FINITE GROUP THEORY: SOLUTIONS

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These are hints/solutions/commentary on the problems. They are not a model for what to actually write on the quals.

1. 2010 FALL MORNING 5

(i) Note that G acts transitively on the set of ℓ. Picking $\ell = [1:0]$, we see that the stabilizer of ℓ is the upper-triangular Borel

$$\mathrm{Stab}_G(\ell) = \begin{pmatrix} * & * \\ & * \end{pmatrix}.$$

Since the stabilizer groups of all the ℓ are conjugate (by transitivity), it suffices to prove that this particular one has a unique p-sylow. By counting its size, the p-Sylow has order $q = p^n$. By inspection, the 'unipotent radical"

$$N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

is a p-Sylow in $\mathrm{Stab}_G(\ell)$. Since all p-Sylows are conjugate, the statement that it is unique is equivalent to it being normal, which we check explicitly.

You should know how to do the computation

$$\#G(\mathbf{F}_q) = (q^2-1)(q^2-q).$$

In particular, the biggest power of p dividing this is $q = p^n$, so N is also a p-Sylow of G. Since all p-Sylows are conjugate, to count the total number of p-Sylows we just have to count the number of N. Alternatively, show that they are in bijection with ℓ. To recover ℓ from N, take the span of the fixed vector of N.

(ii) First assume $\ell_1 = [0,1]$ and $\ell_3 = [1,0]$. Then we are asking for g fixing ℓ_1, ℓ_3 and taking any ℓ_2 to $[1,1]$. Note that we can express $\ell_2 = [a,1]$ with $a \neq 0$. Then take

$$g = \begin{pmatrix} a & \\ & 1 \end{pmatrix}.$$

Now for the general case. Argue that we can find any g taking ℓ_1 to $[0,1]$ and ℓ_3 to $[1,0]$. Then g might not send ℓ_2 to $[1,1]$, but by the first special case there is an h sending $g(\ell_2)$ to $[1,1]$ and fixing the other lines, so $h \circ g$ does the trick.

(iii) We proved in (i) that P_i is the unique p-Sylow subgroup of $\mathrm{Stab}_G(\ell_i)$ for some ℓ_i, and Q_i is the unique p-Sylow subgroup of $\mathrm{Stab}_G(\ell_i')$. By (b) there exists g such that $g(\ell_i) = \ell_i'$. Therefore $g\,\mathrm{Stab}_G(\ell_i)g^{-1} = \mathrm{Stab}_G(g\ell_i)$, and so the unique p-Sylow subgroups match.

2. 2011 Spring Morning 1

(a) Proceed by contradiction. We assume that

$$G = \bigcup_{g \in G} gHg^{-1}. \tag{2.0.1}$$

Let's count how many distinct gHg^{-1} appear above. By orbit-stabilizer, it is $G/N_G(H)$. Now note that $H \subset N_G(H)$, so $|G/N_G(H)| \leq |G/H|$.

Each conjugate of H has $|H|$ elements, and each contains the identity of G. So the total number of elements of G accounted for by the right side of (2.0.1) is

$$1 + |G/N_G(H)| \cdot (|H|-1) \leq 1 + |G/H| \cdot (|H|-1) < |G|$$

if $|H| < |G|$.

(b) The stabilizers are all all conjugate. By (a), there is some $g \in G$ not in any stabilizer.

3. 2013 Fall Afternoon 1

(a) In this case $gHg^{-1} \cap H$ are elements in G fixing both x and gx. If G is Frobenius then this has only the identity. Conversely, if G is not Frobenius then some $g \in G$ lies in $\mathrm{Stab}_G(kx)$ and $\mathrm{Stab}_G(k'x)$. Then $kHk^{-1} \cap (k')H(k')^{-1}$ is non-trivial, so $(k')^{-1}kHk^{-1}(k') \cap H$ is non-trivial.

(b) Take $S = \mathbf{F}_q$, with G by affine transformations. Then H is the stabilizer of 0.

4. 2012 Fall Afternoon 5

Note that $\mathbf{F}_{p^3}$ as a 3-dimensional vector space over $\mathbf{F}_p = \mathbf{Z}/p$. Picking a basis for it, we can identify $\mathrm{GL}_3(\mathbf{Z}/p)$ with the group of $\mathbf{F}_p$-linear automorphisms on $\mathbf{F}_{p^3}$. The subgroup of $\mathbf{F}_{p^3}$-linear automorphisms gives an inclusion

$$\mathbf{F}_{p^3}^\times \hookrightarrow \mathrm{GL}_3(\mathbf{F}_p).$$

(i) First compute the size of $\mathrm{SL}_3(\mathbf{F}_p)$:

$$\#\mathrm{SL}_3(\mathbf{F}_p) = \frac{(p^3-1)(p^3-p)(p^3-p^2)}{(p-1)} = (p^2+p+1)^2(p-1)^2(p+1).$$

Check that if $\ell \mid p^2+p+1$ then $\ell \nmid p(p-1)(p+1)$. The ℓ-Sylow then comes from $(\mathbf{F}_{p^3}^\times)_{\mathrm{Nm}=1} \hookrightarrow \mathrm{SL}_3(\mathbf{F}_p)$.

(ii) In this case the 3-Sylow is the semidirect product of the 3-Sylow in $(\mathbf{F}_{p^3}^\times)_{\mathrm{Nm}=1}$ with $\mathrm{Gal}(\mathbf{F}_{p^3}/\mathbf{F}_p)$, which is not even commutative.

5. 2014 Spring Morning 3

(i) Note that $xyx^{-1}y^{-1}$ lies in P_2, since it can be written as

$$x \cdot (yx^{-1}y^{-1})$$

with both factors in P_2 by normality. Similarly, it lies in P_7. But any element in the intersection of P_2 and P_7 has order simultaneously a power of 2 and of 7, so the order must be 1.

(ii) Let n_2 denote the number of 2-Sylows and n_7 denote the number of 7-sylows. By the Sylow theorems, we know:

- $n_2 \equiv 1 \pmod 2$ and $n_2 \mid 7$.
- $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 8$.

We want to show that $n_2 = 1$ or $n_7 = 1$. If not, then by inspection we must have $n_2 = 7$ and $n_7 = 8$. We'll show that there are not enough elements in the group to allow this to happen.

Any two Sylow 7-subgroups can intersect in only the identity element, since they are cyclic. So each Sylow 7-subgroup contributes 6 new elements of order 7, for a total of $8 \times 6 = 48$ distinct non-identity elements in G of order 7.

Any two Sylow 2-subgroups can intersect in a group of size at most 4. Therefore, two distinct 2-Sylow subgroups contribute at least $8+4 = 12$ elements not already accounted for by the above count of elements of order 7. But $12+48 = 60$ already exceeds the size of G.

(iii) Make a non-split semi-direct product $\mathbf{Z}/7 \rtimes \mathbf{Z}/8$ by having $\mathbf{Z}/8$ act through the non-trivial homomorphism $\mathbf{Z}/8 \to (\mathbf{Z}/7)^* \cong \mathrm{Aut}(\mathbf{Z}/7)$. (The non-normality follows from the fact that the two subgroups don't commute.)

Make a non-split semi-direct product $(\mathbf{Z}/2)^3 \rtimes \mathbf{Z}/7$ by having $\mathbf{Z}/7$ act through $\mathbf{Z}/7 \xrightarrow{\sim} \mathbf{F}_8^\times \hookrightarrow \mathrm{GL}_3(\mathbf{Z}/2)$.

6. 2015 Spring A2

(i) By assumption, we can find $g \in G$ such that $gxg^{-1} = y$. We want to try to get $g \in N$, the normalizer of P. In other words, we want to choose g so that

$$gPg^{-1} = P.$$

We are free to translate g on the left by $C(y)$ and on the right by $C(x)$. If we can get gPg^{-1} to be in $C(y)$, then by the Sylow theorems applied to $C(y)$ we can left-translate by something in $C(y)$ so that $gPg^{-1} = P$.

Is it the case that $gPg^{-1} \subset C(y)$? This is asking if conjugation by y induces the identity on gPg^{-1}. In other words, does conjugation by $g^{-1}yg$ induce the identity on P? But $g^{-1}yg = x$, which lies in $C(P)$ by assumption!

(ii) The normalizer of N is the group of upper-triangular matrices. The matrices

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \text{ and } \begin{pmatrix} b & \\ & a \end{pmatrix}$$

and conjugate [why?] in G, but not in N.

7. 2011 Fall Morning 1

(i) Arbitrary (entry-wise) choices of lift will lie in $\mathrm{GL}_3(\mathbf{Z}/p^5)$, because the determinant commutes with reduction. The kernel is in bijection with 3×3 matrices with entries modulo p^4, hence has size p^{36}.

(ii) $\#G = p^{36}(p^3-1)(p^3-p)(p^3-p^2)$. An explicit p-Sylow is the pre-image of the unipotent radical.

8. 2012 Fall Morning 2

(i) We claim that the "unipotent group"

$$\begin{pmatrix} 1 & * & * & \dots \\ & 1 & * & \dots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}$$

is a Sylow subgroup. This has size $p^{n(n-1)/2}$. To check that it works, we compute the size of $\mathrm{SL}_n(\mathbf{F}_p)$:

$$\frac{(p^n-1)(p^n-p)\dots(p^n-p^{n-1})}{p-1}.$$

The power of p is $1+2+\dots+n-1 = \frac{n(n-1)}{2}$.

(ii) Lots of possibilities here, e.g. let P_i be the subgroup of matrices supported above the "ith superdiagonal." Alternatively, you could take matrices supported "after the ith column".

9. 2013 Spring Morning 3

(i) Use the exact sequence

$$0 \to K \to \mathrm{GL}_2(\mathbf{Z}/9) \to \mathrm{GL}_2(\mathbf{Z}/3) \to 0.$$

You should be know how to compute $\#\mathrm{GL}_2(\mathbf{Z}/p)$ for any prime p; for $p=3$ it's $(3^2-1)(3^2-3)$. It remains to compute $\#K$. This kernel is the group of matrices with entries in $\mathbf{Z}/9$, congruent to 1 modulo 3. Show that any such matrix is of the form $\mathrm{Id}+3M$. Note that this only depends on the entries of M modulo 3, and any M is possible. Therefore, K is in bijection with $\mathrm{Mat}_{3\times 3}(\mathbf{Z}/3)$, which has size 3^4.

(ii) If g has 3-power order in G, then its image in $\mathrm{GL}_2(\mathbf{Z}/3)$ does as well. If the converse is true, then some 3-power exponent of g lies in K. Argue that the bijection $K \simeq \mathrm{Mat}_{3\times 3}(\mathbf{Z}/3)$ is in fact a group homomorphism, so that K has 3-power order. (Alternatively, this can be seen by pure counting.)

(iii) Any Sylow 2-subgroup of G maps isomorphically onto its image in $\mathrm{GL}_2(\mathbf{Z}/3)$, because the kernel has to have a power of 3, since it lies in K. Therefore, it sufficesto show the same result for G replaced by $\mathrm{GL}_2(\mathbf{Z}/3)$. By counting sizes, check that a Sylow 2-subgroup has size 16. We can view $\mathrm{GL}_2(\mathbf{Z}/3) = \mathrm{GL}_2(\mathbf{F}_3)$ as the group of linear automorphisms of $\mathbf{F}_9$ viewed as a 2-dimensional $\mathbf{F}_3$-vector space (picking a basis for $\mathbf{F}_9$ over $\mathbf{F}_3$). Then the inclusion of the subgroup of automorphisms which are moreover $\mathbf{F}_9$-linear corresponds to an embedding

$$\mathbf{F}_9^\times \hookrightarrow \mathrm{GL}_2(\mathbf{Z}/3).$$

Additionally, the Galois action of $\mathrm{Gal}(\mathbf{F}_9/\mathbf{F}_3)$ is $\mathbf{F}_3$-linear and gives an embedding $\mathbf{Z}/2 \hookrightarrow \mathrm{GL}_2(\mathbf{Z}/3)$, which preserves the subgroup $\mathbf{F}_9^\times$, with the generator acting as $x \mapsto x^3$ by the Galois theory of finite fields.

10. 2010 Fall Afternoon 1

(i) Applying the inductive hypothesis to $G/G \cap N \hookrightarrow H/N$, we find that $G/G \cap N = H/N$. This implies that $G \cdot N = H$.

(ii) First, use the usual argument that $Z \neq 0$ (orbit-stabilizer for the conjugation action of H on itself). If $G \cdot Z = H$ and $G \cap Z = 0$, then $H = G \times Z$, but then G would not surject onto $H/[H,H]$.