Rational Points on Intersections of Quadrics

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This talk will have two halves. The first half is a survey of what is known about intersections of quadrics, and then I'll describe work in progress about intersections of three quadrics.

1 Quadrics

We consider one quadratic form $Q(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$. We denote by $\mathcal{Q}$ the hypersurface $Q = 0$ in $\mathbb{P}^{n-1}$.

The *Hasse principle* plus *weak approximation* are equivalent to the statement that $\mathcal{Q}(K)$ is dense in $\mathcal{Q}(\mathbb{A}_K)$.

We know that this holds for smooth quadrics. In addition, we know that $\mathcal{Q}(\mathbb{A}_k) \neq \emptyset$ if $n \geq 5$ and K is totally complex.

We would like this also to hold for intersections of more quadrics, but that's much more difficult. Let's go straightaway to the general situation: consider the intersection of $Q_1, \ldots, Q_r(\underline{x}) \in K[\underline{x}]$. There are two general approaches.

1. (Leep) Assume that K is totally complex. Suppose we have a number $n(r)$ such that whenever $n \geq n(r)$, implies that $Q_1 \cap \ldots \cap Q_r$ has at least one k-point. (For example, we can take $n(1) = 5$.)

   Theorem 1.1. *If $n \geq n(t) + tr + t$, then $Q_1 \cap \ldots \cap Q_r$ contains a t-plane defined over k.*

   Note that if $t = 0$, then this says there is a 0-plane, i.e. a point, over k; this is just the hypothesis. If we take $t = 4$, then we are considering 4-planes, and any quadric in a 4-plane has a rational point (this is a reformulation of $n(1) = 5$). Therefore, we see that $n(r+1) = n(r) + 4r + 4$ is admissible. An easy induction shows that $n(r) = 2r(r+1) + 1$ is admissible.

2. (Hardy-Littlewood circle method, applied by Birch over $\mathbb{Q}$ and Skinner over K) This gives the Hasse principle, weak approximation, and asymptotic formulae. Here one needs $n \geq \rho + 2r(r+1) + 1$. Here ρ is the dimension of the "Birch singular locus," and for a smooth intersection we have $\rho \in [0, r-1]$.

It's useful to compare the outputs from these two methods.

- For Leep's method, there is no smoothness (or any other geometric) assumption. The bound is slightly better because there is no ρ. There are no local conditions required, at least at the finite places.

  On the other hand, it doesn't give weak approximation and it's only for totally complex fields.

- Birch/Skinner's analytic approach works over an arbitrary field, gives weak approximations and asymptotic formulae.

Local conditions. First, there can always be local obstructions for real completions of K (e.g. if the form is positive-definite).

For finite places, $n \geq 5$ suffices for $r = 1$, and $n \geq 9$ suffices for $r = 2$. By Leep's method, $n \geq 2r(r + 1) + 1$ suffices in general. This isn't even sharp at $r = 2$, so we might ask what is.

Conjecture 1.2. $n \geq 4r + 1$ *always suffices.*

Theorem 1.3. *If the cardinality of the residue field of* K_v *is at least* $(2r)^r$*, then* $n \geq 4r + 1$ *suffices.*

The residue field bound is not sharp: 37 is enough for $r = 3$.

2 Two quadrics

The Hasse principle and weak approximation may fail, even for smooth intersections.

Example 2.1. The intersection

$$\begin{cases} x_1x_2 - (x_3^2 - 5x_4^2) & = 0 \\ (x_1 + x_2)(x_1 + 2x_2) - (x_3^2 - 5x_5^2) & = 0 \end{cases}$$

fails the Hasse principle (example due to BSD). One can conjecture that the HP and WA hold for smooth intersections when $n \geq 6$.

Much of the knowledge about intersections of two quadrics can be found in a seminal paper from 1987 of Colliot-Thelene, S, Swinnerton-Dyer:

- Over a totally complex K, any two quadratic forms in $n \geq 9$ variables have a nontrivial common zero. (The corresponding result by Leep's method would give $n \geq 13$).

  To oversimplify, the argument is by handling smooth intersections, and then considering the singular possibilities case-by-case.

- For any K, the HP and WA hold for smooth intersections of for $n \geq 9$.

The paper proposes a plan of attack for intersections of two quadrics in eight variables.

Theorem 2.2 (Heath-Brown)**.** *For any number field K, the HP and WA hold for smooth intersections of quadratics in $n \geq 8$ variables.*

Proof. This is a purely local question. The argument reduces to the residue field. Since the residue field might have characteristic 2, there is much consideration of quadratic forms in characteristic 2. □

3 Three quadrics

Let $r = 3$.

Leep's method in its basic form handles $n \geq 25$, but there is an easy variant that handles $n \geq 21$. Of course, this only applies for K totally complex.

The Birch/Skinner approach handles $n \geq \rho = 25$, so that tells you that $n \geq 27$ suffices for smooth intersections. Also, $n \geq 17$ is sufficient to satify the local conditions at *all* finite places.

Goal: if $Q_1, Q_2, Q_3 \in k[\underline{x}]$ are quadratic forms over a number field K defining a smooth intersection, then HP and WA hold for $n \geq 19$.

This beats the output from the analytic method by a lot.

3.1 Plan of attack

Let $\mathcal{Q}_i$ be the quadric $Q_i = 0$ and $\mathcal{R} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3$.

1. Replace Q_1, Q_2, Q_3 be more suitable generators for $\langle Q_1, Q_2, Q_3 \rangle$. In particular, we want that

   - $\mathcal{Q}_3$ is smooth and contains a 7-plane defined over K.
   - $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is also smooth.

   Why this 7-plane? Geometrically, $\mathcal{Q}_3$ will contain a t-plane over $\overline{K}$ for $t \leq \frac{n-1}{2}$. Since quadrics have local points at finite places when $n \geq 5$, $\mathcal{Q}_3$ automatically contains a 7-plane for such completions. (The importance behind 7 is that $7 = \frac{19-5}{2}$.) Then you have to do some fiddling at the infinite places, but we understand that well.

2. If $\mathcal{Q}_3$ contains one 7-plane L over K, then it contains a lot of them. Choose one such that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ is smooth, and contains points everywhere locally.

3. Apply the result on pairs of quadratic forms in 8 variables.