^r$$

sending

$$(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}) \to (\mathcal{E}_0, \underline{x}).$$

This map is smooth of fiber dimension $r(n-1)$, so Hk_n^μ is smooth of dimension $n^2(g-1) + nr$.

3.2. Moduli of shtukas for GL_n.

3.2.1. *Definition.*

Definition 3.2. A *shtuka* of type μ and rank n is a "Hecke modification" plus a Frobenius structure. More precisely, $\mathrm{Sht}_n^\mu(S) = \{(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})\}$ together with an isomorphism $\mathcal{E}_r \cong {}^\tau\mathcal{E}_0 := (\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{E}_0$.

We have a cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_n^\mu & \longrightarrow & \mathrm{Hk}_n^\mu \\ \downarrow & & \downarrow{\scriptstyle p_0 \times p_r} \\ \mathrm{Bun}_n & \xrightarrow{\mathrm{Frob} \times \mathrm{Id}} & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{array}$$

Example 3.3. For $n = 1$, the choice of points x_i determines the higher $\mathcal{E}_i$ from $\mathcal{E}_0$, namely $\mathcal{E}_i = \mathcal{E}_{i-1} \otimes \mathcal{O}(x_i)$. So $\mathrm{Hk}_1^\mu \cong \mathrm{Pic}_X \times X^r$.

For a point of H_1^μ to be an element of Sht_1^μ, we also need $\mathcal{E}_r \cong {}^\tau\mathcal{E}_0$, i.e.

$${}^\tau\mathcal{E}_0 \otimes \mathcal{E}_0^{-1} \cong \mathcal{O}(\sum \mu_i x_i).$$

Thus Sht_1^μ is a familiar object, classically known as a "Lang torsor". It is a fiber of the Lang isogeny $\mathrm{Pic}_X \to \mathrm{Pic}_X$, hence a torsor for $\mathrm{Bun}_1(\mathbf{F}_q)$.

Example 3.4. For $r = 0$, $\mathrm{Sht}_n^\mu(S)$ is a vector bundle $\mathcal{E}$ on $X \times S$ and an isomorphism $\mathcal{E} \cong {}^\tau\mathcal{E}$. This looks like part of a descent datum. If $S = \mathrm{Spec}\, \overline{\mathbf{F}}_q$, then such $\mathcal{E}$ come from $\mathcal{E}$ on X itself via pullback.

More generally, in this case

$$\mathrm{Sht}_n^\mu = \coprod_{\mathcal{E}} [\mathrm{Spec}\ \mathbf{F}_q / \operatorname{Aut} \mathcal{E}].$$

What exactly does this mean? Concretely, an element of Sht_n^μ is an $\mathrm{Aut}(\mathcal{E})$-torsor on S, which we can think of as a twisted form of $p_X^*(\mathcal{E})$ on $X \times S$.

3.2.2. *Basic geometric facts about* Sht_n^μ.

(1) Sht_n^μ is a Deligne-Mumford stack, smooth and locally of finite type.
(2) There is a morphism
$$\mathrm{Sht}_n^\mu \to X^r$$
which is separated, smooth, and of relative dimension $r(n-1)$.

3.2.3. *Level structure.*

Definition 3.5. Let $D \subset X$ be a finite closed subscheme (in this case, just a finite collection of points with multiplicities). A *level D structure* on $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$ is an isomorphism

$$\mathcal{E}_0|_{D\times S} \xrightarrow{\sim} \mathcal{O}_{D\times S}^{\oplus n}$$

such that $|D| \cap \{x_1, \ldots, x_r\} = \emptyset$, which is compatible with Frobenius in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}_0|_{D\times S} & \xrightarrow{\sim} & \mathcal{O}_{D\times S}^{\oplus n} \\ \downarrow{\scriptstyle\sim} & & \| \\ {}^\tau\mathcal{E}_0|_{D\times S} & \xrightarrow{\sim} & {}^\tau\mathcal{O}_{D\times S}^{\oplus n} \end{array}$$

Note that there is an action of $\mathrm{GL}_n(\mathcal{O}_D)$ on the set of level structures.

In practice, we'll introduce level structure in order to rigidify the objects.

3.2.4. *Stability conditions.* The components of Bun_n are indexed by $\mathbf{Z}$, via

$$\mathcal{E} \mapsto \deg \det \mathcal{E}.$$

We need to fix this to get something of finite type. But that still won't be enough, since we have things like $\mathcal{O}(ap) \oplus \mathcal{O}(-ap)$. For a vector bundle $\mathcal{E}$, let

$$M(\mathcal{E}) := \max\{\deg \mathcal{L} \mid \mathcal{L} \hookrightarrow \mathcal{E}\}.$$

This is enough to cut down to something of finite type.

Definition 3.6. Define $\mathrm{Sht}_{n,D,d,m}^\mu$ to be the stack whose S-points are

- $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$, $\mathcal{E}_r \xrightarrow{\sim} {}^\tau\mathcal{E}_0$
- A level D structure,
- $\deg(\det \mathcal{E}_i) = d$, $M(\mathcal{E}_0) \leq m$.

Facts:

(1) If $D \gg 0$ (with respect to n, m, d) then $\mathrm{Sht}_{n,D,d,m}^\mu$ is represented by a quasi-projective variety.

(2) The map $[\mathrm{Sht}^{\mu}_{n,D,d,m} / \mathrm{GL}_n(\mathcal{O}_D)] \hookrightarrow \mathrm{Sht}^{\mu}_n$ is an open embedding.
(3) Sht^{μ}_n is the union of these substacks for varying d, m.

This is enough to check that Sht^{μ}_n is a DM stack locally of finite type over $\mathbf{F}_q$.

3.2.5. *Smoothness.* Recall the cartesian square

$$\begin{array}{ccccc} \mathrm{Sht}^{\mu}_n & \longrightarrow & \mathrm{Hk}^{\mu}_n & \longrightarrow & X^r \\ \downarrow & & \downarrow{\scriptstyle p_0 \times p_r} & & \\ \mathrm{Bun}_n & \xrightarrow{\mathrm{Frob}\times\mathrm{Id}} & \mathrm{Bun}_n \times \mathrm{Bun}_n & & \end{array}$$

Note that $d\,\mathrm{Frob} = \mathrm{Frob}_* = 0$, and $\mathrm{Id}_* = \mathrm{Id}$. On the other hand, p_{0*} and p_{r*} are both surjections.

Corollary 3.7. *The maps* $(\mathrm{Frob}, \mathrm{Id})\colon \mathrm{Bun}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *and* $(p_0, p_r)\colon \mathrm{Hk}^{\mu}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *are transverse.*

Corollary 3.8. *The map* $\mathrm{Sht}^{\mu}_n \to X^r$ *is smooth, and so has relative dimension* $(n-1)r$.

3.2.6. *Summary.* Sht^r_G is a DM stack locally of finite type, with a smooth separated morphism $\mathrm{Sht}^r_G \to X^r$ of relative dimension r.

3.3. Moduli of Shtukas for PGL_2**.** Let $G = \mathrm{PGL}_2 = \mathrm{GL}_2 / \mathbf{G}_m$, and let Bun_G be the stack of G-torsors on X, which is isomorphic to $\mathrm{Bun}_2 / \mathrm{Bun}_1$, with the action being $\otimes$. This action lifts to Hk^{μ}_2, by

$$(\underline{\mathcal{E}}, \underline{x}, \underline{f}) \mapsto (\underline{\mathcal{E}} \otimes \mathcal{L}, \underline{x}, \underline{f} \otimes \mathrm{Id}).$$

This action doesn't restrict to Sht^{μ}_2 unless $\mathcal{L} \cong {}^{\tau}\mathcal{L}$. Therefore, only the subgroup $\mathrm{Pic}_X(k)$ acts on Sht^{μ}_2. We have cartesian diagrams

$$\begin{array}{ccc} \mathrm{Pic}_X(\mathbf{F}_q) & \longrightarrow & \mathrm{Pic}_X \\ \downarrow & & \downarrow \\ \mathrm{Pic}_X & \xrightarrow{\mathrm{Frob}\times\mathrm{Id}} & \mathrm{Pic}_X \times \mathrm{Pic}_X\,. \end{array} \tag{3.1}$$

and

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_n & \longrightarrow & \mathrm{Hk}^{\mu}_n \\ \downarrow & & \downarrow{\scriptstyle p_0 \times p_r} \\ \mathrm{Bun}_n & \xrightarrow{\mathrm{Frob}\times\mathrm{Id}} & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{array} \tag{3.2}$$

and the objects for $G = \mathrm{PGL}_2$ are obtained by quotienting the second diagram (3.2) by the action of the corresponding groups in the first diagram (3.1).

3.3.1. *Independence of signs when* $n = 2$. If μ, μ' are r-tuples of signs and $n = 2$, then there is a canonical isomorphism $\mathrm{Sht}^{\mu}_G \xrightarrow{\sim} \mathrm{Sht}^{\mu'}_G$. We'll show this by giving an explicit isomorphism between Sht^{μ}_G, for any μ, and $\mathrm{Sht}^{\mu'}$ where $\mu' = (+1, \ldots, +1)$.

Suppose we are given $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi}, \iota) \in \mathrm{Sht}^{\mu}_G$. The key idea is that we can transform an injection $\mathcal{E}_{i-1} \hookleftarrow \mathcal{E}_i$ with deg 1 cokernel into $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \otimes \mathcal{O}(x_i)$. So we take every instance of $\mathcal{E}_{i-1} \hookleftarrow \mathcal{E}_i$, which is a modification of type μ_-, into $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \otimes \mathcal{O}(x_i)$, which is a modification of type μ_+. Given $(\underline{\mathcal{E}}, \underline{x}, \underline{\phi})$ let

$$D_i := \sum_{\substack{1 \le j \le i \\ \mu_j = \mu_-}} \Gamma_{x_i}.$$

Let $\mathcal{E}'_i = \mathcal{E}_i \otimes \mathcal{O}_{X \times S}(D)$, and note that

$$\mathcal{E}'_0 \hookrightarrow \mathcal{E}'_1 \hookrightarrow \ldots \hookrightarrow \mathcal{E}'_r$$

is an element of $\mathrm{Sht}_G^{\mu'}$.

4. Moduli of Shtukas II (Brian Smithling)

4.1. Goal. We're going to start by stating the formula which is the goal of this work. We'll then spend most of the talk explaining the meaning of some parts of it. For $f \in \mathcal{H}$ the Hecke algebra, define

$$\mathbb{I}_r(f) := \langle \theta^\mu_*[\mathrm{Sht}^\mu_T], f * \theta^\mu_*[\mathrm{Sht}^\mu_T]\rangle_{\mathrm{Sht}'^r_G} \in \mathbf{Q}.$$

Here $\theta^\mu_*[\mathrm{Sht}^\mu_T] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}'^r_G)_{\mathbf{Q}}$, a cycle in the "rational Chow group of dimension r cycles proper over k".

This Chow group has an action of ${}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)_{\mathbf{Q}}$, the "rational Chow group of dimension $2r$ cycles proper over the first factor". This actually has an algebra structure.

The main goals for today are:

- Define a map $\mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G)$.
- Define a map $\theta^\mu \colon \mathrm{Sht}^\mu_T \to \mathrm{Sht}'^r_G$.

4.2. The Hecke algebra. Let $G = \mathrm{PGL}_2$, X/k, and $F = k(X)$. Write

$$K = \prod_{x \in |X|} K_x, \quad K_x = G(\mathcal{O}_x).$$

Definition 4.1. The *spherical Hecke algebra* is

$$\mathcal{H} = C^\infty_c(K\backslash G(\mathbf{A})/K, \mathbf{Q}) = \bigotimes_{x\in|X|}{}' C^\infty_c(K_x\backslash G(F_x)/K_x, \mathbf{Q}).$$

The algebra structure is by convolution.

Let $M_{x,n}$ be the subset of $\mathrm{Mat}_2(\mathcal{O}_x)$ with determinant n, viewed in $G(F_x)$. Thus

$$\begin{aligned} M_{x,0} &= K_x \\ M_{x,1} &= K_x \begin{pmatrix} \varpi_x & \\ & 1 \end{pmatrix} K_x \\ &\vdots \end{aligned}$$

Let $h_{nx} \in \mathcal{H}_x$ be the characteristic function of $M_{X,n}$. By Cartan decomposition, these form a $\mathbf{Q}$-basis of $\mathcal{H}_x$.

Let $D = \sum_{x\in|X|} n_x x$ be an effective divisor. Then $h_D = \otimes_{x\in X} h_{n,x} \in \mathcal{H}$ is a $\mathbf{Q}$-basis for $\mathcal{H}$.

4.3. Hecke correspondences. Let μ be an r-tuple with the same number of μ_+'s and μ_-'s. We define $\mathrm{Sht}^\mu_2(h_D)(S)$ to parametrize

- $(x_1, \ldots, x_r)$ maps $S \to X$.

- a commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{E}_0 & \dashrightarrow & \mathcal{E}_1 & \dashrightarrow & \dots & \dashrightarrow & \mathcal{E}_r & \xrightarrow{\sim} & {}^\tau\mathcal{E}_0 \\
\big\downarrow{\scriptstyle \phi_0} & & \big\downarrow{\scriptstyle \phi_1} & & & & \big\downarrow{\scriptstyle \phi_r} & & \big\downarrow{\scriptstyle {}^\tau\phi_0} \\
\mathcal{E}'_0 & \dashrightarrow & \mathcal{E}'_1 & \dashrightarrow & \dots & \dashrightarrow & \mathcal{E}'_r & \xrightarrow{\sim} & {}^\tau\mathcal{E}'_0
\end{array}$$

  with top and bottom rows in Sht_2^μ, such that:
- the map $\det \phi_i$ has divisor $D \times S$.

For $G = \mathrm{PGL}_2$, we define

$$\mathrm{Sht}_G^\mu(h_D) := \mathrm{Sht}_2^\mu(h_D)/\operatorname{Pic}_X(k).$$

We have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Sht}_G^r(h_D) & & \\
 & \swarrow^{p^\leftarrow} & & \searrow^{p^\rightarrow} & \\
\mathrm{Sht}_G^r & & & & \mathrm{Sht}_G^r \\
 & \searrow & & \swarrow & \\
 & & X^r & &
\end{array}$$

LEMMA 4.2. *The maps $p^\leftarrow$ and $p^\rightarrow$ are representable and proper. The map $(p^\leftarrow, p^\rightarrow)$ is also representable and proper.*

PROOF. The fibers of $p^\rightarrow$ are closed subschemes in a product of Quot schemes. For $p^\leftarrow$, dualize. For $(p^\leftarrow, p^\rightarrow)$, the fibers are closed in a product of Hom schemes. Properness follows from Sht being separable and $p^\leftarrow$ being proper. ☐

LEMMA 4.3. *The geometric fibers of $\mathrm{Sht}_G^r(h_D) \to X^r$ have dimension r. Therefore $\dim \mathrm{Sht}_G^r(h_D) = 2r$.*

We now define

$$H\colon \mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G^r \times \mathrm{Sht}_G^r)_{\mathbf{Q}}$$

sending $h_D \mapsto (p^\leftarrow, p^\rightarrow)_*[\mathrm{Sht}_G^r(h_D)]$.

LEMMA 4.4. *The map H is a ring homomorphism.*

Idea of the proof: we need to show that $H(h_D * h_{D'}) = H(h_D) \cdot H(h_{D'})$. We can reduce to checking this over U^r, where we can see it directly.

Remark 4.5. For $g = (g_x) \in G(\mathbf{A})$, one usually defines a self-correspondence $\Gamma(g)$ of $\mathrm{Sht}_G^r|_{(X \setminus S)^r}$ where $S = \{x \colon g_x \notin K_x\}$. Then $\mathbf{1}_{KgK}|_{(X\setminus S)^r}$ is the same cycle as $\Gamma(g)$. However, in this case the total Hecke algebra only acts on the generic fiber. In the paper, the Hecke action is defined over all of X, using that $\mathrm{Sht}_G^r(h_D)$ is defined over all of X.

Here is a variant: let $v\colon X' \to X$ be an étale cover of degree 2, and X' and X geometrically connected. Define

$$\mathrm{Sht}'^r_G := (X')^r \times_{X^r} \mathrm{Sht}_G^r$$

and

$$\mathrm{Sht}_G'^r(h_D) := (X')^r \times_{X^r} \mathrm{Sht}_G^r(h_D).$$

Base changing the maps from earlier, we obtain the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Sht}_G^r(h_D) & & \\ & \swarrow^{p^{\leftarrow\prime}} & & \searrow^{p^{\rightarrow\prime}} & \\ \mathrm{Sht}_G'^r & & & & \mathrm{Sht}_G'^r \\ & \searrow & & \swarrow & \\ & & X'^r & & \end{array}$$

This induces a map

$$H' \colon \mathcal{H} \to {}_c\mathrm{Ch}_{2r}(\mathrm{Sht}_G'^r \times \mathrm{Sht}_{G'}^r)$$

which is again a ring homomorphism.

Definition 4.6. For $f \in \mathcal{H}$, we have an operator $f * (-) := H'(f)$ acting on $\mathrm{Ch}_{c,*}(\mathrm{Sht}_G'^r)_{\mathbf{Q}}$.

4.4. The Heegner-Drinfeld cycle. Let μ be a balanced r-tuple. Let $\widetilde{T} := \mathrm{Res}_{X'/X}\, \mathbf{G}_m$, and $T := \widetilde{T}/\mathbf{G}_m$.

We have an action of $\mathrm{Pic}_{X'}(k)$ on $\mathrm{Sht}_{\widetilde{T}}^\mu$. In particular, $\mathrm{Pic}_X(k)$ acts through its embedding into $\mathrm{Pic}_{X'}(k)$. Then $\mathrm{Sht}_T^\mu := \mathrm{Sht}_{\widetilde{T}}^\mu / \mathrm{Pic}_X(k)$ has a map

$$\pi_T^\mu \colon \mathrm{Sht}_T^\mu \to (X')^r,$$

which is a $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$-torsor. Thus Sht_T^μ is proper smooth of dimension r over k. The spaces Sht_T^μ are canonically independent of μ, and so is the structure map to X^r (but not the one to X'^r).

Let $\mathcal{L}$ be a line bundle on $X' \times S$. Then we get a vector bundle $\nu_*\mathcal{L}$ of rank 2 on $X \times S$. This induces

$$\mathrm{Sht}_T^\mu \to \mathrm{Sht}_G^r$$

sending $(\underline{x}', \underline{\mathcal{L}}, \underline{f}, \iota) \mapsto (\nu(\underline{x}'), \nu_*\underline{\mathcal{L}}, \nu_* f, \nu_*\iota)$. Thus we get

$$\mathrm{Sht}_T^\mu \to \mathrm{Sht}_G'^r = (X')^r \times_{X^r} \mathrm{Sht}_G^r.$$

This is a finite étale morphism.

Definition 4.7. We define the *Heegner-Drinfeld cycle* $\theta_*^\mu[\mathrm{Sht}_T^\mu] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^{r'})_{\mathbf{Q}}$. Then can define

$$\langle \theta_*^\mu[\mathrm{Sht}_T^\mu], f * \theta_*^\mu[\mathrm{Sht}_T^\mu]\rangle_{\mathrm{Sht}_G'^r}.$$

Lemma 4.8. *This pairing is independent of μ.*

The main result is the following:

Theorem 4.9. *If π is an everywhere unramified automorphic representation of G, then*

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \sim ([\mathrm{Sht}_T^\mu]_\pi, [\mathrm{Sht}_T^\mu]_\pi)_\pi.$$

Part 2

Day Two

5. Automorphic forms over function fields
(Ye Tian)

5.1. Cuspidal automorphic forms.

5.1.1. *Goal.* Let X/k be a curve over a finite field and $F = k(X)$. Let $\mathbf{A} = \mathbf{A}_F$ and $\mathcal{O} = \prod_{x\in|X|} \mathcal{O}_x$, where $\mathcal{O}_x$ is the completed local ring of X at x.

Let $G = \mathrm{GL}_d$ and Z be the center of G. Let $G(\mathcal{O})$ be the maximal compact subgroup of $G(\mathbf{A})$.

Definition 5.1. A function $f\colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is *smooth* if it factors through $G(F)\backslash G(\mathbf{A})/K$ for some open subgroup K of $G(\mathbf{A})$. It is *cuspidal* if for any proper standard parabolic $P \subset G$, with unipotent N, the *constant term*

$$\varphi_P(g) = \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\, dn$$

vanishes.

The main goal of the talk is to prove:

Theorem 5.2 (Harder). *For any compact open $K \subset G(\mathbf{A})$, all cuspidal functions φ acting on $G(F)\backslash G(\mathbf{A})/K \to \mathbf{C}$ have support uniformly finite modulo $Z(\mathbf{A})$.*

5.1.2. *Automorphic representations.*

Definition 5.3. A smooth function $\varphi\colon G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ is called *automorphic* if its space spanned by right translations $G(\mathbf{A})$ of φ is admissible. (A smooth representation is *admissible* if the fixed vectors under any compact subgroup are finite dimensional.)

Definition 5.4. A function $\varphi\colon G(\mathbf{A}) \to \mathbf{C}$ has *central character* χ if $\chi(zg) = \chi(z)\varphi(g)$ for all $z \in Z(\mathbf{A})$.

Remark 5.5. If φ is cuspidal automorphic form with a central character, after twisting by $\mu \circ \det$ for some idele character μ, we may view φ as a function on $G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}}$, where $a \in Z(\mathbf{A}) = \mathbf{A}^\times$ has $\deg a = 1$.

Harder's theorem implies that $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A})/Ka^{\mathbf{Z}})$ is of finite dimension.

Definition 5.6. We define $\mathcal{A}_{G,\mathrm{cusp},\chi}$ to be the space of automorphic cuspidal forms of central character χ. This has an action $G(\mathbf{A})$ by right translation.

Theorem 5.7. *For any $\chi \in \chi_G$, $\mathcal{A}|_{G,\mathrm{cusp},\chi}$ is an admissible representation of $G(\mathbf{A})$. Moreover, it has a countable direct sum decomposition*

$$\mathcal{A}_{G,\mathrm{cusp},\chi} = \bigoplus_{\pi\in\Pi_{G,\mathrm{cusp},\chi}} \pi.$$

Here $\Pi_{G,\mathrm{cusp},\chi}$ is the set of equivalence classes of irreducible automorphic cuspidal representations of central character χ.

What is the content of this statement? It's obvious that π occurs as a subquotient. The theorem says that it actually occur as an honest subrepresentation, and also asserts a multiplicity one statement.

PROOF. Admissibility follows from Harder.

Semisimplicity: after twisting $\mathcal{A}_{G,\mathrm{cusp},\chi} \otimes (\mu \circ \det)$, we can assume that is χ is unitary. Then

$$\langle \varphi_1, \varphi_2 \rangle := \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \overline{\varphi_1}\varphi_2 \, dg$$

defines a $G(\mathbf{A})$-invariant positive definite Hermitian scalar product on $\mathcal{A}_{G,\mathrm{cusp},\chi}$. Since $G(\mathbf{A})$ has a countable open basis at e, this implies

$$\mathcal{A}_{G,\mathrm{cusp},\chi} = \bigoplus \pi^{m(\pi)}$$

with $m(\pi) = \dim \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathcal{A}) \geq 1$.

To see that $m(\pi) = 1$, we use that the Whittaker spaces are 1-dimensional. If $\psi \colon F\backslash \mathbf{A}_F \to \mathbf{C}^\times$ is a non-trivial unitary character, and U is the unipotent radical of the Borel, then we have

$$\mathrm{Hom}_{U(\mathbf{A})}(\pi, \psi) = \mathrm{Hom}_{G(\mathbf{A})}(\pi, \mathrm{Ind}_{U(\mathbf{A})}^{G(\mathbf{A})} \psi)$$

The latter is one-dimensional, which we can prove by passing to the local Whittaker model.

If $\xi \colon \pi \hookrightarrow \mathcal{A}_{G,\mathrm{cusp},\chi}$ then we get a map $\pi \to W_\xi$, sending

$$\varphi \mapsto W_{\xi(\varphi)}(g) := \int_{U(F)\backslash U(\mathbf{A})} \xi(\varphi)(ng)\psi(n)^{-1} dn.$$

From this we can "recover"

$$\xi(\varphi)(g) = \sum_{\gamma \in U_{d-1}(F)\backslash G_{d-1}(F)} W_{\xi(\varphi)}[\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g]$$

so the 1-dimensionality of the Whittaker model for π implies $m(\pi) = 1$. □

5.2. Reduction theory on Bun_G.

Definition 5.8. The slope of $\mathcal{E}$ is defined to be $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rank}\, \mathcal{E}}$. We have $\deg \mathcal{E} = \deg \det \mathcal{E}$.

By Riemann-Roch,

$$\chi(\mathcal{E}) = \deg \mathcal{E} + \mathrm{rank}\, \mathcal{E}(1 - g_X).$$

Definition 5.9. A (non-zero) vector bundle $\mathcal{E}$ over X is said to be *semistable* if for all sub-bundles

$$0 \subsetneq \mathcal{F} \subsetneq \mathcal{E},$$

we have $\mu(\mathcal{F}) \leq \mu(\mathcal{G})$. There is an equivalent formulation in terms of quotients.

Definition 5.10. A filtration of a vector bundle $\mathcal{E}$ on X

$$0 = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \ldots \subset F_s\mathcal{E} = \mathcal{E}$$

is a *Harder-Narasimhan (HN) filtration* if $F_j\mathcal{E}/F_{j-1}\mathcal{E}$ are semistable with slopes μ_j satisfying

$$\mu_1 > \mu_2 > \ldots > \mu_j.$$

Example 5.11. Let $X = \mathbf{P}^1/k$. Then $\mathcal{E} = \bigoplus_{i=1}^s \mathcal{O}(n_i)^{r_i}$ with $n_1 > n_2 > \ldots > n_s$ integers. Then the HN filtration is

$$0 \subset \mathcal{O}(n_1)^{r_1} \subset \mathcal{O}(n_1)^{r_1} \oplus \mathcal{O}(n_2)^{r_2} \subset \ldots \subset$$

THEOREM 5.12 (Harder-Narasimhan). *Any non-zero vector bundle over X admits a unique HN filtration.*

PROOF. Let μ_1 be the maximal slope of a sub-bundle $\mathcal{F} \subset \mathcal{E}$. By Riemann-Roch, we know this to be $< \infty$. We claim that in any HN filtration, $F_1\mathcal{E}$ is the maximal subbundle $\mathcal{E}_1$ with $\mu(\mathcal{E}_1) = \mu_1$. (The result would then follow by induction.)

To see that $\mathcal{E}_1$ exists, suppose you have $\mathcal{E}_1', \mathcal{E}_1''$ which both have r_1 with slope μ_1. Consider $\mathcal{F} := \langle \mathcal{E}_1' + \mathcal{E}_1'' \rangle$, the saturation of the subsheaf of $\mathcal{E}$ spanned by $\mathcal{E}_1'$ and $\mathcal{E}_1''$. Then

$$\deg \mathcal{F} \geq 2r_1\mu_1 - \deg(\mathcal{E}_1' \cap \mathcal{E}_1'')$$

(the inequality comes from the saturation) while

$$\operatorname{rank} \mathcal{F} = 2r_1 - \operatorname{rank}(\mathcal{E}_1' \cap \mathcal{E}_1'') > r_1.$$

So $\mu(\mathcal{F}) \geq \mu_1$ and dominates both $\mathcal{E}_1'$ and $\mathcal{E}_1''$.

To see that $F_1\mathcal{E}$, must be defined in this way, note that the definition of $\mathcal{E}_1$ forces it to be semistable. Therefore, its image in $F_i\mathcal{E}/F_{i+1}\mathcal{E}$ has slope at least $\mu(\mathcal{E}_1) \geq \mu_1$, so this image must be 0.

□

Write $B = TU$. By Weil's adelic uniformization, we can interpret

$$B(F)\backslash B(\mathbf{A})/B(\mathcal{O}) \leftrightarrow \text{ isomorphism classes of flags of rank } (1, \ldots, 1).$$

Let Δ be the set of simple roots of G.

THEOREM 5.13 (Siegel Domain). *Let $c_2 \geq 2g$ be an integer. Then*

$$G(\mathbf{A}) = G(F)U(\mathbf{A})T(\mathbf{A})^{\Delta}_{c_2}G(\mathcal{O})$$

where $T(\mathbf{A})^{\Delta}_{c_2} = \{t \in T(\mathbf{A})\colon \deg \alpha(t) \leq c_2 \forall \alpha \in \Delta\}$. In other words (by Iwasawa decomposition), for every $\mathcal{E}$ of rank d over X, there is at least one flag

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_d = \mathcal{E}$$

such that $\deg(\mathcal{E}_{j+1}/\mathcal{E}_j) - \deg(\mathcal{E}_j/\mathcal{E}_{j-1}) \leq c_2$ for all j.

PROOF. Take a subline bundle $\mathcal{L} \subset \mathcal{E}$ with $\mathcal{E}_1 = \langle \mathcal{L} \rangle$ (the saturation) such that

$$1 \leq \deg \mathcal{E} - d \deg \mathcal{L} + d(1-g) \leq d. \tag{5.1}$$

Why is this possible? The lower bound comes from Riemann-Roch applied to $H^0(\mathcal{E} \otimes \mathcal{L}^\vee)$, which is non-zero as soon as there exists $\mathcal{L} \hookrightarrow \mathcal{E}$. The upper bound comes from the inequality

$$\deg \mathcal{E}_1 \geq \det \mathcal{L} \geq \frac{\deg \mathcal{E}}{d} - g$$

which comes from the non-existence of extensions with too large separation of degree (by Serre duality).

By induction, we can lift a filtration with this property $\mathcal{E}/\mathcal{E}_1$. The only question is to check the desired inequality for $i = 1$. If $\mathcal{E}$ is semistable we can conclude as follows: the analog of (5.1) holds to tell us that

$$\mathcal{E}_2 - d \deg \mathcal{E}_1 + 2(1-g) \leq 2$$

so

$$\begin{aligned} \deg \mathcal{E}_2/\mathcal{E}_1 - \deg \mathcal{E}_1 &= \deg \mathcal{E}_2 - 2 \deg \mathcal{E}_1 \\ &\leq 2 - 2(1-g) = 2g. \end{aligned}$$

If $\mathcal{E}$ is not semistable, take an HN filtration, whose associated subquotients are semistable by definition. We apply the conclusion from the semistable case to each subquotient. The only issue is to check that the inequality still holds at the endpoints. The desired inequalities end up following from the semistability.

□

THEOREM 5.14. *Let $K \subset G(\mathcal{O})$ be a compact open subgroup. There exists an open subset $C_K \subset G(\mathbf{A})$ satisfying*

(1) $Z(\mathbf{A})G(F)C_K = C_K$, i.e. C_K is invariant under $Z(\mathbf{A})G(F)$, and
(2) $Z(\mathbf{A})G(F)\backslash C_K/K$ is finite.

Moreover, $\operatorname{supp} \varphi \subset C_K$ *for all cuspidal φ.*

6. The work of Drinfeld
(Arthur Cesar le Bras)

6.1. Notation. Let $k = \mathbf{F}_q$, X/k a smooth projective, geometrically connected curve over k. Let $F = k(X)$. Chooose a point $\infty \in |X|$, and assume for simplicity that $\deg \infty = 1$.

Let $F = k(X)$, F_∞ be the completion of F at ∞. Let $\mathbf{C}_\infty$ be the completion of a separable closure of F_∞, and $A = H^0(X \setminus \{\infty\}, \mathcal{O})$.

6.2. Elliptic modules.

6.2.1. *Definition.* The seed of shtukas were Drinfeld's "elliptic modules". Let $\mathbf{G}_a$ be the additive group, and K a characteristic p field. We set $K\{\tau\} = K \otimes_{\mathbf{Z}} \mathbf{Z}[\tau]$, with multiplication given by

$$(a \otimes \tau^i)(b \otimes \tau^j) = ab^{p^i} \otimes \tau^{i+j}.$$

We have an isomorphism $K\{\tau\} \cong \mathrm{End}_K(\mathbf{G}_a)$ sending

$$\sum_{i=0}^{m} a_i \otimes \tau^i \mapsto \left(X \mapsto \sum_{i=0}^{m} a_i X^{p^i} \right).$$

If a_m is the largest non-zero coefficient, then the *degree* of $\sum_{i=0}^{m} a_i \in K\{\tau\}$ is defined to be p^m. The *derivative* is defined to be the constant term a_0.

Definition 6.1. Let $r > 0$ be an integer and K a characteristic p field. An *elliptic A-module of rank r* is a ring homomorphism

$$\phi \colon A \to K\{\tau\}$$

such that for all non-zero $a \in A$, $\deg \phi(a) = |a|_\infty^r$.

We can also make a relative version of this definition.

Definition 6.2. Let S be a scheme of characteristic p. An *elliptic A-module of rank r over S* is a $\mathbf{G}_a$-torsor $\mathcal{L}/S$, with a morphism of rings $\phi \colon A \to \mathrm{End}_S(\mathcal{L})$ such that for all points $s \colon \operatorname{Spec} K \to S$, the fiber $\mathcal{L}_s$ is an elliptic A-module of rank r.

Remark 6.3. The function $a \mapsto \phi(a)'$ (the latter meaning the derivative of $\phi(a)$) defines a morphism of rings $i \colon A \to \mathcal{O}_S$, i.e. a morphism $\theta \colon S \to \operatorname{Spec} A$.

6.2.2. *Level structure.* Let I be an ideal of A. Let $(\mathcal{L}, \phi)$ be an elliptic module over S. Assume that S is an $A[I^{-1}]$-scheme, i.e. the map θ factors through $\theta \colon S \to \operatorname{Spec} A \setminus V(I)$.

Let $\mathcal{L}_I$ be the group scheme defined by the equation $\phi(a)(x) = 0$ for all $a \in I$. This is an étale group scheme over S with rank $\#(A/I)^r$. An I-level structure on $(\mathcal{L}, \varphi)$ is an A-linear isomorphism

$$\alpha \colon (I^{-1}/A)_S^r \xrightarrow{\sim} \mathcal{L}_I.$$

Choose $0 \subsetneq I \subsetneq A$. We have a functor

$$F_I^r \colon A[I^{-1}] - \mathbf{Sch} \to \mathbf{Sets}$$

sending S to the set of isomorphism classes of elliptic A-modules of rank r with I-level structure, with θ being the structure morphism.

THEOREM 6.4 (Drinfeld). *F_I^r is representable by a smooth affine scheme M_I^r over $A[I^{-1}]$.*

6.3. Analytic theory of elliptic modules.

6.3.1. *Description in terms of lattices.* Let Γ be an A-lattice in $\mathbf{C}_\infty$. (This means a discrete additive subgroup of $\mathbf{C}_\infty$ which is an A-module.) Then we define

$$e_\Gamma(x) = x \prod_{x \in \Gamma - 0} (1 - x/\gamma).$$

Drinfeld proved that this is well-defined for all $x \in \mathbf{C}_\infty$, and induces an additive surjection:

$$e_\Gamma \colon \mathbf{C}_\infty/\Gamma \xrightarrow{\sim} \mathbf{C}_\infty.$$

Given Γ, we define a function

$$\phi_\Gamma \colon A \to \mathrm{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$$

by the following rule. For $a \in A$, there exists $\phi_\Gamma(a)$ such that

$$\phi_\Gamma(a) e_\Gamma(x) = e_\Gamma(ax) \text{ for all } x \in \mathbf{C}_\infty.$$

If Γ is replaced by $\lambda\Gamma$, for $\lambda \in \mathbf{C}_\infty^*$, then ϕ_Γ doesn't change. Therefore, ϕ_Γ is a function on homothety classes of A-lattices.

THEOREM 6.5 (Drinfeld). *The function $\Gamma \mapsto \phi^\Gamma$ induces a bijection between*

$$\left\{ \begin{array}{c} \textit{rank } r \textit{ projective } A\textit{-lattices} \\ \textit{in } \mathbf{C}_\infty/\textit{homothety} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{rank } r \textit{ elliptic } A\textit{-modules} \\ \textit{over } \mathbf{C}_\infty \textit{ such that } \phi(a)' = a \\ /\textit{isomorphism} \end{array} \right\}$$

Remark 6.6. Under this bijection, an I-level structure equivalent to an A-linear isomorphism $(A/I)^r \cong \Gamma/I\Gamma$ for the lattices.

6.3.2. *Uniformization.* We now try to parametrize the objects on the left hand side of (6.5). First we parametrize the isomorphism classes. Let Y be a projective A-module of rank r. Then we have a bijection

$$\left\{ \begin{array}{c} \text{homothety classes of } A\text{-lattices in } \mathbf{C}_\infty \\ \text{isomorphic to } Y \text{ as } A\text{-modules} \end{array} \right\} \leftrightarrow \mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)/\, \mathrm{GL}_A(Y).$$

Next we observe that there is a bijection

$$\mathbf{C}_\infty^\times \backslash \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty) \leftrightarrow \mathbf{P}^{r-1}(\mathbf{C}_\infty) \setminus \bigcup (F_\infty\text{-rational hyperplanes}),$$

given by sending $u \in \mathrm{Inj}(F_\infty \otimes_A Y, \mathbf{C}_\infty)$ to $[u(e_1), \ldots, u(e_r)]$. This is called the *Drinfeld upper half plane* Ω^{r-1}.

As Spec $A = X \setminus \infty$, a projective A-module of rank r is the same as a vector bundle of rank r on $X \setminus \infty$. We saw yesterday that there is an isomorphism (Weil's uniformization)

$$\left\{ \begin{array}{c} \text{rank } r \text{ vector bundles on } X \setminus \infty \\ \text{plus generic trivialization} \end{array} \right\} /\text{isom.} \leftrightarrow \mathrm{GL}_r(\mathbf{A}_F^\infty)/ \prod_{v \neq \infty} \mathrm{GL}_r(\mathcal{O}_v).$$

Set $\mathrm{GL}_r(\widehat{A}) := \prod_{v\neq\infty} \mathrm{GL}_r(\mathcal{O}_v)$, and

$$\mathrm{GL}_r(\widehat{A}, I) := \ker\left(\mathrm{GL}_r(\widehat{A}) \to \mathrm{GL}_r(\widehat{A}/I)\right).$$

In conclusion, there is a natural bijection

$$M_I^r(\mathbf{C}_\infty) \cong \mathrm{GL}_r(F)\backslash(\mathrm{GL}_r(\mathbf{A}_F^\infty)/\mathrm{GL}_r(\widehat{A}, I) \times \Omega^r(\mathbf{C}_\infty))$$

This can be upgraded into an isomorphism of rigid analytic spaces:

Theorem 6.7 (Drinfeld). *We have an isomorphism of rigid analytic spaces over F_∞:*

$$(M_I^r)^{\mathrm{an}} = \mathrm{GL}_r(F)\backslash(\mathrm{GL}_r(\mathbf{A}_F^\infty)/\mathrm{GL}_r(\widehat{A}, I) \times \Omega^r(\mathbf{C}_\infty)).$$

6.4. Cohomology of M_I^2 and global Langlands for GL_2.

6.4.1. *Cohomology of the Drinfeld upper half plane.* We now outline Drinfeld's proof of global Langlands for GL_2 using the moduli space of elliptic modules. Set $r = 2$, and $\Omega := \Omega^2$. Then one has

$$\Omega(\mathbf{C}_\infty) = \mathbf{P}^1(\mathbf{C}_\infty)\backslash\mathbf{P}^1(F_\infty).$$

There is a map λ from $\Omega(\mathbf{C}_\infty)$ to the Bruhat-Tits tree, sending (z_0, z_1) to the homothety class of the norm on F_∞^2 defined by

$$(a_0, a_1) \in F_\infty^2 \mapsto |a_0 z_0 + a_1 z_1|.$$

The pre-image of a vertex is $\mathbf{P}^1$ minus $q+1$ open unit disks, and the pre-image of an open edge is an annulus (which can be thought of as $\mathbf{P}^1$ minus 2 open disks). There is an admissible covering of Ω given by $\{U_e := \lambda^{-1}(e)\}$ as e runs over the closed edges. We have an exact sequence

$$H^1(\Omega, \mathbf{Z}/n) \to \prod_{e\in E} H^1(U_e, \mathbf{Z}/n) \to \prod_{v\in V} H^1(U_v, \mathbf{Z}/n)$$

and using this, we get that for $\ell \neq p$, the vector space $H^1_{\text{ét}}(\Omega, \overline{\mathbf{Q}}_\ell)$ is naturally the space of harmonic cochains on the Bruhat-Tits tree, which is the set of functions c from oriented edges to $\overline{\mathbf{Q}}_\ell$ satisfying

(1) $c(-e) = -c(e)$ and
(2) $\sum_{e\in E(v)} c(e) = 0$.

Any such harmonic cochain defines a ($\overline{\mathbf{Q}}_\ell$-valued) measure on $\partial\Omega = \mathbf{P}^1(F_\infty)$. In other words, we have

$$H^1_{\text{ét}}(\Omega, \overline{\mathbf{Q}}_\ell) = (C^\infty(\mathbf{P}^1(F_\infty), \overline{\mathbf{Q}}_\ell)/\overline{\mathbf{Q}}_\ell)^* \cong \mathrm{St}^*.$$

The isomorphism is $\mathrm{GL}_2(F_\infty)$-invariant.

6.4.2. *Cohomology of M_I^2.* Now we use the uniformization of M_I^2. We can rewrite it as follows:

$$M_I^{2,\mathrm{an}} = \left(\Omega \times \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)/\mathrm{GL}_2(\widehat{A}, I)\right)/\mathrm{GL}_2(F_\infty).$$

(Some elementary trickery is required to go from the previous formulation to the one above.) Now you use the Hochschild-Serre spectral sequence for $Y \to Y/\Gamma$ to get a long exact sequence

$$0 \to H^1(\Gamma, H^0(Y, \overline{\mathbf{Q}}_\ell)) \to H^1(Y/\Gamma, \overline{\mathbf{Q}}_\ell) \to H^1(X, \overline{\mathbf{Q}}_\ell)^\Gamma \to \dots.$$

From this we deduce

$$H^1_{\text{ét}}(M_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) \cong \mathrm{Hom}_{\mathrm{GL}_2(F_\infty)}(\mathrm{St}, C^\infty(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)/\mathrm{GL}_2(\widehat{A}, I)) \otimes \mathrm{sp}$$

where sp is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}_\infty/F_\infty)$, which should be the Galois representation corresponding to Steinberg. This isomorphism is compatible for the action of $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{Gal}(\overline{F}_\infty/F_\infty)$.

Remark 6.8. This is cheating a little; we really need to work with a compactification of M_I^2 instead.

Drinfeld shows that

$$\varinjlim_I H^1(\overline{M}_I^2 \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell) = \bigoplus_\pi \pi^\infty \otimes \sigma(\pi)$$

where π runs over cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ with $\pi_\infty \cong$ St. Here $\sigma(\pi)$ is a $\mathrm{Gal}(\overline{F}/F)$ representation. Moreover, Drinfeld shows that at unramified places, π_v and $\sigma(\pi_v)$ correspond by local Langlands.

6.4.3. *The local Langlands correspondence.* Using this, one can construct the local Langlands correspondence for GL_2 over K, a characteristic p local field. Indeed, let π be a supercuspidal representation of $\mathrm{GL}_2(K)$. Write $K = F_v$ for a global F. Choose a global automorphic representation Π such that $\Pi_v \cong \pi$ and $\Pi_\infty \cong \mathrm{St}$. By the work of Drinfeld, we get $\sigma(\Pi)$ and we know that Π_w and $\sigma(\Pi)_w$ have the same ϵ-factors and L-functions at all w outside some finite set S. Then for any global Hecke character χ, we have

$$\prod_w L_w(\Pi_w \otimes \chi_w) = \prod_w \epsilon_w(\Pi_w \otimes \chi_w) \prod_w L_w(\Pi_w^\vee \otimes \chi_w^{-1} \otimes \omega_{\Pi_w})$$

and similarly for $\sigma(\Pi)$. We can divide these two equalities by the product for $w \notin S$, getting an equality of two finite products

$$\prod_{w \in S} \epsilon'_w(\Pi_w \otimes \chi_w) = \prod_{w \in S} \epsilon'_w(\sigma(\Pi)_w \otimes \chi_w)$$

where $\epsilon'_w(\tau) = \epsilon_w(\tau)\frac{L(\tau^\vee \otimes \omega_\tau)}{L(\tau)}$.

Now for a trick: we can choose χ such that $\chi_v = 1$ and χ_w is very ramified for all other $w \in S - v$, thus forcing the L-factors at those w to be 1. Then $\epsilon'_w(\Pi_w \otimes \chi_w) = \epsilon(\Pi_w \otimes \chi_w)$ only depends on χ_w. In this way one can isolate an equality for the ϵ and L-factors of $\Pi_v = \pi$.

6.5. Elliptic sheaves. (This material is from a discussion session.) We will explain the connection between elliptic modules and shtukas. The relation passes through an intermediate object called an "elliptic sheaf".

Definition 6.9. An *elliptic sheaf of rank* $r > 0$ *with pole at* ∞ is a diagram

$$\begin{array}{ccccccccc} \cdots & \hookrightarrow & \mathcal{F}_{i-1} & \overset{j_i}{\hookrightarrow} & \mathcal{F}_i & \overset{j_{i+1}}{\hookrightarrow} & \mathcal{F}_{i+1} & \hookrightarrow & \cdots \\ & \nearrow & & \nearrow^{t_i} & & \nearrow^{t_{i+1}} & & \nearrow & \\ \cdots & \hookrightarrow & {}^\tau\mathcal{F}_{i-1} & \overset{{}^\tau j_i}{\hookrightarrow} & {}^\tau\mathcal{F}_i & \overset{{}^\tau j_{i+1}}{\hookrightarrow} & {}^\tau\mathcal{F}_{i+1} & \hookrightarrow & \cdots \end{array}$$

(here as usual $\tau^*\mathcal{F} = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{F})$ with $\mathcal{F}_i$ bundles of rank r, such that j and t are $\mathcal{O}_{X\times S}$-linear maps satisfying

1. (1) $\mathcal{F}_{i+r} = \mathcal{F}_i(\infty)$ and $j_{i+r} \circ \ldots \circ j_{i+1}$ is the natural map $\mathcal{F}_i \hookrightarrow \mathcal{F}_i(\infty)$.
2. (2) $\mathcal{F}_i/j(\mathcal{F}_{i-1}$ is an invertible sheaf along Γ_∞.
3. (3) For all i, $\mathcal{F}_i/t_i(\tau^*\mathcal{F}_{i-1}) =$ is an invertible sheaf along Γ_z for some $z\colon S \to X \setminus \infty$ (independent of i).
4. (4) For all geometric points $\overline{s}$ of S, the Euler characteristic $\chi(\mathcal{F}_0|_{X_{\overline{s}}}) = 0$.

Definition 6.10. Let $J \subset A$ be an ideal cutting out the closed subset $I \subset \operatorname{Spec} X$. An *$I$-level structure on an elliptic sheaf* is a diagram

$$\begin{array}{ccc} & & \mathcal{F}_0|_{I\times S} \\ & \overset{\sim}{\underset{f}{\nearrow}} & \| \\ \mathcal{O}^r_{I\times S} & & \| \\ & \overset{\sim}{\underset{{}^\tau f}{\searrow}} & \| \\ & & {}^\tau\mathcal{F}_0|_{I\times S} \end{array}$$

THEOREM 6.11. *Let $z\colon S \to \operatorname{Spec} A \setminus I$. Then there exists a bijection, functorial in S,*

$$\left\{ \begin{array}{c} \textit{rank } r \textit{ elliptic } A\textit{-modules} \\ \textit{with } J\textit{-level structure} \\ \textit{such that } \phi(a)' = z(a) \end{array} \right\} /\textit{isom.} \leftrightarrow \left\{ \begin{array}{c} \textit{rank } r \textit{ elliptic sheaves over } S \\ \textit{with zero } z \\ \textit{and } I\textit{-level structure} \end{array} \right\} /\textit{isom.}$$

We'll define the map for $S = \operatorname{Spec} K$. Let $(\mathcal{F}_i, j_\cdot, t_\cdot)$ be an elliptic sheaf. Define $M_i = H^0(X \otimes_k K, \mathcal{F}_i)$, and

$$M = \varinjlim M_i = H^0((X - \infty) \otimes K, \mathcal{F}_i).$$

This is a module over $A \otimes_k K$.

The $t_\cdot$ induce a map $t\colon M \to M$ which satisfies

- $t(am) = at(m)$, for $a \in A$
- $t(\lambda m) = \lambda^q t(m)$, for $\lambda \in K$.

This t makes M a module over $K\{\tau\}$. Furthermore:

- Because the zero and pole are distinct, t induces an injection

$${}^\tau(\mathcal{F}_i/j(\mathcal{F}_{i-1})) \to \mathcal{F}_{i+1}/j(\mathcal{F}_i).$$

This implies that $\tau\colon M_i/M_{i-1} \to M_{i+1}/M_i$ is injective.

- We claim that $M_0 = 0$. Otherwise, we would have for some $i < 0$ a non-zero x in $M_i \setminus M_{i-1}$. The previous bullet point implies that for all $m \geq 0$, we have $t^m x \in M_{i+m} \setminus M_{i+m-1}$, so $\dim M_m \geq m+1$ because there are independent vectors $(x, tx, \ldots, t^m x)$. For really large m, we would then have $\chi(\mathcal{F}_m) = m = \dim M_m \geq m+1$, a contradiction to $\chi(\mathcal{F}_0) = 0$.
- For all i, M_i/M_{i-1} is 1-dimensional by similar estimates as in the previous bullet point. Finally, if u is a non-zero element of M_1, we have $M \cong K\{\tau\}u$.

The action of A gives a ring homomorphism $A \xrightarrow{\phi} \mathrm{End}_{K\{\tau\}}(M) = K\{\tau\}$.

- The action of A on $M/K\tau(M) \cong M_1$ being in the fiber of $\mathcal{F}_1$ at z implies $\phi(a)' = z(a)$.

One can show that if $(\mathcal{F}_i, t, j)$ is an elliptic sheaf, then for all i

$$t(\tau^* \mathcal{F}_{i-1}) = \mathcal{F}_i \cap t(\tau^* \mathcal{F}_i) \text{as subsheaves of } \mathcal{F}_{i+1}.$$

You can actually reconstruct the entire elliptic sheaf from the triangle

$$\begin{array}{ccc} \mathcal{F}_0 & \overset{j}{\hookrightarrow} & \mathcal{F}_1 \\ & \overset{t}{\nearrow} & \\ {}^{\tau}\mathcal{F}_i & & \end{array}$$

which is just a shtuka!

You can't go in the other direction - shtukas are more general. (You need to impose special conditions, namely "supersingular", on shtukas to construct elliptic sheaves.)

7. Analytic RTF: Geometric Side (Jingwei Xiao)

7.1. The big picture. Yesterday we defined a certain "geometric" quantity $\mathbb{I}_r(f)$. Today we will define an "analytic" quantity $\mathbb{J}_r(f)$. Both of these have two expansion:

$$\textstyle\sum_{u\in\mathbf{P}^1(F)-0} I_\gamma(u,f) \;=\!=\!=\; I_r(f) \;=\!=\!=\; \sum_\pi I_r(\pi,f). \tag{7.1}$$

and

$$\textstyle\sum_{u\in\mathbf{P}^1(F)-0} J_\gamma(u,f) \;=\!=\!=\; J_r(f) \;=\!=\!=\; \sum_\pi J_r(\pi,f). \tag{7.2}$$

The left sides of (7.1), (7.2) are expansions in terms of orbital integrals. The right side are the quantities that we want to compare: $\mathbb{J}_r(\pi,f) \sim L^{(r)}(\pi_F,1/2)$, and $\mathbb{I}_r(\pi,f) = \langle[\mathrm{Sht}_T]_\pi, f*[\mathrm{Sht}_T]_\pi\rangle$. The functions f are "test functions" which provide the flexibility to isolate the terms of interest.

7.2. The relative trace formula. Let F be a global field, the function field of X for $X/\mathbf{F}_q$. Let G/F be a reductive group, and $H_1, H_2 \hookrightarrow G$ subgroups over F. We'll write $[G] := G(F)\backslash G(\mathbf{A})$, and similarly for H_i.

7.2.1. *The kernel.* Let $\mathbf{A} = \mathbf{A}_F$, $[G] = G(F)\backslash G(\mathbf{A})$. For $f \in C_c^\infty(G(\mathbf{A}))$, we define the kernel function

$$\mathbb{K}_f(g_1,g_2) := \sum_{\gamma\in G(F)} f(g_1^{-1}\gamma g_2).$$

The point is that $G(\mathbf{A})$ acts on $C^\infty(G(F)\backslash G(\mathbf{A}))$, and for $\phi \in C^\infty(G(F)\backslash G(\mathbf{A}))$ we have

$$\pi(f)\cdot\phi = \int_{G(F)\backslash G(\mathbf{A})} \mathbb{K}_f(g_1,g_2)\phi(g_2)\,dg_2. \tag{7.3}$$

The relative trace formula involves the quantity

$$\int_{[H_1]\times[H_2]} \mathbb{K}_f(h_1,h_2)\,dh_1dh_2 \tag{7.4}$$

7.2.2. *The geometric expansion.* The geometric expansion of (7.4) is

$$\int_{[H_1]\times[H_2]} \sum_{\gamma\in G(F)} f(h_1^{-1}\gamma h_2)\,dh_1dh_2$$
$$= \sum_{\gamma\in H_1(F)\backslash G(F)/H_2(F)} \int_{[H_1]\times[H_2]} \sum_{\delta\in H_1(F)\gamma H_2(F)} f(h_1^{-1}\delta h_2)\,dh_1dh_2$$

Rearranging, one rewrites this as

$$= \sum_{\gamma\in H_1(F)\backslash G(F)/H_2(F)} \int_{(H_1\times H_2)_\gamma(F)\backslash H_1(\mathbf{A})\times H_2(\mathbf{A})} f(h_1^{-1}\gamma h_2).$$

Here $_\gamma$ denotes the stabilizer of γ:

$$(H_1\times H_2)_\gamma := \{(h_1,h_2)\colon h_1^{-1}\gamma h_2 = \gamma\}.$$

7.2.3. *The spectral expansion.* Now we rewrite (7.3) in a different way. The idea is to decompose

$$\mathbb{K}_f \approx \sum_{\pi \text{ cuspidal}} \sum_{\phi} \pi(f)\phi \otimes \overline{\phi}$$

where ϕ runs over orthogonal basis of π. This is a bit of lie, as one also needs to consider the residual and Eisenstein parts, but it roughly works. Using this, we can rewrite

$$(7.3) = \sum_{\pi} \sum_{\phi} \int_{[H_1]} \pi(f)\phi \, dh_1 \cdot \int_{[H_2]} \overline{\phi} \, dh_2$$

Here the term $\int_{[H_1]} \pi(f)\phi$ is a "period" $\mathcal{P}_{H_1}(\pi(f)\phi)$.

7.2.4. *The split subtorus.* For $G = \mathrm{PGL}_2$, set $H_1 = H_2 = A$ to be the diagonal torus of G. For $f \in C_c^\infty(G(\mathbf{A}))$ we get a kernel function $\mathbb{K}_f$. We consider the integral

$$\int_{[A]\times[A]} \mathbb{K}_f(h_1, h_2)|h_1 h_2|^s \eta(h_2)$$

where if $h = \begin{pmatrix} x & \\ & y \end{pmatrix}$ then $|h| = |x/y|$, and $\eta \colon \mathbf{A}_F^\times \to \{\pm 1\}$ is the character corresponding by class field theory to F'/F.

There's an issue with this integral. Since $A \cong \mathbf{G}_m/F$, $[A] := F^\times \backslash \mathbf{A}_F^\times$ is not compact, since

$$[A]/\prod_x \mathcal{O}_x^\times = \mathrm{Pic}(X).$$

This has infinitely many connected components, which are finite since they are isomorphic to $\mathrm{Pic}^0(X)$. To regularize the integral, define

$$[A]_n = \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix} : v(x/y) = n \right\}.$$

We have a map

$$v \colon \mathbf{A}_F^\times / \prod \mathcal{O}_x^\times \to \mathbf{Z}$$

sending $v(\pi_x) = \log_q(q_x)$ for any $x \in |X|$.

The $[A_n]$ are compact, so we can talk about

$$\int_{[A]_{n_1}\times[A]_{n_2}} \mathbb{K}(h_1, h_2)|h_1 h_2|^s \eta(s) \, ds.$$

This is actually a polynomial in q^s.

Proposition 7.1. *For each f, there exists N such that $|n_1| + |n_2| \geq N$ implies*

$$\int_{[A]_{n_1}\times[A]_{n_2}} \mathbb{K}_f(h_1, h_2)\eta(h_2)|h_1 h_2|^s = 0.$$

Assuming this claim, we can define the regularized integral

$$\int_{[A_1]\times[A_2]}^{\mathrm{reg}} := \sum_{n_1, n_2} \int_{[A]_{n_1}\times[A]_{n_2}}$$

7.3. Spectral expansion. The goal of this section is to establish the identity

$$\mathbb{J}_r(f) = \sum_{u \in \mathbf{P}^1(F) - 0} J_\gamma(u, f).$$

7.3.1. *The invariant map.* We have seen that in the RTF, we care about the double coset space

$$H_1(F) \backslash G(F) / H_2(F).$$

For $G = \mathrm{PGL}_2$, $H_1 = H_2 = A$ we can define an invariant map

$$A(F) \backslash \mathrm{PGL}_2(F) / A(F) \to \mathbf{P}^1 - \{1\}$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{bc}{ad}.$$

PROPOSITION 7.2. *For $x \in \mathbf{P}^1(F) - \{1\}$, we have*

$$\mathrm{inv}^{-1}(x) = \begin{cases} \text{single orbit} & x \neq 0, \infty \\ \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & x = 0 \\ \begin{pmatrix} & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & x = \infty \end{cases}$$

Also, γ is regular semisimple iff $\mathrm{inv}(\gamma) \neq 0, \infty$.

7.3.2. *Expansion.* This lets us write

$$\mathbb{J}(f, s) = \sum_{\gamma \in A(F) \backslash G(F) / A(F)} J(\gamma, f, s)$$

where

$$J(\gamma, f, s) = \int_{[A] \times [A]} \mathbb{K}_{f,\gamma}(h_1, h_2) |h_1 h_2|^s \eta(h_2) \, dh_2$$

and

$$\mathbb{K}_{f,\gamma}(h_1, h_2) = \sum_{\delta \in A(F) \gamma A(F)} f(h_1^{-1} \delta h_2^{-1}).$$

For $u \in \mathbf{P}^1(F) - \{1\}$, we define

$$\mathbb{J}(u, f, s) = \sum_{\substack{\gamma \in A(F) \backslash G(F) / A(F) \\ \mathrm{inv}(\gamma) = u}} \mathbb{J}(\gamma, f, s).$$

7.3.3. *Higher derivatives.* Now we define

$$\mathbb{J}_r(f) = \left(\frac{d}{ds}\right)^r J(f,s)|_{s=0}.$$

Similarly, we have a decomposition

$$\mathbb{J}_r(u,f) = \left(\frac{d}{ds}\right)^r \mathbb{J}(u,f,s)|_{s=0}.$$

Also

$$\mathbb{J}_r(f) = \sum_{u \in \mathbf{P}^1(F)-\{1\}} \mathbb{J}_r(u,f).$$

7.4. The case $r=0$. The goal is to establish the identity

$$\sum_{u \in \mathbf{P}^1(F)-0} I_\gamma(u,f) = \mathbb{I}_r(f).$$

Yesterday we defined

$$\mathbb{I}_0(f) := \langle \mathrm{Sht}_T^0, f * \mathrm{Sht}_T^0 \rangle.$$

Let F'/F be a quadratic extension. Define $T = \mathrm{Res}_{F'/F}\, \mathbf{G}_{m,F'}/\mathbf{G}_{m,F}$. It turns out that we also have an equality

$$\mathbb{I}_0(f) = \int_{[T]\times[T]} \mathbb{K}_f(h_1,h_2).$$

The Waldspurger formula can be reinterpreted in these terms:

$$\sum_{\gamma \in A(F)\backslash G(F)/A(F)} \mathbb{J}_0(\gamma,f) = \mathbb{J}_0(f) \sim L(\pi,0).$$

While

$$\sum_{\gamma \in T(F)\backslash G(F)/T(F)} \mathbb{I}_0(\gamma,f) = \mathbb{I}_0(f) \sim \int_{[T]} \phi_\pi.$$

7.5. The equality $\mathbb{I}_0(f) = \mathbb{J}_0(f)$. The strategy to relate the things is to relate the orbital integrals. So we first need to relate the orbits.

7.5.1. *Matching double cosets.* Let $G = \mathrm{PGL}_2$ or $D^\times/F^\times$, where D is a quaternion algebra over F (with an embedding $F' \hookrightarrow D$).

THEOREM 7.3. *We have*

$$A(F)\backslash \mathrm{PGL}_2(F)^{\mathrm{rss}}/A(F) = \coprod_{G=\mathrm{PGL}_2 \text{ or } D^\times/F^\times} T(F)\backslash G(F)^{\mathrm{rss}}/T(F).$$

PROOF. We consider $G = \mathrm{PGL}_2$ or $D^\times$. Let $H = M_2(F)$ or D, so $G = H^\times$. We have an embedding $F' \hookrightarrow H$.

There exists $\epsilon \in H(F)$ such that $\epsilon x \epsilon^{-1} = \overline{x}$ for $x \in F'$. The choice of ϵ is unique up to multiplication by $(F')^\times$. By computation, $\epsilon^2 \in Z(H) = F$, so $[\epsilon^2] \in F^\times/\mathrm{Nm}(F')^\times$ is well-defined.

PROPOSITION 7.4. *The element $[\epsilon^2] \in F^\times/\mathrm{Nm}(F')^\times$ determines H.*

We have an invariant map

$$T(F)\backslash H^{\times}/T(F) \xrightarrow{\text{inv}} \mathbf{P}^1(F) - \{1\}$$

sending

$$h_1 + \epsilon h_2 \mapsto \frac{h_2\overline{h_2}}{h_1\overline{h_1}}\epsilon^2.$$

The image lands in $\epsilon^2 \cdot \mathrm{Nm}((F')^{\times})$, and is equal in the regular semisimple case since this is equivalent to $h_1 h_2 \neq 0$. $\square$

7.5.2. *Matching orbital integrals.* Now that we've matched up the double cosets, we turn to showing that $\mathbb{I}_0(f) = \mathbb{J}_0(f)$ for $f = \prod_{v\in|X|} f_v \in \mathcal{H}_G$ a bi-K-invariant function. Writing out the expansion, this comes down to

$$\sum_u \mathbb{I}_0(u, f) = \sum_u \mathbb{J}_0(u, f)$$

This becomes a fundamental lemma type statement.

Consider F_v'/F_v a quadratic extension. Let $f \in C^{\infty}(\mathrm{PGL}_2(F_v))$ be bi-K-invariant. We have

$$A(F_v)\backslash \mathrm{PGL}_2(F)/A(F_v) = T(F_v)\backslash \mathrm{PGL}_2(F_v)/T(F_v) \coprod T(F_v)\backslash D^{\times}/T(F_v)$$

By Theorem 7.3, each γ on the left side matches up with a $\gamma_1 \in T(F_v)\backslash \mathrm{PGL}_2(F_v)/T(F_v)$ or $\gamma_2 \in T(F_v)\backslash D^{\times}/T(F_v)$. One can then compute by hand that the corresponding orbital integrals are equal.

- If $\gamma \leftrightarrow \gamma_1 \in T(F_v)\backslash \mathrm{PGL}_2(F_v)/T(F_v)$, then
$$\pm \int_{A(F)\times A(F)} f(h_1^{-1}\gamma h_2)\eta(h_2) = \int_{T(F)\times T(F)} f(h_1^{-1}\gamma h_2)$$
so $\mathbb{I}_0(u, f) = \mathbb{J}_0(u, f)$.
- If $\gamma \leftrightarrow \gamma_2 \in T(F_v)\backslash D^{\times}/T(F_v)$, then
$$\int_{A(F)\times A(F)} f(h_1^{-1}\gamma h_2)\eta(h_2)\, dh_2 = 0$$
so the extra double cosets do not contribute.

This shows that

$$\mathbb{I}_0(f) = \mathbb{J}_0(f).$$

8. Analytic RTF: Spectral Side (Ilya Khayutin)

8.1. Decomposition of the kernel. Recall that we defined

$$\mathbb{J}(f,S)=\int_{[A]\times[A]}\mathbb{K}_f(h_1,h_2)|h_1h_2|^s\eta(h_2)\,dh_1dh_2.$$

We have an action of $G(\mathbf{A})$, and hence $C_c^\infty(G(\mathbf{A}))$, on the space of automorphic functions $L_0^2([G])$. We are going to try to decompose the kernel functions into three parts:

$$\mathbb{K}_f(x_1,x_2)=\mathbb{K}_{f,\mathrm{cusp}}+\mathbb{K}_{f,\mathrm{sp}}+\mathbb{K}_{f,\mathrm{Eis}}$$

corresponding to cuspidal, "special", and Eisenstein. This idea is essentially due to Selberg.

8.1.1. *The cuspidal part.* We have

$$\mathbb{K}_{f,\mathrm{cusp}}=\sum_\pi \mathbb{K}_{f,\pi}$$

where

$$\mathbb{K}_{f,\pi}(x,y)=\sum_\phi \pi(f)\phi(x)\overline{\phi(y)}.$$

where ϕ runs over an orthonormal basis.

8.1.2. *The special part.* Using the determinant map, we have a map

$$\chi\colon [G]\to F^\times\backslash\mathbf{A}^\times/(\mathbf{A}^\times)^2\to\{\pm1\}.$$

Then

$$\mathbb{K}_{f,\mathrm{sp},\chi}(x,y)=\pi(f)\chi(x)\overline{\chi(y)}.$$

This is the same expression as for the cuspidal part, actually - it just looks much simpler because it is 1-dimensional.

8.1.3. *The Eisenstein part.* The Eisenstein part will be defined later.

8.1.4. *Goals:*

(1) Identify $f\in\mathcal{H}$ such that $\mathbb{K}_{f,\mathrm{Eis}}=0$.
(2) For such f, show that

$$\mathbb{J}_\pi(f)=\sum_\phi\frac{\mathcal{P}(\pi(f)\varphi)\overline{\mathcal{P}_\eta(\varphi)}}{\langle\varphi,\varphi\rangle}.$$

8.2. Satake isomorphism. Let $\mathcal{H}_G$ be the spherical Hecke algebra of G. By definition,

$$\mathcal{H}_G=\bigotimes_{x\in|X|}{}' \mathcal{H}_x.$$

For $A\subset G$ the split torus, we have $A\cong\mathbf{G}_m$. Then $\mathcal{H}_A=\bigotimes'\mathcal{H}_{A,x}$, and the local Hecke algebras are all isomorphic to

$$\mathcal{H}_{A,x}\cong\mathbf{Q}[F_x^\times/\mathcal{O}_x^\times]\cong\mathbf{Q}[t_x^{-1},t_x]$$

where $t_x=\mathbf{1}_{\varpi_x^{-1}\mathcal{O}_x^\times}$.

The Weyl group action is, in this normalization,

$$\iota_x(t_x) = q_x t_x^{-1}.$$

The *Satake homomorphism*

$$\mathrm{Sat}_x \colon \mathcal{H}_x \to \mathcal{H}_{A,x}$$

sends $h_x \mapsto t_x + q_x t_x^{-1}$. In fact Sat_x is an isomorphism onto the subgroup of Weyl invariants.

The local Satake homomorphisms extend to a global one:

$$\mathrm{Sat} \colon \mathcal{H} \to \mathcal{H}_A^{\iota}.$$

8.3. Eisenstein ideal.

8.3.1. *Definition of Eisenstein ideal.* We can identify $\mathbf{A}^\times/\mathcal{O}^\times \cong \mathrm{Div}(X)$. There is a map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. Now, $\mathcal{H}_A \cong \mathbf{Q}[\mathrm{Div}(X)] \to \mathbf{Q}[\mathrm{Pic}(X)]$.

The Weyl involution descends to ι_{Pic} on $\mathbf{Q}[\mathrm{Pic}(X)]$,

$$\mathbf{1}_{\mathcal{L}} \mapsto q^{\deg \mathcal{L}} \mathbf{1}_{\mathcal{L}^{-1}}.$$

Thus we have a map

$$a_{\mathrm{Eis}} \colon \mathcal{H} \xrightarrow{\mathrm{Sat}} \mathcal{H}_A^{\iota} \to \mathbf{Q}[\mathrm{Pic}(X)]^{\iota}.$$

Definition 8.1. We define the Eisenstein ideal to be $\mathcal{I}_{\mathrm{Eis}} := \ker a_{\mathrm{Eis}}$.

THEOREM 8.2. *For $f \in \mathcal{I}_{\mathrm{Eis}}$,*

$$\mathbb{K}_{f,\mathrm{Eis}}(x, y) = 0.$$

Before we can prove this, we need to say what $\mathbb{K}_{f,\mathrm{Eis}}$ is. And before that, we need to define the Eisenstein series.

8.3.2. *Eisenstein series.* The Eisenstein representations are induced from A, so we should first parametrize the representations of A, which are necessarily characters. Note that

$$A(\mathbf{A}) = \mathbf{A}^\times \cong \mathbf{A}^1 \times \alpha^{\mathbf{Z}}$$

for some choice of $\alpha \in \mathbf{A}$ with $|\alpha| = q$. For a character

$$\chi \colon F^\times \backslash \mathbf{A}^1 \to \mathbf{C}^\times$$

we can extend to a character

$$\chi_0 \colon F^\times \backslash \mathbf{A} \to \mathbf{C}^\times$$

by sending $\chi(\alpha) = 1$.

More generally, for any $u \in \mathbf{C}$ we get a character

$$\chi_u \colon F^\times \backslash \mathbf{A}^1 \to \mathbf{C}^\times$$

sending $\chi_u(a) = \chi_0(a)|a|^u$.

For $B = A \ltimes U$ and $U = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$, we define the modular character

$$\delta_B \colon B(\mathbf{A}) \to \mathbf{A}^\times$$

by

$$\delta_B \begin{pmatrix} a & b \\ & d \end{pmatrix} = a/d.$$

Finally, we define

$$\phi \colon B(\mathbf{A}) \to \mathbf{C}^\times$$

by $b \mapsto \chi_0(a/d)$.

Definition 8.3. We define the *induced (Eisenstein) representation* $V_{\chi,u}$ by

$$V_{\chi,u} = \{\varphi \in C^\infty(G(\mathbf{A})) \mid \varphi(bg) = \chi(b)|\delta_B(b)|^{1/2+u}\varphi(g) \forall b \in B(\mathbf{A})\}.$$

8.3.3. *The Eisenstein kernel.* Take φ_i orthogonal basis of V_χ. Then

$$\mathbb{K}_{f,\mathrm{Eis}}(x,y) = \sum_\chi \mathbb{K}_{f,\mathrm{Eis},\chi}(x,y)$$

and

$$\mathbb{K}_{f,\mathrm{Eis},\chi}(x,y) = \frac{\log q}{2\pi i} \int_{0+0i}^{0+2\pi i/\log q} \sum_{i,j} (\rho_\chi \varphi_j, \varphi_i) E(x, \varphi_i, u, \chi) \overline{E(y, \varphi_j, u, x)}\, du$$

8.3.4. *Proof of Theorem 8.2.* Any $f \in \mathcal{H}$ is unramified, so $f_{\chi,u}$ is periodic under $u \mapsto u + \frac{2\pi i}{\log q}$. If χ is unramified, then

$$\mathbb{K}_{f,\mathrm{Eis},\chi} = \frac{\log q}{2\pi i} \int_0^{\frac{2\pi i}{\log q}} (\rho_{\chi,u}(f)\mathbf{1}_K, \mathbf{1}_K) \ldots\, du.$$

It is a property of the Satake transform that

$$\mathrm{Tr}(\rho_{\chi,u}(f)) = \chi_{u+1/2}(\mathrm{Sat}(f)) \tag{8.1}$$

Inflate the character $\chi_{u+1/2} \colon F^\times \backslash \mathbf{A}^\times / \mathbf{O}^\times \to \mathbf{C}^\times$ to $\chi_{u+1/2} \colon \mathbf{A}^\times / \mathbf{O}^\times \to \mathbf{C}^\times$. Thus we get a character of $\mathcal{H}_A$. Then (8.1) reads

$$\mathrm{Tr}(\rho_{\chi,u}(f)) = \chi_{u+1/2}(a_{\mathrm{Eis}}(f)) = 0.$$

□

8.4. Relation to L-functions.

8.4.1. *Normalization of L-function.* We have

$$L(\pi_{F'}, s) = L(\pi, s) L(\pi \otimes \eta, s).$$

The functional equation reads

$$L(\pi_{F'}, s) = \epsilon(\pi, s) L(\pi_{F'}, 1-s)$$

where

$$\epsilon(\pi, s) = q^{-8(q-1)(s-1/2)}.$$

Definition 8.4. We define the *normalized L-function*

$$\mathcal{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)}.$$

We write

$$\mathbb{J}(f,s) = \sum_{\pi} \mathbb{J}_\pi(f,s)$$

where

$$\mathbb{J}_\pi(f,s) = \sum_{\varphi} \frac{\mathcal{P}(\pi(f)\varphi, s)\mathcal{P}_\eta(\overline{\varphi}, s)}{\langle \varphi, \varphi \rangle}$$

for $\varphi \in \pi^K$. Here for any character χ,

$$\mathcal{P}_\chi(\varphi, s) = \int_{[A]} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \chi(h)|h|^s \, dh$$

If we write

$$I(s,\varphi,\chi) = \int_{F^\times \backslash \mathbf{A}^\times} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \chi(h)|h|^{s-1/2} \, dh \tag{8.2}$$

and $\widetilde{\varphi}(g) = \varphi({}^t g^{-1})$, then we have a functional equation

$$I(s,\varphi,\chi) = I(1-s, \widetilde{\varphi}, \chi).$$

8.4.2. *Whittaker model.* To relate $\mathbb{J}_r(f)$ with derivatives of L-functions, we use "Whittaker models", which are automorphic variants of Fourier coefficients.

Let $\varphi \in V_\pi$. Then we get

$$\varphi \colon U(F)\backslash U(\mathbf{A}) \to \mathbf{C}$$

as follows. Note that $U \cong \mathbf{G}_a$, since

$$U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

Since $U(F)\backslash U(\mathbf{A}) = \mathbf{F}\backslash\mathbf{A}$ for a character $\psi \colon F\backslash\mathbf{A} \to \mathbf{C}^\times$ we can identify

$$\widehat{(F\backslash\mathbf{A})} \cong \{\psi(\gamma x) \mid x \in \mathbf{A}, \gamma \in F\}.$$

Then we can define a *Whittaker function*

$$W_{\varphi,\psi_\gamma}(g) = \int_{F\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g \right) \psi^{-1}(\gamma h) \, dn \tag{8.3}$$

Now we use a trick: by a change of variables, (8.3) is equal to

$$= \int_{F\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} \gamma^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right) \varphi\psi^{-1}(n) \, dn$$

Call this $W_{\varphi,\psi}(\gamma g)$. Then we have a "Fourier expansion"

$$\varphi = \sum_{\gamma \in F} W_{\varphi,\psi}(\gamma g).$$

Since f is cuspidal, the 0th Fourier coefficient vanishes. Also we have the identity

$$W_{\varphi,\psi}(ng) = \psi(n) W_\varphi(g).$$

In fact, this whole discussion applies locally, and we can define the local Whittaker function $W_{\varphi,\psi,x}$. The Whittaker function decomposes locally:

$$W_{\varphi,\psi} = \prod_{x \in |X|} W_{\varphi,\psi,x}.$$

If we write (8.2) as

$$\begin{aligned} \mathbb{I}(s,\varphi,\chi) &= \int_{F^\times \backslash \mathbf{A}^\times} \sum_{\gamma \in F^\times} W_\varphi(\gamma g) \begin{pmatrix} h & \\ & 1 \end{pmatrix} |h|^{s-1/2} \chi(h) dh \\ &= \int_{\mathbf{A}^\times} W_\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \ldots dh. \end{aligned}$$

Here we have used that since we are integrating over $F^\times \backslash \mathbf{A}^\times$ a sum over $F^\times$, we just get an integral over $\mathbf{A}^\times$ of something that decomposes locally as a product of local integrals. That's basically what an L-function is, so it is not surprising that the result is related to an L-function. However, there's an issue of test vectors. For almost all places, you get the right local factor. But at the finitely many bad places, you need to calculate a constant factor.

Part 3

Day Three

9. Geometric Interpretation of Orbital Integrals (Yihang Zhu)

9.1. Geometric expansion. This is is a talk about "geometrization of the geometric side of the analytic RTF". Yesterday we introduced $\mathbb{J}(f,s)$. This has a geometric expansion and a spectral expansion; we will focus on the geometric expansion:

$$\mathbb{J}(f,s) = \sum_{u \in \mathbf{P}^1(F) - \{1\}} \mathbb{J}(u,f,s).$$

9.1.1. *Orbital integrals.* The regular semisimple orbital integrals correspond to $u \neq 0, \infty$:

$$\mathbb{J}(u,f,s) = \mathbb{J}(\gamma,f,s) = \int_{A(\mathbf{A}) \times A(\mathbf{A})} f(h_1^{-1}\gamma h_2)|h_1h_2|^s \eta(h_2)\, dh_1 dh_2$$

where $\operatorname{inv}(\gamma) = u$. For these γ, there are no convergence issues because the conjugacy class is closed in $G(\mathbf{A})$, and f has compact support in $G(\mathbf{A})$. So no regularization is needed in this case.

We can restrict our attention to Hecke functions of the form $f = h_D$, for D an effective divisor.

9.1.2. *Observation.* We can compute the orbital integral on GL_2, as follows. If $\widetilde{\gamma}$ is a lift of γ, and $D = \sum n_x x$, we can define

$$\widetilde{h} := \bigotimes_x \widetilde{h}_{n_x,x} \in \mathcal{H}_x(\mathrm{GL}_2)$$

where

$$\widetilde{h}_{n_x,x} = \mathbf{1}_{\mathrm{Mat}_2(\mathcal{O}_x)_{\mathrm{val}(\det)=n_x}} \in \mathcal{H}_x(\mathrm{GL}_2).$$

Remark 9.1. The $\widetilde{h}_D$ is not a pullback of h_D; rather, it is a lift.

LEMMA 9.2. *We have*

$$\mathbb{J}(\gamma,h_D,s) = \int_{\Delta(Z(\mathbf{A}))\backslash(\widetilde{A}\times\widetilde{A})(\mathbf{A})} \widetilde{h}_D(h_1^{-1}\widetilde{\gamma}h_2)|\alpha(h_1)\alpha(h_2)|^s \eta(\alpha(h_2))\, dh_1 dh_2.$$

Here $\widetilde{A}$ is the diagonal torus in GL_2, and $\alpha\colon \begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto a/d$.

PROOF. Clear. □

9.1.3. *Geometrization.* Note that $\widetilde{h}_D(h_1^{-1}\widetilde{\gamma}h_2)$ only depends on the value of h_1 and h_2 in $\widetilde{A}(\mathbf{A})/\widetilde{A}(\mathbf{O})$. Since $\widetilde{A} \cong \mathbf{G}_m^2$, we have

$$\widetilde{A}(\mathbf{A})/\widetilde{A}(\mathbf{O}) \cong (\mathbf{G}_m(\mathbf{A})/\mathbf{G}_m(\mathbf{O}))^2 = (\operatorname{Div} X)^2.$$

The condition that $\widetilde{h}_D = 1$ defines a subset of $\Delta(\operatorname{Div} X)\backslash(\operatorname{Div} X)^4$. We'll first describe the subset in $(\operatorname{Div} X)^4$ before quotienting by center. It will be denoted

Definition 9.3. We define $\widetilde{\mathcal{N}}_{D,\widetilde{\gamma}} \subset (\operatorname{Div} X)^4$ to be the set of $(E_1, E_2, E_1', E_2') \in \operatorname{Div}(X)^4$ which are all effective, such that the rational map $\mathbf{O}^2 \xrightarrow{\widetilde{\gamma}} \mathbf{O}^2$ induces a holomorphic map

$$\begin{array}{ccc} \mathbf{O}^2 & \xrightarrow{\widetilde{\gamma}} & \mathbf{O}^2 \\ \uparrow & & \uparrow \\ \mathcal{O}(-E_1) \oplus \mathcal{O}(-E_2) & \xrightarrow{\phi_{\widetilde{\gamma}}} & \mathcal{O}(-E_1') \oplus \mathcal{O}(-E_2') \end{array}$$

such that $\operatorname{Div} \varphi_{\widetilde{\gamma}} = D$. Finally, we define

$$\mathcal{N}_{\widetilde{\gamma},D} := \widetilde{\mathcal{N}}_{\widetilde{\gamma},D} / \Delta(\operatorname{Div} X).$$

The upshot is

$$\mathbb{J}(\gamma, h_D, s) = \sum_{E_1, E_2, E_1', E_2' \in \mathcal{N}_{\widetilde{\gamma},D}} q^{-\deg(E_1 - E_2 + E_1' - E_2')s} \eta(E_1)\eta(E_2)$$

Since η is a quadratic character, we can rewrite this as

$$\mathbb{J}(\gamma, h_D, s) = \sum_{E_1, E_2, E_1', E_2' \in \mathcal{N}_{\widetilde{\gamma},D}} q^{-\deg(E_1 - E_2 + E_1' - E_2')s} \eta(E_1 - E_1')\eta(E_2 - E_1') \qquad (9.1)$$

Here $h_1 \leftrightarrow (E_1', E_2')$ and $h_2 \leftrightarrow (E_1, E_2)$.

The idea of geometrization is that the formula (9.1) should be expressible as the sum, over k-points of a scheme, of the value at that point of the function associated to a sheaf on the scheme. Through this we can relate the formula to Lefschetz cohomology.

9.2. The moduli spaces.

Definition 9.4. Let $\widehat{X}_d \to \operatorname{Pic}_X^d$ be the moduli space of sections, i.e.

$$\widehat{X}_d(S) = \left\{ (\mathcal{L}, s) \colon \begin{array}{c} \mathcal{L} = \text{degree } d \text{ line bundle on } X \times S \\ s \in H^0(X \times S, \mathcal{L}) \end{array} \right\}.$$

Let $X_d = \operatorname{Sym}^d X = X^d // S_d$. This is a scheme, with a natural embedding $X_d \hookrightarrow \widehat{X}_d$ sending

$$(t_1, \ldots, t_d) \mapsto (\mathcal{O}(t_1 + \ldots + t_d), 1).$$

This is an isomorphism onto the open subscheme of X^d where the section is not the zero section.

Note that $\widehat{X}_d \setminus X_d \xrightarrow{\sim} \operatorname{Pic}_X^d$. The composition

$$X_d \hookrightarrow \widehat{X}_d \to \operatorname{Pic}_X^d$$

is the Abel-Jacobi map.

Definition 9.5. For $d = \deg D$, let

$$\Sigma_d = \left\{ \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \mid d_{ij} \in \mathbf{Z}_{\geq 0}, d_{11} + d_{22} = d_{12} + d_{21} = d \right\}$$

Given $\underline{d} \in \Sigma_d$, we define the moduli space $\widetilde{\mathcal{N}}_{\underline{d}}$ classifying

- four line bundles K_1, K_2, K_1', K_2' such that
$$\deg K_i' - \deg K_j = d_{ij}.$$
- A map $\varphi\colon K_1 \oplus K_2 \to K_1' \oplus K_2'$, which we can write as
$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$
with $\varphi_{ij}\colon K_i \to K_j'$, satisfying some technical conditions. One example is if $d_{11} < d_{22}, d_{12} < d_{21}$
$$\varphi_{11} \neq 0, \varphi_{12} \neq 0 \tag{9.2}$$
and $\varphi_{21}, \varphi_{22}$ are not both 0.

There is an obvious action of Pic_X on $\widetilde{\mathcal{N}}_{\underline{d}}$, and we define
$$\mathcal{N}_{\underline{d}} = \widetilde{\mathcal{N}}_{\underline{d}} / \operatorname{Pic}_X .$$

Definition 9.6. We define the moduli space $\mathcal{A}_d$ classifying (Δ, a, b) where
- $\Delta \in \mathrm{Pic}_X^d$ and
- $a, b \in H^0(X, \Delta)$ are global sections not vanishing simultaneously.

Remark 9.7. The scheme $\mathcal{A}_d$ is covered by two pieces
$$X_d \times_{\mathrm{Pic}_X^d} \widehat{X}_d$$
and
$$\widehat{X}_d \times_{\mathrm{Pic}^d} X_d.$$
The morphism $X_d \to \mathrm{Pic}_X^d$ is representable, the fibers being vector spaces, hence $\mathcal{A}_d$ is a scheme.

Definition 9.8. We define a map
$$f_{\underline{d}}\colon \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$
sending
$$(K_1, K_2, K_1', K_2') \mapsto (K_1' \otimes K_2' \otimes K_1^\vee \otimes K_2^\vee, \varphi_{11} \otimes \varphi_{22}, \varphi_{12} \otimes \varphi_{21}).$$

PROPOSITION 9.9. *$\mathcal{N}_{\underline{d}}$ enjoys the following properties.*

(1) $\mathcal{N}_{\underline{d}}$ is a geometrically connected scheme over k.
(2) If $d \geq 4g - 3$, $\mathcal{N}_{\underline{d}}$ is smooth of dimension $2d - g + 1$.
(3) The morphism $f_{\underline{d}}$ is proper.

PROOF. Use the non-vanishing conditions to find a covering of $\mathcal{N}_{\underline{d}}$ analogous to the covering of $\mathcal{A}_d$ discussed above. This imnplies (1) + (3). (Properness reduces to properness of $X_{d_{ij}}$.) For (2), by Riemann-Roch the map $\widehat{X}_{d_{ij}} \to \mathrm{Pic}_X^{d_{ij}}$ is smooth of relative dimension $1 - g + d_{ij}$ if d_{ij} is large. If d is large then at least one of the relevant d_{ij} is large, and you use that one to run this argument. ☐

9.3. Geometrization of the analytic RTF. We now define a crucial local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$. By geometric class field theory there is a rank 1 local system on the Picard scheme of X corresponding to the quadratic character η. We first define a local system L_d on $\widehat{X}_d$ as the pullback of this local system via the map $\widehat{X}_d \to \mathrm{Pic}_X \to \mathrm{Pic}_X^{\mathrm{coarse}}$.

There is an open embedding

$$\mathcal{N}_{\underline{d}} \hookrightarrow (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \tag{9.3}$$

given by the universal φ_{ij}'s. Finally, we define the rank 1 local system $L_{\underline{d}}$ on $\mathcal{N}_{\underline{d}}$ to be the restriction of the local system

$$L_{d_{11}} \boxtimes \mathbf{Q}_\ell \boxtimes L_{d_{12}} \boxtimes \mathbf{Q}_\ell \text{ on } (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}_X^d} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$$

to $\mathcal{N}_{\underline{d}}$ via (9.3).

Definition 9.10. We define

$$\delta \colon \mathcal{A}_d \to \widehat{X}_d$$

to be the morphism sending

$$(\Delta, a, b) \mapsto (\Delta, a - b).$$

We also define

$$\mathcal{A}_D := \delta^{-1}(\mathcal{O}(D), 1) \cong \Gamma(X, \mathcal{O}_X(D)).$$

and the invariant map

$$\mathrm{inv}_D \colon \mathcal{A}_D(k) \to \mathbf{P}^1(F) - \{1\}$$

sending $a \mapsto 1 - a^{-1}$, viewing a as a rational function in F.

PROPOSITION 9.11. *Assume $u \neq 0, \infty$.*

(1) If $u \notin \mathrm{Im}\,\mathrm{inv}_D$, then $\mathbb{J}(u, h_D, s) = 0$.

(2) If $u = \mathrm{inv}_D(a)$ for $a \in \mathcal{A}_D(k)$, then

$$\mathbb{J}(u, h_D, s) = \sum_{\underline{d} \in \Sigma_d} q^{(2d_{12} - d)s} \,\mathrm{Tr}(\mathrm{Frob}_a, Rf_{\underline{d},*} L_{\underline{d}})_{\overline{a}}$$

PROOF. (2) We have a bijection

$$\mathcal{N}_{D,\widetilde{\gamma}} \xrightarrow{\sim} \mathcal{N}_a(k)$$

where $\mathcal{N}_a(k) = \coprod_{\underline{d} \in \Sigma_d} f_{\underline{d}}^{-1}(a)$ sending

$$(E_1, E_2, E_1', E_2') \mapsto (\mathcal{O}(-E_1), \mathcal{O}(-E_2), \mathcal{O}(-E_1'), \mathcal{O}(-E_2'), \varphi_{\widetilde{\gamma}}).$$

Use (9.1) and the definition of $L_{\underline{d}}$. □

10. Definition and properties of $\mathcal{M}_d$ (Jochen Heinloth)

10.1. Goal. Let X be a (smooth, projective, geometrically connected) curve over a finite field k, and $\nu\colon X' \to X$ a degree 2 étale cover.

Let $T := (\mathrm{Res}_{X'/X}\, \mathbf{G}_m)/\mathbf{G}_m$. We can embed T into "PGL_2" $= \mathrm{Aut}(\nu_* \mathcal{O}_{X'})/\mathcal{O}_X^*$.

Remark 10.1. We can also view T as the norm-1 subgroup of the Weil restriction:

$$1 \to T \to \mathrm{Res}_{X'/X}\, \mathbf{G}_m \xrightarrow{\mathrm{Nm}} \mathbf{G}_m \to 1$$

We can put these two definitions together to get an exact sequence

$$1 \to \mathbf{G}_m \to \mathrm{Res}_{X'/X}\, \mathbf{G}_m \xrightarrow{t \mapsto t(\sigma^* t)^{-1}} \mathrm{Res}_{X'/X}\, \mathbf{G}_m \xrightarrow{\mathrm{Nm}} \mathbf{G}_m \to 1. \tag{10.1}$$

The goal is to compute the intersection number

$$\langle \mathrm{Sht}_T, h_D * \mathrm{Sht}_T \rangle_{\mathrm{Sht}_G}$$

where $G = \mathrm{PGL}_2$.

10.2. The moduli space $\mathcal{M}_d$.

10.2.1. *Relation to intersection numbers.* Recall the shtuka space is

$$\begin{array}{ccc} \mathrm{Sht}_T & \longrightarrow & \mathrm{Hk}^\mu \\ \downarrow & & \downarrow \\ \mathrm{Bun}_T & \xrightarrow{\mathrm{Id},\mathrm{Frob}} & \mathrm{Bun}_T \times \mathrm{Bun}_T \end{array}$$

The idea is that the shtuka construction is complicated, so we should try to do the Bun_T intersection first. So we should try to compute "$\mathrm{Bun}_T \cap h_D * \mathrm{Bun}_T$".

Every time we want to do a PGL_2-computation we actually push it to a GL_2-computation. So as usual, set $\widetilde{T} = \mathrm{Res}_{X'/X}\, \mathbf{G}_m$ and $\widetilde{G} = \mathrm{GL}_2$. Note that $\mathrm{Bun}_{\widetilde{T}}$ can just be thought of as parametrizing line bundles on X, by the definition of Weil restriction.

So we want to compute the intersection

$$\begin{array}{ccc} ? & \longrightarrow & \mathrm{Hk}^d \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\widetilde{T}} \times \mathrm{Bun}_{\widetilde{T}} & \longrightarrow & \mathrm{Bun}_2 \times \mathrm{Bun}_2 \end{array}$$

$$(\mathcal{L}, \mathcal{L}') \longrightarrow (\nu_* \mathcal{L}, \nu_* \mathcal{L}')$$

In terms of the previous talks, $d = \deg D$. Recall that Hk^d parametrizes maps of vector bundles $\mathcal{E} \hookrightarrow \mathcal{E}'$ with quotient a torsion sheaf of degree d.

Definition 10.2. We define $\widetilde{\mathcal{M}}_d$ to be the fibered product

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_d & \longrightarrow & \mathrm{Hk}^d \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\widetilde{T}} \times \mathrm{Bun}_{\widetilde{T}} & \longrightarrow & \mathrm{Bun}_2 \times \mathrm{Bun}_2 \end{array}$$

Remark 10.3. What we are calling $\widetilde{\mathcal{M}}_d$ is called $\widetilde{\mathcal{M}}_d^{\heartsuit}$ in the paper, but we're going to omit it because we'll be working with it most often.

10.2.2. *The functor of points.* Let's try to compute the functor of points of $\widetilde{\mathcal{M}}_d$. View $\mathrm{Bun}_{\widetilde{T}}$ as $\mathrm{Pic}_{X'}$. The bottom horizontal map sends

$$(\mathcal{L}, \mathcal{L}') \mapsto (\nu_*\mathcal{L}, \nu_*\mathcal{L}').$$

The space $\widetilde{\mathcal{M}}_d$ (which is analogous to a "Hitchin space") parametrizes

$$\{(\mathcal{L}, \mathcal{L}', \psi \colon \nu_*\mathcal{L} \to \nu_*\mathcal{L}') \mid \deg \operatorname{coker} \psi = d\}.$$

Let's digest this. We need $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}_X^* \times \mathrm{Pic}_X^{*+d}$, and ψ is equivalent to (by adjunction)

$$\varphi \colon \nu^*\nu_*\mathcal{L} \to \mathcal{L}'$$

We have $\nu^*\nu_*\mathcal{L} = \mathcal{L} \oplus \sigma^*\mathcal{L}$. So φ is equivalent to

$$\nu^*\nu_*\mathcal{L} = \mathcal{L} \oplus \sigma^*\mathcal{L} \xrightarrow{\alpha,\beta} \mathcal{L}'$$

which amounts to the data of two maps

$$\mathcal{L} \xrightarrow{\alpha} \mathcal{L}'$$
$$\sigma^*\mathcal{L} \xrightarrow{\beta} \mathcal{L}'$$

10.2.3. *Compactification.* We now introduce a compactification of $\mathcal{M}_d$.

Definition 10.4. We define $\overline{\widetilde{\mathcal{M}}}_d$ to be the moduli space classifying

- $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}_{X'}^* \times \mathrm{Pic}_{X'}^{*+d}$,
- Maps

$$\alpha \colon \mathcal{L} \to \mathcal{L}'$$
$$\beta \colon \mathcal{L} \to \sigma^*\mathcal{L}'$$

  such that α, β are not both 0.

Remark 10.5. The bar on $\overline{\widetilde{\mathcal{M}}}_d$ is because we haven't imposed an injectivity condition on ψ. This space is just called $\widetilde{\mathcal{M}}_d$ in the paper.

There is an action of Pic_X on $\overline{\widetilde{\mathcal{M}}}_d$, and we finally define $\overline{\mathcal{M}}_d := \overline{\widetilde{\mathcal{M}}}_d / \mathrm{Pic}_X$.

Remark 10.6. Obviously $\overline{\widetilde{\mathcal{M}}}_d$ isn't of finite type, since it has infinitely many components. Since $\nu^* \colon \mathrm{Pic}_X^* \to \mathrm{Pic}_{X'}^{2*}$ hits "half" the components, $\overline{\mathcal{M}}_d$ is of finite type. In fact it has exactly 2 components.

The map

$$\psi\colon \nu_*\mathcal{L} \to \nu_*\mathcal{L}'$$

when pulled back to X' becomes

$$\nu^*\psi\colon \nu^*\nu_*\mathcal{L} \to \nu^*\nu_*\mathcal{L}'$$

and is given by

$$\nu^*\psi = \begin{pmatrix} \alpha & \sigma^*\beta \\ \beta & \sigma^*\alpha \end{pmatrix}$$

so $\det \nu^*\psi = \operatorname{Nm}\alpha - \operatorname{Nm}\beta$. We have

$$\widetilde{\mathcal{M}}_d = \overline{\widetilde{\mathcal{M}}_d} \setminus \{\operatorname{Nm}\alpha = \operatorname{Nm}\beta\},$$

and

$$M_d = [\widetilde{\mathcal{M}}_d / \operatorname{Pic}_X].$$

10.2.4. *The moduli space $\mathcal{A}_d$.*

Definition 10.7. We define the moduli space $\overline{\mathcal{A}_d}$ parametrizing

- $\Delta \in \operatorname{Pic}_X$,
- $a, b \in H^0(X, \Delta)$ where a and b never simultaneously vanish.

Thus

$$\overline{\mathcal{A}}_d = \widehat{X}_d \times_{\operatorname{Pic}_X} \widehat{X}_d - Z(\operatorname{Pic}_X^d)$$

where $Z(\operatorname{Pic}_X^d) = (\operatorname{Pic}_X \times_{\operatorname{Pic}_X} \widehat{X}_d \cup \widehat{X}_d \times_{\operatorname{Pic}_X} \operatorname{Pic}_X)$, embedding as the locus where a or b vanish.

Remark 10.8. Again we point out that the notation has changed from the paper and previous talks. What is being called $\overline{\mathcal{A}}_d$ used to be called $\mathcal{A}_d$, and what is being called $\mathcal{A}_d$ is called $\mathcal{A}_d^{\heartsuit}$ in the paper.

10.2.5. *The map f.* There is a map

$$f\colon \overline{\mathcal{M}}_d \to \overline{\mathcal{A}}_d = \widehat{X}_d \times_{\operatorname{Pic}_X} \widehat{X}_d - Z(\operatorname{Pic}_X^d)$$

sending

$$(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\operatorname{Nm}(\mathcal{L}') \otimes \operatorname{Nm}(\mathcal{L})^{-1}, a := \operatorname{Nm}(\alpha), b := \operatorname{Nm}(\beta))$$

So $\overline{\mathcal{M}}_d$ is the pre-image of $\mathcal{A}_d := \langle(\mathcal{L}, a, b)\colon a = b)$.

10.3. Properties of $\mathcal{M}_d$. We begin with an important alternate description of $\overline{\mathcal{M}}_d$. There is a map

$$\iota\colon \overline{\mathcal{M}}_d \to \widehat{X'}_d \times_{\operatorname{Pic}_X} \widehat{X'}_d.$$

Recall that $\widehat{X'}_d \times_{\operatorname{Pic}_X} \widehat{X'}_d$ parametrizes

- $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}_{X'}$,
- $\alpha \in H^0(X', \mathcal{L}), \beta \in H^0(X', \mathcal{L}')$ not both 0,
- $c\colon \operatorname{Nm}(\mathcal{L}) \cong \operatorname{Nm}(\mathcal{L}')\}$.

In these terms, ι sends

$$(\mathcal{L}, \mathcal{L}', \alpha, \beta) \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathcal{L}' \otimes \sigma^*\mathcal{L}^{-1}, \alpha, \beta, \text{canonical}).$$

Proposition 10.9. *Keeping the notation above, the map ι is an isomorphism onto the open subset where a, b don't both vanish.*

PROOF. We can ignore the sections; the interesting part is to keep track of the map on bundles, which looks like

$$(\mathrm{Pic}^*_{X'} \times \mathrm{Pic}^{*+d}_{X'})/\,\mathrm{Pic}_X \to (\mathrm{Pic}^d_{X'} \times_{\mathrm{Pic}_X} \mathrm{Pic}^d_{X'})$$

sending

$$(\mathcal{L}, \mathcal{L}') \mapsto (\mathcal{L}' \otimes \mathcal{L}^{-1}, \mathcal{L}' \otimes \sigma^*\mathcal{L}^{-1}). \tag{10.2}$$

We'll show that this is an isomorphism by describing the inverse.

By "looping" the sequences

$$1 \to T \to \mathrm{Res}_{F'/F}\,\mathbf{G}_m \xrightarrow{\mathrm{Nm}} \mathbf{G}_m \to 1$$

and

$$1 \to \mathbf{G}_m \to \mathrm{Res}_{F'/F}\,\mathbf{G}_m \to T \to 1$$

we obtain exact sequences of groups stacks

$$1 \to \mathrm{Bun}_T \to \mathrm{Pic}_{X'} \xrightarrow{\mathcal{L}\otimes\sigma^*\mathcal{L}} \mathrm{Pic}_X \to 1 \tag{10.3}$$

and

$$1 \to \mathrm{Pic}_X \to \mathrm{Pic}_{X'} \xrightarrow{\mathcal{L}\otimes\sigma^*\mathcal{L}^{-1}} \mathrm{Bun}_T \to 1. \tag{10.4}$$

Suppose we have a point $(\mathcal{M}, \mathcal{M}', c\colon \mathrm{Nm}(\mathcal{M}) \cong \mathrm{Nm}(\mathcal{M}'))$ on the right hand side of (10.2). Then (10.3) tells us that since $\mathcal{M}$ and $\mathcal{M}'$ have the same norm, they differ by a T-bundle. By (10.4), there exist $\mathcal{L}, \mathcal{L}'$ such that $\mathcal{M} = \mathcal{L}' \otimes \mathcal{L}^{-1}$ and $\mathcal{M}' = \mathcal{L}' \otimes \sigma^*\mathcal{L}^{-1}$, and this choice is unique up to multiplication by an element of Pic_X.

□

PROPOSITION 10.10. *If* $\mathrm{char} k \neq 2$ *then* $\overline{\mathcal{M}}_d$ *is a Deligne-Mumford stack.*

PROOF. $\overline{\mathcal{M}}_d$ is covered by the open stacks $X'_d \times_{\mathrm{Pic}_X} \widehat{X'}_d$ and $\widehat{X'}_d \times_{\mathrm{Pic}_X} X'_d$, describing when the sections α and β don't vanish, respectively. By symmetry, it suffices to show that one of these is Deligne-Mumford. Consider the cartesian diagram

$$\begin{array}{ccccc}
X'_d \times_{\mathrm{Pic}_X} \widehat{X'}_d & \longrightarrow & & & \widehat{X'}_d \\
 & & & & \downarrow \pi \\
\downarrow & & & & \mathrm{Pic}_{X'} \\
 & & & & \downarrow \mathrm{Nm} \\
X'_d & \longrightarrow & \mathrm{Pic}_{X'} & \xrightarrow{\mathrm{Nm}} & \mathrm{Pic}_X
\end{array}$$

The map π is representable, since the fiber over $\mathcal{L}$ is $H^0(X', \mathcal{L})$.

The map Nm is a torsor under $\ker(\mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}_X)$, which is the Prym variety $\mathrm{Prym}(X'/X)/\mu_2$, since μ_2 is precisely the group of automorphisms of the norm map on line bundles.

This implies that the fibered product is Deligne-Mumford. □

Remark 10.11. Alternatively, we can establish the Deligne-Mumford property by showing that the automorphisms groups are étale, i.e. have vanishing tangent space. We can compute the tangent space to the map

$$\mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}_X$$

as follows. The map on tangent spaces is

$$T_{\mathrm{Nm}} = H^1(X', \mathcal{O}_{X'}) = H^1(X, \nu_* \mathcal{O}_X) \xrightarrow{\text{trace}} H^1(X, \mathcal{O}_X)$$

and the infinitesimal deformations of this map is the kernel of $H^0(X', \mathcal{O}_{X'}) \xrightarrow{\text{trace}} H^0(X, \mathcal{O}_X)$, which is just multiplication by 2.

Corollary 10.12. *$\overline{\mathcal{M}}_d$ is smooth if $d > 2g' - 1$.*

Proof. The map $\widehat{X'}_d \to \mathrm{Pic}^d_{X'}$ is a vector bundle if $d > 2g' - 1$ by Riemann-Roch, and $\widehat{X'}_d \times_{\mathrm{Pic}_X} X'_d = \overline{\mathcal{M}}_d$. □

Proposition 10.13. *The morphism $f \colon \overline{\mathcal{M}}_d \to \overline{\mathcal{A}}_d$ is proper. Therefore its restriction to $f \colon \mathcal{M}_d \to \mathcal{A}_d$ is also proper.*

Proof. Recall that f is the map

$$\widehat{X'}_d \times_{\mathrm{Pic}_X} \widehat{X'}_d - (\text{both } 0) \xrightarrow{\mathrm{Nm}} \widehat{X}_d \times_{\mathrm{Pic}_X} \widehat{X}_d - (\text{both } 0)$$

where (both 0) refers to the substack where both global sections vanish. So it suffices to show that the norm map $\widehat{X'}_d \to \widehat{X}_d$ is proper. Note that this is obvious on fibers, since both $X'_d \xrightarrow{\nu_d} X_d$ and $\mathrm{Pic}_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}_X$ are proper, the first map being even finite and the second map having the Prym variety as its kernel.

To give a formal proof, we compactify. If we define

$$\overline{\widehat{X}_d} = \{\mathcal{L} \in \mathrm{Pic}^d_X, s \in \mathbf{P}H^0(X, \mathcal{L} \oplus \mathcal{O}_X)\}$$

then the natural map $\overline{\widehat{X}_d} \to \mathrm{Pic}_X$ is obviously proper, so $\overline{\widehat{X}_d}$ is proper. We have an open embedding $\widehat{X}_d \hookrightarrow \overline{\widehat{X}_d}$ sending $(\mathcal{L}, s) \mapsto (\mathcal{L}, [s : 1])$. Note that

$$\overline{\widehat{X'}_d} = [(\widehat{X'}_d \times \mathbf{A}^1 - \text{both } 0)/\mathbf{G}_m]$$
$$\overline{\widehat{X}_d} = [(\widehat{X}_d \times \mathbf{A}^1 - \text{both } 0)/\mathbf{G}_m]$$

where $\widehat{X}_d \times \mathbf{A}^1$ parametrizes $(\mathcal{L}, s \in H^0(\mathcal{L}), f \in H^0(\mathcal{O}_X))$, and similarly for X'. The substack (both 0) refers to the locus where $s = f = 0$. Then we have a cartesian diagram

$$\begin{array}{ccc} \widehat{X'}_d & \hookrightarrow & \overline{\widehat{X'}_d} \\ \downarrow & & \downarrow \\ \widehat{X}_d & \longrightarrow & \overline{\widehat{X}_d} \end{array}$$

and $\overline{\widehat{X'}_d} \to \overline{\widehat{X}_d}$ is proper, so $\widehat{X'}_d \to \widehat{X}_d$ is proper. □

Part 4

Day Four

11. Intersection theory on stacks
(Michael Rapoport)

The aim is to introduce intersection theory on stacks which are only locally of finite type, like the moduli stack of shtukas. Fortunately, we only need the **Q**-theory, which makes things easier.

11.1. Definition of $\mathrm{Ch}(X)_{\mathbf{Q}}$.

11.1.1. *Chow groups for finite type.*

Definition 11.1. Let X/k be a DM stack, finite type over k. Then we define

$$\mathrm{Ch}_*(X)_{\mathbf{Q}} = Z_*(X)_{\mathbf{Q}}/\partial W_*(X)_{\mathbf{Q}}$$

where

- $Z_*(X)_{\mathbf{Q}} = \bigoplus_V \mathbf{Q}$ with V running over irreducible reduced closed substacks of dimension $*$, and
- $W_*(X)_{\mathbf{Q}} = \bigoplus_W k(W)^* \otimes_{\mathbf{Z}} \mathbf{Q}$ with the same index set, and $k(W)$ viewed as a rational function to $\mathbf{A}^1_k$; the inclusion into $Z_*(X)$ is by the "boundary" as in the usual case for schemes.

11.1.2. *Generalization to locally finite type.* When X is *locally* finite type over k, we replace $Z_*(X)_{\mathbf{Q}}$ with $Z_{c,*}(X)_{\mathbf{Q}}$ and $W_{c,*}(X)_{\mathbf{Q}}$, where the subscript c indicates that we only take substacks *proper* over Spec k. We have

$$\mathrm{Ch}_c(X) = \varinjlim_{Y \text{ f.t. } \subset X} \mathrm{Ch}_*(Y)_{\mathbf{Q}} = \varinjlim_{U \text{ open } \subset X} \mathrm{Ch}_{*,c}(U)_{\mathbf{Q}}.$$

11.1.3. *Degree map.* We want to define a map

$$\deg\colon \mathrm{Ch}_{c,0}(X)_{\mathbf{Q}} \to \mathbf{Q}.$$

Since we are working with stacks, we need to account for stabilizers.

Definition 11.2. Let $x \in X$ be represented by a geometric point $\overline{x}\colon \operatorname{Spec} k^s \to X$. We define

$$\deg x := [(k^{\mathrm{sep}})^{\Gamma_x} : k] \cdot \frac{1}{|\operatorname{Aut}(x^s)|}.$$

11.1.4. *Intersection pairing.* Now let X be smooth, locally of finite type, and pure dimension n. Then we have an intersection product

$$\mathrm{Ch}_{c,i}(X)_{\mathbf{Q}} \times \mathrm{Ch}_{c,j}(X)_{\mathbf{Q}} \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}} \qquad (11.1)$$

defined as follows. Let Y_1, Y_2 be closed substacks of X, which are proper over k. Then (11.1) is the colimit of the finite-type intersection products

$$\mathrm{Ch}_i(Y_1)_{\mathbf{Q}} \times \mathrm{Ch}_j(Y_2)_{\mathbf{Q}} \to \mathrm{Ch}_{i+j-n}(Y_1 \cap Y_2) \to \mathrm{Ch}_{c,i+j-n}(X)_{\mathbf{Q}}.$$

The first map is subtle to define: it is the *refined intersection product*

$$(\zeta_1, \zeta_2) \mapsto X \times_{(X,X)} (\zeta_1 \times \zeta_2).$$

What does this mean? It is a special case of the *refined Gysin morphism*. Start with the fibered product diagram

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ i\ } & Y \end{array}$$

where i is a regular embedding of codimension e. Then we get a *refined Gysin morphism*

$$i^!\colon \operatorname{Ch}_i(V)_{\mathbf{Q}} \to \operatorname{Ch}_{i+e}(W)_{\mathbf{Q}}$$

and we define

$$X \times_{(X,X)} (\zeta_1 \times \zeta_2) := \Delta^!(\zeta_1 \times \zeta_2).$$

Thus we have finally constructed the product

$$\operatorname{Ch}_{c,i}(X)_{\mathbf{Q}} \times \operatorname{Ch}_{c,j}(X)_{\mathbf{Q}} \to \operatorname{Ch}_{c,i+j-n}(X)_{\mathbf{Q}}$$

Then composing with the degree map, we get an intersection pairing

$$\langle , \rangle_X\colon \operatorname{Ch}_{c,j}(X)_{\mathbf{Q}} \times \operatorname{Ch}_{c,n-j}(X)_{\mathbf{Q}} \to \mathbf{Q}.$$

Remark 11.3. (i) We have a cycle class map

$$\operatorname{cl}_X\colon \operatorname{Ch}_{c,j}(X)_{\mathbf{Q}} \to H_c^{2n-2j}(X \otimes_k \overline{k}, \mathbf{Q}_\ell(n-j))$$

and the intersection product is compatible with cup product.

(ii) Consider

$$_c\operatorname{Ch}_n(X \times X)_{\mathbf{Q}} = \varinjlim_{Z \subset X \times X} \operatorname{Ch}_*(Z)_{\mathbf{Q}}$$

such that $\operatorname{pr}_1|_Z$ is proper. This is a $\mathbf{Q}$-algebra. It acts on each $\operatorname{Ch}_{c,j}(X)_{\mathbf{Q}}$ via

$$(\xi, \zeta) = \operatorname{pr}_{2*}(\xi \cdot_{(X \times X)} \operatorname{pr}_1^* \zeta).$$

Now that we have a definition, the problem is that we can't really calculate. So instead we pass to K groups.

11.2. Relation to K-theory. For technical reasons, we need to relate the Chow groups to K-theory. First we recall K-theory of schemes of finite type over k. Let $K_0'(X)$ be the Grothendieck group of the abelian category of coherent $\mathcal{O}_X$-modules. Let $K_0'(X)_{\mathbf{Q}}$ be the rationalization.

11.2.1. *The naïve filtration.* We have a filtration

$$K_0'(X)^{\text{naive}}_{\mathbf{Q},\le m} = \operatorname{Im}\left(K_0(\operatorname{Coh}(X)_{\le m})_{\mathbf{Q}} \to K_0'(X)_{\mathbf{Q}}\right)$$

where $\operatorname{Coh}(X)_{\le m}$ is the subcategory of coherent sheaves with support of dimension at most m.

We have a natural graded map

$$\phi_X\colon \operatorname{Ch}_*(X)_{\mathbf{Q}} \to \operatorname{Gr}_*^{\text{naive}}(K_0'(X))_{\mathbf{Q}}$$

sending

$$[V]\colon \mapsto \text{class of } \mathcal{O}_V.$$

This is an isomorphism: we have a commutative diagram

$$\begin{array}{ccc} K_0(\mathrm{Coh}(X)_{\le m})_{\mathbf{Q}} & \longrightarrow & \mathrm{Gr}_m^{\mathrm{naive}}(X)_{\mathbf{Q}} \\ \big\downarrow{\scriptstyle \mathrm{supp}} & & \big\downarrow{\scriptstyle \psi_X} \\ Z_m(X)_{\mathbf{Q}} & \longrightarrow & \mathrm{Ch}_m(X)_{\mathbf{Q}} \end{array}$$

where the map supp sends $\mathcal{F} \mapsto \sum_{\dim V = m} \mu_V(\mathcal{F}) \cdot [V]$.

This discussion was for schemes. For stacks, all definitions extend but it's not clear if the map

$$\begin{array}{ccc} K_0(\mathrm{Coh}(X)_{\le m})_{\mathbf{Q}} & & \\ \big\downarrow & & \\ Z_m(X)_{\mathbf{Q}} & \longrightarrow & \mathrm{Ch}_m(X)_{\mathbf{Q}} \end{array}$$

factors through $K_0'(X)^{\mathrm{naive}}_{\le m}$.

11.2.2. *The not-so-naïve filtration.* This problem is solved in the paper under the assumption

> (*) there exists a finite flat presentation $U \to X$ where U is an algebraic space of finite type over k.

Define $K_0'(X)_{\mathbf{Q},\le m}$ to be the set of $\alpha \in K_0'(X)_{\mathbf{Q}}$ such that there exists a finite presentation $\pi\colon U \to X$ with $\pi^*(\alpha) \in K_0'(U)^{\mathrm{naive}}_{\mathbf{Q},\le m}$.

Example 11.4. It may happen that $K_0'(X)_{\mathbf{Q},\le m}$ is non-zero for $m < 0$. (Of course, this doesn't happen for the naïve filtration.) Let $X = [*/G]$. Then $K_0'(X)_{\mathbf{Q}} = \mathrm{Rep}_{\mathbf{Q}}(G)$, and $K_0'(X)_{\mathbf{Q},\le -1}$ is the augmentation ideal (in particular, non-zero). Indeed, when we pull back via the cover $* \to [*/G]$, anything in the augmentation ideal becomes 0 in $K_0(*)$.

In general, we have an inclusion $K_0'(X)^{\mathrm{naive}}_{\mathbf{Q},\le m} \subset K_0'(X)_{\mathbf{Q},\le m}$, which is an equality if X is an algebraic space.

The filtration just defined enjoys expected functoriality properties: compatibility with flat pullback and under proper pushforward.

Let X be a DM stack satisfying (*). Then there is a homomorphism

$$\psi_X\colon \mathrm{Gr}_m(K_0'(X)_{\mathbf{Q}}) \to \mathrm{Ch}_*(X)_{\mathbf{Q}}$$

induced by a commutative diagram

$$\begin{array}{ccccc} K_0(\mathrm{Coh}(X)_{\le m})_{\mathbf{Q}} & \longrightarrow & K_0'(X)^{\mathrm{naive}}_{\mathbf{Q},\le m} & \longrightarrow & K_0'(X)_{\mathbf{Q},\le m} \\ \big\downarrow & & & & \big\downarrow{\scriptstyle \psi_X} \\ Z_m(X)_{\mathbf{Q}} & & \longrightarrow & & \mathrm{Ch}_m(X)_{\mathbf{Q}} \end{array}$$

We now come to a key technical point, which the compatibility of K-theory with the refined Gysin homomorphism. We will describe two situations in which we can deduce a good compatibility relationship.

11.2.3. *(A): Compatibility with the refined Gysin homomorphism.* Consider the cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & Y \end{array}$$

Assumptions (A).

- Assume that X' satisfies (*).
- Assume that f is the composition of a regular embedding of codimension e and smooth morphism of relative dimension $e-d$. (Note that this is automatic if X and Y are smooth.)

We have two maps: the refined Gysin morphism

$$f^!\colon \operatorname{Ch}_*(Y')_{\mathbf{Q}} \to \operatorname{Ch}_{*-d}(X')_{\mathbf{Q}}$$

and the pullback on K-theory

$$f^*\colon K_0'(Y')_{\mathbf{Q}} \to K_0'(X')_{\mathbf{Q}}$$

sending $\mathcal{F} \mapsto (f')^{-1}(\mathcal{F}) \overset{\mathbf{L}}{\otimes}_{(f\circ g)^{-1}\mathcal{O}_Y} (f')^{-1}(\mathcal{O}_Y)$.

Proposition 11.5. *Under the assumptions (A):*

(1) The pullback f^* sends $K_0'(Y')^{\text{naive}}_{\mathbf{Q},\le m}$ to $K_0'(X')_{\mathbf{Q},\le m}$ and hence induces a map

$$\operatorname{Gr}_m^{\text{naive}} f^*\colon \operatorname{Gr}_m^{\text{naive}} K_0'(Y')_{\mathbf{Q}} \to \operatorname{Gr}_{m-d} K_0'(X')_{\mathbf{Q}}.$$

(2) We have a commutative diagram

$$\begin{array}{ccccc} & & \operatorname{Gr}_m^{\text{naive}} K_0'(Y')_{\mathbf{Q}} & & \\ & \nearrow & & \searrow{\scriptstyle \operatorname{Gr}^{\text{naive}}(f')^*} & \\ K_0(\operatorname{Coh}(X)_{\le m})_{\mathbf{Q}} & & & & \operatorname{Gr}_{m-d}(X')_{\mathbf{Q}} \\ \downarrow{\scriptstyle \text{supp}} & & & & \downarrow \\ Z_m(Y')_{\mathbf{Q}} & & \longrightarrow & & \operatorname{Ch}_{m-d}(X')_{\mathbf{Q}} \end{array}$$

If we also assume that Y' satisfies $*$, then we can fill this in to

$$\begin{array}{ccccc} & & \operatorname{Gr}_m^{\text{naive}} K_0'(Y')_{\mathbf{Q}} & & \\ & \nearrow & \downarrow & \searrow{\scriptstyle \operatorname{Gr}^{\text{naive}}(f')^*} & \\ K_0(\operatorname{Coh}(X)_{\le m})_{\mathbf{Q}} & \dashrightarrow & \operatorname{Gr}_m K_0'(Y')_{\mathbf{Q}}s & \dashrightarrow & \operatorname{Gr}_{m-d}(X')_{\mathbf{Q}} \\ \downarrow{\scriptstyle \text{supp}} & & \downarrow & & \downarrow \\ Z_m(Y')_{\mathbf{Q}} & \longrightarrow & \operatorname{Ch}_m(Y')_{\mathbf{Q}} & \longrightarrow & \operatorname{Ch}_{m-d}(X')_{\mathbf{Q}} \end{array}$$

11.2.4. *(B): Compatibility with Gysin map.* Again consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & Y \end{array}$$

Assumptions (B).

- Assume h is representable.
- Assume that the normal cone of f is a vector bundle of constant virtual dimension. (We will apply this to $(\mathrm{Id}, \mathrm{Frob})\colon X \to X \times X$, where X is smooth, so this is certainly satisfied.)
- Assume that there exists a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

  where U and V are smooth surjective maps from schemes of finite type and i is a regular embedding.

Write $\dim Y' = n$ and $\dim X' = n - d$.

Proposition 11.6. *Under the assumptions (B), the following diagram is commutative:*

$$\begin{array}{ccc} K_0'(Y')_{\mathbf{Q}} & \xrightarrow{\qquad f^* \qquad} & K_0'(X')_{\mathbf{Q}} \\ \downarrow{\scriptstyle supp} & & \downarrow \\ \mathrm{Ch}_n(Y')_{\mathbf{Q}} = Z_n(Y')_{\mathbf{Q}} & \longrightarrow & Z_{n-d}(X')_{\mathbf{Q}} = \mathrm{Ch}_{n-d}(X')_{\mathbf{Q}}. \end{array}$$

11.3. The octahedron lemma. Consider a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longleftarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & S & \longleftarrow & V \\ \uparrow & & \uparrow & & \uparrow \\ C & \longrightarrow & Y & \longleftarrow & D \end{array}$$

Let N be the fiber product as in

$$\begin{array}{ccc} N & \longrightarrow & A \times B \times C \times D \\ \downarrow & & \downarrow \\ X \times_S Y \times_S U \times_S V & \longrightarrow & (X \times_S U) \times (X \times_S Y) \times (Y \times_S U) \times (X \times_S V) \end{array}$$

Lemma 11.7. *There are canonical isomorphisms*

$$(C \times_Y D) \times_{U \times_S V} (A \times_X B) \cong N \cong (C \times_U A) \times_{Y \times_S X} (D \times_V B).$$

Theorem 11.8. *Assume everybody is smooth, except* B *(the "bad" object) of dimension* $d_A, d_B, \ldots$*. Also assume that the fiber products (on the left)* $C \times_Y D$*,* $U \times_S V$*,* $C \times_U A, Y \times_S X$ *have the expected dimension. Further assume that each of the fiber diagrams*

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

and

$$\begin{array}{ccc} D \times_V B & \longrightarrow & B \\ \downarrow & & \downarrow \\ D & \longrightarrow & V \end{array}$$

satisfy the compatibility conditions (A) or (B). Finally assume that both fiber diagrams

$$\begin{array}{ccc} N & \longrightarrow & A \times_X B \\ \downarrow & & \downarrow \\ C \times_Y D & \longrightarrow & U \times_S V \end{array}$$

and

$$\begin{array}{ccc} N & \longrightarrow & D \times_V B \\ \downarrow & & \downarrow \\ C \times_U A & \longrightarrow & Y \times_S X \end{array}$$

satisfies the compatibility condition (A). Let $n = \dim N$*. For the diagram*

$$\begin{array}{ccccc} N & \xrightarrow{\alpha} & D \times_V B & \xrightarrow{d} & B \\ \| & & & & \| \\ N & \xrightarrow{\delta} & A \times_X B & \xrightarrow{a} & B \end{array}$$

we have $\delta^! a^! [B] = d^! \alpha^! [B]$*.*

Roughly speaking, the proof proceeds by using the relation to K-theory, and lifting the statement to the level of derived stacks.

12. LTF for Cohomological Correspondences (Davesh Maulik)

12.1. Motivation. We want to compute an intersection number

$$\mathbb{I}_r(h_D) = \langle \mathrm{Sht}'_T, \mathrm{Sht}'_T, \rangle_{\mathrm{Sht}'_G}.$$

The shtuka involves some sort of Frobenius.

$$\begin{array}{ccc} \mathrm{Sht} & \longrightarrow & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Gamma := \mathrm{Id} \times \mathrm{Frob}} & M \times M \end{array}$$

We'll rewrite this intersection in another order, so that at the end the answer will be presented as a refined Gysin pullback via Frobenius, which we can then compute in terms of a cohomological trace.

Recall the "usual" Grothendieck-Lefschetz trace formula.

THEOREM 12.1. *Let X_0 be a variety over $\mathbf{F}_q$, and $X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}_q}$.*

$$\sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob} \mid H^i_c(X, \mathcal{E})) = \sum_{x \in X_0(\mathbf{F}_q)} \operatorname{Tr}(\mathrm{Frob}_x \mid \mathcal{E}_{\overline{x}}).$$

Outline of

(1) Cohomological correspondences.
(2) Trace formula.
(3) Application.

12.2. Cohomological correspondences.

12.2.1. *Setup.* To convey the idea, we're just going to work with schemes. Let $k = \overline{k}$ be an algebraically closed field, and X a scheme of finite type over k. Let $D(X) := D^b_c(X, \mathbf{Q}_\ell)$. If $f \colon X \to Y$ is a map then we have functors

$$f_*, f_! \colon D^b(X) \to D^b(Y)$$

and

$$f^*, f^! \colon D^b(Y) \to D^b(X)$$

and adjunctions

$$\mathrm{Id} \to f_* f^*$$
$$f_! f^! \to \mathrm{Id}\,.$$

12.2.2. *Borel-Moore homology.* Let $\pi \colon X \to \operatorname{Spec} k$, then $K_X = \pi^! \mathbf{Q}_\ell$ is the dualizing sheaf.

Example 12.2. If X is smooth of dimension n, then $K_X = \mathbf{Q}_\ell[2n](n)$.

Definition 12.3. We define the *Borel-Moore homology*

$$H^{BM}_d(X) := H^{-d}(K_X).$$

If $f \colon X \to Y$ is proper, then we have a trace map

$$\operatorname{Tr} \colon H^{BM}_0(X) \to H^{BM}_0(Y)$$

via

$$f_! K_X = f_! f^! K_Y \to K_Y.$$

using that $K_X = f^! K_Y$.

Think of H_0^{BM} as being a receptacle for 0-cycles (it is the target of cycle class map from Ch_0) , and this as being pushforward of cycles. In particular, if X is proper over k then the pushforward for the structure map $X \to \operatorname{Spec} k$ is the degree map

$$H_0^{BM}(X) \xrightarrow{\deg} \mathbf{Q}_\ell.$$

12.2.3. *Cohomological correspondences.*

Definition 12.4. Given X_1, X_2 a *correspondence* between X_1, X_2 is a diagram

$$\begin{array}{ccc} & C & \\ {\scriptstyle c_1}\swarrow & & \searrow{\scriptstyle c_2} \\ X_1 & & X_2 \end{array}$$

Given $\mathcal{F}_i \in D(X_i)$, a *cohomological correspondence* is an element

$$u \in \operatorname{Hom}_C(c_1^* \mathcal{F}_1, c_2^! \mathcal{F}_2) = \operatorname{Hom}_{X_2}(c_{2!} c_1^* \mathcal{F}_1, \mathcal{F}_2).$$

Example 12.5. For a morphism $f\colon X \to Y$, we have $f^* \mathbf{Q}_\ell = \mathbf{Q}_\ell = \mathrm{Id}^! \mathbf{Q}_\ell$. This gives a cohomological correspondence, which is admittedly trivial.

Example 12.6. Let X_2 be smooth of dimension n. Then $K_{X_2} = \mathbf{Q}_\ell[2n](n)$, so $c_2^! \mathbf{Q}_\ell = K_C[-2n](-n)$. So the cohomological correspondences between $\mathbf{Q}_\ell$ and $\mathbf{Q}_\ell$ are maps

$$\mathbf{Q}_\ell \to K_C[-2n](-n) = H_{2n}^{BM}(c)(-n).$$

We get a Borel-Moore homology class from any cycle, which gives a map

$$\mathrm{Ch}(C) \to \mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell).$$

12.2.4. *Maps on cohomology.* If c_1 is proper, then from a cohomological correspondence u we can define a map

$$R\Gamma_c(u)\colon R\Gamma_c(X_1, \mathcal{F}_1) \to R\Gamma_c(X_2, \mathcal{F}_2).$$

Indeed, we have a map of sheaves

$$\mathcal{F}_1 \to c_{1*} c_1^* \mathcal{F}_1 = c_{1!} c_1^* \mathcal{F}_1$$

(using that c_1 is proper in the second equality) which induces on cohomology

$$R\Gamma_c(X_1, \mathcal{F}_1) \to R\Gamma_c(C, c_1^* \mathcal{F}_1) \xrightarrow{u} R\Gamma_c(C, c_2^! \mathcal{F}_2) = R\Gamma_c(X_2, c_{2!} c_2^! \mathcal{F}_2) \to R\Gamma_c(X_2, \mathcal{F}_2).$$

More generally, given a diagram of correspondences

$$\begin{array}{ccccc} X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 \end{array}$$

if (a) f and f_1 are proper, and (b) c_1 and d_1 are proper then we can define a pushforward

$$[f]_!\colon \mathrm{Corr}_C(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Corr}_D(f_{1!}\mathcal{F}_1, f_{2!}\mathcal{F}_2).$$

This generalizes the previous construction, which is the special case with $Y_1 = D = Y_2 = \mathrm{Spec}\ k$ sending a correspondence $u \mapsto R\Gamma_C(u) \in \mathrm{Corr}_{\mathrm{pt}}(R\Gamma_c(\mathcal{F}_1), R\Gamma_c(\mathcal{F}_2))$.

12.3. Trace formula.

12.3.1. *Self correspondences.* Suppose we have a correspondence between X and itself:

$$\begin{array}{ccc} & C & \\ {}^{c_1}\swarrow & & \searrow^{c_2} \\ X & & X \end{array}$$

If c_1 is proper, then we have an endomorphism of $R\Gamma_c(u)$ on $R\Gamma_c(X, F)$. The fundamental question is: *what is its trace*?

In a relative situation, if we have a map of correspondences

$$\begin{array}{ccccc} & & C & & \\ & {}^{c_1}\swarrow & \downarrow & \searrow^{c_2} & \\ X & & \downarrow & & X \\ \downarrow{\scriptstyle f} & & S & & \downarrow{\scriptstyle f} \\ & \swarrow & & \searrow & \\ S & & & & S \end{array}$$

then $[f]_!(u)$ is an endomorpism of $f_!\mathcal{F}$.

12.3.2. *The trace.* Consider the cartesian square

$$\begin{array}{ccc} \mathrm{Fix}(c) & \xrightarrow{\Delta'} & C \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c=c_1\times c_2} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Definition 12.7. We define a *trace map*

$$\mathrm{R}\mathscr{H}om_C(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \to \Delta'_* K_{\mathrm{Fix}(c)}. \tag{12.1}$$

as follows. We have

$$\mathrm{R}\mathscr{H}om_C(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \cong c^!(\mathbf{D}(\mathcal{F}) \boxtimes \mathcal{F}) \to c^!(\Delta_* K_X)$$

where $\mathbf{D}(-) = \mathrm{R}\mathscr{H}om(-, K_C)$ is Verdier duality, and then we apply base change.

Applying H^0 to (12.1), we get

$$\mathrm{Tr}\colon \mathrm{Corr}_C(\mathcal{F}, \mathcal{F}) \to H^0(\mathrm{Fix}, K_{\mathrm{Fix}(c)}) = H_0^{BM}(\mathrm{Fix}(c)).$$

Now suppose β is a connected component of $\mathrm{Fix}(c)$, so we have

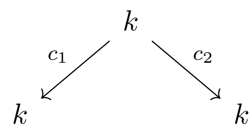
$$H_0^{BM}(\mathrm{Fix}) = \bigoplus_{\beta \in \pi_C(\mathrm{Fix})} H_0^{BM}(\mathrm{Fix}_\beta).$$

Assume further that β is proper over k. Then we can push forward to k and take the degree.

Definition 12.8. In the situation above, we define the *local terms*

$$LT_\beta(u) = \deg(\mathrm{Tr}(u)_\beta) \in \mathbf{Q}_\ell.$$

Example 12.9. For the correspondence



the cohomological correspondences are just $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$ and the trace as defined above coincides with the usual trace.

12.3.3. *The local-global formula.*

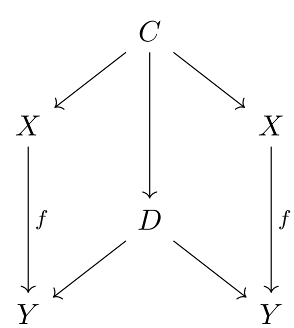
Example 12.10. For X smooth of dimension n and $\mathcal{F} = \mathbf{Q}_\ell$, we have $\mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell) = H_{2n}^{BM}(C)(-n)$. There is a cycle class map

$$\mathrm{Ch}_n(C) \to \mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell) = H_{2n}^{BM}(C)(-n) \xrightarrow{\mathrm{Tr}} H_0^{BM}(\mathrm{Fix})$$

The claim is that the diagram commutes:

$$\begin{array}{ccc} \mathrm{Ch}_n(C) & \longrightarrow & \mathrm{Corr}_C(\mathbf{Q}_\ell, \mathbf{Q}_\ell) = H_{2n}^{BM}(C)(-n) \\ \downarrow{\scriptstyle \Delta^!} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Ch}_0(\mathrm{Fix}) & \longrightarrow & H_0^{BM}(\mathrm{Fix}) \end{array}$$

Theorem 12.11. *The trace commutes with proper pushforward. In other words, if*



is a map of correspondences, with f proper, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Corr}_C(F,F) & \xrightarrow{\mathrm{Tr}} & H_0^{BM}(\mathrm{Fix}(c)) \\ \Big\downarrow{\scriptstyle [f]_!} & & \Big\downarrow{\scriptstyle f_!} \\ \mathrm{Corr}_D(f_!F, f_!F) & \xrightarrow{\mathrm{Tr}} & H_0^{BM}(\mathrm{Fix}(d)) \end{array} \tag{12.2}$$

COROLLARY 12.12. *If C, X are proper over k, then*

$$\mathrm{Tr}(R\Gamma_c(u)) = \sum_\beta LT_\beta(u).$$

PROOF. The left side corresponds to the left path of the commutative diagram in (12.2), and the right side corresponds to the right path in (12.2). ☐

This is what is usually called the Lefschetz-Verdier trace formula.

12.3.4. *The naïve local terms.* There are two issues with the trace formula. First, how do you actually compute the local terms? Consider a correspondence

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle c_1}\swarrow & & \searrow{\scriptstyle c_2} & \\ X & & & & X \end{array}$$

with c_2 is quasifinite. Given $y \in \mathrm{Fix}(c)$, with $x = c_1(y) = c_2(y)$, we can define

$$u_y \colon F_x \to F_x$$

as follows. We have a cohomological correspondence

$$(c_{2!}, c_1^*F) = \bigoplus_{z \mapsto x} c_1^*F|_z \to F_x$$

by adjunction from

$$F_x \hookrightarrow \bigoplus_{z \mapsto x} F|_{c_1(z)}.$$

Definition 12.13. The $\mathrm{Tr}(u_y)$ defined above is called the *naïve local term.*

Example 12.14. The naïve local term does not necessarily coincide with the local terms computed above. Consider translation $x \mapsto x+1$ on $\mathbf{P}^1 \to \mathbf{P}^1$. Then

$$LT_\infty(u) = 2$$

whereas the naive local term is $\mathrm{Tr}(u_\infty) = 1$. The naïve local term doesn't know that the fixed point ∞ should have multiplicity 2; it only counts the physical fixed points. An example in the same spirit is the map $x \mapsto x+1$ on $\mathbf{A}^1$.

Another issue is that we need properness. We can solve that by compactifying everything, but then you get local terms at infinity, which may be non-zero (as we saw in the preceding example).

12.3.5. *A special case.* Let X_0 be a variety over $k = \overline{\mathbf{F}}_q$ and $X = X_0 \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. Consider the correspondence

$$X \xleftarrow{\quad} X^{\mathrm{Frob}} \xrightarrow{\quad} X$$

Let $u = \mathrm{Frob}^* \mathcal{E} \to \mathcal{E}$. Then the local terms coincide with the naïve local terms. In other words,

(1) For all $s \in X_0(\mathbf{F}_q)$, we have
$$LT_s(u) = \mathrm{Tr}(u_s).$$

(2) We have
$$\mathrm{Tr}(R\Gamma_c(u)) = \sum_s \mathrm{Tr}(u_s, E_s).$$

Why? The idea is that Frobenius is contracting near fixed points. For $s \in \mathrm{Fix}$,
$$\mathrm{Frob}^{-1}(\mathfrak{m}_x^n)\mathcal{O}_X \subset \mathfrak{m}_x^{n+1}\mathcal{O}_X$$
for some $n \geq 0$. Geometrically, this means that if we pass to the normal cone we get an endomorphism which contracts everything to the origin.

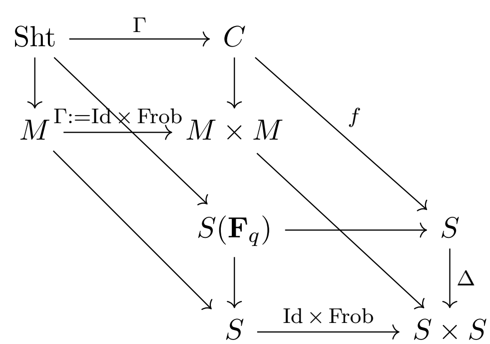
12.4. Applications to the appendix. Consider a correspondence

$$M \xleftarrow{c_1} C \xrightarrow{c_2} M$$

Assume

- c_1 is proper, and
- M is smooth of dimension n, and
- we have a proper map $f\colon C \to S$.

Let $\gamma \in \mathrm{Ch}_n(C)_{\mathbf{Q}}$. Suppose we have a map of cartesian squares



Then we can write
$$\mathrm{Sht} = \coprod_{s \in S(\mathbf{F}_q)} \mathrm{Sht}_s\,.$$

We can pull back $(\Gamma^! \gamma)_s$ = contribution of Sht_s. This is in $\mathrm{Ch}_0(\mathrm{Sht}_S)_{\mathbf{Q}}$, which is proper, so we can apply the degree map to get something in $\mathbf{Q}$. We want a formula for it, so set

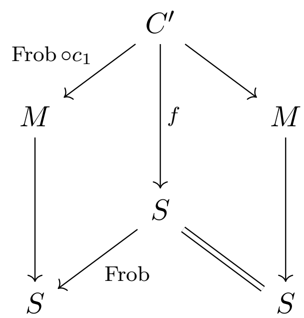
$$\langle \gamma, \Gamma_{\mathrm{Frob}} \rangle_s := \deg(\Gamma^! \gamma)_s.$$

Theorem 12.15. *We have*

$$\langle \gamma, \Gamma_{\mathrm{Frob}} \rangle_s = \mathrm{Tr}((f_! \mathrm{cl}(\gamma))_s \circ \mathrm{Frob}_s \mid (f_! \mathbf{Q}_\ell)_{\bar{s}}).$$

The argument has two steps: compatibility of trace with proper pushforward, and the special case discussed in §12.3.5.

The first idea is to replace the correspondence C with Frobenius, by composing $C \xrightarrow{c_1} M$ with $C \xrightarrow{c_1} M \xrightarrow{\mathrm{Frob}} M$. This gives a C' which lives over the Frobenius correspondence for S.



The second idea is to use the compatibility of trace with proper pushforward to express this as a trace on S, from which one gets the answer.

13. Definition and description of $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$: expressing $\mathbb{I}_r(h_D)$ as a trace (Liang Xiao)

13.1. New moduli spaces.

13.1.1. *Goal.* Recall that $\nu\colon X' \to X$ is an étale (geometrically connected) double cover. Let D be an effective divisor on X of degree d. We have constructed a map

$$\theta^{\mu}\colon \mathrm{Sht}^{\mu}_{T} \to \mathrm{Sht}'^{\mu}_{G} := \mathrm{Sht}^{\mu}_{G} \times_{X^r} (X')^r.$$

The goal is to understand the intersection number

$$\mathbb{I}_r(h_D) := \langle \theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}], h_D * \theta^{\mu}_{*}[\mathrm{Sht}^{\mu}_{T}]\rangle_{\mathrm{Sht}'^{\mu}_{G}} \in \mathbf{Q}.$$

13.1.2. *The stack* $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$. For formal reasons, $\mathbb{I}_r(h_D)$ coincides with the intersection number in the product:

$$\begin{array}{ccc}
\mathrm{Sht}^{\mu}_{\mathcal{M},D} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G}(h_D) \\
\downarrow & & \downarrow \\
\mathrm{Sht}^{\mu}_{T} \times \mathrm{Sht}^{\mu}_{T} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G} \times \mathrm{Sht}'^{\mu}_{G}
\end{array}$$

To be clear, let us flesh out the definition of $\mathrm{Sht}^{\mu}_{\mathcal{M},D}$.

Definition 13.1. We first define the moduli stack $\widetilde{\mathrm{Sht}}^{\mu}_{\mathcal{M},D}$ parametrizing

(1) Modifications of line bundles

$$\mathcal{L}_0 \overset{f_0}{\dashrightarrow} \mathcal{L}_1 \overset{f_1}{\dashrightarrow} \dots \overset{f_r}{\dashrightarrow} \mathcal{L}_r \overset{\tau}{\cong} \mathcal{L}_0,$$

with modification points at $x'_1, \dots, x'_r$.

(2) Modifications of line bundles

$$\mathcal{L}'_0 \overset{f'_0}{\dashrightarrow} \mathcal{L}'_1 \overset{f'_1}{\dashrightarrow} \dots \overset{f'_r}{\dashrightarrow} \mathcal{L}'_r \overset{\tau}{\cong} \mathcal{L}'_0,$$

with modifications points also at the same $x'_1, \dots, x'_r$ as above, because by definition the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Sht}^{\mu}_{\mathcal{M},D} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G}(h_D) & \longrightarrow & (X')^r \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Sht}^{\mu}_{T} \times \mathrm{Sht}^{\mu}_{T} & \longrightarrow & \mathrm{Sht}'^{\mu}_{G} \times \mathrm{Sht}'^{\mu}_{G} & \longrightarrow & (X')^r \times (X')^r
\end{array}$$

(3) Compatible modifications

$$c_i\colon \nu_*\mathcal{L}_i \hookrightarrow \nu_*\mathcal{L}'_i$$

such that $\det(\nu_*\mathcal{L}'_i / c(\nu_*\mathcal{L}_i))$ is an invertible sheaf on $D \times S$.

The stack $\widetilde{\mathrm{Sht}}^{\mu}_{\mathcal{M},D} / \mathrm{Pic}_X(k)$ has an action of $\mathrm{Pic}_X(k)$ as usual. Finally, we have

$$\mathrm{Sht}^{\mu}_{\mathcal{M},D} = \widetilde{\mathrm{Sht}}^{\mu}_{\mathcal{M},D} / \mathrm{Pic}_X(k).$$

As we saw yesterday, datum (3) in Definition 13.2 is equivalent to the data of

$$(\alpha_\bullet, \beta_\bullet)\colon \mathcal{L}_\bullet \oplus \sigma^*\mathcal{L}_\bullet \to \mathcal{L}'_\bullet.$$

The central object of this talk is a "Hecke version" of this moduli space.

13.1.3. *The stack* $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$.

Definition 13.2. Define $\widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d}$ whose S-point are:

(1) $x'_1, \dots, x'_r \in X'(S)$,

(2) $\mathcal{L}_0 \xrightarrow{f_0} \mathcal{L}_1 \xrightarrow{f_1} \dots \xrightarrow{f_r} \mathcal{L}_r$,

(3) $\mathcal{L}'_0 \xrightarrow{f'_0} \mathcal{L}'_1 \xrightarrow{f'_1} \dots \xrightarrow{f'_r} \mathcal{L}'_r$,

(4) A commutative diagram

$$\begin{array}{ccccccc}
\mathcal{L}'_0 & \xrightarrow{f'_1} & \mathcal{L}'_1 & \xrightarrow{f'_2} & \dots & \xrightarrow{f'_r} & \mathcal{L}'_r \\
\alpha_0\uparrow & & \alpha_1\uparrow & & \uparrow & & \alpha_r\uparrow \\
\mathcal{L}_0 & \xrightarrow{f_1} & \mathcal{L}_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_r} & \mathcal{L}_r \\
\downarrow\beta_0 & & \downarrow\beta_1 & & \downarrow & & \downarrow\beta_r \\
\sigma^*\mathcal{L}'_0 & \xrightarrow{f'_1} & \sigma^*\mathcal{L}'_1 & \xrightarrow{f'_2} & \dots & \xrightarrow{f'_r} & \sigma^*\mathcal{L}'_r
\end{array} \tag{13.1}$$

such that each row in (13.1) gives a point of $\widetilde{\mathrm{Hk}}^{\mu}_{T}$ over $x'_1, \dots, x'_r$, and each column

$$\begin{array}{c}
\mathcal{L}'_i \\
\alpha_i\uparrow \\
\mathcal{L}_i \\
\downarrow\beta_i \\
\sigma^*\mathcal{L}'_i
\end{array}$$

gives a point of $\widetilde{\mathcal{M}}_d$, which really just means that

$$\deg \mathcal{L}'_i - \deg \mathcal{L} = d$$

and

$$\mathrm{Nm}(\alpha_i) \neq \mathrm{Nm}(\beta_i)$$

(this is the $\heartsuit$ condition in the paper).

Finally, we define

$$\mathrm{Hk}^{\mu}_{\mathcal{M},d} := \widetilde{\mathrm{Hk}}^{\mu}_{\mathcal{M},d} / \operatorname{Pic}_X .$$

There is a map

$$\begin{array}{ccc}
\mathrm{Hk}^{\mu}_{\mathcal{M},d} & & (x', \mathcal{L}'_\bullet \xleftarrow{\alpha_\bullet} \mathcal{L}_\bullet \xrightarrow{\beta_\bullet} \sigma^*\mathcal{L}'_\bullet) \\
\downarrow & & \downarrow \\
\mathcal{M}_d & & (\mathcal{L}'_\bullet \xleftarrow{\alpha_\bullet} \mathcal{L}_\bullet \xrightarrow{\beta_\bullet} \sigma^*\mathcal{L}'_\bullet)
\end{array}$$

Remark 13.3. We have a cartesian diagram

$$\begin{CD} \mathrm{Sht}^{\mu}_{\mathcal{M},d} @>>> \mathrm{Hk}^{\mu}_{\mathcal{M},d} \\ @VVV @VV{\gamma_0\times\gamma_r}V \\ \mathcal{M}_d @>{\mathrm{Id}\times\mathrm{Frob}}>> \mathcal{M}_d\times\mathcal{M}_d \end{CD}$$

13.1.4. *Relation to Hitchin fibration.* Set $\mathcal{H} := \mathrm{Hk}^{1}_{\mathcal{M},d}$. Then we have

$$\text{(13.1)}\ \mathrm{Hk}^{\mu}_{\mathcal{M},d} = \underbrace{\mathcal{H}\times_{\mathcal{M}_d}\mathcal{H}\times_{\mathcal{M}_d}\ldots\times_{\mathcal{M}_d}\mathcal{H}}_{r\ \text{terms}} \tag{13.2}$$

where the maps $\mathcal{H}\to\mathcal{M}_d$ are γ_1, and the maps $\mathcal{M}_d\leftarrow\mathcal{H}$ are γ_0.

Lemma 13.4. *The composition*

$$\begin{array}{ccc} \mathrm{Hk}^{\mu}_{\mathcal{M},d} & & (x',\mathcal{L}'_{\bullet}\xleftarrow{\alpha_\bullet}\mathcal{L}_{\bullet}\xrightarrow{\beta_\bullet}\sigma^*\mathcal{L}'_{\bullet}) \\ \downarrow{\scriptstyle\gamma_i} & & \downarrow \\ \mathcal{M}_d & & (\alpha_i\colon\mathcal{L}_i\to\mathcal{L}'_i;\beta_i\colon\mathcal{L}_i\to\sigma^*\mathcal{L}'_i) \\ \downarrow{\scriptstyle f_{\mathcal{M}}} & & \downarrow \\ \mathcal{A}_d & & (\Delta:=\mathrm{Nm}(\mathcal{L}'_i)\otimes\mathrm{Nm}(\mathcal{L}_i)^{-1},\mathrm{Nm}(\alpha_i),\mathrm{Nm}(\beta_i)) \end{array}$$

is independent of i.

Proof. We have $\mathcal{A}_d\subset\widehat{X}_d\times_{\mathrm{Pic}^d_X}\widehat{X}_d$, included as the open locus where the sections take distinct values. Consider

$$\begin{CD} \mathcal{L}'_i @<{\alpha_i}<< \mathcal{L}_i @>{\beta_i}>> \sigma^*\mathcal{L}'_i \\ @VV{\text{at } x'}V @VV{\text{at } x'}V @VV{\text{at } x'}V \\ \mathcal{L}'_{i+1} @<{\alpha_{i+1}}<< \mathcal{L}_{i+1} @>{\beta_{i+1}}>> \sigma^*\mathcal{L}'_{i+1} \end{CD}$$

so

$$\mathcal{L}'_{i+1}\otimes\mathcal{L}_{i+1}^{-1}\cong\mathcal{L}'_i(x')\otimes(\mathcal{L}_i(x'))^{-1}\cong\mathcal{L}'_i\otimes\mathcal{L}_i^{-1}$$

and $\alpha=\alpha_{i+1}$ under this identification, and $\mathrm{Nm}(\beta_i)=\mathrm{Nm}(\beta_{i+1})$. (So in fact, all the maps agree to a slightly more refined space, $\widehat{X'}_d\times_{\mathrm{Pic}^d_X}\widehat{X}_d$.) $\square$

13.1.5. *The $\diamond$ locus.* Consider the following "nice locus".

$$\begin{array}{ccccc} & & \mathrm{Hk}^{\mu}_{\mathcal{M},d} & & \\ & \swarrow{\scriptstyle\gamma_0} & & \searrow{\scriptstyle\gamma_r} & \\ \mathcal{M}_d & & & & \mathcal{M}_d \\ & \searrow{\scriptstyle f_{\mathcal{M}}} & & \swarrow{\scriptstyle f_{\mathcal{M}}} & \\ & & \mathcal{A}_d & & \end{array}$$

We denote by $\mathcal{A}_d^\diamond \subset \mathcal{A}_d$ the open substack (Δ, a, b) where $b \neq 0$, and for our other moduli spaces we use $\diamond$ to denote the full pre-image of $\mathcal{A}_d^\diamond$. Thus we have the commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Hk}^{\mu}_{\mathcal{M}^\diamond,d} & & \\ & \swarrow^{\gamma_0} & & \searrow^{\gamma_r} & \\ \mathcal{M}_d^\diamond & & & & \mathcal{M}_d^\diamond \\ & \searrow_{f_{\mathcal{M}}} & & \swarrow_{f_{\mathcal{M}}} & \\ & & \mathcal{A}_d^\diamond & & \end{array}$$

LEMMA 13.5 (Description of $\mathcal{H}^\diamond$). *We have a diagram of cartesian squares*

$$\begin{array}{ccccc} \mathcal{M}_d^\diamond & \xleftarrow{\gamma_0} & \mathcal{H}^\diamond & \xrightarrow{\gamma_r} & \mathcal{M}_d^\diamond \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{X}'_d \times_{\mathrm{Pic}_X^d} X'_d & \longleftarrow & \widehat{X}'_d \times_{\mathrm{Pic}_X^d} I_d & \longrightarrow & \widehat{X}'_d \times_{\mathrm{Pic}_X^d} X'_d \\ \downarrow \mathrm{pr}_2 & & \downarrow \mathrm{pr}_2 & & \downarrow \mathrm{pr}_2 \\ X'_d & \longleftarrow & I'_d & \longrightarrow & X'_d \end{array}$$

where $I'_d = \{(D, x) \in X'_d \times X' \mid x' \in D\}$ *and the maps are*

$$\begin{array}{ccccc} X'_d & \longleftarrow & I'_d & \longrightarrow & X'_d \\ & & & & \\ D & \longleftarrow & (D, x') & \longrightarrow & D - x' + \sigma(x') \end{array}$$

PROOF. A point of $\mathcal{H}^\diamond$ is a diagram

$$\begin{array}{ccccc} \mathcal{L}'_i & \xleftarrow[\alpha_i]{} & \mathcal{L}_i & \xrightarrow[\beta_i]{} & \sigma^*\mathcal{L}'_i \\ \downarrow \text{at } x' & & \downarrow \text{at } x' & & \downarrow \text{at } \sigma(x') \\ \mathcal{L}'_{i+1} & \xleftarrow[\alpha_{i+1}]{} & \mathcal{L}_{i+1} & \xrightarrow[\beta_{i+1}]{} & \sigma^*\mathcal{L}'_{i+1} \end{array}$$

The content of the statement is that we can resconstruct this diagram from the top row, plus the point x'. If we know where x' is. Indeed, from this information we get α_i for free, as $\mathcal{L}_{i+1} = \mathcal{L}_i(x')$. The map β_i is also unique if it exists, but we do not get its existence for free, since we need it to be a *regular* (rational than rational) map. Its divisor is determined by the condition

$$\mathrm{Div}(\beta_i) + \sigma(x') = \mathrm{Div}(\beta_{i+1}) + x'$$

and we need $\mathrm{Div}(\beta_1)$ to be effective, so since the $\diamond$ locus forces $\sigma(x') \neq x'$, the requirement for β_i to exist is

$$x' \in \mathrm{Div}(\beta_i).$$

□

Corollary 13.6. *The map*

$$\gamma = \gamma_i \colon \mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d} \to \mathcal{M}^{\diamond}_d$$

is finite surjective (because $I'_d \to X'_d$ is). Therefore

$$\dim \mathrm{Hk}^{\mu}_{\mathcal{M},d} = \dim \mathcal{M}^{\diamond}_d = 2d - (g-1).$$

13.2. Trace formula for intersection number. Let $[\mathcal{H}^{\diamond}]$ be the class of the Zarisi closure of $\mathcal{H}^{\diamond}$ in $\mathrm{Ch}_{2d-g+1}(\mathcal{H})_{\mathbf{Q}}$. As we have seen, this gives a cohomological correspondence in $\mathrm{Corr}(\mathbf{Q}_{\ell,\mathcal{M}_d}, \mathbf{Q}_{\ell,\mathcal{M}_d})$ via the diagram

$$\begin{array}{ccc} & \mathcal{H} & \\ \swarrow & & \searrow \\ \mathcal{M}_d & & \mathcal{M}_d \end{array}$$

We can then push this down via

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{M}_d & & & & \mathcal{M}_d \\ \downarrow{\scriptstyle f_{\mathcal{M}}} & & \downarrow & & \downarrow{\scriptstyle f_{\mathcal{M}}} \\ \mathcal{A}_d & = & \mathcal{A}_d & = & \mathcal{A}_d \end{array}$$

to obtain a cohomological correspondence on the Hitchin space

$$f_{\mathcal{M}!}[\mathcal{H}^{\diamond}] \colon Rf_{\mathcal{M}!}\mathbf{Q}_{\ell} \to Rf_{\mathcal{M}!}\mathbf{Q}_{\ell}.$$

We have a map $\delta \colon \mathcal{A}_d \to X_d$ sending $(\Delta, a, b) \mapsto (\Delta, a-b)$. The preimage of a divisor $D \in X_d$ will be denoted $\mathcal{A}_D$.

Theorem 13.7. *Suppose D is an effective divisor of degree $d \geq \max\{4g-3, 2g\}$. Then*

$$\mathbb{I}_r(h_D) = \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!} \circ [\mathcal{H}^{\diamond}]_a)^r \,\mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_{\ell})_{\overline{a}})$$

Proof. Recall the diagram

$$\begin{array}{ccccc} \mathrm{Sht}^{\mu}_{\mathcal{M},d} & \longrightarrow & \mathrm{Hk}^{\mu}_{\mathcal{M},d} & \dashrightarrow & \\ \downarrow & & \downarrow{\scriptstyle \gamma_0 \times \gamma_r} & & \\ \mathcal{M}_d & \xrightarrow{\mathrm{Id} \times \mathrm{Frob}} & \mathcal{M}_d \times \mathcal{M}_d & & \mathcal{A}_d \\ \downarrow{\scriptstyle f_{\mathcal{M}}} & & \downarrow{\scriptstyle f_{\mathcal{M}} \times f_{\mathcal{M}}} & & \\ \mathcal{A}_d & \xrightarrow{\mathrm{Id} \times \mathrm{Frob}} & \mathcal{A}_d \times \mathcal{A}_d & \xleftarrow{\Delta} & \end{array}$$

The map from $\mathrm{Hk}^{\mu}_{\mathcal{M},d}$ factors through the diagonal of $\mathcal{A}_d \times \mathcal{A}_d$, which implies that $\mathrm{Sht}^{\mu}_{\mathcal{M},d}$ is fibered over $\mathcal{A}_d(k)$:

$$\mathrm{Sht}^{\mu}_{\mathcal{M},d} \cong \bigsqcup_{a \in \mathcal{A}_d(k)} \mathrm{Sht}^{\mu}_{\mathcal{M},d}(a).$$

So we have a map

$$\begin{array}{ccc} \bigoplus_{D \in X_d(k)} \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbf{Q}} \cong \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}} & \longleftarrow & \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}} \\ (\mathrm{Id}, \mathrm{Frob})^! \zeta & \longleftarrow & \zeta \end{array}$$

□

In the next talk the proof of the following theorem will be sketched:

Theorem 13.8 (Theorem 6.6). *There exists* $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathcal{H})$ *such that* $\zeta|_{\mathcal{H}^{\diamond}}$ *is the fundamental cycle, and*

$$\mathbb{I}_r(h_D) = \deg(\mathrm{Id}, \mathrm{Frob})^! \zeta$$

Then it follows from the trace formula that

$$\begin{aligned} \mathbb{I}_r(h_D) &= \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}\mathrm{cl}(\zeta))_a \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_{\ell})_{\overline{a}}) \\ &= \sum_{a \in \mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}\mathrm{cl}([\mathcal{H}^{\diamond}]))_a \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_{\ell})_{\overline{a}}) \end{aligned}$$

Remark 13.9. There's a technical issue that ζ and $[\mathcal{H}^{\diamond}]$ aren't the same, but at least they're the same on the $\diamond$ locus. You can show by dimension estimate that the difference on the boundary doesn't contribute.

14. Alternative calculation of $\mathbb{I}_r(h_D)$ (Yakov Varshavsky)

14.1. Overview. The goal is to sketch the proof of Theorem 6.6, which was stated last time:

THEOREM 14.1 (Theorem 6.6). *Let D be an effective divisor on X, of degree $d \geq \max\{2g'-1, 2g\}$. Then there exists $\zeta \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}^{\mu}_{\mathcal{M},d})_{\mathbf{Q}}$ such that*

(1) $\zeta|_{\mathrm{Hk}^{\mu}_{\mathcal{M}^{\diamond},d}}$ is the fundamental class, and
(2) $\deg((\mathrm{Id}, \mathrm{Frob}^!)\zeta)_D = \mathbb{I}_r(h_D) = \langle [\mathrm{Sht}^{\mu}_T], h_D * [\mathrm{Sht}^{\mu}_T]\rangle_{\mathrm{Sht}^{\mu}_G}$.

The strategy of the proof is to:

(1) Give a formula for ζ.
(2) Prove that ζ satisfies properties (1), (2) of Theorem (14.1).

The basic idea is that Sht is the intersection of something with the graph of Frobenius. On the right hand side of Theorem 14.1(2), we are taking an intersection of objects obtained by intersecting with the graph of Frobenius. On the left hand side, we are intersecting first and then intersecting with the graph of Frobenius. That these coincide is the substance of the "octahedron lemma" from Rapoport's talk.

14.2. The fundamental diagram. Consider the commutative diagram of algebraic stacks

$$\begin{array}{ccccc} A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} \\ \downarrow & & \downarrow & & \downarrow \\ A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle\beta} \\ A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} \end{array}$$

Think of A_{13} as the "bad" object. Thus, also fibered products involving it are also "bad", while all objects not involving it are "good". We'll be more precise about this shortly. The point is that we can take the fiber product of the rows and then columns, or columns and then rows.

If we take the fibered products of rows first, then we get

$$\begin{array}{ccccccc} A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} & & A_{1*} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} & & A_{2*} \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\delta} \\ A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} & & A_{3*} \end{array}$$

If we take the fibered products of columns first, then we get and

$$\begin{array}{ccccc} A_{11} & \xrightarrow{\alpha} & A_{12} & \longleftarrow & A_{13} \\ \downarrow & & \downarrow & & \downarrow \\ A_{21} & \longrightarrow & A_{22} & \longleftarrow & A_{23} \\ \uparrow & & \uparrow & & \uparrow \beta \\ A_{31} & \longrightarrow & A_{32} & \longleftarrow & A_{33} \end{array}$$

$$A_{*1} \xrightarrow{\gamma} A_{*2} \longleftarrow A_{*3}$$

PROPOSITION 14.2. *The two fiber products* $A_{1*} \times_{A_{2*}} A_{3*}$ *and* $A_{*1} \times_{A_{*2}} A_{*3}$ *are canonically equivalent.*

From the diagrams we have refined Gysin maps

$$\mathrm{Ch}(A_{13}) \xrightarrow{\alpha^!} \mathrm{Ch}(A_{1*}) \xrightarrow{\delta^!} \mathrm{Ch}(A_{1*} \times_{A_{2*}} A_{3*})$$
$$\mathrm{Ch}(A_{13}) \xrightarrow{\beta^!} \mathrm{Ch}(A_{*3}) \xrightarrow{\gamma^!} \mathrm{Ch}(A_{*1} \times_{A_{*2}} A_{*3}).$$

Thanks to Proposition 14.3 it is meaningful to ask if they agree.

PROPOSITION 14.3. *Assume that*

- *the* A_{ij} *are smooth equidimensional all* $i, j \neq 1, 3$*,*
- *the* $A_{2*}, A_{3*}, A_{*1}, A_{*2}$ *are smooth of expected dimension,*
- *the map* $\alpha, \beta, \gamma, \delta$ *satisfy assumptions (A) and (B) from Rapoport's talk.*

Then we have

$$\delta^! \alpha^! [A_{13}] = \gamma^! \beta^! [A_{13}] \in \mathrm{Ch}(A_{1*} \times_{A_{2*}} A_{3*}) = \mathrm{Ch}(A_{*1} \times_{A_{*2}} A_{*3}).$$

The content of Proposition 14.3, as compared to Proposition 14.2, is that Proposition 14.3 involves both "derived fiber products" and classical fiber products. Therefore, the content of Proposition 14.3 is that the "derived fiber product = usual fiber product". The assumptions are there to make that true.

14.3. Application to shtukas.

14.3.1. *The fundamental diagram.* We are interested in the specialization of the fundamental diagram to shtukas:

$$\begin{array}{ccccc} \mathrm{Hk}_T^{\mu} \times \mathrm{Hk}_T^{\mu} & \xrightarrow{\Pi^{\mu} \times \Pi^{\mu}} & \mathrm{Hk}_G'^{r} \times \mathrm{Hk}_G'^{r} & \longleftarrow & \mathrm{Hk}_{G,d}'^{r} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_T^2 \times \mathrm{Bun}_T^2 & \longrightarrow & \mathrm{Bun}_G^2 \times \mathrm{Bun}_G^2 & \longleftarrow & H_d \times H_d \\ \uparrow {\scriptstyle \mathrm{Id} \times \mathrm{Frob}} & & \uparrow {\scriptstyle \mathrm{Id} \times \mathrm{Frob}} & & \uparrow {\scriptstyle \mathrm{Id} \times \mathrm{Frob}} \\ \mathrm{Bun}_T \times \mathrm{Bun}_T & \xrightarrow{\Pi \times \Pi} & \mathrm{Bun}_G \times \mathrm{Bun}_G & \longleftarrow & H_d \end{array}$$

Some of the objects of this diagram haven't even been defined yet; we will give the definitions (and recall old ones) shortly.

The fibered products column-wise will be

$$\begin{array}{ccccc}
\mathrm{Hk}^\mu_T \times \mathrm{Hk}^\mu_T & \xrightarrow{\Pi^\mu \times \Pi^\mu} & \mathrm{Hk}'^r_G \times \mathrm{Hk}'^r_G & \longleftarrow & \mathrm{Hk}'^r_{G,d} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}^2_T \times \mathrm{Bun}^2_T & \longrightarrow & \mathrm{Bun}^2_G \times \mathrm{Bun}^2_G & \longleftarrow & H_d \times H_d \\
\uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} & & \uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} & & \uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} \\
\mathrm{Bun}_T \times \mathrm{Bun}_T & \xrightarrow{\Pi \times \Pi} & \mathrm{Bun}_G \times \mathrm{Bun}_G & \longleftarrow & H_d
\end{array}$$

$$\mathrm{Sht}^\mu_T \times \mathrm{Sht}^\mu_T \xrightarrow{\theta^\mu \times \theta^\mu} \mathrm{Sht}'^r_G \times \mathrm{Sht}'^r_G \longleftarrow \mathrm{Sht}'_{G,d}$$

The fibered products row-wise will be

$$\begin{array}{cccccc}
\mathrm{Hk}^\mu_T \times \mathrm{Hk}^\mu_T & \xrightarrow{\Pi^\mu \times \Pi^\mu} & \mathrm{Hk}'^r_G \times \mathrm{Hk}'^r_G & \longleftarrow & \mathrm{Hk}'^r_{G,d} & \mathrm{Hk}'^r_{G,d} \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
\mathrm{Bun}^2_T \times \mathrm{Bun}^2_T & \longrightarrow & \mathrm{Bun}^2_G \times \mathrm{Bun}^2_G & \longleftarrow & H_d \times H_d & \mathcal{M}_d \times \mathcal{M}_d \\
\uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} & & \uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} & & \uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} & \uparrow{\scriptstyle \mathrm{Id}\times\mathrm{Frob}} \\
\mathrm{Bun}_T \times \mathrm{Bun}_T & \xrightarrow{\Pi \times \Pi} & \mathrm{Bun}_G \times \mathrm{Bun}_G & \longleftarrow & H_d & \mathcal{M}_d
\end{array}$$

Definition 14.4. The total fiber product is denoted $\mathrm{Sht}^\mu_{\mathcal{M},d}$. This can be viewed as the fiber product of the column-wise fiber products, or as the fiber product as the row-wise fiber products, thanks to Proposition 14.2.

14.3.2. *Explication of the terms.* Now we're going to tell you what these things are.

- H_d is the Hecke stack. We can view $H_d = \widetilde{H_d}/\operatorname{Pic}_X$, where $\widetilde{H}_d$ parametrizes modifications
$$\phi \colon \mathcal{E} \hookrightarrow \mathcal{E}'$$
with $\operatorname{coker}\phi$ is finite over S and flat of rank d.
- The map $\mathrm{Bun}_T \to \mathrm{Bun}_G$ sends $\mathcal{L} \mapsto \nu_*\mathcal{L}$.
- The stacks Hk^μ_T and $\mathrm{Hk}^\mu_G \cong \mathrm{Hk}^r_G$ parametrize modifications of type μ between chains of bundles. It is perhaps easier to phrase things in terms of Hk^μ_2, which parametrize chains of modifications
$$\mathcal{E}_0 \xrightarrow{\text{at } x_1} \mathcal{E}_1 \xrightarrow{\text{at } x_2} \dots \xrightarrow{\text{at } x_r} \mathcal{E}_r$$
Then we define $\mathrm{Hk}^r_G = \widetilde{\mathrm{Hk}}^r_G/\operatorname{Pic}_X$, and $\mathrm{Hk}'^r_{G,d} = \mathrm{Hk}^r_G \times_{X^r} (X')^r$.

- The stack $\widetilde{\mathrm{Hk}}^r_{G,d}$ parametrizes "modifications of (modifications of type μ) of degree d": that is, chains of rank 2 vector bundles

$$\begin{array}{ccccccc}
\mathcal{E}_0 & \xrightarrow{\text{at } x_1} & \mathcal{E}_1 & \xrightarrow{\text{at } x_2} & \dots & \xrightarrow{\text{at } x_r} & \mathcal{E}_r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{E}'_0 & \xrightarrow{\text{at } x_1} & \mathcal{E}'_1 & \xrightarrow{\text{at } x_2} & \dots & \xrightarrow{\text{at } x_r} & \mathcal{E}'_r
\end{array}$$

  such that the rows are in Hk^r_G and the columns are in H_d. As usual, we require the modifications of both at $x_1, \dots, x_r$. We define $\mathrm{Hk}^r_G = (X')^r \times_{X^r} \mathrm{Hk}'^r_G$.
- Finally we set $\mathrm{Hk}^r_{G,d} = \widetilde{\mathrm{Hk}}^r_{G,d}/\operatorname{Pic}_X$ and $\mathrm{Hk}'^r_{G,d} = \widetilde{\mathrm{Hk}'}^r_{G,d}/\operatorname{Pic}_X$.

The potentially bad objects in the diagram are $\mathrm{Hk}'^r_{G,d}$ and everything that involves it: $\mathrm{Sht}'_{G,d}$ and $\mathrm{Hk}'^r_{\mathcal{M},d}$. Everything else is smooth. To show this, you first study the objects in the big diagram. For example, the map $\mathrm{Hk} \xrightarrow{\mathrm{pr}_1} \mathrm{Bun}_G$ is smooth of relative dimension $2d$. (We've already seen this for $d = 1$: there is one dimension for the choice of point and one dimension coming from the $\mathbf{P}^1$ parametrizing the choice of modification type at that point.)

14.3.3. *Intersection numbers.* Specializing Propositions 14.2, 14.3 we get:

Corollary 14.5. *We have*

$$(\mathrm{Id}, \mathrm{Frob})^!(\Pi^\mu \times \Pi^\mu)^!([\mathrm{Hk}'^r_{G,d}]) = (\theta^\mu \times \theta^\mu)^!(\mathrm{Id} \times \mathrm{Frob})^!([\mathrm{Hk}'^r_{G,d}]) \in \mathrm{Ch}_0(\mathrm{Sht}^\mu_{\mathcal{M},d})$$

Definition 14.6. We now define $\zeta := (\Pi^\mu \times \Pi^\mu)^!(\mathrm{Hk}'^r_G) \in \mathrm{Ch}_{2d-g+1}(\mathrm{Hk}'^r_{\mathcal{M}_d})$.

We then need to check that ζ enjoys the following two properties promised in Theorem 14.1.

(1) Consider the fiber product

$$\begin{array}{ccc}
\mathrm{Hk}^\mu_{\mathcal{M},d} & \longrightarrow & \mathrm{Hk}'^r_{G,d} \\
\downarrow & & \downarrow \\
\mathrm{Hk}^\mu_T \times \mathrm{Hk}^\mu_T & \longrightarrow & \mathrm{Hk}'^r_G \times \mathrm{Hk}'^r_G
\end{array}$$

The total space $\mathrm{Hk}_{\mathcal{M},d}$ is bad, and hard to understand. However, we claim that the open substack $\mathrm{Hk}_{\mathcal{M}^\diamond,d}$ has the expected dimension. Then the claim is that $[\zeta] \in \mathrm{Ch}(\mathrm{Hk}_{\mathcal{M}^\diamond,d})$ is the fundamental class.

(2) We know that we can write

$$\mathrm{Sht}^\mu_{\mathcal{M}_d} = \bigsqcup_{D \in X_d(k)} \mathrm{Sht}^\mu_{\mathcal{M},D},$$

which implies that $\mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M}_d})_{\mathbf{Q}} = \bigoplus \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},D})_{\mathbf{Q}}$. Using Corollary 14.5, we have

$$\begin{aligned}\deg((\mathrm{Id}, \mathrm{Frob}^!)\zeta)_D &= \deg[(\mathrm{Id}, \mathrm{Frob})^!(\Pi^{\mu} \times \Pi^{\mu})^!([\mathrm{Hk}'^{r}_{G,d}]) \mid_{\mathrm{Sht}^{\mu}_{\mathcal{M},D}}] \\ &= (\theta^{\mu} \times \theta^{\mu})^!(\mathrm{Id} \times \mathrm{Frob})^!([\mathrm{Hk}'^{r}_{G,d}]) \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d})\end{aligned}$$

To establish Theorem 14.1 (2), we need to show that

$$(\theta^{\mu} \times \theta^{\mu})^!(\mathrm{Id} \times \mathrm{Frob})^!([\mathrm{Hk}'^{r}_{G,d}]) \in \mathrm{Ch}_0(\mathrm{Sht}^{\mu}_{\mathcal{M},d}) = \mathbb{I}_r(h_D).$$

Note that $(\theta^{\mu} \times \theta^{\mu})^!(\mathrm{Id} \times \mathrm{Frob})^! = [\mathrm{Sht}'^{r}_{G}]$. We use that we have the cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_{\mathcal{M}_D} & \longrightarrow & \mathrm{Sht}_G(h_D) \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\mu}_T \times \mathrm{Sht}^{\mu}_T & \longrightarrow & \mathrm{Sht}'^{r}_G \times \mathrm{Sht}'^{r}_G \end{array}$$

We have the compatibility relation "pullback then restrict to DS is the same as restrict to D and then pullback", so

$$\begin{aligned}\mathbb{I}_r(h_D) &:= \langle \theta^{\mu}_*[\mathrm{Sht}^{\mu}_T], h_D * \theta^{\mu}_*[\mathrm{Sht}^{\mu}_T]\rangle \\ &= \langle (\theta^{\mu} \times \theta^{\mu})_*([\mathrm{Sht}^{\mu}_T] \times [\mathrm{Sht}^{\mu}_T]), \mathrm{pr}_*[\mathrm{Sht}'_G(h_D)]\rangle_{\mathrm{Sht}'^{r}_G \times \mathrm{Sht}'^{r}_G} \\ &= \deg(\theta^{\mu} \times \theta^{\mu})^![\mathrm{Sht}'^{r}_G]\end{aligned}$$

as desired.

Part 5

Day Five

15. Comparison of $\mathcal{M}_d$ and $\mathcal{N}_d$; the weight factors (Ana Caraiani)

15.1. Review.

15.1.1. *Goal.* Let D be an effective divisor on X of degree $d \geq \max\{2g'-1, 2g\}$. We have an associated Hecke function h_D. The goal is to prove the *key identity* (Theorem 8.1 in the paper)

$$(\log q)^{-r}\mathbb{J}_r(h_D) = \mathbb{I}_r(h_D)$$

We'll use the notation in the paper [YZ]:

$$\mathcal{A}_d = (\widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d) \setminus \text{both sections vanish.}$$

(This is consistent with talks up to Wednesday morning, but inconsistent with the ones afterwards.)

To prove this we'll use the geometrization of both sides that we have been developing, and which we now review.

15.1.2. *The analytic side.* On the analytic side, the geometrization takes the form

$$(\log q)^{-r}\mathbb{J}_r(h_D) = \sum_{\underline{d}\in\Sigma_d}\sum_{a\in\mathcal{A}_D(k)} (2d_{12}-d)^r \cdot \mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}})_{\overline{a}}). \qquad (15.1)$$

Remark 15.1. We obtained this formula from geometrization of $\mathrm{Tr}(u, h_D)$ by taking the sum over invariants $u \in \mathbf{P}^1(F) - \{1\}$

The analytic side was geometrized by the moduli space $\mathcal{N}_{\underline{d}}$. Recall that we defined a map

$$Rf_{\mathcal{N}_{\underline{d}}*} \colon \mathcal{N}_{\underline{d}} \to \mathcal{A}_d$$

in the following way: it is the restriction to an open substack of the "addition" map

$$\mathrm{add}_{d_{11},d_{22}} \times \mathrm{add}_{d_{12},d_{21}} \colon (\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}^d_X} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}}) \to \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d.$$

Also, recall that the local system $L_{\underline{d}}$ from (15.1) was the restriction to $\mathcal{N}_{\underline{d}}$ from $(\widehat{X}_{d_{11}} \times \widehat{X}_{d_{22}}) \times_{\mathrm{Pic}^d_X} (\widehat{X}_{d_{12}} \times \widehat{X}_{d_{21}})$ of

$$(L_{d_{11}} \boxtimes \mathbf{Q}_\ell) \boxtimes (L_{d_{12}} \boxtimes \mathbf{Q}_\ell)$$

where for $d' \geq 0$, $L_{d'}$ is the local system on $\widehat{X}$ pulled back from the local system on $\mathrm{Pic}^{d'}_X$ corresponding to

$$L := (\nu_*\mathbf{Q}_\ell)^{\sigma=-1}, \quad \nu \colon X' \to X.$$

Note that $L_d|_{X_{d'}\subset\widehat{X}_{d'}}$ is descended from $L^{\boxtimes d'}$ on $X^{d'}$. The upshot is that on the open substack $X_{d'} \subset \widehat{X}_{d'}$ we understand the local system L_d very concretely, so we have a chance of computing $H^*(X_{d'}, L_{d'})$.

15.1.3. *The geometric side.* On the geometric side, the geometrization takes the form

$$\mathbb{I}_r(h_D) = \sum_{a\in\mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}[\mathcal{H}^\diamond]_a)^r \circ \mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_\ell)_{\overline{a}}).$$

Here the point is that one can understand $[\mathcal{H}^\diamond]$ as the fundamental class over the "nice" locus $\diamond$. The map $f_{\mathcal{M}}\colon \mathcal{M}_d \to \mathcal{A}_d$ was the norm map

$$\widehat{X'}_d \times_{\mathrm{Pic}^d_X} \widehat{X'}_d \xrightarrow{\widehat{\nu}_d\times\widehat{\nu}_d} \widehat{X}_d \times_{\mathrm{Pic}^d_X} \widehat{X}_d.$$

15.1.4. *The comparison.* We have reduced to a comparison of traces on cohomology:

$$\sum_{\underline{d}\in\Sigma_d}\sum_{a\in\mathcal{A}_D(k)} (2d_{12}-d)^r\cdot\mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}})_{\overline{a}}) \sim \sum_{a\in\mathcal{A}_D(k)} \mathrm{Tr}((f_{\mathcal{M}!}[\mathcal{H}^\diamond]_a)^r\circ\mathrm{Frob}_a, (Rf_{\mathcal{M}!}\mathbf{Q}_\ell)_{\overline{a}}).$$

To tackle this, we'll first compare the local systems. So the strategy is:

(1) Compute $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$.
(2) Compute $Rf_{\mathcal{N}_{\underline{d}}!}L_{\underline{d}}$.
(3) Compute the action of $f_{\mathcal{M}!}[\mathcal{H}^\diamond]$.

The idea is to use the "perverse continuation principle". This tells us that if we know that $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$ and $Rf_{\mathcal{N}_{\underline{d}}!}L_{\underline{d}}$ satisfy certain special properties, then we can establish an "identity" between them globally if we can do it over a "nice" open set. This is important technically because the geometrization process was that we understood the relevant moduli spaces well over a nice open subset, but not everywhere.

For this we need to show that the sheaves are (shifted) perverse, and moreover that they are the middle extension of their restriction to a "nice" open subset of $\mathcal{A}_d$. The poins is that middle extensions are completely determined by their restriction to the open subset. So we'll compare $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$ and $Rf_{\mathcal{N}_{\underline{d}}!}L_{\underline{d}}$ on an open subset, using the representation theory of finite groups.

We probably won't have time to do everything, so we'll focus on (1).

15.2. The geometric side. We want to compute $Rf_{\mathcal{M}!}\mathbf{Q}_\ell$. Let

$$j\colon X_d^\circ \hookrightarrow X_d \hookrightarrow \widehat{X}_d$$

to be the locus of multiplicity-free divisors.

Taking pre-images of X_d°, we get a étale Galois covers

$$\underbrace{(X')^{d,\circ} \xrightarrow{\{\pm1\}^d} X^{d,\circ} \xrightarrow{S_d} X_d^\circ}_{\mathrm{Gal}=\{\pm1\}^d\rtimes S_d}.$$

Let $\chi_i\colon \{\pm1\}^d \to \{\pm1\}$ be the character which is non-trivial on the first i factors, and trivial on the last $d-i$ factors.

Let $S_{i,d-i} \cong S_i \times S_{d-i}$ be the stabilizer of the first i elements. Let $\Gamma_d(i) = \{\pm1\}^i \rtimes S_{i,d-i} \subset \Gamma_d := \{\pm1\}^d \rtimes S_d$. Since the $S_{i,d-i}$-action on $\{\pm1\}^i$ stabilizes the character χ_i, we can inflate χ_i to $\Gamma_d(i)$ as $\chi_i \boxtimes \mathbf{1}$, and then we set

$$\rho(i) = \mathrm{Ind}_{\Gamma_d(i)}^{\Gamma_d}(\chi_i \boxtimes \mathbf{1}).$$

Note that this has dimension $\binom{d}{i}$. It determines an irreducible local system $L(\rho_i)$ on X_d°. We want to extend this to a (shifted) perverse sheaf $K_i = (j_{!*}L(\rho_i)[d])[-d]$. This is called the *middle extension*: it is a perverse extension of $L(\rho_i)[d]$ to $\widehat{X}_d$, which is characterized by the following property:

> If $Z := \widehat{X}_d - X_d^\circ$, and $i\colon Z \hookrightarrow \widehat{X}_d$, then $(j_{!*}L(\rho_i)[d])[-d]$ is the unique (shifted) perverse extension of $L(\rho_i)[d]$ such that it has no subobjects or quotients of the form i_*M where M is perverse on Z.

This condition can be rephrased in terms of "support and co-support" conditions. Perverse sheaves form an abelian subcategory of $D = D_c^b(\widehat{X}_d)$. They are defined by support and co-support conditions. The support condition cuts out a subcategory ${}^pD^{\leq 0} \subset D$, and the co-support condition is the Verdier dual of the support condition, cutting out ${}^pD^{\geq 0} \subset D$.

The *middle extension*

$$K_i = j_{!*}(L(\rho_i)[d])$$

is the unique perverse extension such that $i^*K \in {}^pD^{\leq -1}(Z)$, and $i^!K \in {}^pD^{\geq 1}(Z)$.

PROPOSITION 15.2. *Assume that $d \geq 2g' - 1$. Then we have a canonical isomorphism of shifted perverse sheaves on $\mathcal{A}_d$:*

$$Rf_{\mathcal{M}!}\mathbf{Q}_\ell \cong \bigoplus_{i,j=0}^{d} (K_i \boxtimes K_j)|_{\mathcal{A}_d}$$

PROOF. Recall that $f_{\mathcal{M}}$ is the restriction of

$$\widehat{\nu}_d \times \widehat{\nu}_d\colon \widehat{X'}_d \times \widehat{X'}_d \to \widehat{X}_d \times \widehat{X}_d.$$

Since these maps are proper, we can use base change and the Künneth formula to reduce to showing:

$$R\widehat{\nu}_{d*}\mathbf{Q}_\ell \cong \bigoplus_{i=0}^{d} K_i.$$

We argue this by showing that $R\widehat{\nu}_{d*}\mathbf{Q}_\ell$ is a middle extension (up to shift) and then computing over the "good" open set.

Why is $R\widehat{\nu}_{d*}\mathbf{Q}_\ell$ a middle extension? Actually this follows from a general principle: $\widehat{\nu}_d\colon \widehat{X'}_d \to \widehat{X}_d$ is a *small map*, which means that

$$\operatorname{codim}\{y \in \widehat{X}_d \mid \dim \widehat{\nu}_d^{-1}(y) \geq r\} > 2r \text{ for } r \geq 1.$$

We can check this explicitly: the map $\widehat{\nu}_d$ is a union of $\nu_d\colon X'_d \to X_d$ and $\mathrm{Nm}\colon \mathrm{Pic}^d_{X'} \to \mathrm{Pic}^d_X$. Since $\nu_d\colon X'_d \to X_d$ is finite, the only positive dimensional fibers live over $\mathrm{Pic}^d_X \hookrightarrow \widehat{X}_d$. The codimension here is $d - g + 1$, while the dimension of the fibers in this locus is $g - 1$. So the map will clearly be small for large enough d.

Now, it is formal that if $\widehat{\nu}_d$ is proper and and the source $\widehat{X'}_d$ is smooth and geometrically irreducible, then

$$R\widehat{\nu}_{d*}\mathbf{Q}_\ell = j_{!*}(R\nu^\circ_{d*}\mathbf{Q}_\ell)$$

where $\nu_d^\circ \colon (X')_d^\circ \to X_d^\circ$ (Why? You check the support condition by bounding the cohomological dimension of fibers, and the complex is automatically self-dual because the map is proper, so you get the cosupport condition for free. The strictness of the inequality gets you the middle extension property).

Now it's enough to show that

$$R\nu_{d*}^\circ \mathbf{Q}_\ell \cong \bigoplus_{i=0}^{d} L(\rho_i)$$

This is just an equality of local systems, so it follows from a purely representation-theoretic fact:

$$\mathrm{Ind}_{S_d}^{\Gamma_d} \mathbf{1} = \bigoplus_{i=0}^{d} (\mathrm{Ind}_{\Gamma_d(i)}^{\Gamma_d} \chi_i \boxtimes \mathbf{1}).$$

To prove this, make a dimension count and show that there is a $\Gamma_d(i)$-equivariant embedding $\chi_i \boxtimes \mathbf{1} \hookrightarrow \mathbf{Q}_\ell[\Gamma_d/S_d]$. This can be done explicitly: send

$$\chi_i \boxtimes 1 \mapsto \mathbf{1}_{\chi_i} = \sum_{\varepsilon \in \Gamma_d/S_d} \chi(\varepsilon)\varepsilon.$$

□

15.3. The analytic side. Next we want to compute $Rf_{\mathcal{N}_{\underline{d}}!}L_{\underline{d}}$ on $\mathcal{A}_d$. Here one wrinkle is that the map $f_{\mathcal{N}_{\underline{d}}} \to \mathcal{A}_d$ is actually not small. It is obviously finite on $\mathcal{B} := X_d \times_{\mathrm{Pic}_X^d} X_d \subset \mathcal{A}_d$. So the problem occurs on

$$A_d \setminus \mathcal{B} = \underbrace{(\{0\} \times X_d)}_{=:\mathcal{C}} \sqcup \underbrace{(X_d \times \{0\})}_{=:\mathcal{C}'}.$$

Let's think about what the fibers look like over $\mathcal{C}$. A point of $\mathcal{C}$ is just a divisor, say $(0, D)$. Assume $d_{11} < d_{22}$; then by the definition of $\mathcal{N}_d$ (which we admittedly skated over) $\varphi_{11} \neq 0$ and $\varphi_{22} = 0$. So the fiber is

$$f_{\mathcal{N}_{\underline{d}}}^{-1}(0, D) = X_{d_{11}} \times \mathrm{add}_{d_{12},d_{21}}^{-1}(D). \tag{15.2}$$

This has dimension d_{11}, which can go up to about $d/2$. You can check that the smallness just fails by a constant factor of about g for all d.

However, we don't really need smallness. If you think about the argument we just made, you'll see that it's enough to bound the *cohomological dimension* (as opposed to dimension) of fibers of $f_{\mathcal{N}_{\underline{d}}}$ to conclude that the pushforward is a middle extension.

By (15.2) the cohomology of the geometric fiber over $(0, D)$ is then $H^*(X_{d_{11}} \otimes_k \overline{k}, L_{d_{11}}) \otimes H^0(\mathrm{add}_{d_{12},d_{21}}^{-1}(D) \otimes_k \overline{k}, L_{d_{12}})$. We can ignore the second factor, since it doesn't effect the cohomological dimension. Since $L_{d_{11}}$ is a non-trivial local system,

$$H^*(X_{d_{11}} \otimes_k \overline{k}, L_{d_{11}}) \cong \bigwedge^{d} (H^1(X, L_{d_{11}})[-d_{11}]).$$

which vanishes if $d_{11} > 2g - 2$.

The upshot is that $Rf_{\mathcal{N}_{\underline{d}}*}L_{\underline{d}}$ is the middle extension for large enough d. Then you want to show that

$$\mathrm{add}^{\circ}_{j,n-j*}(L_j \boxtimes \mathbf{Q}_\ell) \cong L(\rho_j) \text{ on } X^{\circ}_d.$$

where

$$\mathrm{add}^{\circ}_{j,n-j} \colon (X_j \times X_{d-j})^{\circ} \to X^{\circ}_d.$$

The local system $L_j \boxtimes \mathbf{Q}_\ell$ corresponds to $\chi_j \boxtimes \mathbf{1}$ on $(X_j \times X_{d-j})^{\circ}$.

15.4. Weight factors. We've completely run out of time to discuss the weight factors. The punchline is that, by using perversity, one can compute the Hecke action on $\mathcal{A}^{\diamond}_d$ using Lemma 6.3 of Liang Xiao's talk. This ends up giving the weight factors $d - 2j$ on $K_i \boxtimes K_j$.

16. Horocycles
(Lizao Ye)

16.1. Outlook. Let $G = \mathrm{PGL}_2$, $B \subset G$ be the Borel, and $H \subset B$ be the torus. (Actually, it is better to regard H as a quotient of B.)

Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Sht}_B & & \\ & \swarrow & & \searrow & \\ \mathrm{Sht}_G & & & & \mathrm{Sht}_H \\ & \searrow & & \swarrow & \\ & & X^r & & \end{array}$$

We basically want to prove

$$\mathbb{J}_r(\pi) = \mathbb{I}_r(\pi).$$

What we have is $\mathbb{J}_r = \mathbb{I}_r$. So we need to have some spectral decomposition. This has been done for the analytic side. The geometric side rests on spectral decomposition of the cohomology of shtukas, $H^{2r}(\mathrm{Sht}_G)$. This is achieved by an analysis of the Hecke action.

16.2. Hecke action. Let $D \hookrightarrow X$ be a divisor. We have defined a correspondence

$$\begin{array}{ccccc} & & \mathrm{Sht}_G(h_D) & & \\ & \swarrow & & \searrow & \\ \mathrm{Sht}_G & & & & \mathrm{Sht}_G \end{array}$$

The stack $\mathrm{Sht}_G(h_D)$ parametrizes modifications of iterated shtukas

$$(\mathcal{E}_{\cdot} \hookrightarrow \mathcal{E}'_{\cdot}).$$

In fact, for every $g = \bigotimes g_v \in G(\mathbf{A}_F)$, we get a correspondence $\mathrm{Sht}_G(g)$. This defines an algebra homomorphism

$$\mathcal{H}_G \to \mathrm{End}\, H^i(\mathrm{Sht}_G).$$

We sketch why this is the case.

Recall that the ring structure on the Hecke algebra is defined by convolution:

$$\mathbf{1}_{Kg_1K} * \mathbf{1}_{Kg_2K} = \sum_{g_3 \in K\backslash G/K} [g_3^{-1}Kg_1K \cap Kg_2^{-1}K : K] \cdot \mathbf{1}_{Kg_3K}$$

The fibered product of $\mathrm{Sht}_G(g_1)$ and $\mathrm{Sht}_G(g_2)$ is basically several copies of $\mathrm{Sht}_G(g_3)$:

$$\begin{array}{ccccccc} & & & ? & & & \\ & & \swarrow & & \searrow & & \\ & \mathrm{Sht}_G(g_1) & & & & \mathrm{Sht}_G(g_2) & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathrm{Sht}_G & & & \mathrm{Sht}_G & & & \mathrm{Sht}_G \end{array}$$

In fact the number $[g_3^{-1}Kg_1K \cap Kg_2^{-1}K : K]$ is the number of copies of $\mathrm{Sht}_G(g_3)$ appearing in the fiber product.

16.3. The constant term map. We are going to define a constant term map

$$H_c^r(\mathrm{Sht}_G) \to H_c^0(\mathrm{Sht}_H).$$

We begin by considering the diagram

$$\begin{array}{ccc} & \mathrm{Sht}_{B,\eta} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{Sht}_{G,\eta} & & \mathrm{Sht}_{H,\eta} \end{array}$$

where η is the generic point of X^r, and the subscript η denotes restriction to the generic fiber. Since we have restricted everything to the generic fiber, we have $\dim \mathrm{Sht}_{G,\eta} = r$, $\dim \mathrm{Sht}_{B,\eta} = r/2$, and $\dim \mathrm{Sht}_{H,\eta} = 0$.

THEOREM 16.1 (Drinfeld-Varshavsky). *The map* $\mathrm{Sht}_{B,\eta} \to \mathrm{Sht}_{G,\eta}$ *is finite unramified.*

We need this properness, because to define a map on cohomology from a cohomological correspondence requires properness of the first map.

Definition 16.2. The *constant term map* is the composition

$$CT\colon H_c^r(\mathrm{Sht}_G) \xrightarrow{p^*} H_c^r(\mathrm{Sht}_B) \xrightarrow{q_*} H_c^0(\mathrm{Sht}_H).$$

The key point is that the constant term map is compatible with the Satake homomorphism.

PROPOSITION 16.3. *For* $h \in \mathcal{H}_G$*, we have*

$$CT \circ h = \mathrm{Sat}(h) \circ CT.$$

PROOF. Consider the diagram

$$\begin{array}{ccccc} \mathrm{Sht}_H & \longleftarrow & \mathrm{Sht}_H(h_x) & \longrightarrow & \mathrm{Sht}_H \\ \uparrow & & & & \uparrow \\ \mathrm{Sht}_B & & & & \mathrm{Sht}_B \\ \downarrow & & & & \downarrow \\ \mathrm{Sht}_G & \longleftarrow & \mathrm{Sht}_G(h_x) & \longrightarrow & \mathrm{Sht}_G \end{array}$$

Define a middle term $\mathrm{Sht}_B(h_x)$ to make the bottom right square cartesian. We claim that it automatically makes the bottom left square cartesian.

$$\begin{array}{ccccc}
\mathrm{Sht}_H & \longleftarrow & \mathrm{Sht}_H(h_x) & \longrightarrow & \mathrm{Sht}_H \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Sht}_B & \longleftarrow & \mathrm{Sht}_B(h_x) & \longrightarrow & \mathrm{Sht}_B \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Sht}_G & \longleftarrow & \mathrm{Sht}_G(h_x) & \longrightarrow & \mathrm{Sht}_G
\end{array}$$

Let's unravel the claim. The stack $\mathrm{Sht}_B(h_x)$ parametrizes modifications

$$\begin{array}{ccc}
\mathcal{L} & \longrightarrow & \mathcal{L}' \\
\big\downarrow & & \big\downarrow \\
\mathcal{E} & \xhookrightarrow{\text{at } x} & \mathcal{E}'
\end{array}$$

The fact that both diagrams are cartesian amounts to saying that given the datum

$$\begin{array}{ccc}
\mathcal{L} & & \\
\big\downarrow & & \\
\mathcal{E} & \xhookrightarrow{\text{at } x} & \mathcal{E}'
\end{array}$$

we can fill it in to

$$\begin{array}{ccc}
\mathcal{L} & \dashrightarrow & \mathcal{L}' \\
\big\downarrow & & \big\downarrow \\
\mathcal{E} & \xhookrightarrow{\text{at } x} & \mathcal{E}'
\end{array}$$

Indeed, we take $\mathcal{L}'$ to be the saturation of the image of $\mathcal{L}$ in $\mathcal{E}'$.

Thanks to the cartesian-ness, base change applies to the squares in the diagram. Therefore, we get an obvious compatibility relation by following two maps $H_c^*(\mathrm{Sht}_G) \to H_c^*(\mathrm{Sht}_H)$.

To finish, we recall that the constant term

$$\mathcal{H}_G \to \mathcal{H}_H$$

sends

$$h_x \mapsto t_x + q_x t_x^{-1}.$$

The middle row

$$\mathrm{Sht}_B \dashleftarrow S \dashrightarrow \mathrm{Sht}_B \tag{16.1}$$

maps points as in

$$\begin{array}{ccccccc}
\mathcal{L} & \dashleftarrow & \mathcal{L} & \hookrightarrow & \mathcal{L}' & \dashrightarrow & \mathcal{L}' \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{E} & \dashleftarrow & \mathcal{E} & \hookrightarrow & \mathcal{E}' & \dashrightarrow & \mathcal{E}'
\end{array}$$

The middle object $\mathrm{Sht}_B(h_x)$ is a disjoint union of two things. One is where the modification occurs in the sub, and one is where it doesn't. In that latter case, it occurs in the quotient. Write $\mathrm{Sht}_B(h_x) = S_1 \sqcup S_2$, where S_1 parametrizes the modifications with $\mathcal{L} \cong \mathcal{L}'$ and S_2 parametrizes the modifications with $\mathcal{M} \cong \mathcal{M}'$. Then we can separate the correspondence (16.1) into two ones:

$$\mathrm{Sht}_B \xleftarrow{\sim} S_2 \xrightarrow{q_x:1 \text{ étale}} \mathrm{Sht}_B \tag{16.2}$$

and

$$\mathrm{Sht}_B \xleftarrow{1:q_x \text{ étale}} S_1 \xrightarrow{\sim} \mathrm{Sht}_B \tag{16.3}$$

□

16.4. Statement of the main theorem. There's a finite type substack of Sht_G outside of which the map from Sht_B is an isomorphism. Thus Sht_B is the "infinite part" of Sht_G. So the cohomology of Sht_G on this infinite part is the same as on the corresponding part of Sht_B, which can then be calculated by pushforward to Sht_H. The Sht_H is a $\mathrm{Pic}^0(\mathbf{F}_q)$-torsor over X, which we understand well. So the issue is in understanding the fibers of $\mathrm{Sht}_B \to \mathrm{Sht}_H$.

THEOREM 16.4. *For large enough degrees, fibers of* $\mathrm{Sht}_B^d \to \mathrm{Sht}_H^d$ *are isomorphic to an affine space* $\mathbf{G}_a^{r/2}$ *divided by a finite étale group scheme* Z.

COROLLARY 16.5. *Let* $\pi_G \colon \mathrm{Sht}_G \to X^r$. *For large* d, *the cone of*

$$R\pi_{G!}(\mathrm{Sht}_G^{<d}) \to R\pi_{G!}(\mathrm{Sht}_G^{\leq d})$$

has cone some locally constant sheaf on X^r, *concentrated in degree* r.

17. Cohomological spectral decomposition and finishing the proof (Chao Li)

17.1. Overview. The goal is to prove the main theorem:

THEOREM 17.1. *For all* $f \in \mathcal{H}_G$,

$$(\log q)^{-r}\mathbb{J}_r(f) = \mathbb{I}_r(f).$$

If we have proved this for all f, then we can apply spectral decomposition. On the left side, we use the analytic spectral decomposition to extract the term

$$\lambda_\pi(f)\mathcal{L}^{(r)}(\pi_{F'}, 1/2).$$

On the right side we use the cohomological spectral decomposition to extract the term

$$\langle [\mathrm{Sht}_T]_\pi, f * [\mathrm{Sht}_T]_\pi \rangle.$$

If we then take $f = \mathbf{1}_K \in \mathcal{H}$ then we get

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) = \langle [\mathrm{Sht}_T]_\pi, [\mathrm{Sht}_T]_\pi \rangle.$$

So we arrive at the desired higher Gross-Zagier formula.

The most difficult intersection is the self-intersection. Allowing the choice of h_D for large D lets us take the intersection in easier cases, and then deduce it for the one we really want.

The strategy is to show that validity of the identity for h_D for all D with $d := \deg D \gg 0$ (in fact, $d \geq \max\{2g'-1, 2g\}$) is enough to deduce it for all $f \in \mathcal{H}$. This implication uses only elementary commutative algebra. What makes it possible is finiteness properties of the action of $\mathcal{H}$ on the middle-dimensional cohomology of shtukas.

17.2. Key finiteness theorems. Let $\mathcal{H}_G = \bigotimes_{x \in |X|} \mathcal{H}_x$ be the Hecke algebra. It acts on the vector space $V := H_c^{2r}(\mathrm{Sht}_G, \mathbf{Q}_\ell)$. The difficulty is that V is infinite-dimensional, because Sht_G is only locally of finite type.

Example 17.2. For $r = 0$, $V = \mathcal{A} = C_c^\infty(G(F)\backslash G(\mathbf{A})/K, \mathbf{Q}_\ell) = \mathcal{A}_{\mathrm{Eis}} \oplus \mathcal{A}_{\mathrm{cusp}}$. The cuspidal part is finite-dimensional, but the Eisenstein part is infinite-dimensional.

Therefore, we bring in the Eisenstein ideal to kill the Eisenstein part. Recall we have the Satake transform

$$a_{\mathrm{Eis}} \colon \mathcal{H}_G \xrightarrow{\mathrm{Sat}} \mathcal{H}_{\mathbf{G}_m} = \mathbf{Q}_\ell[\mathrm{Div}_X(k)] \twoheadrightarrow \mathbf{Q}_\ell[\mathrm{Pic}_X(k)].$$

This is compatible with the local Satake transforms

$$\begin{array}{ccc} \mathcal{H}_G & \xrightarrow{\mathrm{Sat}} & \mathcal{H}_{\mathbf{G}_m} \\ \uparrow & & \uparrow \\ \mathcal{H}_{G,x} & \xrightarrow{\mathrm{Sat}} & \mathcal{H}_{\mathbf{G}_m,x} \end{array}$$

$$h_x \longrightarrow t_x + q_x t_x^{-1}.$$

Definition 17.3. We define

$$\mathcal{I}_{\mathrm{Eis}} := \ker(\mathcal{H}_G \to \mathbf{Q}_\ell[\mathrm{Pic}_X(k)]).$$

The map a_{Eis} is surjective:

$$\mathcal{H}_G/\mathcal{I}_{\mathrm{Eis}} \cong \mathbf{Q}_\ell[\mathrm{Pic}_X(k)]^\iota.$$

Thus $Z_{\mathrm{Eis}} := \mathrm{Spec}\ \mathcal{H}/\mathcal{I}_{\mathrm{Eis}} \hookrightarrow \mathrm{Spec}\ \mathcal{H}$ is a reduced, 1-dimensional subvariety.

Remark 17.4. A $\overline{\mathbf{Q}}_\ell$-point of Spec $\mathcal{H}$ is a map

$$\mathcal{H}_G \xrightarrow{s} \overline{\mathbf{Q}}_\ell.$$

The condition that this is in Z_{Eis} says that it factors through

$$\begin{array}{ccccc} \mathcal{H}_G & \longrightarrow & \mathcal{H}_G/\mathcal{I}_{\mathrm{Eis}} & \longrightarrow & \overline{\mathbf{Q}}_\ell \\ & \searrow & \| & \nearrow & \\ & & \mathbf{Q}_\ell[\mathrm{Pic}_X(k)]^\iota & & \end{array}$$

So it factors through a character $\chi\colon \mathrm{Pic}_X(k) \to \overline{\mathbf{Q}}_\ell^*$, and the definition of the Eisenstein ideal implies that

$$s(h_x) = \chi(t_x) + q_x\chi(t_x^{-1}).$$

Example 17.5. For $\chi = 1$, we see that $s(h_x) = 1 + q_x$. This is analogous to Mazur's Eisenstein ideal, $T_p \mapsto 1 + p$.

THEOREM 17.6. *$\mathcal{I}_{\mathrm{Eis}} \cdot V$ is finite-dimensional over $\mathbf{Q}_\ell$.*

PROOF. We have a stratification

$$\mathrm{Sht}_G = \bigcup_d \mathrm{Sht}_G^{\leq d}$$

where each $\mathrm{Sht}_G^{\leq d}$ is an open substack of finite type. Then

$$V = \varinjlim_d H_c^{2r}(\mathrm{Sht}_G^{\leq d}).$$

The difference between the cohomology of $\mathrm{Sht}_G^{<d}$ and $\mathrm{Sht}_G^{\leq d}$ can be explicitly understood in terms of horocycles when $d > 2g-2$. By the discussion of horocycles, for $\pi_G\colon \mathrm{Sht}_G \to X^r$,

$$\mathrm{Cone}(R\pi_{G!}^{<d}\mathbf{Q}_\ell \to R\pi_{G!}^{\leq d}\mathbf{Q}_\ell) = R\pi_{\mathbf{G}_m!}^{d}\mathbf{Q}_\ell[-r],$$

which is moreover a local system concentrated in degree r.

So when $d \gg 0$, we understand how the cohomology grows, and we also understand the Hecke action. By the local constancy, it then suffices to show that on the geometric generic fiber $\overline{\eta}$ (so the middle dimension is r) the vector space

$$\mathcal{I}_{\mathrm{Eis}} \cdot H_c^r(\mathrm{Sht}_{G,\overline{\eta}})$$

is finite-dimensional over $\mathbf{Q}_\ell$.

For any finite-type substack of $\mathrm{Sht}_{G,\overline{\eta}}$, we get finiteness of cohomology for free. So it would be great to show that $\mathcal{I}_{\mathrm{Eis}} \cdot V$ lies in $H_c^r(U)$ for some finite type $U \subset \mathrm{Sht}_{G,\overline{\eta}}$.

To this end it suffices to show that if we take $U = \mathrm{Sht}_G^{\leq d}$ for sufficiently large d, then for all $f \in \mathcal{I}_{\mathrm{Eis}}$ the composition

$$H_c^r(\mathrm{Sht}_G) \xrightarrow{f*} H_c^r(\mathrm{Sht}_G) \to H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U)$$

is 0, as this implies that $\mathcal{I}_{\mathrm{Eis}} \cdot V$ is in the finite-dimensional vector space $\operatorname{Im} H_c^r(U)$.

We can extend the map through the injection (for large enough d)

$$H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U) \hookrightarrow \prod_{d' > 2g-2} H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d'})$$

which comes from the study of horocycles

$$\begin{array}{ccccc} H_c^r(\mathrm{Sht}_G) & \xrightarrow{f*} & H_c^r(\mathrm{Sht}_G) & \longrightarrow & H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U) \\ & \searrow & & & \big\downarrow \\ & & & & \prod_{d'>2g-2} H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d'}) \end{array}$$

Since the last map in the composition is an injection, it suffices to show that the dashed arrow is 0. To this end, we extend the diagram

$$\begin{array}{ccccc} H_c^r(\mathrm{Sht}_G) & \longrightarrow & H_c^r(\mathrm{Sht}_G) & \longrightarrow & H_c^r(\mathrm{Sht}_G)/\operatorname{Im} H_c^r(U) \\ \big\downarrow & & \big\downarrow & \searrow & \big\downarrow \\ \prod_d H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d}) & \longrightarrow & \prod H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d}) & \longrightarrow & \prod_{d'>2g-2} H_0(\mathrm{Sht}_{\mathbf{G}_m}^{d'}) \end{array}$$

If $f \in \mathcal{I}_{\mathrm{Eis}}$, then $\mathrm{Sat}(f)* = 0$ by the definition of $\mathcal{I}_{\mathrm{Eis}}$ and the compatibility of the Satake homomorphism with the constant term map, so

$$f * (H_c^r(\mathrm{Sht}_G)) \subset \operatorname{Im}(H_c^r(U) \to H_c^r(\mathrm{Sht}_G))$$

is finite-dimensional. ☐

Here is another important theorem, which we don't have time to prove.

THEOREM 17.7. *V is a finitely generated module over $\mathcal{H}_x$ for all $x \in |X|$.*

17.3. Cohomological spectral decomposition. We're only interested in the Hecke action on V and through the Satake transform, so we make the following definition:

Definition 17.8. We define $\overline{\mathcal{H}} = \operatorname{Im}(\mathcal{H}_G \to \operatorname{End}(V) \times \mathbf{Q}_\ell[\mathrm{Pic}_X(k)])$.

COROLLARY 17.9. *$\overline{\mathcal{H}}$ is a finitely generated algebra over $\mathbf{Q}_\ell$.*

PROOF. Since $\overline{\mathcal{H}} \hookrightarrow \operatorname{End}_{\mathcal{H}_x}(V \oplus \mathbf{Q}_\ell[\mathrm{Pic}_X(k)])$ and V and $\mathbf{Q}_\ell[\mathrm{Pic}_X(k)]$ are finitely generated $\mathcal{H}_x$-modules, we deduce that $\overline{\mathcal{H}}$ is a finitely generated module over $\mathcal{H}_x$. Therefore, it is a finitely generated algebra over $\mathbf{Q}_\ell$. ☐

THEOREM 17.10. *We have*

(1) $\operatorname{Spec} \overline{\mathcal{H}}^{\mathrm{red}} = Z_{\mathrm{Eis}} \cup Z_0^r$ *where* Z_0^r *is finite. Here* Z_{Eis} *is 1-dimensional and* Z_0^r *is a finite set of closed points.*

(2) $V = V_{\mathrm{Eis}} \oplus V_0$ *as* $\mathcal{H}_G$*-modules, with*

$$\operatorname{supp} V_{\mathrm{Eis}} \subset Z_{\mathrm{Eis}}$$

and

$$\operatorname{supp} V_0 = Z_0^r.$$

17.4. Proof of the main identity.

LEMMA 17.11. *Let* $I \subset \mathcal{H}_x$ *be a non-zero ideal. Then for any* $m \geq 1$,

$$I + \{h_{nx}, n \geq m\} = \mathcal{H}_x.$$

PROOF. We can identify $\mathcal{H}_x \cong \mathbf{Q}_\ell[t, t^{-1}]^\iota \cong \mathbf{Q}_\ell[t + t^{-1}]$. Under this identification,

$$h_{nx} = t^n + t^{n-2} + \ldots + t^{-n}.$$

So the proof reduces to showing that

$$I + \{t^n + t^{-n} \colon n \geq m\} = \mathbf{Q}_\ell[t + t^{-1}].$$

This is an elementary exercise in algebra. □

Note that the validity of $\mathbb{I}(f) = \mathbb{J}(f)$ only depends on the image of f inside $\overline{\mathcal{H}} := \operatorname{Im}(\mathcal{H} \hookrightarrow \operatorname{End}(V) \oplus \operatorname{End}(\mathcal{A}))$ which is a finitely generated $\mathbf{Q}_\ell$-algebra by Corollary 17.9.

Definition 17.12. Let $\mathcal{H}'_{d_0} \subset \mathcal{H}_G$ be the subalgebra of $\mathcal{H}$ generated by the elements h_D for all $\deg D \geq d_0$.

LEMMA 17.13. *For any* $d_0 \geq 1$*, there exists an ideal* $I \subset \overline{\mathcal{H}}$ *such that*

(1) $I \subset \operatorname{Im}(\mathcal{H}'_{d_0} \to \overline{\mathcal{H}})$*, and*

(2) $\overline{\mathcal{H}}/I$ *is finite dimensional.*

PROOF. Commutative algebra using the key finiteness theorems. □

Using Lemmas 17.11 and 17.13, we deduce the result needed for the main identity.

COROLLARY 17.14. *For any* $d_0 \geq 1$*, the composition*

$$\mathcal{H}'_{d_0} \hookrightarrow \mathcal{H} \to \overline{\mathcal{H}}$$

is surjective.

PROOF. Take I as in Lemma 17.13. Since I is generated by h_D for $\deg D \geq d_0$, it suffices to show that the composition

$$\mathcal{H}'_{d_0} \hookrightarrow \mathcal{H} \to \overline{\mathcal{H}} \to \overline{\mathcal{H}}/I$$

is surjective.

Consider the corresponding local statement: for all $x \in |X|$, we have a map

$$\mathcal{H}_x \cap \mathcal{H}' \hookrightarrow \mathcal{H}_x \to \operatorname{Im}(\mathcal{H}_x) \subset \overline{\mathcal{H}}/I.$$

Lemma 17.13 (2) tells us that $\overline{\mathcal{H}}/I$ is finite-dimensional. Therefore $\operatorname{Im}(\mathcal{H}_x)$ is finite-dimensional. By Lemma 17.11, noting that the image of I in $\mathcal{H}_x$ is non-zero because $\operatorname{Im}(\mathcal{H}_x)$ is finite-dimensional, $\mathcal{H}_x \cap \mathcal{H}' \twoheadrightarrow \operatorname{Im}(\mathcal{H}_x)$.

Since this works for every x, we get the surjectivity of the global map

$$\mathcal{H}' \hookrightarrow \mathcal{H} \twoheadrightarrow \overline{\mathcal{H}} \twoheadrightarrow \overline{\mathcal{H}}/I$$

as desired.

$\square$