1. Problem Set 1

(1) Let $\alpha = \phi_1\, dx + \phi_2\, dy + \phi_3\, dz$, where $\phi, \phi_2, \phi_3 \in \mathbb{C}[x,y,z]$, be a polynomial 1-form on $\mathbb{C}^3$. Define a skew-symmetric bracket $\{-,-\}_\alpha :\ \mathbb{C}[x,y,z] \times \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z]$, by the formula

$$\{-,-\}_\alpha :\ \mathbb{C}[x,y,z] \times \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z], \quad \{f,g\}_\alpha := \frac{\alpha \wedge df \wedge dg}{dx \wedge dy \wedge dz}. \tag{1.0.1}$$

(Here the 3-form in the numerator is necessarily of the form $h\, dx \wedge dy \wedge dz$ for some polynomial h, and the fraction stands for that h).

- Show that the Jacobi identity holds for $\{-,-\}_\alpha$ iff the $\alpha \wedge d\alpha = 0$. (More generally, let X be a smooth variety, $\mathsf{vol} \in \mathit{Poly}^3(X)$ a nonwhere vanishing 3-polyvector, and α a 1-form. Then, the bivector $\Pi = i_\alpha \mathsf{vol}$ is Poisson iff $\alpha \wedge d\alpha = 0$.)
- Take $\alpha := d\phi$ for some nonzero polynomial ϕ Thus, $\{-,-\}_{d\phi}$ is a nonzero Poisson bracket on $\mathbb{C}[x,y,z]$. Find $\{x,y\}_{d\phi}$.
- Show that $\mathbb{C}[\phi]$, the subalgebra generated by ϕ, is contained in the Poisson center of $\mathbb{C}[x,y,z]$.
- Show that any sufficiently general level set of ϕ is a symplectic leaf.
- Deduce that the Poisson center equals

$$\{f \in \mathbb{C}[x,y,z] \mid f \text{ is algebraic over } \mathbb{C}[\phi]\}.$$

- The Poisson bracket $\{-,-\}_{d\phi}$ descends to the quotient $A_\phi := \mathbb{C}[x,y,z]/(\phi)$. Classify all symplectic leaves in $\operatorname{Spec} A_\phi$ in the case where ϕ has isolated critical points in $\mathbb{C}^3$.

(2) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$.

- Show that symplectic leaves in $\mathfrak{g}^*$ are precisely the coadjoint G-orbits.
- Let $\lambda \in \mathfrak{g}^*$ and $x, y \in \mathfrak{g}$, The vectors $u = \operatorname{ad}^* x(\lambda), v = \operatorname{ad}^* y(\lambda)$, where ad^* denotes the (co)adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$, are tangent to the G-orbit O of the element λ. Find $\omega(u,v)$, the value of the canonical symplectic 2-form ω on the leaf O at the vectors (u,v).

(3) Let G be a Lie group and $H \subset G$ a Lie subgroup. Let $\mathfrak{g} = \operatorname{Lie} G$, resp. $\mathfrak{h} = \operatorname{Lie} H$, and identify $(\mathfrak{g}/\mathfrak{h})^*$ with $\mathfrak{h}^\perp \subset \mathfrak{g}^*$, the annihilator of the subspace $\mathfrak{h} \subset \mathfrak{g}$. We have natural identifications of vector bundles

$$T^*(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})^* \cong G \times_H \mathfrak{h}^\perp. \tag{‡}$$

- Let $\lambda, \alpha, \beta \in \mathfrak{h}^\perp$ and $x, y \in \mathfrak{g}$. We view λ as an element of the fiber of $T^*(G/H)$ over the base point $1.H/H$, and α, β as 'vertical tangent vectors' at $\lambda \in T^*(G/H)$, i.e. as elements of the tangent space $T_\lambda(T^*(G/H))$ which are tangent to the fiber of the projection $T^*(G/H) \to G/H$. Similarly, write $x(\lambda), y(\lambda)$ for the elements of $T_\lambda(T^*(G/H))$ tangent to the G-orbit of λ under the G-action on $T^*(G/H)$.
  Let ω be the canonical symplectic 2-form on $T^*(G/H)$. Express each of the numbers:

$$\omega(\alpha,\beta),\ \ \omega(x(\lambda), y(\lambda)),\ \ \omega(x(\lambda), \alpha)$$

  in terms of λ, α, β and x, y.
- The group G acts on G/H by left translations, so we have a Hamiltonian action of G on $T^*(G/H)$. Give a formula for the corresponding moment map, viewed as a map $G \times_H \mathfrak{h}^\perp \to \mathfrak{g}^*$.

(4) Let (V, ω) be a (finite dimensional) symplectic vector space and $\Gamma \subset Sp(V,\omega)$ a finite subgroup.

- Classify all symplectic leaves in V/Γ.
- More difficult: Let U be the unique open dense symplectic leaf in V/Γ and ω the corresponding symplectic 2-form on U. Prove that for any resolution of singularities

$\pi : X \to V/\Gamma$ the 2-form $\pi^*\omega$, on $\pi^{-1}(U)$, extends to a regular (possibly degenerate) 2-form on the whole of X.
Hint: Given a resolution of singularities $\pi : X \to V/\Gamma$, consider a resolution of singularities of the variety $(X \times_{V/\Gamma} V)_{\text{red}}$.

2. Problem set 2

(1) Let $\langle e, h, f\rangle$ be the standard basis of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. We identify $\mathbb{C}[\mathfrak{g}^*]$ with $\mathbb{C}[e, h, f]$.
  - Write an explicit formula for the Poisson bracket on $\mathbb{C}[e, h, f]$ transported from the canonical one on $\mathbb{C}[\mathfrak{g}^*]$.
  - Show that the Poisson center of the algebra $\mathbb{C}[e, h, f]$ is generated by the polynomial $P = h^2 + 2ef$.

(2) Find explicit formulas for each of the maps μ defined below:
  - Let (V, ω) be a symplectic vector space. The natural action of $Sp(V, \omega)$, the symplectic group, on V is Hamiltonian. Identify $(\operatorname{Lie} Sp(V, \omega))^*$ with $\operatorname{Lie} Sp(V, \omega)$ via the trace pairing. Let $\mu : V \to \operatorname{Lie} Sp(V, \omega)$ be the map obtained, using this identification, from the moment map $V \to (\operatorname{Lie} Sp(V, \omega))^*$.
  - Let G, a Lie group, act on its Lie algebra $\mathfrak{g}$ by the adjoint action. This gives a Hamiltonian action of G on $T^*\mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g}$ with moment map $T^*\mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g}^*$. Given an invariant nondegenerate symmetric bilinear form $\langle -, -\rangle$ on $\mathfrak{g}$ let $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the map obtained from the moment map via the identifications $\mathfrak{g}^* \cong \mathfrak{g}$, resp. $T^*\mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}$.
  - Let $\operatorname{Rep}(Q, d)$ be the variety of d-dimensional representations of a quiver Q. The group G_d acts on $\operatorname{Rep}(Q, d)$. Let $\mu : \operatorname{Rep}(\bar{Q}, d) \to \mathfrak{g}_d$ be the map obtained from the moment map $T^* \operatorname{Rep}(Q, d) \to (\mathfrak{g}_d)^*$ via the identifications $T^* \operatorname{Rep}(Q, d) \cong \operatorname{Rep}(\bar{Q}, d)$, resp. $(\mathfrak{g}_d)^* \cong \mathfrak{g}_d$.

(3) Fix $n \geq 2$ and let $\Gamma_n \cong \mathbb{Z}/(n)$ be the group of n-th roots of unity. We have an imbedding $\Gamma_n \hookrightarrow SL_2(\mathbb{C})$, $\zeta \mapsto \operatorname{diag}(\zeta, \zeta^{-1})$. Since $SL_2(\mathbb{C}) = Sp(\mathbb{C}^2)$, this gives a Γ_n-action on $\mathbb{C}^2$ that preserves the standard symplectic form. Hence, the induced Γ_n-action on the polynomial algebra $\mathbb{C}[u, v] = \mathbb{C}[\mathbb{C}^2]$, respects the Poisson bracket that comes from the symplectic form. Thus, the algebra $\mathbb{C}[u, v]^{\Gamma_n}$ is a Poisson subalgebra of $\mathbb{C}[u, v]$.
Construct an isomorphism of Poisson algebras
$$\mathbb{C}[u, v]^{\Gamma_n} \cong A_{\phi_n}, \quad \text{where} \quad \phi_n := x^2 + y^2 + z^n.$$
(we've used the notation of Problem 1 from Problem Set 1.)

3. Problem set 3

(1) Let V be a finite dimensional vector space. The group $GL(V)$ acts naturally on V and it also acts on $\mathfrak{gl}(V) = \operatorname{Lie} GL(V)$ by conjugation. We let $GL(V)$ act diagonally on the vector space $\mathfrak{gl}(V) \oplus V$. This gives the Hamiltonian $GL(V)$-action on
$$T^*(\mathfrak{gl}(V) \oplus V) = \mathfrak{gl}(V) \oplus \mathfrak{gl}(V) \oplus V \oplus V^*. \tag{†}$$
We will write an element of the cotangent space as a quadruple (x, y, i, j) where $x, y \in \mathfrak{gl}(V)$, $i \in V$, $j \in V^*$.
  - Find an explicit formula for the moment map
$$\mu : \mathfrak{gl}(V) \oplus \mathfrak{gl}(V) \oplus V \oplus V^* \to \mathfrak{gl}(V) \cong \mathfrak{gl}(V)^*.$$
  - Show that the $GL(V)$-action on $\mu^{-1}(\mathrm{Id})$, the fiber of μ over the identity $\mathrm{Id} \in \mathfrak{gl}(V)$, is free. So, $M := \operatorname{Spec}(\mathbb{C}[\mu^{-1}(\mathrm{Id})]^{GL(V)})$, the corresponding Hamiltonian reduction, is a smooth symplectic affine variety.
  - Find $\dim M$.

- Consider a collection of functions on the vector space (†) given by the formulas

  $$a_n(x,y,i,j) = \mathrm{Tr}(x^n), \quad b_n(x,y,i,j) = \mathrm{Tr}(y^n), \qquad n = 1,2,\dots.$$

  These functions are $GL(V)$-invariant, hence they descend to regular functions $\bar{a}_n, \bar{b}_n \in \mathbb{C}[M]$. Show that

  $$\{\bar{a}_n, \bar{a}_m\} = \{\bar{b}_n, \bar{b}_m\} = 0 \qquad \forall m,n \geq 1.$$

  and find $\{\bar{a}_n, \bar{b}_m\}$ for all $1 \leq m,n \leq 2$.
- Use the First Fundamental Theorem of Invariant Theory to prove that if $\dim V = 2$ then the functions $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2$ generate $\mathbb{C}[M]$ as a Poisson algebra, i.e. that the only Poisson subalgebra of $\mathbb{C}[M]$ that contains all four functions $\bar{a}_n, \bar{b}_n,\ n = 1,2$, is $\mathbb{C}[M]$ itself. (These functions do *not* generate $\mathbb{C}[M]$ as a commutative algebra !).

(2) Let $W = \mathbb{Z}/(2) = \{1, s\}$ be the Weyl group of the root system $\mathbf{A}_1$ and $\mathbf{e} := \frac{1}{2}(1+s)$, an idempotent in the group algebra. Let $\mathsf{H}_{t,c}(\mathbf{A}_1)$ be the corresponding symplectic reflection algebra with parameters $t, c \in \mathbb{C}$ and $\mathbf{e}\mathsf{H}_{t,c}(\mathbf{A}_1)\mathbf{e}$ its spherical subalgebra.

- Show that, for $t \neq 0$, the algebra $\mathbf{e}\mathsf{H}_{t,c}(\mathbf{A}_1)\mathbf{e}$ is generated by the elements $\mathbf{e}x^2$ and $\mathbf{e}y^2$
- Establish an algebra isomorphism

  $$\mathbf{e}\mathsf{H}_{1,c}(\mathbf{A}_1)\mathbf{e} \cong \mathcal{U}\mathfrak{g}/(\Delta),$$

  for an appropriate central element $\Delta \in \mathcal{U}\mathfrak{g}$.
- Show that $\mathbf{e}\mathsf{H}_{0,c}(\mathbf{A}_1)\mathbf{e}$ is a commutative algebra and, moreover, the assignment $\mathbf{e}x^2 \mapsto x,\ \mathbf{e}y^2 \mapsto y$, extends uniquely to a Poisson algebra isomorphism

  $$\mathbf{e}\mathsf{H}_{0,c}(\mathbf{A}_1)\mathbf{e} \xrightarrow{\sim} A_\phi,$$

  where $\phi = x^2 + y^2 + z^2 - \frac{c(c+1)}{2}$ (notation of Problem 1 from Problem Set 1).

(3) Let G be a connected semisimple group with trivial center, and (e,h,f) an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, the Lie algebra of $\mathfrak{g}$. The $\mathrm{ad}\, h$-action on $\mathfrak{g}$ is semisimple with integer eigenvalues, hence it gives a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\, \mathfrak{g}_i$. Put $\mathfrak{p} = \oplus_{i\geq 0}\, \mathfrak{g}_i$. This is a parabolic subalgebra of $\mathfrak{g}$. Let P be the corresponding parabolic subgroup of G. The subspace $\mathfrak{g}_{\geq 2} := \oplus_{i\geq 2}\, \mathfrak{g}_i$, of $\mathfrak{g}$, is $\mathrm{Ad}\, P$-stable, so we define

$$X := G \times_P \mathfrak{g}_{\geq 2}.$$

The group G acts on X by $g : (h,x) \mapsto (gh,x)$ for all $g,h \in G, x \in \mathfrak{g}_{\geq 2}$. Further, the assignment $(h,x) \mapsto \mathrm{Ad}\, h(x)$ gives a G-equivariant map $\pi : X \to \mathfrak{g}$.

- Check that $\mathfrak{g}_{\geq 2}$ is an $\mathrm{Ad}\, P$-stable subspace of $\mathfrak{p}$ and the P-orbit of the element $e \in \mathfrak{g}_{\geq 2}$ is Zariski open and dense in $\mathfrak{g}_{\geq 2}$. [Hint: Use representation theory of $\mathfrak{sl}_2$ to prove that the tangent space to this orbit equals $\mathfrak{g}_{\geq 2}$.]
- Show that π is proper and its image equals $\overline{\mathrm{Ad}\, G(e)}$, the closure of the G-orbit of e in $\mathfrak{g}$.
- Show that π restricts to an isomorphism $\pi^{-1}(\mathrm{Ad}\, G(e)) \xrightarrow{\sim} \mathrm{Ad}\, G(e)$, hence, it is a birational isomorphism of X and $\overline{\mathrm{Ad}\, G(e)}$.
- Identify $\mathfrak{g}^* \cong \mathfrak{g}$ and view $\mathrm{Ad}\, G(e)$ as a coadjoint orbit in $\mathfrak{g}^*$. Let ω be the canonical symplectic 2-form on that orbit.
  Show that the 2-form $\pi^*\omega$ on $\pi^{-1}((\mathrm{Ad}\, G(e))$ extends to a regular, possibly degenerate, 2-form ω_X on X.
- Show that in the case where $\mathfrak{g}_i = 0$ for all odd i the 2-form ω_X is in fact nondegenerate.