FACTORIZATION ALGEBRAS AND CHIRAL ALGEBRAS (OCT 29, 2020)

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Contents

1. Review of factorization algebras

Let X be a curve. Jacob defined *factorization algebras* as: a collection of quasicoherent sheaves $A^{(n)}$ on X^n, equipped with isomorphisms upon restricting to the diagonal or to the disjoint locus, e.g.

- $\Delta^*(\mathcal{A}^{(2)}) \xrightarrow{\sim} \mathcal{A}^{(1)}$ and
- $j^*(\mathcal{A}^{(2)}) \xrightarrow{\sim} \mathcal{A}^{(1)} \boxtimes \mathcal{A}^{(1)}|_{X \times X - \Delta}$.

To give more examples, on X^3 the data of a factorization algebra includes isomorphisms

(1) $\Delta^*_{x_1=x_2}(\mathcal{A}^{(3)}) \xrightarrow{\sim} \mathcal{A}^{(2)}$,
(2) $\Delta^*_{x_1=x_3}(\mathcal{A}^{(3)}) \xrightarrow{\sim} \mathcal{A}^{(2)}$
(3) $\Delta^*_{x_2=x_3}(\mathcal{A}^{(3)}) \xrightarrow{\sim} \mathcal{A}^{(2)}$
(4) $\Delta_{x_1=x_2=x_3}(\mathcal{A}^{(3)}) \xrightarrow{\sim} \mathcal{A}^{(1)}$.
(5) $j^*_{x_1 \neq x_2, x_3}(\mathcal{A}^{(3)}) \xrightarrow{\sim} \mathcal{A} \boxtimes \mathcal{A}^{(2)}|_{x_1 \neq x_2, x_3}$
(6) $j^*(\mathcal{A}^{(3)}) \xrightarrow{\sim} \mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}|_{X^3 - \Delta}$.

There are also compatibilities on these isomorphisms when the loci intersect, e.g. (2) and (5).

If $\mathcal{A}^{(n)}$ are in the abelian category of quasi-coherent sheaves, then everything is recovered from the data over X^2.

1.1. Unital factorization algebras.

Unital factorization algebras. Jacob also introduced the notion of a *unital structure* on a factorization algebra. That is a system of maps $\mathcal{O}_{X^n} \to \mathcal{A}^{(n)}$ compatible with the identifications above. He also explained that the unital structure equips all the $\mathcal{A}^{(n)}$ with a connection, making them into left D-modules on X^n.

For $f\colon Y_1 \to Y_2$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Dmod}^l(Y_1) \arrow[d] & \mathrm{Dmod}^l(Y_2) \arrow[l, "f^\dagger"'] \arrow[d] \\ \mathrm{QCoh}(Y_1) & \mathrm{QCoh}(Y_2) \arrow[l, "f^*"'] \end{tikzcd}$$

Example 1.1. Let $\mathcal{A} \in \mathrm{CommAlg}(\mathrm{Dmod}^l(X))$. Then we can associate a factorization algebra $\mathrm{Fact}(\mathcal{A})$ such that $\mathcal{A}^{(1)} = \mathcal{A}$.

Suppose you have a closed embedding $f\colon Y_1 \hookrightarrow Y_2$. There is $f^\dagger\colon \mathrm{Dmod}^l(Y_2) \to \mathrm{Dmod}^l(Y_1)$. There is also $f_* = f_!\colon \mathrm{Dmod}(Y_1) \to \mathrm{Dmod}(Y_2)$, normalized to be t-exact. Unfortunately, with these normalizations the functors are not adjoint. Rather, $f^\dagger$ is right adjoint to $f_* = f_!$, up to shift. (So $f^\dagger$ agrees with $f^!$ up to a shift.)

Example 1.2. For an open/closed decomposition, the Cousin sequence is

$$f_! f^\dagger(M)[-\mathrm{codim}] \to M \to j_* j^* M$$

Consider this applied to $\mathcal{M} = \mathcal{A}^{(2)}$ on X^2. Then the Cousin complex is

$$\Delta_! \Delta^\dagger[-1]\mathcal{A}^{(2)} \to \mathcal{A}^{(2)} \to j_* j^*(\mathcal{A}^{(2)}).$$

So unwinding the normalizations and identifications gives a distinguished triangle

$$\mathcal{A}^{(2)} \to j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_! \mathcal{A}^{(1)}.$$

So $\mathcal{A}^{(2)} = \ker(j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_! \mathcal{A}^{(1)})$.

1.2. **Left vs right D-modules.** The category of left D-modules is a bit inconvenient, so we'll want to switch to right D-modules. This is given by tensoring with the dualizing sheaf.

A better way to think about this is that there is one category of D-modules, but it has two forgetful functors to quasicoherent sheaves, called oblv^l and oblv^r.

$$\begin{tikzcd} & \mathrm{Dmod}(X) \arrow[dl, "\mathrm{oblv}^l"'] \arrow[dr, "\mathrm{oblv}^r"] & \\ \mathrm{QCoh}(X) \arrow[rr, "-\otimes \Omega_X"] & & \mathrm{QCoh}(X) \end{tikzcd}$$

The right normalization is better for functoriality. Given $f\colon Y_1 \to Y_2$, the diagram commutes

$$\begin{tikzcd} \mathrm{Dmod}(Y_1) \arrow[d, "\mathrm{oblv}^r"] & \mathrm{Dmod}(Y_2) \arrow[l, "f^!"'] \arrow[d, "\mathrm{oblv}^r"] \\ \mathrm{QCoh}(Y_1) & \mathrm{QCoh}(Y_2) \arrow[l, "f^!"'] \end{tikzcd}$$

For a factorization algebra $\mathcal{A}^{(n)}$, we denote $\mathcal{A}^{(n)}_{\mathrm{Fact}} \in \mathrm{Dmod}^r(X^{(n)})$ the corresponding right D-modules. So now the factorization axioms read

$$j^*(\mathcal{A}^{(2)}_{\mathrm{Fact}}) \xrightarrow{\sim} \mathcal{A}^{(1)}_{\mathrm{Fact}} \boxtimes \mathcal{A}^{(1)}_{\mathrm{Fact}}|_{X^2 - \Delta}$$

and

$$\Delta^!(\mathcal{A}^{(2)}_{\mathrm{Fact}}) \cong \mathcal{A}^{(1)}_{\mathrm{Fact}}.$$

Also

$$\mathcal{A}^{(2)}_{\mathrm{Fact}} = \ker\left(j_* j^*(\mathcal{A}^{(1)}_{\mathrm{Fact}} \boxtimes \mathcal{A}^{(1)}_{\mathrm{Fact}}) \to \Delta_! \mathcal{A}^{(1)}_{\mathrm{Fact}}[1]\right).$$

2. Chiral algebras

2.1. Chiral algebras from factorization algebras. We will now define the chiral algebra associated to the factorization algebra $\mathcal{A}^{(n)}$: it is the collection $\mathcal{A}_{\mathrm{ch}}^{(n)} := \mathcal{A}_{\mathrm{Fact}}^{(1)}[-n]$, so in particular $\mathcal{A}_{\mathrm{ch}}^{(1)} = \mathcal{A}_{\mathrm{Fact}}^{(1)}[-1]$. Hence in this normalization

$$\mathcal{A}_{\mathrm{ch}}^{(2)} = \ker\left(j_*j^*(\mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)}) \to \Delta_!\mathcal{A}_{\mathrm{ch}}^{(1)}\right).$$

The point is that this can be normalized to lie in the heart of the t-structure.

The map

$$j_*j^*(\mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)}) \to \Delta_!\mathcal{A}_{\mathrm{ch}}^{(1)} \tag{2.1}$$

is almost like a thing with a binary operation, so that's reminiscent of what an algebra is. Let's call (2.1) the "chiral bracket".

Proposition 2.1. *The chiral bracket satisfies the conditions of a Lie bracket.*

Why is it skew-symmetric? Because the shift by 1 affects the sign rule. (Symmetry under S_2 goes to skew-symmetric under S_2 after shifting.)

Now for the "Jacobi identity", we'll write three ways to go from $j_*j^*(\mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)})$ to $\Delta_!(\mathcal{A}_{\mathrm{ch}}^{(1)})$ on X^3. First we can map via chiral bracket $\boxtimes\,\mathrm{Id}$ to $\Delta_{(1=2)\neq 3!}(j_*j^*(\mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)}))$[1]. Then we map via chiral bracket again to $\Delta_!(\mathcal{A}_{\mathrm{ch}}^{(1)})$.

So we have

$$\mathcal{A}^{(3)} \to j_*j^*(\mathcal{A}^{(3)}) \to \Delta_{1=2!}j_*(j^*\Delta_{12}^!(\mathcal{A}^{(3)})) \oplus \ldots \to \Delta_{123!}\Delta_{123}^!(\mathcal{A}^{(3)})$$

This is acyclic because it is just the Cousin complex. The fact that the composition

$$j_*j^*(\mathcal{A}^{(3)}) \to \Delta_{1=2!}j_*(\mathcal{A} \boxtimes \mathcal{A}) \oplus \ldots \to \Delta_!(\mathcal{A})$$

is 0 is just the fact that Cousin complex is a complex, and is the Jacobi identity in this context.

(This is related to the fact that there is a functor from $\mathbb{E}_2$ algebras to Lie algebras, by shifting.)

Definition 2.2. A *chiral algebra* is a right D-module $\mathcal{A}_{\mathrm{ch}}$ on X equipped with a bracket

$$j_*j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_!\mathcal{A}_{\mathrm{ch}}$$

satisfying the Lie axioms.

A *unit* in a chiral algebra $\mathcal{A}$ is a map

$$u\colon \Omega_X \to \mathcal{A}_{\mathrm{ch}}$$

such that the diagram below commutes:

$$\begin{tikzcd} j_*j^*(\Omega_X \boxtimes A) \arrow[r, "\mathrm{can}"] \arrow[d, "u\boxtimes\mathrm{Id}"] & \Delta_!\mathcal{A}_{\mathrm{ch}} \arrow[d, "\mathrm{Id}"] \\ j_*j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \arrow[r] & \Delta_!\mathcal{A}_{\mathrm{ch}} \end{tikzcd}$$

[1]Explanation of intermediate steps: we have a map $j_*j^*(\mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)}) \to (j_{1,2\neq 3})_*((j_{1\neq 2})_*(j_{1\neq 2})^*\mathcal{A}_{\mathrm{ch}}^{(1)} \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)}) \boxtimes \mathcal{A}_{\mathrm{ch}}^{(1)})$. Via chiral bracket on the first two factors, this maps to $(j_{1,2\neq 3})_*(\Delta_{1=2!}\mathrm{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}})$. Then using the chiral bracket again goes to $\Delta_{1=2=3!}\mathcal{A}_{\mathrm{ch}}$. So I'm not sure we actually pass through the intermediate step above.

Here Ω_X is the line bundle of 1-forms on X, which is the shift of the dualizing sheaf, viewed right as a right D-module. Explanation of "can": note that the Cousin sequence for $\Omega_X \boxtimes \mathcal{A}_{\mathrm{ch}}$ is

$$\Omega_X \boxtimes \mathcal{A}_{\mathrm{ch}} \to j_* j^*(\Omega_X \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_! \mathcal{A}_{\mathrm{ch}}$$

The rightmost map is "can".

Theorem 2.3. *The functor from unital factorization algebras to unital chiral algebras is an equivalence.*

Proof. Let's explain the inverse. We can recover

$$\mathcal{A}_{\mathrm{ch}}^{(2)} := \ker\left(j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \twoheadrightarrow \Delta_! \mathcal{A}_{\mathrm{ch}}\right).$$

Remark: the unit implies the right term is surjection. This also exhibits $j^* \mathcal{A}_{\mathrm{ch}}^{(2)} \xrightarrow{\sim} \mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}$ and $\Delta^!(\mathcal{A}_{\mathrm{ch}}^{(2)}) = \mathcal{A}_{\mathrm{ch}}^{(1)}[-1]$.

To recover $\mathcal{A}^{(3)}$, we set

$$\mathcal{A}_{\mathrm{ch}}^{(3)} := H^0\left(j_*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_{1=2!} j_*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \oplus \ldots \to \Delta_{123!}(\mathcal{A}_{\mathrm{ch}})\right).$$

□

Remark 2.4. This explains why the factorization algebra was determined by the data on X^2, satisfying the conditions on X^3. The data on X^2 gives the chiral bracket, and the data on X^3 showed that it satisfies the Lie axioms.

Remark 2.5. The theorem stated only applies to unital chiral algebras. There are certainly other kinds of chiral algebras; for example, you can take the chiral bracket $j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_! \mathcal{A}_{\mathrm{ch}}$ to just be 0. However, it works without the unit if we consider derived objects. So the unit is something that allows you to stay in the heart of the t-structure.

2.2. **Commutative chiral algebras.** Let $\mathcal{A}^l \in \mathrm{CommAlg}(\mathrm{Dmod}^l(X))$. We then turn it into a right D-module in the same as before: $\mathcal{A}_{\mathrm{ch}} = (\mathcal{A}^l \otimes \text{dualizing})[-1] = \mathcal{A}^l \otimes \Omega_X$. We will then construct a chiral bracket on it.

$$j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \dashrightarrow \Delta_! \mathcal{A}_{\mathrm{ch}}.$$

We have an exact triangle

$$\Delta_!(\mathcal{A}_{\mathrm{ch}} \overset{!}{\boxtimes} \mathcal{A}_{\mathrm{ch}})[-1] \to \mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}} \to j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}})$$

This gives

$$j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_!(\mathcal{A}_{\mathrm{ch}} \overset{!}{\boxtimes} \mathcal{A}_{\mathrm{ch}}) \xrightarrow{\text{mult}} \Delta_!(\mathcal{A}_{\mathrm{ch}}).$$

We then define $\mathcal{A}_{\mathrm{ch}}^{(2)} = \ker(j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_!(\mathcal{A}_{\mathrm{ch}}))$.

Definition 2.6. A chiral algebra is said to be *commutative* if the composition

$$\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}} \to j_* j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}}) \to \Delta_!(\mathcal{A}_{\mathrm{ch}})$$

is zero. (The composite map is called the "Lie-$*$" bracket, so in other words a chiral algebra is commutative iff the Lie-$*$ bracket vanishes.)

Proposition 2.7. *The above construction is an equivalence between* $\mathrm{ComAlg}(\mathrm{Dmod}^l(X))$ *and commutative chiral algebras.*

Warning 2.8. In a higher categorical situation, commutativity becomes a *structure* – one needs to specify nullhomotopies.

3. Examples

3.1. **The Beilinson-Drinfeld Grassmannian.** Recall our first example of factorization algebra: for the Beilinson-Drinfeld $\pi\colon \mathrm{Gr}_{X^n} \to X^n$ and $\mathcal{L}_{X^n}$ a line bundle with factorization structure, we have $\mathcal{A}^{(n)} := \pi_!(\mathcal{L}_{X^n}) \in \mathrm{QCoh}(X^n)$.

For an ind-scheme $\mathcal{Y} = \varinjlim Y_i$, the cohomology is a pro-object $\Gamma(\mathcal{Y};\mathcal{L}) = \varprojlim \Gamma(Y_i,\mathcal{L})$. It's better to use cosections to get an ind-object, $\Gamma(\mathcal{Y};\mathcal{L})^\vee = \varinjlim \Gamma(Y_i,\mathcal{L})^\vee$.

We can rewrite this succintly: $\Gamma(Y_i,\mathcal{L})^\vee = \Gamma(Y_i,\mathcal{L}^{-1}\otimes\omega_{Y_i})$. We have that $\omega_{\mathcal{Y}} \cong \varinjlim \omega_{Y_i}$. So $\Gamma_c(\mathcal{Y},\mathcal{L})^\vee = \Gamma(\mathcal{Y},\mathcal{L}^{-1}\otimes\omega_{\mathcal{Y}})$.

Remark 3.1. Jacob assumed that $\mathcal{L}$ was ample, so we get sections and no higher cohomology. The point is if Y_i are smooth, the dualizing is a shift of the canonical bundle. But for an ind-scheme, ω lives in cohomological degree $-\infty$.

The Gr_{G,X^n} comes with a section (trivial bundle with tautological trivialization). Consider the formal completion $\mathrm{Gr}^\wedge_{G,X^n}$ along this section. We can then restrict $\mathcal{L}^{-1}$ to the formal completion. Define $\mathcal{A}^{(n)} = \pi^\wedge_!(\mathcal{L}^{-1})$ to be the direct image of $\mathcal{L}^{-1}$ on $\mathrm{Gr}^\wedge_{G,X^n}$ to X^n.

3.2. **Lie-$*$ algebras.**

Definition 3.2. A *Lie-$*$ algebra* is a D-module L on X equipped with

$$L\boxtimes L \to \Delta_! L$$

satisfying the Lie axioms.

There is a forgetful functor from chiral algebras to Lie-* algebras, as given a chiral bracket $j_*j^*(\mathcal{A}\boxtimes\mathcal{A}) \to \Delta_!\mathcal{A}$ we can just inflate via $\mathcal{A}\boxtimes\mathcal{A} \to j_*j^*(\mathcal{A}\boxtimes\mathcal{A})$ to get a Lie-* bracket. This is analogous to the forgetful functor from $\mathbb{E}_2$ algebras to shifted Lie algebras.

Key observation: the above functor admits a left adjoint $L \mapsto U_{\mathrm{ch}}(L)$, called the "chiral universal envelope". That gives many important examples of chiral algebras. This fact also has a topological analog.

Example 3.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and consider $L := \mathfrak{g}\otimes D_X$ as a right D-module. We have to write down

$$(\mathfrak{g}\boxtimes D_X)\boxtimes(\mathfrak{g}\boxtimes D_X) \to \Delta_!(\mathfrak{g}\boxtimes D_X)$$

In other words we have to give a map

$$\mathfrak{g}\otimes\mathfrak{g} \to \Gamma(X\times X, \Delta_!(\mathfrak{g}\boxtimes D_X)).$$

There is a map $\Gamma(X,\mathfrak{g}\otimes D_X) \to \Gamma(X\times X,\Delta_!(\mathfrak{g}\boxtimes D_X))$. We then compose this with $\mathfrak{g}\otimes\mathfrak{g} \xrightarrow{[\cdot,\cdot]} \mathfrak{g} \to \Gamma(X,\mathfrak{g}\otimes D_X)$.

Example 3.4. We consider $T_X\otimes_{\mathcal{O}_X} D_X$. We try repeating the same construction, but the problem is that the Lie bracket on T_X is not $\mathcal{O}_X$-linear.

Given two quasicoherent sheaves M_1, M_2 on X we can talk about differential operators $M_1 \to M_2$; these are the same as $\mathcal{O}_X$-linear maps $M_1 \to M_2\otimes_{\mathcal{O}_X} D_X$.

We have $\Delta_!(M\otimes_{\mathcal{O}_X} D_X) = \Delta_*(M)\otimes_{\mathcal{O}_{X\times X}}\mathcal{D}_{X\times X}$. So we want to define

$$(T_X\otimes_{\mathcal{O}_X} D_X)\boxtimes(T_X\otimes_{\mathcal{O}_X} D_X) \to \Delta_!(T_X\otimes_{\mathcal{O}_X} D_X) \xrightarrow{\sim} \Delta_*(T_1)\otimes_{\mathcal{O}_{X\times X}} D_{X\times X}.$$

Such a latter map $T_X\boxtimes T_X \to \Delta_*(T_X)\otimes_{\mathcal{O}_{X\times X}} D_{X\times X}$ is the same as a differential operator, which we can take to be the Lie bracket $T_X\boxtimes T_X \to \Delta_*(T_X)$.

Lemma 3.5. $U_{\mathrm{ch}}(\mathfrak{g}\otimes D_X) = \pi^\wedge_!\mathcal{O}$ *where* $\pi^\wedge_!\colon \mathrm{Gr}^\wedge_{X^n} \to X^n$.

Remark 3.6. For this construction $\mathfrak{g}$ can be any Lie algebra (not necessarily reductive).

4. Modules for chiral algebras

Let $\mathcal{A}_{\mathrm{ch}}$ be a chiral algebra.

Definition 4.1. A *chiral* $\mathcal{A}_{\mathrm{ch}}$-module on X is a D-modules M on X equipped a map[2]

$$j_*j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes M) \to \Delta_! M$$

plus a Jacobi identity (Lie algebra acting on a module) for the three maps

$$j_*j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes \mathcal{A}_{\mathrm{ch}} \boxtimes M) \to \Delta_! M.$$

We say that M is *unital* if the diagram below commutes:

$$\begin{tikzcd} j_*j^*(\Omega_X \boxtimes M) \arrow[r] \arrow[d] & \Delta_! M \arrow[d, "\mathrm{Id}"] \\ j_*j^*(\mathcal{A}_{\mathrm{ch}} \boxtimes M) \arrow[r] & \Delta_! M \end{tikzcd}$$

Example 4.2. $\mathcal{A}_{\mathrm{ch}}$ is a chiral module over itself.

Remark 4.3. An important class of examples comes from chiral A-modules supported at $x \in X$.

Example 4.4. If $M \in \mathcal{A}\text{−}\mathrm{Mod}^{\mathrm{ch}}(X)$, then $i_{x!}i_x^!(M)[1]$ is a chiral $\mathcal{A}$-module at a point x. So "fibers of chiral modules are chiral modules."

Example 4.5. How do $\mathcal{A}_{\mathrm{ch}}$-modules look like for $\mathcal{A}_{\mathrm{ch}}$ commutative?

Let $A^l \in \mathrm{CommAlg}(\mathrm{Dmod}^l(X))$. Then $i_x^*(\mathcal{A}) \in \mathrm{CommAlg}(\mathrm{Vect})$. We claim that there is a functor from $i_x^*(A^l)$-modules to chiral $\mathcal{A}_{\mathrm{ch}}$-modules.

"Chiral modules supported at $x \in X$ only care about the chiral algebra away from x." To see an example of what this means we can modify A^l at a point. Let $\phi\colon (\mathcal{A}')^l \hookrightarrow \mathcal{A}^l$ be an isomorphism away from x. Then we also get a functor from $i_x^*((A')^l)$-modules to chiral $\mathcal{A}_{\mathrm{ch}}$-modules.

Lemma 4.6. *Any chiral $\mathcal{A}_{\mathrm{ch}}$-module is a union of ones of the above form.*

Example 4.7. If $\mathcal{A}$ is the (commutative) chiral algebra of functions on jets into a space, then functions on loops are chiral modules.

5. Lie-$*$ modules

Definition 5.1. Let L be a Lie-* algebra on X. A *Lie-$*$ module* over L is a D-module M on X with a map $L \boxtimes M \to \Delta_! M$ satisfying a Jacobi identity. We denote the category of such as $L\text{−}\mathrm{Mod}^{\mathrm{Lie}-*}(X)$.

A *chiral* L-module is a D-module M on X with a map $j_*j^*(L \boxtimes M) \to \Delta_! M$ satisfying a Jacobi identity for the three maps

$$j_*j^*(L \boxtimes L \boxtimes M) \to \Delta_! M$$

given by $j_*j^*(L \boxtimes L \boxtimes M) \to \Delta_{x_1=x_2!}(j_*j^*(L \boxtimes M)) \to \Delta_!(M)$ and its variations obtained by permuting $\{1,2,3\}$. We denote the category of such by $L - \mathrm{Mod}^{\mathrm{ch}}(X)$.

There is an obvious functor $L - \mathrm{Mod}^{\mathrm{Lie}-*}(X) \leftarrow L - \mathrm{Mod}^{\mathrm{ch}}(X)$, given by pre-composing with $L \to j_*j^*L$. This functor has a left adjoint.

Given a chiral module, we can form the de Rham cohomology $\Gamma_{\mathrm{dR}}(D_x, M)$. This is a topological vector space, since D_x is an ind-scheme.

[2] Another approach: it is a split square-zero extension $\mathcal{A}_{\mathrm{ch}} \oplus \epsilon M$.

Example 5.2. Suppose $M = M_0 \otimes_{\mathcal{O}_X} \mathcal{D}_X$ for M_0 a quasicoherent sheaf. Then $\Gamma_{\mathrm{dR}}(D_x, M) = \Gamma(D_x, M_0)$. In particular, if $M = \mathcal{D}_X$ then $\Gamma_{\mathrm{dR}}(D_x, M) = \widehat{\mathcal{O}}_x$.

We can also take $\Gamma_{\mathrm{dR}}(D_x^\circ, M)$. This is also a topological vector space.

Example 5.3. We have $\Gamma_{\mathrm{dR}}(D_x, M_0 \otimes D_X) = \Gamma(D_x^\circ, M_0)$. For $M = \mathcal{D}_X$, we have $\Gamma_{\mathrm{dR}}(D_x, M) = \widehat{K}_x$.

Remark 5.4. If L is a Lie-$*$ algebra, then $\Gamma_{\mathrm{dR}}(D_x, L)$ and $\Gamma_{\mathrm{dR}}(D_x^\circ, L)$ are topological Lie algebras.

Lemma 5.5. *Let $L - \mathrm{Mod}_x^{\mathrm{Lie}-*}$ be the category of Lie-$*$ modules over L which are supported at $x \in X$. Then $L - \mathrm{Mod}_x^{\mathrm{Lie}-*}$ is equivalent to the category of discrete modules for $\Gamma_{\mathrm{dR}}(D_x, L)$ and $L - \mathrm{Mod}_x^{\mathrm{ch}}$ is equivalent to the category of discrete modules for $\Gamma_{\mathrm{dR}}(D_x^\circ, L)$.*

Example 5.6. For $L = \mathfrak{g} \otimes \mathcal{D}_x$, we have

$$\Gamma_{\mathrm{dR}}(D_x, L) = \mathfrak{g} \otimes \widehat{\mathcal{O}}_x = \mathfrak{g}[[t]],$$

and

$$\Gamma_{\mathrm{dR}}(D_x^\circ, L) = \mathfrak{g} \otimes \widehat{K}_x = \mathfrak{g}((t)).$$

Then $(\mathfrak{g} \otimes \mathcal{D}_x) - \mathrm{Mod}_x^{\mathrm{Lie}-*}$ is equivalent to the category of discrete $\mathfrak{g}[[t]]$-modules, and $\mathfrak{g} \otimes \mathcal{D}_x - \mathrm{Mod}_x^{\mathrm{ch}}$ is equivalent to the category of discrete $\mathfrak{g}((t)) - \mathrm{Mod}$.

Example 5.7. There is an equivalence between $T_x \otimes \mathcal{D}_X - \mathrm{Mod}_x^{\mathrm{Lie}-*}$ and modules over $\mathrm{Span}(L_{-1}, L_0, L_1 \ldots)$-modules, where $L_{-1} = \partial_t$, $L_0 = t\partial_t$, $L_1 = t^2 \partial_t$, etc.

On the other hand, chiral modules over $T_x \otimes \mathcal{D}_X$ are modules over span of L_i for all $i \in \mathbf{Z}$.

The cokernel of $\Gamma_{\mathrm{dR}}(D_x, L) \hookrightarrow \Gamma_{\mathrm{dR}}(D_x^\circ, L)$ is $i_x^!(L)[1]$. The map $\Gamma_{\mathrm{dR}}(D_x, L) \hookrightarrow \Gamma_{\mathrm{dR}}(D_x^\circ, L)$ describes restriction and induction functors.

(We are working our ways up to the description of the chiral universal envelope.)

Let L be a Lie-$*$ algebra and $U_{\mathrm{ch}}(L)$ its universal envelope. We have a map as Lie algebras $L \to U_{\mathrm{ch}}(L)$.

Proposition 5.8. *The restriction functor $U_{\mathrm{ch}}\, L{-}\mathrm{Mod}^{\mathrm{ch}}(X) \to L{-}\mathrm{Mod}^{\mathrm{ch}}(X)$ is an equivalence.*

Suppose $\mathcal{A}$ is a chiral algebra. Write $\mathcal{A}_x = i_x^!(\mathcal{A})[1]$. We can think of $\mathcal{A}_x$ as a chiral module for $\mathcal{A}$ at x. We want to describe it what it looks like for $U_{\mathrm{ch}}(L)_x$, as a module for $\Gamma_{\mathrm{dR}}(D_x^\circ, L)$. The unit gives a map $\mathbf{C} \to U_{\mathrm{ch}}(L)_x$, and the axioms (namely, that the unit is killed by Lie-$*$ bracket) show that it is annihilated by $\Gamma_{\mathrm{dR}}(D_x, L)$.

So we get a map

$$\mathrm{Ind}_{\Gamma_{\mathrm{dR}}(D_x, L)}^{\Gamma_{\mathrm{dR}}(D_x^\circ, L)}(\mathbf{C}) \to U_{\mathrm{ch}}(L)_x. \tag{5.1}$$

The LHS is a "vacuum representation".

Theorem 5.9. *This map* (5.1) *is an isomorphism.*

Remark 5.10. Jacob asks: given a chiral algebra $\mathcal{A}$ on a punctured curve $X - x$, what is the relation between extending $\mathcal{A}$ to X and putting a module at the puncture? Answer: given an extension, the fiber $(\mathcal{A}_x) \in \mathcal{A}_{X-x} - \mathrm{Mod}_x^{\mathrm{ch}}$ has the universal property:

$$\mathrm{Hom}_{\mathcal{A}_{X-x} - \mathrm{Mod}_x^{\mathrm{ch}}}((\mathcal{A}_x), M) = \{m \in M \colon \text{ annihilated by the Lie-}* \text{ bracket}\}.$$

6. Factorization modules

We have a correspondence between $\mathcal{A}_{\mathrm{Fact}}$ and $\mathcal{A}_{\mathrm{ch}}$.

Definition 6.1. A *factorization module* over a factorization algebra $\mathcal{A}_{\mathrm{Fact}}$ is a sequence of D-modules (because we're over a curve) $M = M^{(0,1)} \in \mathrm{Dmod}(\mathrm{pt} \times X)$, $M^{(1,1)} \in \mathrm{Dmod}(X \times X)$, $M^{2,1} \in \mathrm{Dmod}(X^2 \times X)$, ... with isomorphisms

$$\Delta^!(M^{(1,1)}) \xrightarrow{\sim} M^{0,1}$$

$$j^! M^{(1,1)} \xrightarrow{\sim} \mathcal{A}^{(1)}_{\mathrm{Fact}} \boxtimes M^{(0,1)}.$$

plus identifications $M^{(2,1)}|_{x_1 \neq x_3, x_2 \neq x_3} = \mathcal{A}^{(2)} \boxtimes M^{(0,1)}$ and $M^{(2,1)}|_{x_2 = x_3} = M^{(1,1)}$, etc. (Think of the last coordinate as being the module coordinate, and the rest being algebra coordinates.)

So we have an exact triangle

$$M^{(1,1)}[\text{shift?}] \to j_* j^* (\mathcal{A}_{\mathrm{Fact}} \boxtimes M) \to \Delta_! M$$

The equivalence between $\mathcal{A}_{\mathrm{Fact}}$ and $\mathcal{A}_{\mathrm{ch}}$ intertwines an equivalence between factorization modules and chiral modules.

We can define $\mathcal{A}_{\mathrm{Fact}} - \mathrm{Mod}^{\mathrm{Fact}}(X^2)$ as a sequence $M^{(0,2)}, M^{(1,2)}, M^{(2,2)}$ on X^2, $X \times X^2$, $X^2 \times X^2$, etc. with isomorphisms

$$M^{(1,2)}|_{x_1 \neq x_2, x_1 \neq x_3} = \mathcal{A}_{\mathrm{Fact}} \boxtimes M^{(0,2)}.$$

and

$$M^{(1,2)}|_{x_1 = x_2} = M^{(0,2)}.$$

etc.

There is a lot of structure on the category of factorization modules.

- We have $\Delta_! \colon \mathcal{A}\text{--}\mathrm{Mod}(X) \to \mathcal{A}\text{--}\mathrm{Mod}(X^2)$.
- We also have $\mathcal{A}\text{--}\mathrm{Mod}(X) \otimes \mathcal{A}\text{--}\mathrm{Mod}(X) \to \mathcal{A}\text{--}\mathrm{Mod}(X^2)$ taking M_1, M_2 to $j_* j^*(M_1 \boxtimes M_2)$.
- For $M \in \mathcal{A}\text{--}\mathrm{Mod}(X^n)$ and $\mathcal{F} \in \mathrm{Dmod}(X^n)$, we have $\mathcal{F} \overset{!}{\otimes} M \in \mathcal{A}\text{--}\mathrm{Mod}(X^n)$.
- We also have maps $j_* j^*(M_1 \boxtimes M_2) \to \Delta_! M_3$ in $\mathcal{A}\text{--}\mathrm{Mod}(X^2)$. This is mimicking the formalism of nearby cycles. We want to define $M_1 \dot{\otimes} M_2 = \Psi(j^*(M_1 \boxtimes M_2))$. But we don't know that this is co-representable. And the associativity seems to fail, $(M_1 \dot{\otimes} M_2) \dot{\otimes} M_3 \neq M_1 \dot{\otimes} (M_2 \dot{\otimes} M_3)$.

Instead of trying to force a monoidal category, it's better to consider the union of $\mathcal{A}_{\mathrm{Fact}} - \mathrm{Mod}^{\mathrm{Fact}}(X^n)$ as a *factorization category*.

7. Chiral universal envelope

We will now say more about the chiral universal envelope of a Lie-$*$ algebra L. Recall that a Lie-$*$ module is a D-module M with a map $L \boxtimes M \to \Delta_!(M)$, satisfying axioms. A chiral module is an M with a map $j_* j^*(L \boxtimes M) \to \Delta_!(M)$, satisfying Lie axioms. There is an obvious functor from chiral modules to Lie-$*$ modules, it has a left adjoint.

7.1. **Fiberwise description.** We said restriction induces $L\text{−Mod}^{\mathrm{ch}} \xleftarrow{\sim} U_{\mathrm{ch}}L\text{−Mod}^{\mathrm{ch}}$. We said $L\text{−Mod}_x$ is equivalent to discrete modules for $\Gamma_{\mathrm{dR}}(D_x, L)$. This gives a description of $L\text{−Mod}_x^{\mathrm{Lie}-*} \to L\text{−Mod}_x^{\mathrm{ch}}$, as induction functor $M \mapsto \mathrm{Ind}_{\Gamma_{\mathrm{dR}}(D_x,L)}^{\Gamma_{\mathrm{dR}}(D_x^\circ,L)}(M)$.

We have an exact sequence

$$0 \to \Gamma_{\mathrm{dR}}(D_x, L) \to \Gamma_{\mathrm{dR}}(D_x^\circ, L) \to i_x^!(L)[1] \to 0$$

for $i_x\colon \mathrm{pt} \to X$. Hence $\mathrm{Ind}_{\Gamma_{\mathrm{dR}}(D_x,L)}^{\Gamma_{\mathrm{dR}}(D_x^\circ,L)}(M)$ has a PBW filtration with associated graded $M \otimes \mathrm{Sym}(i_x^!(L)[1])$.

The unit $\mathbf{C} \to U_{\mathrm{ch}}(L)$ induces by adjunction a map

$$\mathrm{Ind}_{\Gamma_{\mathrm{dR}}(D_x,L)}^{\Gamma_{\mathrm{dR}}(D_x^\circ,L)}(\mathbf{C}) \to U_{\mathrm{ch}}(L)_x, \tag{7.1}$$

and it turns out to be an isomorphism. This gives:

Theorem 7.1. *$U_{\mathrm{ch}}(L)$ has a PBW filtration with associated graded $\mathrm{Sym}^!(L[1])[-1]$ as commutative chiral algebras.*

7.2. **Construction of the chiral universal envelope as a factorization algebra.** Now we give a construction of $U_{\mathrm{ch}}(L)$. It is local on the curve, so we may assume X is affine. It will be easier to construct the incarnation as a factorization algebra $U_{\mathrm{ch}}(L)^{(1)}_{\mathrm{Fact}} \in \mathrm{Dmod}(X)$.

We consider $p_1, p_2\colon X \times X \to X$. Let $j\colon X \times X - \Delta \hookrightarrow X$ be the inclusion of the complement of the diagonal. We consider $p_{1*}(p_2^!(L)) = \omega_X \otimes \Gamma_{\mathrm{dR}}(X, L)$. So the Lie-* algebra on L gives this a structure of Lie algebra in the category $\mathrm{Dmod}(X)$. Let's call it L_X.

We can also consider $p_{1*}(j_*j^*p_2^!(L)) \in \mathrm{LieAlg}(\mathrm{Dmod}(X))$, which we call L_X°. What does it look like? By base change the fiber over $x \in X$ is $i_x^!(L_X^\circ) = \Gamma_{\mathrm{dR}}(X - x, L)$. We have $U_{\mathrm{ch}}(L)^{(1)}_{\mathrm{Fact}} = \mathrm{Ind}_{L_X}^{L_X^\circ}(\omega_X)$.

Why is this compatible with (7.1)? We have a fiber square

$$\begin{tikzcd}
\Gamma_{\mathrm{dR}}(X, L) \arrow[r] \arrow[d] & \Gamma_{\mathrm{dR}}(X - x, L) \arrow[d] \\
\Gamma_{\mathrm{dR}}(D_x, L) \arrow[r] & \Gamma_{\mathrm{dR}}(D_x^\circ, L)
\end{tikzcd}$$

This implies that induction along the top is compatible with induction along the bottom.

We also need to specify $U_{\mathrm{ch}}(L)^{(n)}_{\mathrm{Fact}} \in \mathrm{Dmod}(X^n)$. We consider a similar setup with $p_1\colon X^n \times X \to X^n$ and $p_2\colon X^n \times X \to X$, and j be the complement of the incidence divisor. We define $L_{X^n} = p_{1*}p_2^!(L)$ and $L_{X^n}^\circ = p_{1*}j_*j^*p_2^!(L)$. These lie in $\mathrm{LieAlg}(\mathrm{Dmod}(X^n))$, and we set $U_{\mathrm{ch}}(L)^{(n)}_{\mathrm{Fact}} := \mathrm{Ind}_{L_{X^n}}^{L_{X^n}^\circ}(\omega_{X^n})$.

The PBW filtration in Theorem 7.1 comes from the exact triangle $0 \to L_X \to L_X^\circ \to L \to 0$ in $\mathrm{Dmod}(X)$.

7.3. **Local definition.** If you like dislike the choice of global curve X involved in the above construction, we will give a curve-free definition.

We can define a formal scheme $D^\wedge_{X^n} = (X^n \times X)^\wedge_H$. We also have the "spec version" D_{X^n}. Finally, we can define $D^\circ_{X^n} = D_{X^n} - H$. We can replace the global situation with the one in the diagram below.

$$\begin{tikzcd} D^\circ_{X^n} \arrow[dr, hook, "j"] & & \\ & D_{X^n} \arrow[r, "\widehat{p}_2"] \arrow[d, "\widehat{p}_1"] & X \\ & X^n & \end{tikzcd}$$

We can define $\widehat{L}_{X^n}$ and $\widehat{L}^\circ_{X^n}$ by the analogous formulas, and then $U_{\mathrm{ch}}(L)^{(n)}_{\mathrm{Fact}} = \mathrm{Ind}^{\widehat{L}^\circ_{X^n}}_{\widehat{L}_{X^n}}(\omega_{X^n})$.

Example 7.2. We have

$$\begin{tikzcd} \mathrm{Gr}_{X^n} \arrow[d, "\pi"] & \mathrm{Gr}^\wedge_{X^n} \arrow[l, hook'] \arrow[dl, "\widehat{\pi}"] \\ X^n & \end{tikzcd}$$

We claim that

$$\widehat{\pi}_!(\omega_{\mathrm{Gr}^\wedge_{X^n}}) \cong U_{\mathrm{ch}}(\mathfrak{g} \otimes \mathcal{D}_X)^{(n)}_{\mathrm{Fact}}. \tag{7.2}$$

We have that $\widehat{L}^\circ_{X^n}$ acts on $\mathrm{Gr}^\wedge_{X^n}$, with $L^\wedge_{X^n}$ preserving the unit section. We have a map $\omega_{X^n} \to \widehat{\pi}^\wedge_!(\omega_{\mathrm{Gr}^\wedge_{X^n}})$ as modules for $\widehat{L}^\circ_{X^n}$. The image of ω_{X^n} is annihilated by $\widehat{L}_{X^n}$. So that gives by adjunction

$$\mathrm{Ind}^{\widehat{L}^\circ_{X^n}}_{\widehat{L}_{X^n}}(\omega_{X^n}) \xrightarrow{\sim} \widehat{\pi}_!(\omega_{\mathrm{Gr}^\wedge_{X^n}}).$$

The fact that it is an isomorphism follows from PBW: for $y \hookrightarrow Y$, the associated graded on $\omega_{Y^\wedge_y}$ is $\mathrm{Sym}(T_y Y)$. Concretely in this example, on the fiber over a point of X we are saying $\mathrm{Ind}^{\mathfrak{g}((t))}_{\mathfrak{g}[[t]]}(\mathbf{C})$ is the dualizing sheaf on $\mathrm{Gr}^\wedge_{X^n}$. Take the associated graded, it looks like Sym of the "tangent space" to Gr_G.

8. Central charge

Give a central extension

$$0 \to \mathbf{C}\zeta \to \mathfrak{g}' \to \mathfrak{g} \to 0.$$

We consider $U(\mathfrak{g})' := U(\mathfrak{g}') \otimes_{\mathbf{C}[\mathbf{C}\zeta]} \mathbf{C}$, where $\mathbf{C}\zeta$ is the center. In other words, it is $U(\mathfrak{g}')/$"$1 = 1$". So we have $\mathrm{gr}(U(\mathfrak{g})') = \mathrm{Sym}(\mathfrak{g})$. There is an equivalence $U(\mathfrak{g})' - \mathrm{Mod}$ with $\mathfrak{g}'$-modules on which $1 \in \mathbf{C}$ acts as the identity.

There is an analogous construction for Lie algebras. Suppose we have

$$0 \to \Omega_X \to L' \to L \to 0.$$

We can then form $U_{\mathrm{ch}}(L')/$"$\Omega_X = \Omega_X$" $=: U_{\mathrm{ch}}(L)'$. We have $\mathrm{gr}(U_{\mathrm{ch}}(L)') = \mathrm{Sym}^!(L[1])[-1]$.

Note that $U_{\mathrm{ch}}(L)' - \mathrm{Mod}^{\mathrm{ch}}(X)$ can be described as chiral modules for L' on which Ω_X acts in the canonical way, i.e. the restriction of the action map to $j_* j^*(\Omega_X \boxtimes M) \to \Delta_! M$ is the canonical map.

Example 8.1. Let $L = \mathfrak{g} \otimes \mathcal{D}_X$. We pick an invariant form $\kappa \colon : \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{C}$. We have an extension

$$0 \to \Omega_X \to L'_{\mathfrak{g}} \to \mathfrak{g} \otimes \mathcal{D}_X \to 0.$$

As a D-module it's split, $L'_{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{D}_X \oplus \omega_X$. To specify the Lie-* algebra structure, we just have to specify the map

$$(\mathfrak{g} \otimes \mathcal{D}_X) \boxtimes (\mathfrak{g} \otimes \mathcal{D}_X) \to \Delta_! \Omega_X \text{ on } X^2.$$

This is equivalent to specifying $\mathfrak{g} \otimes \mathfrak{g} \to \Gamma(X \times X, \Delta_! \Omega_X)$. It will factor through $\kappa \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{C}$, so it is specified by specifying an element of $\Gamma(X \times X, \Delta_! \Omega_X)$. Recall $\Delta_! \Omega_X$ is $j_* j^* (\Omega_X \boxtimes \Omega_X)/\Omega_X \boxtimes \Omega_X$ as a quasicoherent sheaf. We consider the short exact sequence for the diagonal

$$0 \to \Omega_X \to \frac{\Omega_X \boxtimes \Omega_X(2\Delta_X)}{\Omega_X \boxtimes \Omega_X} \to \mathcal{O}_X \to 0.$$

There is a canonical element which projects to $1 \in_X$ and whose residues along the diagonal are 0. In coordinates it is $\frac{dx_1 \wedge dx_2}{(x_1 - x_2)^2}$.

Example 8.2. We can try the same for the Virasoro algebra, but the extension

$$0 \to \Omega_X \to \mathrm{Vir} \to \Gamma_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0$$

is not split as modules, canonically or on a global curve.