Discussion Session on Function Field Analogues

Notes by Tony Feng

April 8, 2016

These are notes on a discussion session given by Urs Hartl on function field analogues.

For references, see articles 7, 12 (dictionary), 13, 15, 17, 23, 24 (survey) on www.math.uni-muenster.de/u/urs.hartl/Public.html.en

1 The Fargues-Fontaine Curve

In constructing the curve, we start with a base local field which is either of mixed or equal characteristics.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$

There is also the input of a perfectoid field. Whereas in the mixed case we have a field $\mathbb{C}_p/\mathbb{Q}_p$ and a $\mathbb{C}_p^\flat$ over a characteristic p field, in the equal case we start with two symmetric local rings in characteristic p.

Number Field	Function Field
$\mathbb{Q}_p$ $\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z))$ $\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$

While in characteristic 0 we build the Witt vectors of some algebra R, in equal characteristic we build the power series ring.

Number Field	Function Field	Remarks
$\mathbb{Q}_p$ $\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$ $W(R^+)$	$\mathbb{F}_p((z))$ $\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$ $R^+[[z]]$	 $R^+ = \text{algebra }/\mathbb{F}_p[[\zeta]]$

We can imagine for simplicity that $R^+ = \mathbb{F}_p[[\zeta]]$, so $R = \mathbb{F}_p((\zeta))$. To build the curve we take the adic spectrum, and then poke out two points.

Number Field	Function Field
$\mathbb{Q}_p$ $\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$ $W(R^+)$ $Y_{(R,R^+)} = \text{Spa}(W(R^+), W(R^+)) \setminus V(p[\varpi^\flat])$	$\mathbb{F}_p((z))$ $\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$ $R^+[[z]]$ (imagine $R^+ = \mathbb{F}_p[[\zeta]]$) $Y_{(R,R^+)} = \text{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta)$

There are two Frobenii acting in the equal characteristic case. We denote by φ_R the one which takes $\zeta \mapsto \zeta^p$ and $z \mapsto z$. In equal characteristic we have $Y^{\mathrm{ad}}_{(R,R^+)} = \operatorname{Spa}(R, R^+) \times_{\mathbb{F}_p} \operatorname{Spa} E$. Now we quotient by Frobenius.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$
$W(R^+)$	$R^+[[z]]$ (imagine $R^+ = \mathbb{F}_p[[\zeta]]$)
$Y_{(R,R^+)} = \operatorname{Spa}(W(R^+), W(R^+)) \setminus V(p[\varpi^\flat])$	$Y_{(R,R^+)} = \operatorname{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta)$
$X^{\mathrm{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$	$X^{\mathrm{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$

Vector bundles on X^{ad} are vector bundles $\mathcal{E}$ on Y^{ad} plus an equivariant structure $\varphi_{\mathcal{E}}\colon \varphi^*\mathcal{E} \cong \mathcal{E}$. In equal characteristic,

$$H^0(Y^{\mathrm{ad}}_{(R,R^+)}, \mathcal{O}_{Y^{\mathrm{ad}}}) = \left\{ \sum_{i=-\infty}^{\infty} b_i z^i, \quad b_i \in R \mid \text{ convergent on } 0 < |z| < 1 \right\}.$$

The radius function is the distance to $z = 0$. We define Y^I to be the (adic) spectrum of the ring of power series convergent on $\{|z| \in I\}$.

We have a bundle $\mathcal{O}(d)$ corresponding to $\mathcal{E} = (\mathcal{O}_{Y^{\mathrm{ad}}}, \varphi_{\mathcal{E}} = \cdot z^{-d})$. It is relatively easy to write down global sections in equal characteristic. In mixed characteristic it is much harder, because the elements are not truly power series.

Number Field	Function Field
$\mathbb{Q}_p$	$\mathbb{F}_p((z))$
$\mathbb{C}_p/\mathbb{Q}_p \leftrightarrow \mathbb{C}_p^\flat/(\text{char p})$	$\mathbb{F}_p((z)) \leftrightarrow \mathbb{F}_p((\zeta))$
$W(R^+)$	$R^+[[z]]$ (imagine $R^+ = \mathbb{F}_p[[\zeta]]$)
$Y_{(R,R^+)} = \operatorname{Spa}(W(R^+), W(R^+)) \setminus V(p[\varpi^\flat])$	$Y_{(R,R^+)} = \operatorname{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta)$
$X^{\mathrm{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$	$X^{\mathrm{ad}}_{(R,R^+)} = Y_{(R,R^+)}/\varphi_R$
$\mathcal{O}(d) = (\mathcal{O}_{Y^{\mathrm{ad}}}, \varphi_{\mathcal{E}} = \cdot p^{-d})$	$\mathcal{O}(d) = (\mathcal{O}_{Y^{\mathrm{ad}}}, \varphi_{\mathcal{E}} = \cdot z^{-d})$
Hard to write sections	Easy to write sections

2 p-divisible groups

The basic analogues are tabulated below.

Number Field	Function Field
p-divisible groups / R^+	divisible local Anderson modules / R^+
Dieudonné modules	effective local shtukas / R^+

However there are some differences. For instance, the functor from p-divisible groups to Dieudonné modules is not known to be fully faithful in general. However, the functor from divisible local Anderson modules to effective local shtukas is fully faithful.

Now what are the things on the right side anyway? Divisible local Anderson modules are too messy to define. However, we can say define an effective local shtukas.

Definition 2.1. An *effective local shtuka* is a pair $\underline{M} = (M, \varphi_M)$ with M a finite projective over $R^+[[z]]$ and

$$\varphi_M \colon \varphi^* M[\frac{1}{z-\zeta}] \cong M[\frac{1}{z-\zeta}].$$

Here

$$\varphi^* M[\frac{1}{z-\zeta}] = M \otimes_{R^+[[z]],\varphi_R} R^+[[z]].$$

Thus φ_M is the linearization of a $\varphi = \varphi_R$-linear map. We think of this as analogous to a modification between a vector bundle and its Frobenius twist on A_{inf}, which is an isomorphism away from the points $z - \zeta$. Finally, the "effective" means that we demand

$$\varphi_M(\varphi^* M) \subset M.$$

We can't really describe the functor from local divisible Anderson modules since we didn't even say what those were, but we remark that if G is such, then it is related to the corresponding M by

$$(\mathrm{Lie}\, G)^\vee = M/\varphi_M(\varphi^* M). \tag{1}$$

Example 2.2. Let $M = (R^+[[z]], \varphi_M = (z-\zeta))$. This is the analogue of the p-divisible group μ_{p^∞}. The divisible local Anderson module would be $\widehat{\mathbb{G}}_{a,R}$ with an $\mathbb{F}_p[[z]]$-action, where z acts by $[z](X) = \zeta X + X^p$ because $\varphi_M = z - \zeta$. This is a Lubin-Tate formal group.

In view of 1 we have $\mathrm{Lie}\, G = R^+[[z]]/(z-\zeta)$, soo $[z]|_{\mathrm{Lie}\, G} = \zeta$. This is plausible if you remember the analogy $z \leftrightarrow p$, and that for a p-divisible group the action of multiplication by p induces multiplication by p on the Lie algebra.

3 B_{dR}

Recall that

$$Y^{\mathrm{ad}}_{(R,R^+)} = \mathrm{Spa}(R^+[[z]], R^+[[z]]) \setminus V(z\zeta).$$

There is a closed subset $V(z-\zeta)$ which induces

$$\theta \colon R^+[[z]] \to R$$

sending $z \mapsto \zeta$.

In mixed characteristic we defined the ring

$$B^+_{\mathrm{dR}}(R) = \varprojlim W(R^+)[1/[\varpi^\flat]]/(\xi^n).$$

We think about this as the completion of $Y^{\mathrm{ad}}_{(R,R^+)}$ along a closed subvariety of codimension one:

$$B^+_{\mathrm{dR}}(R) = (\mathcal{O}_{Y^{\mathrm{ad}}_{(R,R^+)}, V(\xi)})^\wedge.$$

In the equal characteristic side, we define

$$B^+_{\mathrm{dR}} = \varprojlim R^+[[z]][1/\zeta]/(z-\zeta)^n = (\mathcal{O}_{Y^{\mathrm{ad}}_{(R,R^+)}, V(z-\zeta)})^\wedge.$$

Note that this is simply isomorphic to $R[[z-\zeta]]$; this is analogous to how for $R = \mathbb{C}_p^\flat$ then on the left we get in the classical case an isomorphism of rings $B^+_{\mathrm{dR}}(\mathbb{C}_p^\flat) \cong \mathbb{C}_p[[\xi]]$.

4 The Period Map

Let $\underline{M}$ be a local shtuka over (R, R^+). We define its *de Rham cohomology* to be

$$H^1_{\mathrm{dR}}(\underline{M}, B^+_{\mathrm{dR}}(R)) := \varphi^* M \otimes_{R^+[[z]]} R[[z-\zeta]].$$

To put this in its proper context, let's remember the analogies going on here. Imagining $R = \mathbb{F}_q[[\zeta]]$, the ring $R^+[[z]]$ is analogous to $A_{\mathrm{inf}} = W(R^+)$ in mixed characteristic, and $\varphi^* M$ is a vector bundle over it. Then $R[[z-\zeta]]$ is analogous to B^+_{dR} in mixed characteristic, which we can thought of as the completion of A_{inf} along the point at ∞ describing an untilt.

The upshot is that if we think of M as a vector bundle on A_{inf}, then we are defining its de Rham cohomology is the restriction to a formal disk about infinity.

Think about this from the perspective of the curve. For a vector bundle on Y, restricting to a neighborhood of infinity gives a B^+_{dR}-module, while restricting to its complement gives a B^+_{cris}-module. Vector bundles on the curve correspond precisely, by a Beauville-Laszlo uniformization interpretation, to $(B^+_{\mathrm{dR}}, B^+_{\mathrm{cris}})$ modules.

$$H^1_{\mathrm{cris}}(\underline{M}, R^+/(\zeta)[[z]]) := \varphi^* M \otimes_{R^+[[z]]} R^+/(\zeta)[[z]].$$

There is a comparison between crystalline and de Rham cohomology by the Genestevier-Lafforgue Lemma. It says the following. Let $R^+ = k[[\zeta]]$. Then there is a map

$$R^+/(\zeta)[[z]] = k[[z]] \to B^+_{\mathrm{dR}} = k((\zeta))[[z-\zeta]]$$

given by

$$z \mapsto z = \zeta + (z-\zeta)$$

and the de Rham and crystalline cohomologies, as defined above, become isomorphic for this comparison.

There is also an étale cohomology group $H^1_{\text{ét}}(\underline{M}, \mathbb{F}_p[[z]])$. To describe it, first tensor $M \otimes_{k[[z]]} k((\zeta))^{\mathrm{sep}}[[z]]$. Frobenius is an isomorphism after inverting $z-\zeta$, which is indeed invertible here, so we can take

$$H^1_{\text{ét}}(\underline{M}, \mathbb{F}_p[[z]]) := (M \otimes_{k[[z]]} k((\zeta))^{sep}[[z]])^{\varphi=1}.$$

Finally, let's discuss the period morphism. Consider

$$H^1_{\mathrm{dR}}(\underline{M}, B_{\mathrm{dR}}) := \varphi^* M \otimes_{R^+[[z]]} R[[z-\zeta]][\frac{1}{z-\zeta}].$$

There is a submodule

$$\varphi_M^{-1}(M \otimes_{R^+[[z]]} R[[z-\zeta]]) \subset H^1_{\mathrm{dR}}(\underline{M}, B_{\mathrm{dR}}).$$

This is called the *Hodge-Pink lattice*; it corresponds to $\mathrm{Fil}^0\, H^1_{\mathrm{dR}}(\underline{M}, B_{\mathrm{dR}})$ for the Hodge filtration.

The period map takes $\underline{M}$ to its Hodge-Pink lattice. But to make sense of this we have to say in which ambient space this lattice varies - that is, we have to "fix" $H^1_{\mathrm{dR}}(\underline{M}, B_{\mathrm{dR}})$. There is a Rapoport-Zink space of deformations of a fixed $\underline{\mathbb{M}}$, parametrizing pairs

$$(\underline{M}, \underline{M}_{R^+/\zeta} \cong \underline{\mathbb{M}}_{R^+/\zeta}).$$

By the crystalline nature of the cohomology functors, the isomorphism modulo p lifs canonically to an isomorphism $H^1_{\mathrm{dR}}(\underline{M}, B_{\mathrm{dR}}) \cong H^1(\underline{\mathbb{M}}, B_{\mathrm{dR}})$.