Symplectic algebraic geometry and quiver varieties

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1 Deformation quantization

If $\mathfrak{g}$ is a Lie algebra, then $U\mathfrak{g}$ is a deformation of $\operatorname{Sym}\mathfrak{g}$. This has been known for at least 100 years. This is useful motivation for a lot of mathematics, but I don't think it was actually used to rigorously prove anything for a long time.

However, in the last 20 years people have realized that that this analogy can be made rigorous.

> One can think of symplectic resolutions as the semisimple Lie algebras of the twenty-first century.

For a long time, people thought that semisimple Lie algebras are very special - their representations lead to notions like Category $\mathcal{O}$, etc. The new notion is that this is somehow not that unique to semisimple Lie algebras.

1.1 Deformations

A *deformation* of some given object (an algebra, scheme, or whatever) is a flat family of objects over a smooth base, whose fiber at a basepoint is the given object.

$$\begin{array}{ccc} F & \hookrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ b & \hookrightarrow & B \end{array}$$

Let's try to formalize what this means. To do so, it is useful to introduce the notion of a *deformation functor*.

Definition 1.1. A (contravariant) *deformation functor* is a functor

$$\left(\substack{\text{category of} \\ \text{pointed test schemes}}\right) \to \mathbf{Sets}$$

sending S to the set of flat deformations over S, up to isomorphism. Given a map of test schemes $S \to T$, we can pull back any flat family over T to a flat family over S (which explains the functoriality).

A semiuniversal deformation exists if there exists a *moduli space* for the objects, i.e if there exists a formal scheme $\mathcal{M}$ such that

$$\operatorname{Def}(S) = \operatorname{Hom}(S, M) \text{ for all } S.$$

This means that there exists a *universal deformation* over $\mathcal{M}$, such that any other deformation is obtained by pullback.

$$\begin{array}{ccc} \mathcal{F} & \dashrightarrow & \mathcal{F}_{\mathcal{M}} \\ \downarrow & & \downarrow \\ S & \dashrightarrow & \mathcal{M} \end{array}$$

Remark 1.2. The "semi" refers to a lack of uniqueness in our definition.

Usually one takes test schemes to be $S = \text{Spec } R$ where R is a local Artin ring (since we are really interested in "local" issues, not necessarily global ones). If we are over a field, then R is a finite-dimensional algebra with a unique maximal ideal $\mathfrak{m}$, which is nilpotent.

1.2 Quantization

Let A_0 be a commutative algebra over a field k of characteristic 0.

Definition 1.3. A *quantization* of A_0 is an associative algebra A/R, together with an isomorphism $A/\mathfrak{m}A \cong A_0$.

A special case of this, which has been known for a long, is the notion of *one-parameter deformation*. This is the special case $R = k[\hbar]/\hbar^{n+1}$ for some $n \geq 1$. Geometrically, one says that A is an infinitesimal extension on A_0 in a formal neighborhood:

$$\begin{array}{ccccc} A_0 & \hookrightarrow & A & & \\ \downarrow & & \downarrow & & \\ 0 & \hookrightarrow & \text{Spec } R & \hookrightarrow & \text{Spec } k[[\hbar]] = \mathbb{D}. \end{array}$$

Definition 1.4. For any scheme $X = \text{Spec } R$, a *1-parameter quantization* is a sheaf of associative R-algebras $\mathcal{O}_\hbar$ over X such that $\mathcal{O}_\hbar/\hbar \cong \mathcal{O}_X$.

1.3 Poisson algebras

Definition 1.5. Let A be an associative algebra. A *non-commutative Poisson bracket* on A is a bilinear form

$$\{-,-\}\colon A \times A \to A$$

which is a Lie bracket, such that in addition

$$\{ab, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b.$$

Example 1.6. For any A, we can take $\{a, b\} = ab - ba$.

Definition 1.7. A *Poisson algebra* is a commutative algebra with a non-commutative Poisson bracket.

Important construction. Let A be a first-order one-parameter deformation of a commutative algebra A_0, i.e. A is a flat algebra over $k[\hbar]/\hbar^2$, with a given isomorphism $A/\hbar A \cong A_0$. Since A is flat over $k[\hbar]/\hbar^2$, multiplication by $\hbar$ gives an isomorphism $A/\hbar A \cong \hbar A$. Then for any $a, b \in A$, we can take the commutator $[a, b] = ab - ba$. Since the $A/\hbar \cong A_0$ is commutative, we have

$$ab - ba \in \hbar A \cong A/\hbar A \cong A_0.$$

In terms of formulas, this is "division by $\hbar$": $ab - ba \mapsto \frac{1}{\hbar}(ab - ba)$. The claim is that this descends to a well-defined bilinear skew-symmetric bracket

$$\{-,-\}: A_0 \times A_0 \to A_0.$$

This is automatically compatible with the Leibniz formula, but it may not satisfy Jacobi's identity. If the deformation A lifts to *second* order over $k[\hbar]/\hbar^2$, then in fact the Jacobi identity is satisfied, so A_0 has the structure of Poisson algebra. (We need only the first order deformation to define the bracket, but we need *any* extension to a second order deformation to confirm the Jacobi identity.)

So we see that studying deformations naturally leds to studying Poisson algebras.

1.4 Relating quantizations and Poisson structures

We want to consider not only deformations over $R = k[\hbar]/\hbar^{n+1}$. It's not obvious which R are reasonable to consider. The main candidate has motivation from within number theory.

Example 1.8. Let X be a scheme over a finite field $\mathbb{F}_p$. We could ask to look at flat families of schemes $\mathcal{X} \xrightarrow{f} \mathbb{A}^1$ over $\mathbb{F}_p$ such that $f^{-1}(0) = X$.

However, there is a totally different version of the question: you can instead consider lifts of X to a scheme over $\mathbb{Z}_p$. This is also a kind of deformation, where the parameter is p instead. This is a totally different type of test scheme, which leads to totally different types of deformations.

The second problem is more interesting than the first, so that's what we want to consider. What are the test objects? They will be the set of finite dimensional local $k[\hbar]$-algebras R, with maximal ideal $\mathfrak{m}$ such that $\hbar \in \mathfrak{m}$. Moreover, they will be presented with a structure map $S = \operatorname{Spec} R \to \mathbb{D}$ (the formal disk).

Definition 1.9. Fix a Poisson algebra A_0. We define the deformation functor

$$\operatorname{Def}(A_0) = \left\{ A \mid \begin{array}{c} A = \text{associative flat } R\text{-algebra} \\ \text{with } R\text{-linear Poisson bracket} \\ \text{such that } ab - ba = \hbar\{a,b\} \text{ for all } a,b \in A \\ \text{and } A/\mathfrak{m} = A_0 \text{ as a Poisson algebra} \end{array} \right\}.$$

Since A is an R-algebra, $A/\hbar$ is an $R/\hbar$-algebra, equipped with a non-commutative Poisson bracket coming from $ab - ba = \hbar\{a, b\}$.

Let $\mathcal{M}_{quant} \to \mathbb{D}$ be a semiuniversal deformation of $(A_0, \{-,-\}))$. This has a map to $\mathcal{M}_{Poiss}$ (the space of deformations of Poisson structures on A_0), by the association of Poisson algebras to deformations.

Theorem 1.10 (Bezrukavnikov-Kaledin-Verbitsky)**.** *Let X be a smooth algebraic variety over $\mathbb{C}$ such that $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Suppose $\{-,-\}_0$ is a non-degenerate Poisson bracket on X (meaning on $\mathcal{O}_X$), i.e. X has an algebraic symplectic form ω_0. Then*

1. *there exists a semi-universal deformation of X,*

2. *There is an isomorphism* $\mathcal{M}_{quant} \xrightarrow{\sim} \mathcal{M}_{Poiss} \times \mathbb{D}$.

3. *The map* $\mathcal{M}_{Poiss} \to H^2_{\mathrm{dR}}(X)$ *sending* $\beta \mapsto [\beta]$ *induces an isomorphism*

$$\mathcal{M}_{Poiss} \cong H^2(X, \mathbb{C})^{\wedge}_{\omega_0}.$$

♠♠♠ TONY: [so there's a map from the completion of cohomology to cohomology...?]

Remark 1.11. The second assertion says that studying quantizations is the same as studying Poisson deformations plus some formal parameter $\hbar$.

Regarding the third assertion, observe that a deformation of doesn't change the "constant terms," hence preserves non-generacy of a Poisson bracket.

Example 1.12. Consider $U\mathfrak{g}$ where $\mathfrak{g}$ is semisimple. How does this fit into our picture? An important subalgebra of $U\mathfrak{g}$ is $Z(U\mathfrak{g})$. Now, $U\mathfrak{g}$ is has the PBW filtration, so we can form the *Rees algebra*

$$R(U\mathfrak{g}) = \sum \hbar^i U\mathfrak{g}_{\leq i} \subset U\mathfrak{g}[\hbar]$$

and also the Rees algebra of the center,

$$R(Z(U\mathfrak{g})) \subset Z(U\mathfrak{g})[\hbar].$$

Now $A := R(U\mathfrak{g})$ is an algebra over $R(Z(U\mathfrak{g}))$. We can interpret A as a flat family of associative algebras over $\mathcal{M} := \operatorname{Spec} R(Z(U\mathfrak{g}))$. Since $R(Z(U\mathfrak{g})) \supset k[\hbar]$, this has a projection to $\mathbb{A}^1_{\hbar}$. When $\hbar = 0$, the semisimplicity of the algebra implies that $\operatorname{Spec} R(Z(U\mathfrak{g}))$ is the product of $\mathbb{A}^1$ and $\mathfrak{h}/W$ (the Cartan modulo the Weyl action).

In analogy with the theorem, we should think of $\mathcal{M}$ as being analogous to $\mathcal{M}_{Poiss}$ (with A as its universal family), and $\mathbb{A}^1$ as analogous to $\mathbb{D}$. Then $\mathfrak{h}/W$ would be the analogue of $\mathcal{M}_{quant}$.

Example 1.13. Let $A_0 = \mathbb{C}[\mathbb{C}^n \times \mathbb{C}^n]^{S_n}$, the invariants of $\mathbb{C}[\mathbb{C}^n \times \mathbb{C}^n]$ under the diagonal action of S_n. Now, $\mathbb{C}[\mathbb{C}^n \times \mathbb{C}^n]$ has a natural Poisson structure, and A_0 is a Poisson subalgebra.

One has a natural one-parameter Poisson deformation of A_0. This would be really hard to see if you didn't know about the universal enveloping algebra. The idea is the following. Instead of A_0, you look at $B := \mathbb{C}[\mathbb{C}^n \times \mathbb{C}^n] \rtimes S_n$. Then you deform B instead. In particular, B contains the element $e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$. The key observation is that $eBe \cong \mathbb{C}[\mathbb{C}^n \times \mathbb{C}^n]^{S_n}$.

There is a family of deformations of B, denoted $B_{\hbar,c}$ (the c is the "Poisson direction") with generators x_i, y_i for $i = 1, \ldots, n$. It contains $\mathbb{C}[S_n]$. There are some tricky commutation relations

$$\begin{aligned} [x_i, y_i] &= \hbar \cdot 1 \pm c \sum_{j \neq i} s_{ij} \quad s_{ij} = (i, j) \text{ transposition} \\ [x_i, y_j] &= c s_{ij} \\ [x_i, x_j] &= 0 \\ [y_i, y_j] &= 0 \end{aligned}$$

and the commutation relations within $\mathbb{C}[S_n]$ is the natural one.

Theorem 1.14. *In the above notation, $eB_{\hbar,c}e$ is a flat quantization of A_0. For $\hbar = 0$ and any c, $eB_{\hbar=0,c}e$ is commutative.*

How does this fit into what we said before? We can write $A_0 = \mathbb{C}[(\mathbb{C}^n \times \mathbb{C}^n)/S_n]$, which is *singular*. The main theorem applied to smooth symplectic manifolds, but this is neither smooth nor symplectic. Moreover, if you look at $H^2((\mathbb{C}^n \times \mathbb{C}^n)/S_n)$, you'll see that it vanishes. So that looks pretty bad, as far as the theorem goes.

To see how this fits, you have to realize that $(\mathbb{C}^n \times \mathbb{C}^n)/S_n$ has a very special resolution of singularities, namely by the Hilbet scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$, a smooth symplectic variety. It is homotopic to the fiber over 0, which has a second cohomology group of rank 1. After you realize this, you see that it fits perfectly: there is one quantum parameter and one classical parameter to the space of quantizations.

2 Holomorphic deformations

2.1 Algebraic version

Let X be a smooth algebraic variety, O its sheaf of regular functions, T its tangent sheaf, and $T^* = \Omega^1$ its cotangent sheaf. Any Poisson bracket on O has the form

$$\{f, g\} = \langle \Pi, df \wedge dg \rangle$$

for some bivector $\Pi \in \bigwedge^2 T$.

We have a map

$$O \xrightarrow{d} \Omega^1 \xrightarrow{i_\Pi} T$$

sending $f \mapsto i_\Pi df = \xi_f$, where ξ_f is the derivation $\xi_f(g) := \{f, g\}$. Then the Jacobi identity holds for $\{-,-\}$ if and only if the association $f \mapsto \xi_f$ takes $\{-,-\}$ to the commutator of vector fields. For the rest of the lecture, we will assume that Π is such that this holds.

Question: Can one define a natural bracket $\{-,-\}^1$ on Ω^1 that goes to $[-,-]$ under i_Π?

Since α and β are 1-forms, $\alpha \wedge \beta \in \Omega^2$. Abusing notation, we denote by $i_\Pi(\alpha \wedge \beta)$ the contraction of Π with $\alpha \wedge \beta$. This is a function, and we want a 1-form, so we try

$$\boxed{\{\alpha, \beta\}^1 = d(i_\Pi(\alpha \wedge \beta)).}$$

Now, does this do what we want? It is easy to show from the definitions that

$$i_\Pi\{\alpha, \beta\}^1 = [i_\Pi \alpha, i_\Pi \beta]$$

whenever α, β are exact. But this equality can be checked locally, and in the smooth world any closed form is locally exact, so that suggests that it should hold even for closed forms. In the algebraic world, we can prove this by passing to the completion, where an algebraic version of the Poincaré Lemma holds to show that any closed form is exact in a formal neighborhood.

2.2 The twistor deformation

Let X be a smooth, complex-analytic manifold. Let O, T, Ω^1 be the holomorphic sheaves of functions, etc. In the analytic topological any closed form is locally exact, so we have the exact sequence of sheaves

$$0 \to \mathbb{C} \to O \xrightarrow{\bar{\partial}} \Omega^1_{\text{exact}} = \Omega^1_{\text{closed}} \to 0.$$

Corollary 2.1. *If $H^1(X, O_X) = H^2(X, O_X) = 0$ then the boundary map is an isomorphism*

$$H^1(\Omega^1_{closed}) \cong H^2(X, \mathbb{C}).$$

So we have maps

$$H^2(X,\mathbb{C}) = H^1(X,\Omega^1_{\text{closed}}) \to H^1(X,\Omega^1) \xrightarrow{i_\Pi} H^1(X,T).$$

Call this composition map σ. The group $H^1(X,T)$ classifies first-order deformations of the complex structure on X.

Here is a holomorphic version of Theorem 1.10.

Theorem 2.2. *Assume that $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$. For any $v \in H^2(X,\mathbb{C})$, the first-order deformation $\phi_1 := \sigma(v)$ extends to a formal 1-parameter deformation X_t of the complex structure on X as a C^∞-manifold structure, such that the Poisson bracket*

$$\{f,g\}_t = \langle \Pi, df \wedge dg\rangle.$$

is holomorphic for all t.

Note that while the formula $\{f,g\}_t = \langle \Pi, df \wedge dg\rangle$ appears to be independent of t, the class of holomorphic functions from which f,g are drawn changes with the complex structure, and hence with t. Therefore, this is in some sense varying with t.

Remark 2.3. (1) Assume that Π is non-degenerate. Then $\Pi^{-1} = \omega$ is a symplectic form, and we have an isomorphism

$$\mathcal{M}_{Poiss} \xrightarrow{\sim} H^2(X)^\wedge_\omega$$

which sends $(X_t,\Pi) \mapsto [\omega] + tv$.

(2) The theorem describes deformations of the symplectic structure along a "line corresponding to v." These are called "twistor" deformations.

One of the interesting features of the holomorphic case is that the family of smooth varieties is constant, so for instance their cohomology is constant, while the holomorphic structure varies. In the algebraic case, the algebraic varieties honestly vary and there is no notion of "underlying smooth manifold."

Proof. Let $\Omega^{p,q}$ be the sheaf of C^∞-forms on X of type (p,q). Denote by ∂ and $\overline{\partial}$ the usual differentials, so $d = \partial + \overline{\partial}$.

A deformation of our complex structure is specified by deforming the operator $\overline{\partial} \rightsquigarrow \overline{\partial} - \phi_t$, where $\phi_t = t\phi_1 + t^2\phi_2 + \dots$ and each $\phi_k \in \Omega^{0,1}(T)$ has the form

$$\phi_k = \sum a_{ij}(z,\overline{z})\frac{\partial}{\partial z_i}d\overline{z_j}.$$

(Note that the first-order deformation ϕ_1 is already determined.) Then the holomorphic functions on X_t are those smooth functions $f \in C^\infty(X)$ such that

$$\frac{\partial f}{\partial \overline{z_j}} = \sum_j a_{ij}(t;z,\overline{z})\frac{\partial f}{\partial z_i}$$

where

$$\phi_t = \sum a_{ij}(t) \frac{\partial}{\partial z_i} d\overline{z_j}.$$

The integrability condition is that $0 = \overline{\partial}_t^2$, which amounts to the usual Maurer-Cartan equations

$$\overline{\partial}\phi_t = [\phi_t, \phi_t].$$

Recall that $\phi \in \Omega^{0,1}(T)$, so the bracket here means "bracket on the vector field part and wedge on the differential forms part." The heuristic way to think of this is as $T \otimes \Omega^{0,\bullet}$, with T considered as the "interesting" part and $\Omega^{0,\bullet}$ as parameters, i.e. $\Omega^{0,\bullet}$ is the "ground ring." Thus the bracket of the thing with itself may not be zero, because the "coefficients" are not constants.

Then Π is holomorphic with respect to ϕ_t if and only if

$$\mathcal{L}_{\phi_t}\Pi = 0.$$

By the Dolbeault isomorphism theorems, $H^i(X, \mathcal{O}) \cong H^{0,i}_{\overline{\partial}}(X, \mathbb{C})$, which vanish for $i = 1, 2$ by the assumptions. I'll assume for simplification that the ν in the hypothesis is represented by a $(1,1)$-form ν such that $\partial\nu = \overline{\partial}\nu = 0$ (this is without loss of generality because of the cohomological vanishing).

It turns out that if we satisfy the first equation, then the second one basically comes for free, so most of the effort is in constructing ϕ_t. This is done by inductively solving for the higher order terms in terms of the lower ones.

1. For the coefficient of t, we need
$$\overline{\partial}\phi_1 = 0.$$
But
$$\overline{\partial}\phi_1 = \overline{\partial} i_\Pi \nu = i_\Pi \overline{\partial}\nu = 0$$
by the assumption $\overline{\partial}\nu = 0$.

2. For the coefficient of t^2, we need
$$\overline{\partial}\phi_2 = [\phi_1, \phi_1].$$
So what we need to show that $[\phi_1, \phi_1]$ is $\overline{\partial}$-closed, because then it will automatically be exact, by the vanishing of $H^{0,2}_{\overline{\partial}}(X, \mathbb{C})$. Let $\sigma = i_\Pi$ as before, so $\phi_1 = \sigma(\nu)$. Recall that ν is a ∂-closed 1-form, and we discussed that for closed 1-forms i_Π takes the Poisson bracket $\{-,-\}^1$ to the commutator bracket on vector fields (this was done for 1-forms rather than $(1,1)$-forms, but we may clearly just as well ignore the $\overline{z}$ coefficients)
$$\begin{aligned}[\phi_1, \phi_1] &= \overline{\partial}[\sigma(\nu), \sigma(\nu)] \\ &= \sigma(\partial\{\nu \wedge \nu\}^1) \\ &= \sigma(\partial\sigma(\nu^2))\end{aligned}$$

Then taking $\overline{\partial}$, we find that

$$\overline{\partial}[\phi_1, \phi_1] = \overline{\partial}\sigma\partial\sigma(\overline{\partial}\nu \wedge \nu)$$

but this vanishes $\overline{\partial}\nu = 0$. The point was that $\overline{\partial}$ commutes with everything, because σ is defined in terms of the holomorphic coordinates.

It turns out to be important to track the form of ϕ_2. Since $\overline{\partial}(\sigma(\nu \wedge \nu)) = 0$, there exists a β such that $\sigma(\nu \wedge \nu) = \overline{\partial}\beta$. Then taking $\phi_2 = \sigma\partial\beta$ solves the equation.

Let us digress a bit to discuss the bigger picture. If we want to extend a first-order deformation to second-order (say), we would like to show that the obstructions, which may be identified with some cohomology group, vanish. But this is often not true. That doesn't necessarily mean that your particular first-order deformation doesn't extend, but that you have to show that *its* obstruction is 0. It's hard to get a handle on the obstruction, so you have a little extra information to get from each order deformation to the next. For us, that extra bit of leverage is the special form of the answer for ϕ_j.

3. In general, the idea is to look for a solution of the form $\phi_{i+2} = \sigma\partial \cdot \beta_i$. Then we'll set

$$\beta_t = \beta_0 + t\beta_1 + \dots.$$

Once you solve the relevant equations for the 1-forms, you can apply σ to obtain the solutions for vector fields. In the end, we reduce to solving

$$\overline{\partial}\beta_t = \{\beta_t, \beta_t\}.$$

We already have $\beta_0 = \beta$. Now we just have to solve recursively. We have

$$\overline{\partial}\beta_n = \{\beta_1, \beta_{n-1}\} + \{\beta_2, \beta_{n-2}\} + \dots + \{\beta_{n-1}, \beta_1\}.$$

You just have to check that this form is closed. You can show by induction that

$$\overline{\partial}\beta_n = \sum_{i+j+k=n} \{\beta_i, \{\beta_j, \beta_k\}\}$$

and then the Jacobi Identity guarantees that this vanishes.

The conclusion is that you obtain a final answer of the form

$$\phi_t = t\sigma(\nu + t\partial\beta_t).$$

Then you have to check the holomorphicity condition, which is

$$L_\sigma(\nu + t\partial\beta_t)\Pi = 0.$$

But that is an immediate consequence of what $\nu + t\partial\beta_t$ is. Indeed, ν is closed and $t\partial\beta_t$ is exact. But we have that $L_{\sigma ?} = 0$ whenever ? is ∂-closed. □

3 Semi-positive varieties

3.1 Definitions and examples

Let (X, ω) be a holomorphic symplectic manifold. Let ν be the class of a Kähler form on X (the main example is when ν is the first chern class of an ample line bundle).

Suppose we have a (Poisson) deformation

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_t = f^{-1}(t) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \hookrightarrow & \mathbb{C} & \hookleftarrow & t \end{array}$$

Since X is symplectic, its the Poisson structure is non-degenerate at $t = 0$. Then it will be non-degenerate for small t, so we can think of this $\mathcal{X}$ as a family of symplectic manifolds in some neighborhood of $t = 0$.

Now comes a key idea of Mori from the minimal model theory.

Lemma 3.1. *There can be no family $C_t \subset X_t$ of compact complex curves.*

Proof. We show this at the formal level, where X_t is the twistor deformation family from Theorem 2.2.

We choose an identification $H^\bullet(X_t) \cong H^\bullet(X)$ $\spadesuit\spadesuit\spadesuit$ TONY: [what's an easy way to see this?]. The then family C_t defines a family of homology classes $[C_t] \in H_2(X_t, \mathbb{Z})$. But since $H_2(X_t, \mathbb{Z})$ is "discrete" this class $[C_t]$ must be constant, independent of t. Now we consider the integral

$$\int_{[C_t]} \omega_t.$$

On one hand, it vanishes formally because $\omega_t|_{C_t} = 0$ (since it's a holomorphic 2-form on a curve). On the other hand, the construction of the twistor deformation was that $[\omega_t] = [\omega] + t\nu$. Therefore, the above is

$$\int_C \omega + t\nu = t \int_C \nu$$

but the right hand side is positive because ν is a Kähler form. □

Definition 3.2. Call a variety X *semi-positive* if

- X is quasi-projective,
- there is an action of $\mathbb{C}^\times$ on X such that $X^{\mathbb{C}^\times}$ is projective,
- for all $x \in X$, the limit for this action $\lim_{z \to 0} zx$ exists.

Example 3.3. Let X be afffine. Then giving a $\mathbb{C}^\times$ action on X is the same as giving a $\mathbb{Z}$-grading on the coordinate ring of X:

$$\mathbb{C}[X] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[X](i).$$

Then X is semi-positive if and only if $\mathbb{C}[x](i) = 0$ for all $i < 0$ and $\mathbb{C}[X](0) = \mathbb{C}$.

Example 3.4. We can always define the *affinization* of X to be $X^{\text{aff}} = \operatorname{Spec} \Gamma(X, O_X)$. If $\mathbb{C}^\times$ acts on X, then by functoriality $\mathbb{C}^\times$ acts on X^{aff}. If in addition $X \to X^{\text{aff}}$ is proper, then X is semi-positive if and only if X^{aff} is semi-positive. (We probably need to assume that X is quasiprojective to start out.)

The main example is when $X \to X^{\text{aff}}$ is a resolution of singularities.

Example 3.5. There is one notable exception, namely *Higgs bundles*. Let Σ be a smooth projective curve. If G is a reductive group, then we denote by $\text{Bun}_G(\Sigma)$ the moduli stack of G-bundles on Σ, and the *Higgs bundle* is $\text{Higgs}_G(\Sigma) = T^* \text{Bun}_G(\Sigma)$.

I don't want to get into the theory of stacks, so I'll focus on the stable part Higgs^{st}. This is a scheme. There are cases where some connected component is smooth. Then such a Higgs^{ss} is semipositive.

- The $\mathbb{C}^\times$ action comes from dilation of the fibers of the cotangent bundle $T^* \text{Bun}_G(\Sigma)$.
- There is a *Hitchin map*

  $$\text{Higgs}^{ss} \xrightarrow{\text{proper}} \mathbb{C}^N,$$

  This is the affinization map for Higgs^{ss}. The action of $\mathbb{C}^\times$ on $\mathbb{C}^N$ is contracting. The fixed point set is the fiber over 0, and since the map is proper that means that the fixed locus on Higgs^{ss} is proper.

3.2 Properties

Let X be a semi-positive variety.

1. (*Compactification*)

   Definition 3.6. Let X be smooth projective. Define

   $$X^{\text{bad}} := \{x \in X \mid \lim_{z \to \infty} zx \text{ exists}\}.$$

   X has a canonical completion

   $$X \overset{\text{open}}{\hookrightarrow} \overline{X} = X \sqcup (X \setminus X^{\text{bad}})/\mathbb{C}^\times.$$

   Example 3.7. If $X = \mathbb{A}^n$, then $\overline{X} = \mathbb{P}^n$.

   Now $(X \setminus X^{\text{bad}})/\mathbb{C}^\times$ may not be smooth, but it is at least *rationally smooth*: it has orbifold singularities.

2. (*BB Decomposition*) Let $X^{\mathbb{C}^\times} = \bigsqcup_i F_i$ (this is smooth because the action is smooth). There is a BB decomposition $X = \coprod X_i$, where

$$X_i = \{x \in X \mid \lim_{z \to 0} zx \in F_i\}.$$

Moreover, *X_i is a vector bundle over F_i.*

The original theorem of BB applied to proper varieties. Now, our X is not necessarily proper, but it has the canonical compactification from (1), and as $t \to 0$, the two pieces X and $(X \setminus X^{\mathrm{bad}})/\mathbb{C}^\times$ must stay away from each other, so the result can be boostrapped to this case.

3. (*Purity*) We have an isomorphism of cohomology $H^\bullet(X) \cong H^\bullet(X^{\mathbb{C}^\times})$, and the cohomology is pure.

Proposition 3.8. *Let $f\colon X \to \mathbb{C}$ be smooth and equivariant. Suppose that $z \in \mathbb{C}^\times$ act on $\mathbb{C}$ by multiplication by z^m, where $m > 0$. Let $X_t = f^{-1}(t)$. Then:*

1. *the restriction map $H^\bullet(X) \to H^\bullet(X_t)$ is an isomorphism for all t, and*
2. *$H^\bullet(X_t)$ is pure.*

Sketch of proof. By base change on $\mathbb{C}$, we can reduce to the case $m = 1$. We may write

$$X = \underbrace{X_0}_{\text{closed}} \sqcup \underbrace{(\mathbb{C}^\times \times X_1)}_{\text{open}}.$$

Since we know that $H^\bullet(X)$ is pure, and the $\mathbb{C}^\times$ contracts all of the fibers to X_0 (so that the inclusion $H^\bullet(X_0) \to H^\bullet(X)$ is an isomorphism), we know that the statement holds for $t = 0$. Then we have to check the fibers over $t \neq 0$, and by $\mathbb{C}^\times$-equivariance we can reduce to a single value of t, say $t = 1$. We have the triple

$$X_0 \hookrightarrow X \hookleftarrow \mathbb{C}^\times \times X_1.$$

Then we get the standard exact triangle

$$i_* \mathbb{C}_{X_0}[?] \xrightarrow{i_*} \mathbb{C}_X \xrightarrow{j^*} \mathbb{C}_{\mathbb{C}^\times \times X_1}$$

which induces a long exact sequence in cohomology

$$\ldots \to H^i(X_0) \to H^i(X) \to H^i(\mathbb{C}^\times \times X_1) \to \ldots.$$

Since $H^i(X_0)$ and $H^i(X)$ are pure, this splits into short exact sequences. We know the first and second terms are isomorphic, and the third term by the Kunneth formula. So the short exact sequences are

$$0 \to H^\bullet(X_0) \to H^\bullet(\mathbb{C}^\times) \otimes H(X_1) \to H^{\bullet+1}(X_0).$$

Now, we have an isomorphism $H^\bullet(\mathbb{C}^\times) \cong \mathbb{C}[0] \oplus \mathbb{C}[2]$ where the grading refers to the *weights*, so the only possibility is that $H^\bullet(X_0)$ maps isomorphically to $\mathbb{C}[0] \otimes H^\bullet(X_1)$ and $\mathbb{C}[2] \otimes H^\bullet(X_1)$ maps isomorphically $H^{\bullet+1}(X_0)$. Thus, $H^\bullet(X_1)$ is also pure. ◻

Proposition 3.9. *Let X be smooth and semi-positive. Suppose $X \to X^{\text{aff}}$ is proper, and that X has an algebraic symplectic form ω with weight $m > 0$. Finally, assume that $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$. Let X_t be the holomorphic twistor deforming ν (the Kähler class). Then*

1. *If $\mathcal{F}$ is $\mathbb{C}^\times$-equivariant coherent sheaf on X, then*

$$H^\bullet(X, \mathcal{F}) = \bigoplus_{m \in \mathbb{Z}} H^\bullet(X^{an}, \mathcal{O}_X^{an} \otimes_{\mathcal{O}_X} \mathcal{F})(m).$$

2. *The twistor deformation X_t is algebraic.*

3. *The twistor deformation extends to a deformation $\mathcal{X} \to \mathbb{C}$ with fiber X over 0 and X_t over t, which is $\mathbb{C}^\times$-equivariant where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight m.*

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_t = f^{-1}(t) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \hookrightarrow & \mathbb{C} & \hookleftarrow & t \end{array}$$

4. *The map $X_t \to X_t^{\text{aff}}$ is finite for all $t \neq 0$.*

Proof of 4. That π_0 is proper implies π_t is proper for all t ♠♠♠ TONY: [why?]. If X_t had a component of positive dimension, then since it is proper it would contains a projective curve $C_1 \subset X_1$ (without loss of generality, by the equivariance). Moving C_1 moving it by the $\mathbb{C}^\times$ action would produce a family of curves, which is impossible by the result at the beginning. Therfore, the dimension of the fiber must be 0, hence the fiber must be finite, and proper plus quasifinite implies finite.

□

Corollary 3.10. *If $X \to X^{\text{aff}}$ is a resolution of singularities (i.e. birational), then $X_t \cong X_t^{\text{aff}}$ for all $t \neq 0$.*

Proof. By the proposition, $X_t \to X_t^{\text{aff}}$ is birational *and* finite, hence an isomorphism. □

4 Symplectic resolutions

4.1 Main theorem

Definition 4.1. A symplectic variety (X, ω) is called a *symplectic resolution* if $X \to X^{\mathrm{aff}}$ is a resolution of singularities.

For the next set of results, we assume:

Running hypotheses. Let (X, ω) be a symplectic resolution, such that X is semipositive and the symplectic form ω has weight $m > 0$.

Theorem 4.2. *Under the running hypotheses,*

1. *There exists a universal Poisson deformation*

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & X_t \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow \\ X^{\mathrm{aff}} & \longrightarrow & \mathcal{X}^{\mathrm{aff}} & \longleftarrow & X_t^{\mathrm{aff}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^2(X, \mathbb{C}) & \longleftarrow & t \end{array}$$

   which is $\mathbb{C}^\times$-equivariant, and where $\mathbb{C}^\times$ acts on $H^2(X)$ by $z \mapsto z^m$.

2. *For sufficiently generic $t \in H^2(X)$, X_t is affine (so in particular equal to its affinization).*

3. *We have $H^\bullet(\pi^{-1}(0)) \cong H^\bullet(X) \cong H^\bullet(\mathcal{X})$.*

Theorem 4.3. *Under the running hypotheses, there exists a $\mathbb{C}^\times$-equivariant quantization $\mathcal{O}_\hbar$ of $\mathbb{C}[\hbar]$-algebras on $\mathcal{X}$ (the universal symplectic deformation.*

The global sections $\Gamma(\mathcal{O}_\hbar)$ form a free graded associative algebra $\mathbb{C}[H^2(X) \times \mathbb{C}_\hbar]$. For $\nu \in H^2(X)$, the specialization $\in A_\nu := \Gamma(\mathcal{O}_\hbar)|_{\nu,\hbar=1}$ is a filtered associative algebra, with associated graded $\mathbb{C}[X] = \mathbb{C}[X^{\mathrm{aff}}]$.

Theorem 4.4. *Under the running hypotheses, let*

$$\pi \colon X \to X^{\mathrm{aff}}$$

be the affinization map. Then for all $x \in X^{\mathrm{aff}}$, the fibral homology $H_*(\pi^{-1}(x))$ is generated by algebraic cycles. In particular there is no odd homology, and all even homology is pure of Tate type.

4.2 Proof of big theorem

Now we want to sketch the proof of Theorem 4.4. Let $0 \in X^{\mathrm{aff}}$ be the (unique) fixed point (cut out by the augmentation ideal).

- We know that $H^\bullet(\pi^{-1}(0)) \cong H^\bullet(X)$ because the $\mathbb{C}^\times$-action defines a contraction from X to $\pi^{-1}(0)$.
- Denote by $K(X)$ the Grothendieck group of $\mathbf{Coh}(X)$ *tensored with* $\mathbb{Q}$. There is a map

  $$K(X) \xrightarrow{\mathrm{ch}} H^\bullet(X)$$

  given by the Chern character. The image of the Chern character is always contained in the span of the algebraic cycles inside $H^\bullet(X)$, because by definition Chern classes come from algebraic cycles. So it suffices to show that the Chern character is surjective.
- How could one ever show that the Chern character is surjective? There is a useful general criterion which we now discuss.

  Definition 4.5. We say that X has *decomposable diagonal* in K-theory if $[\mathcal{O}_\Delta] \in K(X \times X)$ (where $\mathcal{O}_\Delta$ is the structure sheaf of $\Delta(X) \subset X \times X$) is in the image of the map

  $$K(X) \boxtimes K(X) \to K(X \times X).$$

  Lemma 4.6. *Suppose that Y is a smooth projective variety with decomposable diagonal. Then* $\mathrm{ch} : K(Y) \to H^\bullet(Y)$ *is surjective.*

  Proof. Suppose $[\mathcal{O}_\Delta] = \sum [\mathcal{E}_i \boxtimes \mathcal{F}_i]$. Then we have

  $$\mathrm{ch}\,(\mathcal{O}_\Delta) = \sum \mathrm{ch}\ \mathcal{E}_i \boxtimes \mathrm{ch}\ \mathcal{F}_i \in H^\bullet(Y \times Y).$$

  But $\mathrm{ch}\,(\mathcal{O}_\Delta) = [Y_\Delta] + (\text{higher degree})$. This implies that

  $$[Y_\Delta] = \sum c_i' \boxtimes c_i''$$

  where c_i', c_i'' are algebraic cycles. Now we use a trick, namely that the class of the diagonal is the identity for the convolution operator, i.e. $[Y_\Delta] * c := (p_2)_* p_1^* c = c$. Then, for *any* $c \in H^\bullet(Y)$ we have

  $$\begin{aligned} c &= [Y_\Delta] * c \\ &= \sum (c_i' \boxtimes c'') * c \\ &= \sum \langle c_i'', c \rangle \cdot c_i' \end{aligned}$$

  which is manifestly an algebraic cycle. □

Unfortunately, we can't apply this directly because our X is not projective.

- How could we show that X has a decomposable diagonal? This can be shown to follow from another general property, which we now introduce.

  Definition 4.7. A coherent sheaf $\mathcal{T}$ on Y is called a *tilting generator* if

  $$\mathrm{Ext}^i(\mathcal{T}, \mathcal{T}) = 0 \text{ for all } i > 0$$

  and for any $\mathcal{F} \neq 0$ in $D^b_{coh}(Y)$,

  $$R\,\mathrm{Hom}(\mathcal{T}, \mathcal{F}) \neq 0.$$

  Now comes the crucial point, which is a combination of the theorems of Kaledin, Bezrukavnikov, etc.

  Theorem 4.8 (Bezrukavnikov-Kaledin + Kaledin)**.** *If (X, ω) is a symplectic resolution then there exists a tilting generator.*

  Proof sketch. Reduce to finite characteristic, and then use Roman's favorite trick, which is that quantization in positive characteristic is almost commutative. □

- Fix a tilting generator $\mathcal{T}$. Consider the algebra $A = \mathrm{Hom}(\mathcal{T}, \mathcal{T}) \cong R\,\mathrm{Hom}(\mathcal{T}, \mathcal{T}$ (because the higher Exts vanish by definition). Then we have two facts:

  1. By generalities on homological algebra, the functor $R\,\mathrm{Hom}(\mathcal{T}, -)$ gives an equivalence $D^b_{coh}(Y) \to D^b(A - \mathbf{mod})$.
  2. (Grothendieck's homological characterization of smoothness) Y is smooth if and only if $D^b_{coh}(Y)$ has finite homological dimension.

  Corollary 4.9. *If $\mathcal{T}$ is a tilting generator and Y is smooth, then A has finite homological dimension.*

- We claim that if A is an algebra with finite homological definition, then A has a finite projection resolution by A-bimodules:

  $$A \leftarrow P_1 \boxtimes \mathbb{Q}_1 \leftarrow \ldots \leftarrow P_n \boxtimes Q_n$$

  where the P_i (resp. Q_i) are are projective left (resp. right) A-modules.

  Indeed, by definition A has finite projective resolutions, and one just has to take care that the resolution can be chosen of this form. We'll come back to this later and show why this special form of the resolution can be chosen.

  If we know this, then we're done because it means that $\mathcal{O}_\Delta \cong A$ has a resolution of the desired type. (Apply the equivalence of categories twice, to Y and $Y \times Y$, to see that A as an (A, A)-bimodule must corresponds to $\mathcal{O}_\Delta$.)

  We also still have to deal with the issue that our X is not projective.

- Let X be our resolution. By the $\mathbb{C}^\times$-action, we have a grading

$$\mathbb{C}[X] = \mathbb{C}[X^{\mathrm{aff}}] = \mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_i$$

  such that $\mathcal{A}_0 = \mathbb{C}$ and $\dim_{\mathbb{C}} \mathcal{A}_i < \infty$. (See Example 3.3.)

  It is easy to show that one can choose $\mathcal{T}$ to be $\mathbb{C}^\times$-equivariant. Then we have a grading also on $A := \mathrm{Hom}(\mathcal{T}, \mathcal{T})$, which contains $\mathcal{A}$ in its center, since the homomorphisms are always algebras over the regular functions, and we claim that A is a finitely generated submodule over $\mathcal{A}$. In particular, $A_i = 0$ for almost all $i < 0$.

  Conjecture 4.10 (Bezrukavnikov)**.** *One can choose $\mathcal{T}$ so that $A_i = 0$ for* all $i < 0$.

  We claim that if $A = \mathrm{Hom}(\mathcal{T}, \mathcal{T})$ has no negative degrees, then by some simple homological algebra A is Koszul. Thus, the conjecture would imply that $D^b_{coh}(Y)$ is equivalent to modules over a Koszul algebra, which is a really strong consequence.

- We now want to proved the existence of a resolution of the form

$$A \leftarrow P_1 \boxtimes Q_1 \leftarrow \ldots \leftarrow P_n \otimes Q_n$$

  The situation looks like the completion of a semi-simple algebra, because if you truncate A at very high degrees, then it is a finite-dimensional algebra. We understand the situation of finite-dimensional algebras pretty well: if you quotient by the nilradical then you get a semisimple algebra, and then you can lift projectives across the nilradical. That reduces to proving the statement for a finite-dimensional simple algebra, which is a matrix algebra. Then projectives are free, and it's obvious that free over a tensor product is a tensor product of frees.

  More precisely, $A/A_{>N}$ for all $N \gg 0$ is a finite-dimensional algebra, and one can proved that the desired kind of resolution exists for $A/A_{>N}$. Then one takes a limit over $N \to \infty$. If A has some negative degrees, then one has to take some care, but the argument still works.

- Working equivariantly, one can extend the argument to show that X has decomposable diagonal in equivariant K-theory. (The structure sheaf of the diagonal is an equivariant sheaf, and you show that everything can be taken to be equivariant.)

- We claim that if we have a variety X with decomposable diagonal in equivariant K-theory, then the fixed point set $X^{\mathbb{C}^\times}$ has decomposable diagonal in ordinary K-theory. In the semipositive situation, $X^{\mathbb{C}^\times}$ is a smooth projective variety. So we have proved the theorem for the fixed point set.

- The conclusion is that $H_*(X^{\mathbb{C}^\times})$ is spanned by algebraic cycles, and then BB implies that $H_*(X)$ is also.

Remark 4.11. We've shown the result for X and the central fiber. How do you get the result for the general fiber? Kaledin has proved that you can choose a slice $S_x \subset X^{\mathrm{aff}}$ such that in $\pi^{-1}(S_x) \to S_x$, x is the central fiber (necessarily for a different $\mathbb{C}^\times$-action).

Example 4.12. For the Springer resolution $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$, it was conjectured around 1976 that $H^\bullet(\pi^{-1}x)$ is generated by algebraic cycles. It was proved by DeConcini-Lusztig-Procesi in 1991, but by a "case-by-case" analysis for each possible fixed point set. Kaledin's proof, which we have sketched here, is the only conceptual one that we know, even in the special case of the Springer resolution.