

GEOMETRIC LANGLANDS DUALITY FOR PERIODS

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ABSTRACT. We study conjectures of Ben-Zvi–Sakellaridis–Venkatesh that categorify the relationship between automorphic periods and L -functions in the context of the Geometric Langlands equivalence. We provide evidence for these conjectures in some low-rank examples, by using derived Fourier analysis and the theory of chiral algebras to categorify the Rankin–Selberg unfolding method.

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1. INTRODUCTION

Many arithmetic questions are encoded in the analytic behavior of L -functions, and a powerful tool for understanding L -functions is via integral representations as periods of automorphic forms. Some of the first examples of such integral representations include:

- Riemann’s integral representation of the zeta function, as periods of theta series.
- Hecke’s integral representation of standard L -functions for GL_2 , as periods of modular forms.
- Rankin–Selberg’s integral representation of tensor product L -functions for GL_2 , via Rankin–Selberg convolution of modular forms against mirabolic Eisenstein series.

The purpose of the present paper is to study a *categorification* of automorphic periods and their relation to L -functions, in the context of the Geometric Langlands equivalence. In particular, we will focus on the three examples noted above in terms of the *Relative Langlands duality* conjectures of Ben-Zvi–Sakellaridis–Venkatesh [BZSV23], henceforth abbreviated “BZSV”.

1.1. Relative Langlands duality. Let us briefly summarize the format of the BZSV conjectures. Langlands defined a duality of split reductive groups, $G \leftrightarrow \check{G}$. BZSV extend this to a duality

$$\{\text{hyperspherical } G\text{-varieties}\} \longleftrightarrow \{\text{hyperspherical } \check{G}\text{-varieties}\}.$$

We do not define “hyperspherical” G -variety, but we mention that the cotangent bundles of smooth affine spherical G -varieties are among the main examples. There is also a \mathbb{G}_{gr} -action commuting with the G -action on the objects of both sides (cf. [BZSV23, §1.4]), whose role we suppress in the introduction.

Now we turn to the global period conjecture of BZSV. Let C be a smooth projective curve over \mathbf{C} . Let $(G, M) \leftrightarrow (\check{G}, \check{M})$ be dual pairs of split reductive groups and hyperspherical varieties over \mathbf{C} . Assume furthermore, partially for simplicity, that we have presentations $M = T^*X$ and $\check{M} = T^*\check{X}$ where X (resp.

\check{X}) is a spherical variety for G (resp. \check{G}). Let Bun_G be the moduli stack of G -bundles on C , and let $\text{Loc}_{\check{G}}$ be the (derived) moduli stack of \check{G} -local systems on C .

1.1.1. *Automorphic period sheaf.* From the datum $(G, M = T^*X)$, BZSV define a space Bun_G^X equipped with a map $\pi^{\text{aut}}: \text{Bun}_G^X \rightarrow \text{Bun}_G$,¹ and an associated (un-normalized) period sheaf

$$\mathcal{P}_X := \pi_1^{\text{aut}}(\underline{\mathbf{C}}_{\text{Bun}_G^X} := \text{constant sheaf on } \text{Bun}_G^X) \in \text{Dmod}(\text{Bun}_G).$$

Example 1.1.1 (Relation to period integrals). Viewing Bun_G as the mapping stack $\text{Map}(C, BG)$, Bun_G^X is approximately the mapping stack $\text{Map}(C, X/G)$. Suppose $X = G/H$ is a homogeneous spherical variety. Then it is often the case that

$$\text{Bun}_G^X \cong \text{Map}(C, X/G) \cong \text{Map}(C, BH) = \text{Bun}_H.$$

In the analogous situation of a curve C over a *finite* field \mathbf{F}_q , $\text{Bun}_G(\mathbf{F}_q)$ is canonically identified, by Weil’s uniformization theorem, with a double coset groupoid of the form

$$[G] := G(F) \backslash G(\mathbf{A}_F) / \prod_{c \in |C|} G(\widehat{\mathcal{O}}_{C,c})$$

for $F = \mathbf{F}_q(C)$ the function field of C . In this case \mathcal{P}_X categorifies the pushforward of the constant function along $[H] \rightarrow [G]$, which represents the distribution

$$f \mapsto \int_{[H]} f dh$$

on functions f on $[G]$. In this sense \mathcal{P}_X categorifies the period integral associated to X .

1.1.2. *Spectral period sheaf.* From the datum $(\check{G}, \check{M} = T^*\check{X})$, BZSV define a space $\text{Loc}_{\check{G}}^{\check{X}}$ equipped with a map $\pi^{\text{spec}}: \text{Loc}_{\check{G}}^{\check{X}} \rightarrow \text{Loc}_{\check{G}}$, and an associated (un-normalized) “ L -sheaf”

$$\mathcal{L}_{\check{X}} := \pi_*^{\text{spec}}(\omega_{\text{Loc}_{\check{G}}^{\check{X}}} := \text{dualizing complex on } \text{Loc}_{\check{G}}^{\check{X}})^{\not\sim} \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}).$$

Here $(-)^{\not\sim}$ is the “shearing operation”, which shifts and twists graded components; we will not explain it in the introduction.²

In contrast to the period sheaf \mathcal{P}_X , which is clearly related to classical automorphic periods as explained in Example 1.1.1, it is not immediately obvious how $\mathcal{L}_{\check{X}}$ is related to an L -function. One of the main points of this paper is to explain this precisely. The punchline is that we identify a piece of $\mathcal{L}_{\check{X}}$ as the (relative) *factorization homology* of a certain (relative) *cocommutative factorization coalgebra*, which is a sheaf-theoretic analogue of an Euler product. This will be discussed further in §1.3.2.

Remark 1.1.2. It is essential to consider $\text{Loc}_{\check{G}}^{\check{X}}$ and $\text{Loc}_{\check{G}}$ as *derived* stacks. Morally, one should also consider Bun_G^X as a derived stack, but because we will always work with either D -modules or constructible sheaves or étale sheaves on the automorphic side, it is possible to formulate the period sheaf without doing this. However, we will see in the calculations that it is still useful to incorporate the derived structure.

1.1.3. *The duality conjecture.* There is a conjectural *Geometric Langlands equivalence*

$$\mathbb{L}_G: \text{Dmod}(\text{Bun}_G) \xrightarrow{\sim} \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}), \quad (1.1.1)$$

which (if it exists) is characterized by various compatibility properties, discussed in [AG15] and [Gai15]. For GL_1 , this has been known since the work of Rothstein [Rot96] and Laumon [Lau96] (although in a slightly different formulation), and recently a proof for general groups has been announced by Gaitsgory-Raskin partially in joint work with Arinkin, Beraldo, Chen, Faergeman, and Lin [ABG⁺23].

The global period conjecture of BZSV concerns the behavior of the equivalence \mathbb{L}_G on specific objects. BZSV define “normalized” versions of the period sheaf and L -sheaf. For the period sheaf, the normalized version $\mathcal{P}_X^{\text{norm}}$ is obtained from \mathcal{P}_X by certain shifts and twists. For the L -sheaf, $\mathcal{L}_X^{\text{norm}}$ is obtained from

¹This is an oversimplification: the construction of Bun_G^X depends furthermore on the action of \mathbb{G}_{gr} on X .

²The shearing depends on the action of \mathbb{G}_{gr} on \check{X} .

$\mathcal{L}_{\check{X}}$ by shifting, twisting, and tensoring with a certain line bundle. Then BZSV conjecture (see [BZSV23, Conjecture 12.1.1] for the precise statement) that the Geometric Langlands equivalence \mathbb{L}_G takes³

$$\mathrm{Dmod}(\mathrm{Bun}_G) \ni \mathcal{P}_X^{\mathrm{norm}} \xrightarrow{\mathbb{L}_G} \mathcal{L}_{\check{X}}^{\mathrm{norm}} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}) \quad (1.1.2)$$

1.2. Results. We will study the three examples listed at the beginning of the introduction.⁴ In the language of relative Langlands duality, they are the Tate, Hecke, and singular Rankin-Selberg (for GL_2) examples in Figure 1.

Strictly speaking, singular spherical varieties fall outside the scope of [BZSV23]. However, we expect that there should be some extension of relative Langlands duality that encompasses singular varieties. Part of our motivation for analyzing the singular Rankin-Selberg example is to investigate what such an extension might look like. The work of Chen-Venkatesh [CV24] also investigates examples of singular spherical varieties.

	X	G	\check{X}	\check{G}
Tate	\mathbf{A}^1	\mathbf{G}_m	\mathbf{A}^1	\mathbf{G}_m
Hecke	$\mathrm{GL}_2 / \mathrm{GL}_1$	GL_2	\mathbf{A}^2	GL_2
singular Rankin-Selberg	$\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{GL}_2 \times \mathrm{GL}_2}(\mathbf{A}^2 \setminus 0)$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	$(\mathbf{A}^2 \otimes \mathbf{A}^2)^{\mathrm{rank} \leq 1}$	$\mathrm{GL}_2 \times \mathrm{GL}_2$

FIGURE 1. Examples of dual hyperspherical varieties $(G, M = T^*X) \leftrightarrow (\check{G}, \check{M} = T^*\check{X})$.

Our main results take the following form. In each of the examples of Figure 1, we produce exact triangles

$$\mathcal{A} \rightarrow \mathcal{P}_X^{\mathrm{norm}} \rightarrow \mathcal{B} \in \mathrm{Dmod}(\mathrm{Bun}_G) \quad (1.2.1)$$

and

$$\check{\mathcal{A}} \rightarrow \mathcal{L}_{\check{X}}^{\mathrm{norm}} \rightarrow \check{\mathcal{B}} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}) \quad (1.2.2)$$

and we prove a result of the form:

Theorem 1.2.1. *In each of the examples of Figure 1, assuming the existence of the Geometric Langlands equivalence \mathbb{L}_G for the respective G , it matches $\mathcal{A} \xrightarrow{\mathbb{L}_G} \check{\mathcal{A}}$ and $\mathcal{B} \xrightarrow{\mathbb{L}_G} \check{\mathcal{B}}$ as indicated in the diagram below, where the rows are exact:*

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{P}_X^{\mathrm{norm}} & \longrightarrow & \mathcal{B} & \in \mathrm{Dmod}(\mathrm{Bun}_G) \\ \downarrow \mathbb{L}_G & & \downarrow \mathbb{L}_G? & & \downarrow \mathbb{L}_G & \downarrow \mathbb{L}_G \\ \check{\mathcal{A}} & \longrightarrow & \mathcal{L}_{\check{X}}^{\mathrm{norm}} & \longrightarrow & \check{\mathcal{B}} & \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}) \end{array} \quad (1.2.3)$$

See Theorem 6.1.2, Theorem 7.4.2, and Theorem 8.4.4 for the precise statements (which involve some notation that we do not want to explain here). The existence of a decomposition into two pieces is an artifact of the low rank examples that we study; in general we would expect to find multi-step filtrations of $\mathcal{P}_X^{\mathrm{norm}}$ and $\mathcal{L}_{\check{X}}^{\mathrm{norm}}$ whose associated gradeds can be matched under \mathbb{L}_G by our methods.

Remark 1.2.2. Our results are stated in terms of the Geometric Langlands equivalence (which, as already noted, has been announced in [ABG⁺23]). Note that it is known in the case of the Tate period, since there $G = \mathrm{GL}_1$. For the Hecke and singular Rankin-Selberg periods, we only use the Geometric Langlands equivalence for $G = \mathrm{GL}_2$, whose proof was outlined several years ago in [Gai15].

Theorem 1.2.1 reduces the BZSV Conjecture (1.1.2) (in the cases of Figure 1) to an extension problem. It would be interesting to investigate the meaning of the extension classes. In the Tate case, the group of extensions is relatively easy to describe, and there is a distinguished canonical element on each side that we are able to pin down as the relevant extension classes, thus proving the full relative duality conjecture in that case.

Theorem 1.2.3. *In the Tate case, we have $\mathcal{P}_X^{\mathrm{norm}} \xrightarrow{\mathbb{L}_G} \mathcal{L}_{\check{X}}^{\mathrm{norm}}$ as conjectured in (1.2.3).*

³The version we write here omits the duality involution appearing in *loc. cit.*, which appears for reasons of convention in [BZSV23] regarding right/left actions. See [CV24, §A.1] for a discussion of this point.

⁴Actually, we replace the Rankin-Selberg period by a singular subperiod.

Let us comment on the nature of the exact triangles (1.2.1) and (1.2.2). They are induced by geometric decompositions of the hyperspherical varieties. For example, in the Tate case, one has the stratification of X (resp. \check{X}) by G -orbits (resp. \check{G} -orbits): $\mathbf{A}^1 = 0 \sqcup (\mathbf{A}^1 \setminus 0)$. On the automorphic side, this induces an open-closed decomposition of Bun_G^X , and (1.2.1) comes from the excision triangle for this open-closed decomposition.

On the spectral side, the stratification by \check{G} -orbits induces (in all three examples) an open-closed decomposition

$$V \xrightarrow{\text{closed}} \text{Loc}_G^{\check{X}} \xleftarrow{\text{open}} W$$

and (1.2.2) comes from the corresponding exact triangle for cohomology with supports,

$$\text{R}\Gamma_V(\text{Loc}_G^{\check{X}}, -) \rightarrow \text{R}\Gamma(\text{Loc}_G^{\check{X}}, -) \rightarrow \text{R}\Gamma(W, -).$$

In particular, the term \check{A} in (1.2.2) is the relative local cohomology along the zero-section $\text{Loc}_G^{\check{X}} \hookrightarrow \text{Loc}_G^{\check{X}}$ of the dualizing sheaf of $\text{Loc}_G^{\check{X}}$.

In the case of the Tate period, there is a matching between the G -orbits on X and \check{G} -orbits on \check{X} (reversing the closure relations) such that matching orbits contribute corresponding pieces under the Geometric Langlands equivalence. However, this naive matching does not persist in general: in both the Hecke and singular Rankin-Selberg periods, X is homogeneous for the G -action, while there are two \check{G} -orbits on \check{X} . In those cases, the exact triangle for $\mathcal{P}_X^{\text{norm}}$ comes instead from a stratification *in the Fourier dual space*. A space and its Fourier dual are united in microlocal geometry, suggesting that in general, one should look for a decomposition in the hyperspherical variety $M = T^*X$. Indeed, a closed-open stratification $X = Z \sqcup U$ induces a decomposition of T^*X as

$$T^*X = (Z \times_X T^*X) \sqcup (T^*U). \quad (1.2.4)$$

Note that $Z \times_X T^*X$ can be thought of as the embedding in T^*X of the Lagrangian correspondence

$$T^*Z \longleftarrow Z \times_X T^*X \longrightarrow T^*X$$

associated to the map $Z \hookrightarrow X$. More generally, it happens that T^*X admits a decomposition into Lagrangian correspondences, which may not come from a stratification of X . We then see a corresponding decomposition of the (automorphic or spectral) X -period, as in (1.2.1) or (1.2.2). See §7.5 for how this plays out in the example of the Hecke period.

This suggests that the duality $M \leftrightarrow \check{M}$ often preserves more structure on both sides, which would be interesting to codify.

1.3. Methods. The proof of Theorem 1.2.1 for the Tate case is in some sense by direct computation. For the Hecke and singular Rankin-Selberg periods, the classical approach is via the “unfolding method” of Rankin-Selberg, and our approach can be seen as a categorification of the Rankin-Selberg method. However, we note some significant differences which make our task more difficult. The Rankin-Selberg method analyzes the period of a single cuspidal Hecke eigenform, while we want to analyze the automorphic period sheaf as a distribution on all of Bun_G . If we were only interested in the pairing of the period sheaf against individual cuspidal Hecke eigensheaves, then the relevant analysis has already been carried out by Lysenko [Lys02]. But our task also includes the issue of analyzing periods of Eisenstein series where the corresponding L -function has a pole, a subtle topic which is not understood even in the analysis of classical automorphic forms. Miraculously, we find a very clean answer; in some sense, working at the level of sheaves allows us to bypass convergence problems that would make it difficult to articulate the answer numerically.

We highlight here some ways in which our calculations make interesting contact with other themes in mathematics.

1.3.1. Derived Fourier analysis. The premise of the unfolding method is to “unfold” automorphic periods in terms of Whittaker periods. Classically, this means that one should consider the Fourier expansion of a Hecke eigenform.

But in the geometric context, it turns out that the spaces which arise naturally from trying to geometrize natural vector spaces are not vector bundles, which a priori precludes us from applying the sheaf-theoretic Fourier transform. For example, in the Tate case one is led to consider the space Bun_G^X over Bun_{GL_1} whose fiber over a line bundle L is its space of global sections $H^0(C; L)$. Thus each fiber is a vector space, but the

fibers vary in rank (even on connected components) hence cannot interpolate into a vector bundle. For this reason it is natural to consider *derived vector bundles*, which are derived linear spaces associated to perfect complexes, generalizing how vector bundles are associated to locally free coherent sheaves. For example, one wants to enhance Bun_G^X to a derived vector bundle over $\mathrm{Bun}_{\mathrm{GL}_1}$ whose fibers are, informally speaking, the complexes $\mathrm{R}\Gamma(C; L)$. In general, derived vector bundles have both “derived” and “stacky” aspects, which are interchanged under duality.

In order to execute unfolding at the geometric level, it is therefore necessary to use an expansion of the Fourier transform to derived vector bundles, which we call the *derived Fourier transform*. It turns out that the good properties of the Deligne-Laumon Fourier transform extend from vector bundles to derived vector bundles, although this is technically quite tricky to establish – see [FYZ23, §6 and Appendix A]. The trickiest aspect is to show the near-involutivity of the Fourier transform, for which the crux is the statement that the derived Fourier transform of the constant sheaf is (an appropriate shift and twist of) the delta-sheaf at the origin of the dual bundle. It is this statement which we use to unfold the Hecke and singular Rankin-Selberg period sheaves.

We arrived at the formulation of derived Fourier analysis when contemplating the *functional equation* for the Tate period (§6.5). The theory was written up in [FYZ23], which used it to study the modularity of higher theta functions in the sense of [FYZ21]. Not coincidentally, the modularity of theta functions is closely connected to the functional equation for zeta functions.

1.3.2. *Factorization homology.* Our analysis gives a new perspective on the appearance of L -functions in periods of automorphic forms. Let V be a representation of \check{G} . Then to a Galois representation $\sigma: \mathrm{Gal}(F^s/F) \rightarrow \check{G}(\overline{\mathbf{Q}}_\ell)$ with Frobenii having rational characteristic polynomials, one forms the L -function as an Euler product

$$L(V, \sigma, s) := \prod_{v \in |F|} L_v(V, \sigma, s)$$

where $L_v(V, \sigma, s)$ is some local factor at the place $v \in |F|$ formed using the characteristic polynomial of $\sigma(\mathrm{Frob}_v)$ acting on V .

The geometric counterpart to such Euler products is *factorization homology*. To explain it, let Y be a variety over a field \mathbf{C} . (We will eventually apply this theory to $Y = C$, although there is a version of the theory for any variety over any field.) We write $\mathrm{pt} := \mathrm{Spec} \mathbf{C}$. Then we have a pair of adjoint functors

$$\begin{array}{ccc} & \mathrm{pr}_! & \\ \curvearrowright & & \curvearrowleft \\ \mathrm{Dmod}(Y) & & \mathrm{Dmod}(\mathrm{pt}) \\ \curvearrowleft & & \curvearrowright \\ & \mathrm{pr}^! & \end{array}$$

where the derived categories of D -modules $\mathrm{Dmod}(Y)$ and $\mathrm{Dmod}(\mathrm{pt})$ can be equipped with a symmetric monoidal structure with respect to the $!$ -tensor product (under the usual identification of $\mathrm{Dmod}(\mathrm{pt})$ with the derived category of \mathbf{C} -vector spaces, this is the usual tensor product). The functor $\mathrm{pr}^!$ is symmetric monoidal with respect to these symmetric monoidal structures. Therefore, it induces a functor

$$\mathrm{ComAlg}(\mathrm{Dmod}(Y)) \xleftarrow{\mathrm{pr}^!} \mathrm{ComAlg}(\mathrm{Dmod}(\mathrm{pt})), \quad (1.3.1)$$

where $\mathrm{ComAlg}(-)$ denotes the category of commutative algebras in a symmetric monoidal category. Then the *factorization homology* functor

$$C_c^{\mathrm{Fact}}(Y; -): \mathrm{ComAlg}(\mathrm{Dmod}(Y)) \rightarrow \mathrm{ComAlg}(\mathrm{Dmod}(\mathrm{pt})) \quad (1.3.2)$$

is the *left adjoint* to (1.3.1).

Example 1.3.1. If $Y = \{y_1, \dots, y_n\}$ is a finite disjoint union of points, then $\mathrm{ComAlg}^!(\mathrm{Dmod}(Y)) \cong \bigotimes_{i=1}^n \mathrm{ComAlg}(\mathrm{Dmod}(\mathrm{pt}))$, under which identification we have

$$C_c^{\mathrm{Fact}}(Y; \mathcal{F}) \cong \bigotimes_{i=1}^n \mathcal{F}_{y_i}.$$

This suggests the intuition, emphasized in the work of Gaitsgory-Lurie [GL16], that in general $C_c^{\mathrm{Fact}}(Y; \mathcal{F})$ is a “continuous tensor product of \mathcal{F} over the points of Y ”. The analogy to Euler products is evident.

In the work of Gaitsgory-Lurie [GL16] on Weil’s Tamagawa Number Conjecture over function fields, a certain product of zeta functions arose as the trace of Frobenius on a certain factorization homology group. In our analysis here, general L -functions (and even more general objects) will be categorified by factorization homology. More precisely, let ω_V be the dualizing complex of the \check{G} -representation V regarded as an affine space over \mathbf{C} , and consider the local cohomology at the origin $\mathrm{R}\Gamma_0(V; \omega_V)$; informally, it is the space of distributions on V with set-theoretic support at 0. Then $\mathrm{R}\Gamma_0(V; \omega_V)$ is equipped with the natural structure of a cocommutative coalgebra in $\mathrm{Rep}(\check{G})$, which allows to form a corresponding \check{G} -equivariant *cocommutative factorization coalgebra* on C . The \check{G} -equivariant structure allows to twist this cocommutative factorization coalgebra by a \widehat{G} -local system σ on C , and we regard its factorization homology as the categorification of the L -function $L(V, \sigma, s)$.

As for how such objects arise in our calculation, suppose $\check{X} = V$ is a (linear) \check{G} -representation, as is the case in the Tate and Hecke periods. Then the origin $0 \in V$ contributes a closed \check{G} -stratum, and we will stratify $\mathrm{Loc}_{\check{G}}^{\check{X}}$ by the induced zero-section of $\mathrm{Loc}_{\check{G}}$. Then, using the work of Beilinson-Drinfeld on *chiral algebras*, we prove a local-global principle that relates the derived pushforward of $\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}$, with set-theoretic support along the zero-section, to the relative factorization homology of the cocommutative factorization coalgebra attached to $\mathrm{R}\Gamma_0(V; \omega_V)$ (and twisted by the universal \widehat{G} -local system on C). More generally, the same argument will show that each \check{G} -fixed point $x \in \check{X}$ contributes a term to the L -sheaf $\mathcal{L}_{\check{X}}$ which is the relative factorization homology over $\mathrm{Loc}_{\check{G}}$ of the local cohomology $\mathrm{R}\Gamma_x(\check{X}; \omega_{\check{X}})$ turned into a cocommutative factorization coalgebra and then twisted by the universal local system. However, if x is not a *smooth* point of \check{X} , as happens in the singular Rankin-Selberg period, then the resulting factorization homology is analogous to an Euler product which is *not* an L -function.

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2. NOTATION AND CONVENTIONS

2.1. Coefficient fields. Let \mathbf{F} be an algebraically closed field, which will be the ground field for the “automorphic side”.

Let k be an algebraically closed field of characteristic 0, which will be the ground field for the “spectral side”, as well as the coefficients for sheaves on the automorphic objects. When discussing the “de Rham” setting, we will have a fixed identification $k \cong \mathbf{C}$.

2.2. Reductive groups. Let G be a split reductive group over F and \check{G} its Langlands dual group, regarded as a split reductive group over k .

2.3. Categories and sheaves. All our categories will be k -linear. All our functors are derived, e.g., f_* means $\mathrm{R}f_*$, Γ means $\mathrm{R}\Gamma$, etc. If we need to refer to individual cohomology groups, we will write $\mathrm{R}^i f_*$ and H^i , etc.

Our categories are derived, e.g., Vect is the derived category of k -vector spaces and $\mathrm{Rep}(\check{G})$ is the derived category of \check{G} -representations, unless specified otherwise.

We denote by \underline{k}_S the constant sheaf on a space S with value k . When context is clear, we may omit the subscript.

When $\mathbf{F} = \mathbf{C}$, we denote by $\mathrm{Dmod}(S)$ the derived category of D -modules on S .

2.3.1. Categories of sheaves. Recall that there are (at least) three flavors of the Geometric Langlands equivalence, which we refer to as *de Rham* [AG15], *Betti* [BZN18], and *finite field* [AGK⁺22]. Our final theorems are all about the de Rham setting because this is where the proof of the Geometric Langlands Equivalence is best documented, but many of our calculations are agnostic to the specific sheaf theory, so we introduce a notation that is agnostic to the sheaf theory. For an algebraic stack S over \mathbf{F} , we denote by $\mathrm{Shv}(S)$ any of the three flavors of “topological” sheaf theories [BZSV23, §B.4–B.7]:

- (1) (de Rham) If $F = \mathbf{C}$, then we may take $\mathrm{Shv}(S)$ to be the derived category of D -modules on S , denoted $\mathrm{Dmod}(S)$. In this case we must work with $k = \mathbf{C}$.

- (2) (Betti) If $F = \mathbf{C}$, then we may take $\mathrm{Shv}(S)$ to be the derived category of all sheaves on $S(\mathbf{C})$ equipped with the analytic topology, in the sense of [AGK⁺22, Appendix G]. There are no restrictions on k .
- (3) (finite field) If F has positive characteristic, then we may take $\mathrm{Shv}(S)$ to be the derived category ind-constructible ℓ -adic étale sheaves on S . Then $k = \overline{\mathbf{Q}}_\ell$ for some prime ℓ .

We denote by $\mathrm{Shv}_{\mathrm{hol}}(S) \subset \mathrm{Shv}(S)$ the subcategory of *holonomic* sheaves. In the D -module context this means holonomic D -modules in the usual sense, while in the Betti and ℓ -adic contexts it means constructible objects.

2.3.2. *Artin-Schreier*. The *Artin-Schreier sheaf* refers to a certain object [BZSV23, Definition 10.5.1] in $\mathrm{Shv}(\mathbf{A}^1)$, defined case-by-case in each flavor of sheaf theory:

- (1) (de Rham) The exponential D -module.
- (2) (Betti) The locally constant \mathbf{C}^\times -equivariant (via squaring) sheaf on \mathbf{C} defined as $(j_!k^- \oplus j_*k)[-1]$ where $j: \mathbf{C}^\times \hookrightarrow \mathbf{C}$ and k, k^- are the trivial and non-trivial rank one local systems on $\mathbf{C}^\times/\mathbf{C}^\times \cong \mathrm{B}\mu_2$.
- (3) (Finite Field) The étale Artin-Schreier sheaf induced by a choice of non-trivial additive character $\psi: \mathbf{F}_q \rightarrow k^\times$.

2.3.3. *Shifting and twisting*. For $n \in \mathbf{Z}$, the endofunctor $\mathcal{K} \mapsto \mathcal{K}(n)$ of $\mathrm{Shv}(S)$ is the n th Tate twist. This only has meaning in the finite field context, where we will choose a square root of the cyclotomic character to make sense of half Tate-twists. We use the notation $\langle n \rangle := \Pi^n[n](n/2)$ as in [BZSV23, (2.4)]. Note that “ $\langle n \rangle$ ” in [FYZ23] is instead called $\langle 2n \rangle$ in this paper in order to be consistent with [BZSV23].

2.3.4. *Correspondences*. Consider a correspondence

$$A_1 \xleftarrow{c_1} C \xrightarrow{c_2} A_2$$

Our convention is that “the functor induced by $\mathcal{K} \in \mathrm{Shv}(C)$ ” is

$$c_{2!}(\mathcal{K} \otimes c_1^*(-)): \mathrm{Shv}(A_1) \rightarrow \mathrm{Shv}(A_2).$$

Note that in the de Rham case this is not a priori defined on all of $\mathrm{Shv}(A_1)$; we will only apply it to objects on which it is defined.

2.3.5. *Graded categories*. For a monoid Λ and a category \mathbf{C} , we denote by \mathbf{C}^Λ the category of Λ -graded objects in \mathbf{C} .

2.4. **Geometry**. Let C be a smooth projective curve over \mathbf{F} . For $d \geq 0$, we write $C^{(d)} = \mathrm{Sym}^d C$ for the d th symmetric power of C , with the convention $C^{(0)} := \mathrm{Spec} \mathbf{F}$.

2.4.1. *Spin structure*. Fix a square root $\Omega_C^{1/2}$ of the canonical line bundle Ω_C . The Conjectures of Ben-Zvi–Sakellaridis–Venkatesh are formulated in terms of such a choice – see [BZSV23, §10.1.2].

2.4.2. *Derived mapping stacks*. If X is a proper scheme and Y is locally of finite presentation over a field, then the derived mapping stack $\mathrm{Map}(X, Y)$ is constructed in [TV05, §3.6] and [TV08, §2.2.6.3] (the first reference constructs the internal hom on stacks on any site, and the second establishes its geometricity properties under the given assumptions). On objects, $\mathrm{Map}(X, Y)$ sends an animated ring R to the anima of morphisms $X_R \rightarrow Y$.

Suppose Y is equipped with a map $\pi: Y \rightarrow X$. Then we have the derived space of sections $\mathrm{Sect}(X, Y)$, defined as the derived fibered product

$$\begin{array}{ccc} \mathrm{Sect}(X, Y) & \longrightarrow & \mathrm{Map}(X, Y) \\ \downarrow & & \downarrow \pi \\ \{\mathrm{Id}\} & \longrightarrow & \mathrm{Map}(X, X) \end{array}$$

2.4.3. *Moduli of G -bundles*. By definition, for BG the classifying stack of G , we have

$$\mathrm{Bun}_G := \mathrm{Map}(C, BG).$$

Although defined a priori as a derived stack, this is in fact a classical Artin stack [FYZ21, Example 5.3].

2.4.4. *Moduli of \check{G} -local systems.* Parallel to the sheaf theory, there are several different flavors of moduli spaces of local system – see [BZSV23, §11.1.2] for discussion of them. Our calculations will be somewhat specific to the de Rham setting, so we recall that only.

Recall the *de Rham stack* C_{dR} associated to C from [Gai15, §3.1]. For $B\check{G}$ the classifying stack of \check{G} , we define

$$\mathrm{Loc}_{\check{G}} := \mathrm{Map}(C_{\mathrm{dR}}, B\check{G}).$$

We abbreviate $\mathrm{Loc}_n := \mathrm{Loc}_{\mathrm{GL}_n}$ and $\mathrm{Bun}_n := \mathrm{Bun}_{\mathrm{GL}_n}$.

2.4.5. *Tangent and cotangent complexes.* Let $f: X \rightarrow Y$ be a map of derived Artin stacks. We denote by \mathbf{L}_f the cotangent complex of f , and by \mathbf{T}_f the tangent complex of f if \mathbf{L}_f is a perfect complex (in which case \mathbf{T}_f is defined to be the dual of \mathbf{L}_f).

2.4.6. *Classical truncation.* For a derived stack S , we denote by S_{cl} its classical truncation, which comes equipped with a canonical closed embedding $S_{\mathrm{cl}} \hookrightarrow S$.

2.5. **Coherent sheaves.** We follow [Gai13] in our conventions on (ind-)coherent sheaves. For a map of derived Artin stacks $f: X \rightarrow Y$, we denote by $\omega_{X/Y} := f^! \mathcal{O}_Y$ the relative dualizing sheaf. For Y a point, we abbreviate $\omega_X := \omega_{X/Y}$.

2.5.1. *Perfect complexes.* For a derived stack S , we denote by $\mathrm{Perf}(S)$ the category of perfect complexes on S . This is equivalent to complexes of finite tor-dimension, defined in [Aut, Tag 0652], which we follow in using *cohomological grading* to define tor-amplitude.

For $\mathcal{E} \in \mathrm{Perf}(S)$, we denote by $\mathcal{E}^\vee := \mathcal{R}\mathcal{H}om(\mathcal{E}, S)$ the dual perfect complex.

2.5.2. *Coherent singular support.* On the spectral side, the period sheaves $\pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\check{G}}}^{\mathrm{spec}})$ naturally live in $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G}})$ but typically do not lie in the full subcategory $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ which is the domain of the Geometric Langlands equivalence. The embedding of this full subcategory is left adjoint to a co-localization functor $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G}}) \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$. For an object $\mathcal{F} \in \mathrm{IndCoh}(\mathrm{Loc}_{\check{G}})$, we write $\mathcal{F} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ for its image under this co-localization. That is, we use the specification of the ambient category in order to indicate the application of a singular support co-localization functor.

Part 1. Tools

3. DERIVED FOURIER ANALYSIS

3.1. **Derived Fourier transform.** In this subsection we recall the “Derived Fourier analysis” developed in [FYZ23]. While *loc. cit.* worked in the setting of ℓ -adic étale sheaves, essentially the same constructions and arguments go through more generally for holonomic sheaves in the sense of §2.3.1. We shall give the statements, omitting proofs since these are identical to the ℓ -adic setting.

3.1.1. *Derived vector bundles.* Let S be a derived Artin stack. There is a functor Tot_S from the category $\mathrm{Perf}(S)$ of perfect complexes on S to the category of derived stacks over S , which extends the usual construction of a vector bundle from a locally free coherent sheaf. We normalize this construction as in [FYZ23, §6]. For $\mathcal{E} \in \mathrm{Perf}(S)$, we will call $E := \mathrm{Tot}_S(\mathcal{E})$ the associated *derived vector bundle* associated to E . The *virtual rank* of E , denoted $\mathrm{rank}(E)$, is the locally constant function on S given by $s \mapsto \chi(\mathcal{E}_s)$, the Euler characteristic of the fiber of \mathcal{E} at a geometric point s .

Definition 3.1.1. For a derived vector bundle $E \rightarrow S$, we denote by $z_E: S \rightarrow E$ the zero-section. Note that z_E need not be a closed embedding. We write $\delta_E := z_{E!}(\mathbf{k}_S) \in \mathrm{Shv}(E)$ and call it the *delta-sheaf* of E .

3.1.2. *The derived Fourier transform for derived vector bundles.* Let S be a derived Artin stack, $\mathcal{E} \in \mathrm{Perf}(S)$. For $\mathcal{E}^\vee \in \mathrm{Perf}(S)$ the linear dual of \mathcal{E} , we have a tautological evaluation pairing $\mathcal{E} \otimes \mathcal{E}^\vee \rightarrow \mathcal{O}_S$. Setting $E := \mathrm{Tot}_S(\mathcal{E})$ and $E^\vee := \mathrm{Tot}_S(\mathcal{E}^\vee)$, this induces on total spaces a map

$$\mathrm{ev}: E \times_S E^\vee \rightarrow \mathbf{A}^1.$$

Let $\mathcal{A} \in \mathrm{Shv}(\mathbf{A}^1)$ be the Artin-Schreier sheaf (cf. §2.3.2), which we note is holonomic.

The *derived Fourier transform*

$$\mathrm{FT}_E^\psi: \mathrm{Shv}_{\mathrm{hol}}(E) \rightarrow \mathrm{Shv}_{\mathrm{hol}}(E^\vee)$$

is the functor

$$\mathcal{K} \mapsto \mathrm{pr}_{2!}(\mathrm{pr}_1^*(\mathcal{K}) \otimes \mathrm{ev}^* \mathcal{A})[\mathrm{rank}(E)]$$

where the maps are as in the diagram

$$\begin{array}{ccc} & E \times_S E^\vee & \xrightarrow{\mathrm{ev}} \mathbf{A}^1 \\ \mathrm{pr}_1 \swarrow & & \searrow \mathrm{pr}_2 \\ E & & E^\vee \end{array}$$

We now tabulate some basic properties of the derived Fourier transform. Below we let r be the virtual rank of $E \rightarrow S$.

3.1.3. *Base change.* Let $h: \tilde{S} \rightarrow S$ be a map of derived stacks. For a derived vector bundle $E \rightarrow S$, let $\tilde{E} \rightarrow \tilde{S}$ be its base change along h . So we have derived Cartesian squares

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{h^E} & E \\ \downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{h} & S \end{array} \quad \begin{array}{ccc} \tilde{E}^\vee & \xrightarrow{h^{E^\vee}} & E^\vee \\ \downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{h} & S \end{array}$$

Then there are canonical natural isomorphisms of functors $\mathrm{Shv}_{\mathrm{hol}}(E) \rightarrow \mathrm{Shv}_{\mathrm{hol}}(\tilde{E}^\vee)$

$$\mathrm{FT}_{\tilde{E}} \circ (h^E)^* \cong (h^{E^\vee})^* \circ \mathrm{FT}_E \quad (3.1.1)$$

$$\mathrm{FT}_{\tilde{E}} \circ (h^E)! \cong (h^{E^\vee})! \circ \mathrm{FT}_E \quad (3.1.2)$$

and canonical natural isomorphisms of functors $\mathrm{Shv}_{\mathrm{hol}}(\tilde{E}) \rightarrow \mathrm{Shv}_{\mathrm{hol}}(E^\vee)$

$$\mathrm{FT}_E \circ (h^E)_! \cong (h^{E^\vee})_! \circ \mathrm{FT}_{\tilde{E}} \quad (3.1.3)$$

$$\mathrm{FT}_E \circ (h^E)_* \cong (h^{E^\vee})_* \circ \mathrm{FT}_{\tilde{E}}. \quad (3.1.4)$$

3.1.4. *Involutivity.* There is a canonical natural isomorphism $\mathrm{FT}_{E^\vee} \circ \mathrm{FT}_E \cong \mathrm{mult}_{-1}^*(-r)$ of functors $\mathrm{Shv}_{\mathrm{hol}}(E) \rightarrow \mathrm{Shv}_{\mathrm{hol}}(E)$, where mult_{-1} is multiplication by -1 on E .

Remark 3.1.2. The construction of this natural isomorphism is the central point of derived Fourier analysis. It is significantly more involved than in the situation of classical vector bundles.

3.1.5. *Functoriality.* Let $f: E' \rightarrow E$ be a linear map of derived vector bundles having virtual ranks r', r respectively. This induces a morphism $f^\vee: E^\vee \rightarrow E'^\vee$ of dual derived bundles. Then we have canonical natural isomorphisms of functors $\mathrm{Shv}_{\mathrm{hol}}(E') \rightarrow \mathrm{Shv}_{\mathrm{hol}}(E^\vee)$:

- (1) $f^{\vee*} \circ \mathrm{FT}_{E'} \cong \mathrm{FT}_E \circ f_![r' - r]$,
- (2) $f^{\vee!} \circ \mathrm{FT}_{E'} \cong \mathrm{FT}_E \circ f_*r - r'$,

and canonical natural isomorphisms of functors $\mathrm{Shv}_{\mathrm{hol}}(E) \rightarrow \mathrm{Shv}_{\mathrm{hol}}(E'^\vee)$:

- (3) $\mathrm{FT}_{E'} \circ f^* \cong f_!^\vee \circ \mathrm{FT}_Er - r'$,
- (4) $\mathrm{FT}_{E'} \circ f^! \cong f_*^\vee \circ \mathrm{FT}_E[r' - r]$.

Example 3.1.3. Natural transformation (1) gives an isomorphism

$$\mathrm{FT}_E(\delta_E) \cong \underline{k}_{E^\vee}[r] \in \mathrm{Shv}(E^\vee)$$

and natural transformation (3) gives an isomorphism

$$\mathrm{FT}_{E^\vee}(\underline{k}_{E^\vee}) \cong \delta_E-r \in \mathrm{Shv}(E).$$

4. CHIRAL ALGEBRAS AND FACTORIZATION HOMOLOGY

In this section we recall or develop some tools in the theory of *chiral algebras*. This subject was developed by Beilinson-Drinfeld [BD04], although our presentation instead follows the references [FG12, GL16, Ho17, Ho21a, Ho21b]. Most of this section just reviews material from the literature: factorization (co)algebra theory is reviewed in §4.2 and §4.3, and derived infinitesimal geometry is reviewed in §4.4. The only “new” material is §4.5, which studies factorization homology of cocommutative coalgebras of distributions. Our work suggests that these are the correct categorification of L -functions.

4.1. Non-unital algebras. Let (\mathbf{C}, \otimes) be a symmetric monoidal category. There is an equivalence between the notion of augmented unital commutative algebras in (\mathbf{C}, \otimes) , by which we mean algebras A equipped with an augmentation $\epsilon: A \rightarrow \mathbf{1}$ (morphisms are required to be compatible with the augmentation) and *non-unital* algebras \overline{A} . The equivalence sends an augmented k -algebra A to $\overline{A} := \ker(\epsilon)$, and in the other direction sends a non-unital algebra \overline{A} to $\mathbf{1} \oplus \overline{A}$.

In this section by $\text{ComAlg}(\mathbf{C}, \otimes)$ we mean the category of *non-unital* commutative algebras in the symmetric monoidal category (\mathbf{C}, \otimes) .

A similar discussion applies to augmented cocommutative coalgebras. We denote by $\text{ComCoalg}(\mathbf{C}, \otimes)$ the category of *non-unital* commutative algebras in the symmetric monoidal category (\mathbf{C}, \otimes) .

4.2. Factorization algebras.

4.2.1. The Ran space. Suppose C is proper. Let FinSurj be the category of finite sets, with morphisms being *surjective* maps. The *Ran space* of C is the prestack

$$\text{Ran}(C) := \text{colim}_{I \in \text{FinSurj}^{\text{op}}} C^I.$$

The category of sheaves $\text{Shv}(\text{Ran } C)$ is defined by descent along $!$ -pullback. We have a “union” map

$$\text{Ran}(C) \times \text{Ran}(C) \xrightarrow{\text{union}} \text{Ran}(C).$$

Since the transition maps defining $\text{Ran } C$ are proper, union has a notion of $!$ -direct image.

4.2.2. The convolution tensor structure. The map

$$\text{union}_!: \text{Shv}(\text{Ran}(C) \times \text{Ran}(C)) \rightarrow \text{Shv}(\text{Ran}(C))$$

induces a symmetric monoidal structure on $\text{Shv}(\text{Ran}(C))$, which we denote \otimes^* . We call this the *convolution tensor structure*.

Remark 4.2.1. By its construction, for the map $\text{pr}_{\text{Ran}}: \text{Ran } C \rightarrow \text{pt} := \text{Spec } \mathbf{F}$ the proper pushforward $\text{pr}_{\text{Ran}}^!$ is symmetric monoidal. Consider the commutative diagram

$$\begin{array}{ccc} \text{ComAlg}(\text{Shv}(C), \overset{!}{\otimes}) & \xleftarrow{\Delta^!} & \text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*) \\ & \swarrow \text{pr}^! & \uparrow \text{pr}_{\text{Ran}}^! \\ & & \text{ComAlg}(\text{Shv}(\text{pt})) \end{array}$$

It turns out that the left adjoint to $\Delta^!$ exists (and will be discussed below): it is the formation of the associated *commutative factorization algebra*. Therefore, forming the left adjoint diagram shows that the left adjoint to $\text{pr}^!$, which was called *factorization homology* in §1.3.2, agrees with $\text{pr}_{\text{Ran}}^!$ composed with the left adjoint to $\Delta^!$.

4.2.3. Commutative factorization algebras. Recall that $\text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*)$ denotes the category of commutative (non-unital) algebra objects in the convolution tensor structure. In particular, an object of $\text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*)$ entails $\mathcal{A} \in \text{Shv}(\text{Ran } C)$ plus a map

$$\text{union}_!(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \mathcal{A}.$$

By adjunction, such data is equivalent to $\mathcal{A} \boxtimes \mathcal{A} \rightarrow \text{union}^! \mathcal{A}$. We often abuse notation by referring to an object of $\text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*)$ by “ \mathcal{A} ”, suppressing the multiplication map and higher coherence data.

The disjoint subspace $(\text{Ran } C \times \text{Ran } C)_{\text{disj}}$ is defined as

$$(\text{Ran } C \times \text{Ran } C)_{\text{disj}} = \text{colim}_{\substack{I_1 \in \text{FinSurj}^{\text{op}} \\ I_2 \in \text{FinSurj}^{\text{op}}}} (C^{I_1} \times C^{I_2})_{\text{disj}}$$

where $(C^{I_1} \times C^{I_2})_{\text{disj}}$ parametrizes pairs $(D_1 \in C^{I_1}, D_2 \in C^{I_2})$ such that D_1 and D_2 have disjoint support.

Definition 4.2.2. A *factorization algebra* is an $\mathcal{A} \in \text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*)$ such that the structure map

$$\mathcal{A} \boxtimes \mathcal{A} \rightarrow \text{union}^! \mathcal{A}$$

restricts to an isomorphism on the open subspace $(\text{Ran } C \times \text{Ran } C)_{\text{disj}}$. The full subcategory of factorization algebras is denoted $\text{ComAlg}_{\text{Fact}}^*(\text{Shv}(\text{Ran } C)) \subset \text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*)$.

4.2.4. *Diagonal embedding.* The diagonal map $\Delta: C \rightarrow \text{Ran } C$ induces a symmetric monoidal functor

$$\Delta^!: (\text{Shv}(\text{Ran } C), \otimes^*) \rightarrow (\text{Shv}(C), \overset{!}{\otimes})$$

because of the Cartesian square

$$\begin{array}{ccc} C & \xrightarrow{\Delta \times \Delta} & \text{Ran } C \times \text{Ran } C \\ \downarrow \text{Id} & & \downarrow \text{union} \\ C & \xrightarrow{\Delta} & \text{Ran } C \end{array} \quad (4.2.1)$$

Theorem 4.2.3 ([GL16, Theorem 5.6.1]). *The functor $\Delta^!$ admits a fully faithful left adjoint*

$$\text{Fact}: \text{ComAlg}(\text{Shv}(C), \overset{!}{\otimes}) \rightarrow \text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*),$$

whose essential image is $\text{ComAlg}_{\text{Fact}}^*(\text{Shv}(\text{Ran } C))$. In particular, $\Delta^!$ restricts to an equivalence

$$\Delta^!: \text{ComAlg}_{\text{Fact}}^*(\text{Shv}(\text{Ran } C)) \xrightarrow{\sim} \text{ComAlg}(\text{Shv}(C), \overset{!}{\otimes}).$$

Example 4.2.4 (Free factorization algebras). The symmetric monoidality of $\Delta^!$ implies that we have a commutative diagram

$$\begin{array}{ccc} \text{Shv}(C) & \xleftarrow{\text{oblv}} & \text{ComAlg}(\text{Shv}(C), \overset{!}{\otimes}) \\ \Delta^! \uparrow & & \Delta^! \uparrow \\ \text{Shv}(\text{Ran } C) & \xleftarrow{\text{oblv}} & \text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*) \end{array}$$

Hence the diagram of left adjoints commutes:

$$\begin{array}{ccc} \text{Shv}(C) & \xrightarrow{\text{Sym}^!} & \text{ComAlg}(\text{Shv}(C), \overset{!}{\otimes}) \\ \downarrow \Delta_! & & \downarrow \text{Fact} \\ \text{Shv}(\text{Ran } C) & \xrightarrow{\text{Sym}^*} & \text{ComAlg}(\text{Shv}(\text{Ran } C), \otimes^*) \end{array} \quad (4.2.2)$$

where for $\text{Sym}^?$ refers to the formation of the symmetric algebra with respect to the respective symmetric monoidal structure. Let $M \in \text{Shv}(C)$ and $\mathcal{A} := \text{Sym}^!(M) \in \text{ComAlg}(\text{Shv}(C), \overset{!}{\otimes})$. Then (4.2.2) gives a natural isomorphism $\text{Fact}(\mathcal{A}) \cong \text{Sym}^*(\Delta_! M)$. Explicitly,

$$\text{Sym}^*(\Delta_! M) = \bigoplus_{n>0} \text{union}_!(((\Delta_! M)^{\boxtimes n})_{\Sigma_n}) = \bigoplus_{n>0} \Delta_n!((M^{\boxtimes n})_{\Sigma_n}),$$

where Δ_n is the composition

$$\begin{array}{ccc} C^n & \xrightarrow{\Delta^n} & (\text{Ran } C)^n \\ & \searrow \Delta_n & \downarrow \text{union} \\ & & \text{Ran } C \end{array}$$

and the coinvariants $(-)_{\Sigma_n}$ are for the natural action of the symmetric group Σ_n on n elements. See [Ho17], especially §3.5 of *loc. cit.*, for more discussion on free factorization algebras.

4.2.5. *Chiral algebras.* Consider the correspondence

$$\begin{array}{ccc} & (\text{Ran } C \times \text{Ran } C)_{\text{disj}} & \\ & \swarrow & \searrow \text{union} \\ \text{Ran}(C) \times \text{Ran}(C) & & \text{Ran}(C) \end{array}$$

Then $!$ -pullback and $*$ -pushforward through this diagram equips $\text{Shv}(\text{Ran}(C))$ with another symmetric monoidal structure called the *chiral tensor structure*, which we denote \otimes^{ch} . For us, \otimes^{ch} is an intermediary technical device.

As usual, in any symmetrical monoidal category we have an operadic notion of associative/commutative (co)algebras and Lie algebras. Below we summarize the results on “chiral Koszul duality” from [FG12] that we will need. First, we have the following algebraic structures.

- (*Lie* algebras on Ran C*) We denote by $\text{Lie}^*(\text{Ran } C)$ the category of Lie algebras in the symmetric monoidal category $(\text{Shv}(\text{Ran } C), \otimes^*)$.
- (*Lie* algebras on C*) We denote by $\text{Lie}^*(C) \subset \text{Lie}^*(\text{Ran } C)$ the full subcategory of sheaves supported on $\Delta(C) \subset \text{Ran}(C)$.
- (*Chiral Lie algebras on Ran C*) We denote by $\text{Lie}^{\text{ch}}(\text{Ran } C)$ the category of Lie algebras in $(\text{Shv}(\text{Ran } C), \otimes^{\text{ch}})$.
- (*Commutative algebras on Ran C*) We abbreviate by $\text{ComAlg}^*(\text{Ran } C)$ the category of commutative algebras in $(\text{Shv}(\text{Ran } C), \otimes^*)$. We denote by $\text{ComAlg}^{\text{ch}}(\text{Ran } C)$ the category of commutative algebras in $(\text{Shv}(\text{Ran } C), \otimes^{\text{ch}})$.
- (*Cocommutative coalgebras on Ran C*) We denote by $\text{ComCoalg}^*(\text{Ran } C)$ the category of cocommutative coalgebras in $(\text{Shv}(\text{Ran } C), \otimes^*)$. We denote by $\text{ComCoalg}^{\text{ch}}(\text{Ran } C)$ the category of cocommutative coalgebras in $(\text{Shv}(\text{Ran } C), \otimes^{\text{ch}})$.

Remark 4.2.5. It follows from the Cartesian square (4.2.1) that $\Delta^!$ induces an equivalence between $\text{Lie}^*(C)$ and $\text{Lie}(C) := \text{Lie}(\text{Shv}(C), \otimes^{\dagger})$. See [Ho17, Proposition 4.2.3] for the details.

The identity functor on $\text{Shv}(C)$ promotes to a symmetric monoidal functor

$$(\text{Shv}(\text{Ran } C), \otimes^*) \rightarrow (\text{Shv}(\text{Ran } C), \otimes^{\text{ch}}).$$

This formally induces functors

$$\text{ComAlg}^*(\text{Ran } C) \xleftarrow{\text{oblv}} \text{ComAlg}^{\text{ch}}(\text{Ran } C), \quad (4.2.3)$$

$$\text{Lie}^*(\text{Ran } C) \xleftarrow{\text{oblv}} \text{Lie}^{\text{ch}}(\text{Ran } C), \quad (4.2.4)$$

and

$$\text{ComCoalg}^*(\text{Ran } C) \xrightarrow{\text{oblv}} \text{ComCoalg}^{\text{ch}}(\text{Ran } C). \quad (4.2.5)$$

The left adjoint to (4.2.4) is a functor

$$\text{Lie}^*(\text{Ran } C) \xrightarrow{\text{Ind}_{\text{Lie}}^{\text{ch}}} \text{Lie}^{\text{ch}}(\text{Ran } C).$$

The notion of *cocommutative factorization coalgebras* in $\text{ComCoalg}^{\text{ch}}(\text{Ran } C)$ is defined in [FG12, §2.4.6]⁵. The full subcategory spanned by such is denoted $\text{ComCoalg}_{\text{Fact}}^{\text{ch}}(\text{Ran } C)$. Let us describe the definition informally for intuition. For a non-empty finite set I and $\mathcal{F} \in \text{Shv}(\text{Ran } C)$, let \mathcal{F}_I be the $!$ -restriction of \mathcal{F} along $C^I \rightarrow \text{Ran } C$. If $\mathcal{A} \in \text{ComCoalg}^{\text{ch}}(\text{Ran } C)$, then \mathcal{A} has a system of maps $\mathcal{A}_I \rightarrow \mathcal{A}_{I_1} \otimes^{\dagger} \dots \otimes^{\dagger} \mathcal{A}_{I_n}$ for every (disjoint) partition $I = I_1 \sqcup \dots \sqcup I_n$. We say that \mathcal{A} is a *cocommutative factorization coalgebra* if all such maps restrict to isomorphisms on the open subspaces $(C^{I_1} \times \dots \times C^{I_n})_{\text{disj}}$.

4.3. Graded factorization (co)algebras. Let Λ be a free abelian monoid of finite rank, identified with $\mathbf{Z}_{\geq 0}^r$ via a choice of basis $\alpha_1, \dots, \alpha_r$. Let $\Lambda^+ := \Lambda \setminus \{0\}$. We will consider Λ^+ -graded factorization algebras. These turn out to have more elementary incarnations, as explained in [Gai16, §4.1] and [GL19, §4, 5].

4.3.1. Colored divisors. For $\lambda = \sum_i d_i \alpha_i \in \Lambda$, write

$$C^{(\lambda)} := C^{(d_1)} \times C^{(d_2)} \times \dots \times C^{(d_n)}.$$

We define the space of Λ -colored divisors $\text{Div}^{\Lambda^+}(C)$ to be

$$\text{Div}^{\Lambda^+}(C) = \coprod_{\lambda \in \Lambda^+} C^{(\lambda)}.$$

It can also be described as the moduli space of $(D \in \text{Div } C, \phi: D \rightarrow \Lambda^+)$. We think of ϕ as equipping D with a “coloring” by Λ^+ , and sometimes refer to a point of $\text{Div}^{\Lambda^+}(C)$ as a “ Λ^+ -colored divisor”.

⁵Francis-Gaitsgory call it “commutative factorization coalgebras”.

4.3.2. *Convolution and chiral tensor structures.* Addition of divisors induces maps

$$\text{add}_{\lambda_1, \lambda_2}: C^{(\lambda_1)} \times C^{(\lambda_2)} \rightarrow C^{(\lambda_1 + \lambda_2)}$$

which assembled into a map

$$\text{add}: \text{Div}^{\Lambda^+}(C) \times \text{Div}^{\Lambda^+}(C) \rightarrow \text{Div}^{\Lambda^+} C.$$

This induces a symmetric monoidal structure on $\text{Shv}(\text{Div}^{\Lambda^+} C)$ that we call the *convolution tensor structure* \otimes^* , with

$$\mathcal{F} \otimes^* \mathcal{G} := \text{add}_!(\mathcal{F} \boxtimes \mathcal{G}).$$

The disjoint locus $(\text{Div}^{\Lambda^+} C \times \text{Div}^{\Lambda^+} C)_{\text{disj}}$ is defined as the open subspace of $(D_1, D_2) \in \text{Div}^{\Lambda^+} C \times \text{Div}^{\Lambda^+} C$ such that D_1 and D_2 have disjoint support. Then $!$ -pull and $*$ -push through the diagram

$$\begin{array}{ccc} & (\text{Div}^{\Lambda^+} C \times \text{Div}^{\Lambda^+} C)_{\text{disj}} & \\ & \swarrow & \searrow \text{union} \\ \text{Div}^{\Lambda^+} C \times \text{Div}^{\Lambda^+} C & & \text{Div}^{\Lambda^+} C \end{array}$$

induces another symmetric monoidal structure on $\text{Shv}(\text{Div}^{\Lambda^+} C)$ that we call the *chiral tensor structure* \otimes^{ch} .

As explained in [Ras, §4], these structures allow to imitate the theory of factorization algebras on $\text{Div}^{\Lambda^+} C$ (instead of $\text{Ran} C$). Moreover, while the pro-nilpotence of the chiral symmetric monoidal structure on $\text{Shv}(\text{Ran}(C))$ was a non-trivial theorem in [FG12], it is obvious in $\text{Shv}(\text{Div}^{\Lambda^+} C)$ due to the nature of the grading. Hence all the results of [FG12] pass over to the graded setting, with the same proofs.

4.3.3. *Graded factorization algebras.* The notion of Λ^+ -graded factorization algebras is developed in [GL19, §5]. Informally, this consists of

$$\mathcal{A} = \{\mathcal{A}^\lambda\}_{\lambda \in \Lambda^+} \in \text{Shv}(\text{Div}^{\Lambda^+} C) \cong \prod_{\lambda \in \Lambda^+} \text{Shv}(C^{(\lambda)}) \quad (4.3.1)$$

equipped with a “homotopy-compatible” system of identifications

$$\mathcal{A}^{\lambda_1 + \lambda_2}|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\text{disj}}} \cong \mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2}|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\text{disj}}} \quad (4.3.2)$$

for all $\lambda_1, \lambda_2 \in \Lambda^+$.

Remark 4.3.1 (Ranification). This definition looks rather different to the notion of factorization algebra in Definition 4.2.2. The translation goes through a “Ranification” procedure explained in [Ras, §4.7], which turns a factorization algebra $\mathcal{A} = \{\mathcal{A}^\lambda\}_{\lambda \in \Lambda^+}$ in the sense of (4.3.2) into a factorization algebra $\text{Ranification}(\mathcal{A}) \in \text{Shv}(\text{Ran} C)^{\Lambda^+}$ in the sense of Definition 4.2.2; we note that

$$\Delta^! \text{Ranification}(\mathcal{A}) = \prod_{\lambda \in \Lambda^+} \Delta_\lambda^! \mathcal{A}^\lambda \in \text{Shv}(C)^{\Lambda^+}.$$

Thanks to Ranification, when working with Λ^+ -graded factorization algebras we can replace $\text{Ran}(C)$ with $\text{Div}^{\Lambda^+} C$.

4.3.4. *Commutative graded factorization algebras.* We say that a Λ^+ -graded factorization algebra $\mathcal{A} := \{\mathcal{A}^\lambda\}$ is *commutative* if for all $\lambda_1, \lambda_2 \in \Lambda^+$ the maps

$$\mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2}|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\text{disj}}} \cong \mathcal{A}^{\lambda_1 + \lambda_2}|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\text{disj}}}$$

extend to maps

$$\mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2} \rightarrow \text{add}_{\lambda_1, \lambda_2}^! \mathcal{A}^{\lambda_1 + \lambda_2}. \quad (4.3.3)$$

For each $\lambda \in \Lambda^+$, let $\Delta_\lambda: C \rightarrow C^{(\lambda)}$ be the diagonal map. Let

$$\Delta^! := \prod_{\lambda \in \Lambda^+} \Delta_\lambda^!: \text{Shv}(\text{Div}^{\Lambda^+} C) \rightarrow \text{Shv}(C)^{\Lambda^+}.$$

Parallel to Theorem 4.2.3, we have:

Theorem 4.3.2. *The functor $\mathcal{A} \mapsto \Delta^! \mathcal{A}$ induces an equivalence between the category of commutative factorization algebras in $\text{Shv}^*(\text{Div}^{\Lambda^+} C)$ and commutative algebras in $\text{Shv}(C)^{\Lambda^+}$.*

4.3.5. *Graded factorization coalgebras.* The case of (cocommutative) coalgebras graded by Λ^+ will play an important role for us. A reference is [Gai], although Gaitsgory calls “cocommutative factorization algebras” what we call “cocommutative factorization coalgebras”. Indeed, a *factorization coalgebra* on $\mathrm{Div}^{\Lambda^+} C$ is by definition the same as a factorization algebra: informally speaking, a collection

$$\mathcal{A} = \{\mathcal{A}^\lambda\}_{\lambda \in \Lambda^+} \in \mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C) \cong \bigoplus_{\lambda \in \Lambda^+} \mathrm{Shv}(C^{(\lambda)})$$

equipped with a “homotopy-compatible” system of identifications

$$\mathcal{A}^{\lambda_1 + \lambda_2} \Big|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\mathrm{disj}}} \cong \mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2} \Big|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\mathrm{disj}}}. \quad (4.3.4)$$

(The “moral” difference between graded factorization algebras and coalgebras is that the right side of (4.3.4) is $\mathrm{add}_{\lambda_1, \lambda_2}^*$ for factorization coalgebras, while the analogous expression for factorization algebras should use $\mathrm{add}_{\lambda_1, \lambda_2}^!$, but $\mathrm{add}_{\lambda_1, \lambda_2}^!$ is étale when restricted to the disjoint locus, so these coincide.)

We say that a factorization coalgebra \mathcal{A} is *cocommutative* if the maps come by adjunction from maps

$$\mathcal{A}^{\lambda_1 + \lambda_2} \rightarrow \mathcal{A}^{\lambda_1} \otimes^* \mathcal{A}^{\lambda_2} \quad (4.3.5)$$

on $C^{(\lambda_1 + \lambda_2)}$. If \mathcal{A} is holonomic, so that it lies in the domain of the partially defined adjoint add^* , then (4.3.5) can be formulated equivalently as saying that each (4.3.4) extends to a map

$$\mathrm{add}_{\lambda_1, \lambda_2}^* \mathcal{A}^{\lambda_1 + \lambda_2} \rightarrow \mathcal{A}^{\lambda_1} \boxtimes \mathcal{A}^{\lambda_2}$$

on $C^{(\lambda_1)} \times C^{(\lambda_2)}$.

Remark 4.3.3. It is evident that Verdier duality induces an equivalence between holonomic commutative Λ^+ -graded factorization algebras and holonomic cocommutative Λ^+ -graded factorization coalgebras. Let us say that an object of $\mathrm{Shv}(C)^{\Lambda^+}$ is *graded holonomic* if its component each each grading degree $\lambda \in \Lambda^+$ is holonomic. In turn, Theorem 4.3.2 gives an equivalence between holonomic commutative Λ^+ -graded factorization algebras and graded holonomic commutative algebras in $\mathrm{Shv}(C)^{\Lambda^+}$.

The map

$$\Delta^* := \prod_{\lambda \in \Lambda^+} \Delta_\lambda^*: \mathrm{Shv}_{\mathrm{hol}}(\mathrm{Div}^{\Lambda^+} C) \rightarrow \mathrm{Shv}_{\mathrm{hol}}(C)^{\Lambda^+} \quad (4.3.6)$$

is symmetric monoidal for \otimes^* -monoidal structure on $\mathrm{Shv}_{\mathrm{hol}}(\mathrm{Div}^{\Lambda^+} C)$ and the usual monoidal structure \otimes on $\mathrm{Shv}_{\mathrm{hol}}(C)^{\Lambda^+}$. It is clear that Verdier duality on $\mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C)$ induces an equivalence between commutative algebras in $\mathrm{Shv}_{\mathrm{hol}}(C)^{\Lambda^+}$ and cocommutative coalgebras in $\mathrm{Shv}_{\mathrm{hol}}(C)^{\Lambda^+}$. Applying Remark 4.3.3 to Theorem 4.3.2 then gives:

Theorem 4.3.4. *The functor $\mathcal{A} \mapsto \Delta^* \mathcal{A}$ induces an equivalence between the category of cocommutative factorization coalgebras in $\mathrm{Shv}_{\mathrm{hol}}^*(\mathrm{Div}^{\Lambda^+} C)$ and cocommutative cocolgebras in $(\mathrm{Shv}_{\mathrm{hol}}(C)^{\Lambda^+}, \otimes)$.*

Example 4.3.5. Let V be a vector space and let $\overline{\mathrm{Sym}}(V) = \bigoplus_{n > 0} \mathrm{Sym}^n(V)$ be the free graded (non-unital) commutative symmetric algebra on V . (The grading corresponds to the standard scaling \mathbf{G}_m -action on V .) By Example 4.2.4, the associated factorization algebra $\mathrm{Fact}(\overline{\mathrm{Sym}}(V)) = \{\mathcal{A}^{(n)} \in \mathrm{Shv}(C^{(n)})\}$ has

$$\mathcal{A}^{(n)} := (\pi_n! \mathcal{A}^{\boxtimes n})_{\Sigma_n} \quad (4.3.7)$$

where $\pi_n: C^n \rightarrow C^{(n)}$ is the quotient map and the coinvariants $(-)_\Sigma_n$ are for the natural action of the symmetric group Σ_n on n elements.

Note that the graded dual of $\overline{\mathrm{Sym}}(V)$ is $\overline{\mathrm{Sym}}(V^*)$, which has a graded cocommutative coalgebra structure. We invert the grading so that $\overline{\mathrm{Sym}}(V^*)$ is graded by $\mathbf{Z}_{>0}$. Then the associated $\mathbf{Z}_{>0}$ -graded cocommutative factorization coalgebra (under Theorem 4.3.4) is given levelwise by the Verdier dual of $\mathcal{A}^{(n)}$.

4.3.6. *Factorization homology.* Let $\mathcal{A} = \{\mathcal{A}^\lambda\}$ a Λ^+ -graded factorization algebra on $\text{Div}^{\Lambda^+} C$. Then we define its *factorization homology* to be

$$C_c^{\text{Fact}}(C, \mathcal{A}) := \bigoplus_{\lambda \in \Lambda^+} C_c^*(C^{(\lambda)}, \mathcal{A}^\lambda).$$

If \mathcal{A} is a commutative factorization algebra, then this is compatible with the previous definition of factorization homology (1.3.2) under the equivalence of Theorem 4.3.2. We will also apply this to cocommutative factorization coalgebras.

4.4. **Recollections on derived infinitesimal geometry.** We will now study some constructions related to formal geometry. The theory of formal completion is complicated in the derived setting. For a closed embedding of *classical* schemes $Z \hookrightarrow Y$, one has the notion of “ n th infinitesimal neighborhood” of Z in Y , which is the closed subscheme of Y defined by the n th power of the ideal sheaf of Z . Then the formal completion of Z in Y is the colimit over n of the n th infinitesimal neighborhoods. However, in derived algebraic geometry it does not make sense to take the “ n th power of an ideal sheaf”, so it is subtler to form the n th infinitesimal neighborhood.

In [GR17b, Chapter 9, §5], this problem is solved: for a map $f: Z \rightarrow Y$ of derived stacks, Gaitsgory-Rozenblyum define a sequence of derived stacks

$$Z = Z^{(0)} \rightarrow Z^{(1)} \rightarrow Z^{(2)} \rightarrow \dots \rightarrow Z^{(n)} \rightarrow \dots \rightarrow Y. \quad (4.4.1)$$

When f is a closed embedding, then $Z^{(n)}$ is the derived analogue of the “ n th infinitesimal neighborhood of Z in Y ”. The extensions are produced using the theory of the cotangent complex, and the derived deformation to the normal cone.

Remark 4.4.1. Suppose f is a closed embedding of classical schemes. If f is a *regular* embedding, then $Z^{(n)}$ is the usual n th infinitesimal neighborhood. In general, if f is not a regular embedding then $Z^{(n)}$ can have non-trivial derived structure, even though f is a map of classical schemes. In particular, the $Z^{(n)}$ constructed above may not agree with the classical notion of n th infinitesimal neighborhood.

Definition 4.4.2. The *formal completion* of Y along f is $\text{colim}_n Z^{(n)}$, the colimit formed in the category of derived stacks. We will denote it by Y_f^\wedge , or sometimes just Y_Z^\wedge by abuse of notation.

4.4.1. *Ind-coherent sheaves on the formal completion.* For the notions of ind-coherent sheaves on derived ind-stacks and formal completions, we refer to [GR14, §6,7]. Suppose $Z \hookrightarrow Y$ is a closed embedding of stacks. Let $\text{IndCoh}_Z(Y) \subset \text{IndCoh}(Y)$ be the full subcategory of ind-coherent sheaves with set-theoretic support on Z . This admits a co-localization functor $\text{IndCoh}(Y) \rightarrow \text{IndCoh}_Z(Y)$. Letting $U = Y \setminus Z$ be the open complement of Z and $j: U \hookrightarrow Y$ be its open embedding, the co-localization functor sends an ind-coherent sheaf \mathcal{F} to the derived fiber of $\mathcal{F} \rightarrow j_* j^* \mathcal{F}$.

Let $\mathcal{Z} := Y_Z^\wedge = \text{colim}_n Z^{(n)}$ be the formal completion of Y along Z . Let $\widehat{i}: \mathcal{Z} \rightarrow Y$ be the colimit of the $i^{(n)}: Z^{(n)} \rightarrow Y$. Then $\widehat{i}^!$ factors through the co-localization map $\text{IndCoh}(Y) \rightarrow \text{IndCoh}_Z(Y)$, and induces an equivalence (cf. [GR14, §7.4])

$$\begin{array}{ccc} & \widehat{i}_* & \\ & \curvearrowright & \\ \text{IndCoh}(\mathcal{Z}) & & \text{IndCoh}_Z(Y) \\ & \curvearrowleft & \\ & \widehat{i}^! & \end{array}$$

where the left adjoint \widehat{i}_* is the colimit of the $i_*^{(n)}$. The co-localization functor $\text{IndCoh}(Y) \rightarrow \text{IndCoh}_Z(Y) \cong \text{IndCoh}(\mathcal{Z})$ is monoidal (with respect to $!$ -tensor product). In particular, it sends the unit $\omega_Y \in \text{IndCoh}(Y)$ to $\omega_{\mathcal{Z}} \in \text{IndCoh}(\mathcal{Z})$.

4.4.2. *Description as modules over a Lie algebroid.* Let $f: Z \rightarrow Y$ be any map of derived stacks. Then [GR17b, Chapter 8, Section 3.2] constructs a *Lie algebroid structure* on the relative tangent complex $\mathbf{T}_{Z/Y}$. We denote this Lie algebroid by $\mathcal{L}(Z/Y)$. We will only use the notion of Lie algebroid as a formal device used to access $\text{IndCoh}(Y_f^\wedge)$, so we will not elaborate on it.

Recall that every derived stack has an underlying reduced sub-stack, and a *nil-isomorphism* is a map that is an isomorphism on underlying reduced stacks [GR17a, Chapter 1, Definition 8.1.2]. Equivalently, f is representable in derived schemes and is an isomorphism on reduced subschemes after any base change from a derived scheme S to Y .

Let $i: Z \rightarrow \mathcal{Z}$ be a nil-isomorphism, for example the natural map $Z \rightarrow \mathcal{Z} := Y_Z^\wedge$ induced by a map $f: Z \rightarrow Y$. Then by [GR17b, Chapter 5, §2.2.6, and Chapter 8, §4.1] we have an equivalence

$$\mathcal{L}(Z/\mathcal{Z})\text{-Mod}(\text{IndCoh}(Z)) \cong \text{IndCoh}(\mathcal{Z}), \quad (4.4.2)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(Z/\mathcal{Z})\text{-Mod}(\text{IndCoh}(Z)) & \xleftarrow{\sim} & \text{IndCoh}(\mathcal{Z}) \\ \downarrow \text{oblv} & & \downarrow i^! \\ \text{IndCoh}(Z) & \xlongequal{\quad} & \text{IndCoh}(Z) \end{array}$$

Since $\mathcal{L}(Z/\mathcal{Z})$ is a Lie algebroid over Z , there is the *Lie algebra homology* functor

$$\text{Chev}(\mathcal{L}(Z/\mathcal{Z}), -): \mathcal{L}(Z/\mathcal{Z})\text{-Mod}(\text{IndCoh}(Z)) \rightarrow \text{IndCoh}(Z).$$

It is the left adjoint of the functor $\text{triv}: \text{IndCoh}(Z) \rightarrow \mathcal{L}(Z/\mathcal{Z})\text{-Mod}(\text{IndCoh}(Z))$ equipping an ind-coherent sheaf on Z with the trivial $\mathcal{L}(Z/\mathcal{Z})$ -module structure. The functor $\text{Chev}(\mathcal{L}(Z/\mathcal{Z}), -)$ can be modeled explicitly in terms of the bar construction.

Suppose we have a retraction $\pi: \mathcal{Z} \rightarrow Z$. Then from the construction of the equivalence (4.4.2), the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(Z/\mathcal{Z})\text{-Mod}(\text{IndCoh}(Z)) & \xleftarrow{\sim} & \text{IndCoh}(\mathcal{Z}) \\ \text{Chev}(\mathcal{L}(Z/\mathcal{Z}), -) \left(\uparrow \text{triv} \right) & & \pi_* \left(\uparrow \pi^! \right) \\ \text{IndCoh}(Z) & \xlongequal{\quad} & \text{IndCoh}(Z) \end{array} \quad (4.4.3)$$

Example 4.4.3. Consider $\omega_Z \in \text{IndCoh}(Z)$. Then $\omega_{\mathcal{Z}} \cong \pi^!(\omega_Z) \in \text{IndCoh}(\mathcal{Z})$ corresponds under (4.4.2) to $\omega_Z \in \text{IndCoh}(Z)$ equipped with the trivial action of $\mathcal{L}(Z/\mathcal{Z})$, and then $\pi_*(\omega_{\mathcal{Z}}) \in \text{IndCoh}(Z)$ is naturally isomorphic to the Lie algebra homology $\text{Chev}(\mathcal{L}(Z/\mathcal{Z}), \omega_Z)$.

4.5. Factorization homology of distribution coalgebras. In this subsection we prove the key local-global principle that will be used to calculate spectral periods. In this subsection we assume that $\mathbf{F} = k$.

4.5.1. Twisted mapping stack. Let Y be an affine scheme acted upon by \check{G} and L be a \check{G} -local system on C , thought of as a \check{G} -bundle $L \rightarrow C_{\text{dR}}$. Then we may form the space $\pi: Y \times^G L \rightarrow C_{\text{dR}}$. The derived space of sections $\text{Sect}(C_{\text{dR}}, Y \times^G L)$ is defined in §2.4.2; informally speaking, it parametrizes flat maps from C to Y twisted by the local system L .

4.5.2. Distribution coalgebras. For a closed subscheme $Z \hookrightarrow Y$ we may form $\Gamma_Z(Y; \omega)$, the derived global sections of the dualizing sheaf of Y with set-theoretic support in Z . The formation of $\Gamma_Z(Y; \omega)$ is functorial in maps of pairs (Y, Z) which are proper in the second factor.

In particular, let $y \in Y$ and consider $\Gamma_y(Y, \omega)$. Informally, it is the space of “distributions on Y supported at y ”. The diagonal map $Y \xrightarrow{\Delta} Y \times Y$ equips $\Gamma_y(Y, \omega)$ with the structure of a unital cocommutative coalgebra, the unit coming from $(Y, y) \rightarrow (\text{pt}, \text{pt})$. We denote by $\bar{\Gamma}_y(Y; \omega)$ the corresponding non-unital cocommutative coalgebra (cf. §4.1).

Suppose $y \in Y$ is a \check{G} -fixed point. Then the \check{G} -action on Y equips $\bar{\Gamma}_y(Y; \omega)$ with an action of \check{G} . Since the diagonal map is \check{G} -equivariant (as is the projection to a point), the commutative coalgebra structure is compatible with the \check{G} -action. This promotes $\bar{\Gamma}_y(Y; \omega)$ to a cocommutative coalgebra in $\text{Rep}(\check{G})$.

Suppose furthermore that $\bar{\Gamma}_y(Y; \omega)$ has a $\mathbf{Z}_{>0}$ -grading.⁶ We may view $\bar{\Gamma}_y(Y; \omega)$ as a constant cocommutative coalgebra in $\text{Shv}(C)^{\mathbf{Z}_{>0}}$ by $*$ -pullback, and then by Theorem 4.3.4 there is an associated $\mathbf{Z}_{>0}$ -graded cocommutative factorization coalgebra, which we denote $\text{Fact}(\bar{\Gamma}_y(Y; \omega)) \in \text{ComAlg}_{\text{Fact}}^*(\text{Div}^{\mathbf{Z}_{>0}} C)$.

⁶For example, this happens if Y has a commuting \mathbf{G}_m -action for which y is an attracting point (i.e., y is a \mathbf{G}_m -fixed point and in some neighborhood $U \ni y$ the action extends to $U \times \mathbf{A}^1 \rightarrow U$ and y). In the applications to spherical varieties, the \mathbf{G}_m -action on Y will be the \mathbf{G}_{gr} -action on \check{X} .

4.5.3. *Twisting a factorization algebra by local systems.* Recall that $B\check{G}$ is the classifying stack $[\mathrm{Spec} k/\check{G}]$. In [Gai15, §4.2] the space $(B\check{G})_{\mathrm{Ran} C}$ is defined. For a derived affine scheme S , the S -points of $(B\check{G})_{\mathrm{Ran} C}$ is the groupoid of pairs of $\underline{c} \in \mathrm{Ran}(C)_{\mathrm{dR}}(S)$ and a map

$$(\mathcal{D}_{\underline{c}})_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S \rightarrow B\check{G}$$

where $\mathcal{D}_{\underline{c}}$ is the formal completion of $S \times C$ along its closed subscheme $\Gamma_{\underline{c}}$, defined as the union of the graphs of the maps $(S_{\mathrm{cl}})_{\mathrm{red}} \rightarrow C$ comprising \underline{c} .

A \check{G} -equivariant factorization algebra is $\mathcal{A} \in \mathrm{QCoh}((B\check{G})_{\mathrm{Ran} C})$ such that $\mathrm{oblv}(\mathcal{A}) \in \mathrm{QCoh}((\mathrm{Ran} C)_{\mathrm{dR}}) \cong \mathrm{Dmod}(\mathrm{Ran} C)$ is a factorization algebra. A \check{G} -equivariant factorization algebra can be twisted by a \check{G} -local system on C . To explain this, recall from [Gai15, §4.3] that there is an evaluation map

$$\mathrm{ev}: \mathrm{Ran}(C)_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}} \rightarrow (B\check{G})_{\mathrm{Ran} C}. \quad (4.5.1)$$

For a derived affine scheme S , the map (4.5.1) sends $(\underline{c} \in \mathrm{Ran}(C)_{\mathrm{dR}}(S), L \in \mathrm{Loc}_{\check{G}}(S))$ to the pair of \underline{c} and the composition of the maps

$$(\mathcal{D}_{\underline{c}})_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S \xrightarrow{\mathrm{pr}_1} (\mathcal{D}_{\underline{c}})_{\mathrm{dR}} \rightarrow C_{\mathrm{dR}} \xrightarrow{L} B\check{G}.$$

We denote by $\mathcal{A}^{\mathrm{univ}}$ the $!$ -pullback of $\mathcal{A} \in \mathrm{QCoh}((B\check{G})_{\mathrm{Ran} C})$ to $\mathrm{Ran}(C)_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}}$ along ev . Informally, it is the universal twisting of \mathcal{A} by local \check{G} -local systems on C ; restricting to $L \in \mathrm{Loc}_{\check{G}}$ giving the twist of \mathcal{A} by a particular \check{G} -local system L .

Example 4.5.1. Let $y \in Y$ be as in §4.5.2. Since $\bar{\Gamma}_y(Y; \omega)$ is a cocommutative coalgebra in $\mathrm{Rep}(\check{G})$, this construction applies to give $\mathrm{Fact}(\bar{\Gamma}_y(Y; \omega))^{\mathrm{univ}} \in \mathrm{QCoh}(\mathrm{Ran}(C)_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}})$.

4.5.4. *Local-global principle for distributions.* Let $\mathrm{Loc}_{\check{G}}^Y$ be the derived mapping stack $\mathrm{Map}(C_{\mathrm{dR}}, Y/\check{G})$. The tautological map $Y/\check{G} \rightarrow B\check{G}$ induces a map

$$\pi: \mathrm{Loc}_{\check{G}}^Y \rightarrow \mathrm{Loc}_{\check{G}}.$$

The \check{G} -fixed point $y \in Y$ induces a distinguished section $\sigma_y: \mathrm{Loc}_{\check{G}} \rightarrow \mathrm{Loc}_{\check{G}}^Y$ to π . Let $j: U \hookrightarrow \mathrm{Loc}_{\check{G}}^Y$ be the complement of σ_y . A key formula for us will be the following ‘‘local-global principle’’ for calculating

$$\pi_{\sigma_y^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^Y}) := \mathrm{fib}(\pi_* \omega_{\mathrm{Loc}_{\check{G}}^Y} \rightarrow \pi_* j_* \omega_U),$$

the direct image of $\omega_{\mathrm{Loc}_{\check{G}}^Y}$ under π , with supports along the section σ_y .

Proposition 4.5.2. *Assume that the cotangent complex of Y is perfect (i.e., has bounded tor-dimension). Let $\mathrm{pr}_2: \mathrm{Ran}(C)_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}} \rightarrow \mathrm{Loc}_{\check{G}}$ be the second projection map. Then there is a natural isomorphism*

$$\overline{\pi_{\sigma_y^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^Y})} \cong \mathrm{pr}_{2*}(\mathrm{Fact}(\bar{\Gamma}_y(Y; \omega))^{\mathrm{univ}}) \otimes \omega_{\mathrm{Loc}_{\check{G}}} \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}) \quad (4.5.2)$$

where on the left side $\overline{(-)}$ refers to removing the unit (§4.1).

Remark 4.5.3. The factor of $\omega_{\mathrm{Loc}_{\check{G}}}$ on the right side of (4.5.2) comes from the fact that tensoring with $\omega_{\mathrm{Loc}_{\check{G}}}$ induces a fully faithful symmetric monoidal functor $(\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}), \otimes) \rightarrow \mathrm{IndCoh}(\mathrm{Loc}_{\check{G}}, \overset{!}{\otimes})$.

Proof. Recall that $\mathbf{T}_{y/Y} \cong \mathbf{T}_y Y[-1]$ has a canonical Lie algebra structure. This can be thought of as coming from the group structure of the derived loop space (i.e., inertia group) $y \times_Y y$, whose associated Lie algebra is $\mathbf{T}_y Y[-1]$. Moreover, the \check{G} -action on (Y, y) equips $\mathbf{T}_y Y[-1]$ with the structure of a Lie algebra in $\mathrm{Rep}(\check{G})$.

Regarding $\mathbf{T}_{y/Y} \in \mathrm{Shv}(\mathrm{pt})$, this equips its $!$ -pullback $\underline{\mathbf{T}}_{y/Y} \in \mathrm{Shv}(C)$ with the structure of a \check{G} -equivariant Lie algebra in $(\mathrm{Shv}(C), \overset{!}{\otimes})$, so we can view it as a \check{G} -equivariant Lie * algebra on C by Remark 4.2.5.

Let $\mathrm{pr}_2: C_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}} \rightarrow \mathrm{Loc}_{\check{G}}$. The relative tangent complex of the map $\sigma_y: \mathrm{Loc}_{\check{G}} \rightarrow \mathrm{Loc}_{\check{G}}^Y$ is

$$\mathbf{T}_{\sigma_y} \cong \mathbf{T}_{\pi}[-1] \cong \mathrm{pr}_{2*}(\underline{\mathbf{T}}_{y/Y}^{\mathrm{univ}})$$

where $\underline{\mathbf{T}}_{y/Y}^{\mathrm{univ}} \in \mathrm{QCoh}(C_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}})$ is the twist of $\mathbf{T}_{y/Y}$ by the universal family of \check{G} -local systems. The \check{G} -equivariant Lie algebra structure on $\underline{\mathbf{T}}_{y/Y}$ equips $\mathrm{pr}_{2*}(\underline{\mathbf{T}}_{y/Y}^{\mathrm{univ}})$ with a Lie algebra structure in $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}})$. Then according to Example 4.4.3, we have a natural isomorphism

$$\pi_{\sigma_y^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^Y}) \cong \mathrm{Chev}(\mathrm{pr}_{2*}(\underline{\mathbf{T}}_{y/Y}^{\mathrm{univ}})). \quad (4.5.3)$$

For sanity of notation we will present the rest of the proof over the trivial local system only. The full version over $\text{Loc}_{\tilde{G}}$ can be done by repeating the argument below (and the theory of chiral algebras) with C_{dR} replaced by $C_{\text{dR}} \times \text{Loc}_{\tilde{G}}$, $\text{Ran}(C)_{\text{dR}}$ replaced by $\text{Ran}(C)_{\text{dR}} \times \text{Loc}_{\tilde{G}}$, etc.

A formula of Beilinson-Drinfeld, reproved in [FG12, Proposition 6.3] via chiral Koszul duality⁷, gives for any Lie^{*} algebra \mathcal{L} on C a natural isomorphism⁸

$$\text{Chev}(\Gamma_{\text{dR}}(C; \mathcal{L})) \cong C_c^*(\text{Ran } C; \text{Chev}(\text{Ind}_{\text{Lie}}^{\star \rightarrow \text{ch}} \mathcal{L})).$$

Applying this to $\mathcal{L} = \underline{\mathbf{T}}_{y/Y}$, we obtain a natural isomorphism

$$\text{Chev}(\Gamma_{\text{dR}}(C; \underline{\mathbf{T}}_{y/Y})) \cong C_c^*(\text{Ran } C; \text{Chev}(\text{Ind}_{\text{Lie}}^{\star \rightarrow \text{ch}} \mathcal{L})). \quad (4.5.4)$$

Another application of Example 4.4.3 to $y \in Y$ gives a natural isomorphism between the cocommutative coalgebra $\overline{\Gamma}_y(Y; \omega)$ and $\text{Chev}(\underline{\mathbf{T}}_{y/Y})$. Comparing (4.5.3) and (4.5.4), we see that it suffices to construct a natural isomorphism $\text{Chev}(\text{Ind}_{\text{Lie}}^{\star \rightarrow \text{ch}} \mathcal{L}) \cong \text{Fact}(\text{Chev}(\underline{\mathbf{T}}_{y/Y}))$. This follows from [FG12, Lemma 6.2.6], which we note is dual to [Gai, Proposition 2.3.5] (which is proved by the same general argument of [FG12, Lemma 6.2.6]). \square

5. GEOMETRIC LANGLANDS EQUIVALENCE

Let G be a split reductive group. The Geometric Langlands equivalence is an equivalence of categories

$$\mathbb{L}_G: \text{Dmod}(\text{Bun}_G) \xrightarrow{\sim} \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\tilde{G}}) \quad (5.0.1)$$

characterized by various properties [Gai15, Conjecture 3.4.2]. Some of the properties (including all those that we will use later) are listed in this section. For GL_1 this has been known since the work of Rothstein [Rot96] and Laumon [Lau96] (although in a slightly different formulation), and recently a proof for general groups has been announced by Gaitsgory-Raskin partially in joint work with Arinkin, Beraldo, Chen, Faergeman, and Lin [ABG⁺23].

We will use the existence of (5.0.1), satisfying the properties mentioned in this section, as a black box – that is to say, we will not use any internal details of the proof. Furthermore, we will only use (5.0.1) for $G = \text{GL}_1$ (where it is classical) and $G = \text{GL}_2$ (where the proof was outlined several years ago in [Gai15]).

5.1. Whittaker compatibility. Fix a Borel subgroup $B < G$ with unipotent radical N , and write $T := B/N$. This induces a natural map $\text{Bun}_B \rightarrow \text{Bun}_T$, and for $\mathcal{F}_T \in \text{Bun}_T(\mathbf{F})$ we write $\text{Bun}_N^{\mathcal{F}_T}$ for the fibered product

$$\begin{array}{ccc} \text{Bun}_N^{\mathcal{F}_T} & \longrightarrow & \text{Bun}_B \\ \downarrow & & \downarrow \\ \text{pt} & \xrightarrow{\mathcal{F}_T} & \text{Bun}_T \end{array}$$

Let $\{\alpha_i\}_{i \in \mathcal{I}}$ be the simple (positive) roots corresponding to N . We write $\mathcal{F}_T^{\alpha_i}$ for the line bundle obtained from \mathcal{F}_T via the homomorphism $\alpha_i: T \rightarrow \mathbf{G}_m$. There is an evaluation morphism

$$\text{ev}: \text{Bun}_N^{\mathcal{F}_T} \rightarrow \prod_{i \in \mathcal{I}} \underline{\mathbb{H}^1(C, \mathcal{F}_T^{\alpha_i})}$$

constructed in [FGV01, §4.1.1], where for a vector space V/\mathbf{F} we write \underline{V} for the associated affine space regarded as an \mathbf{F} -scheme.

Let $X := G/U$ regarded as a $G \times T$ -space via $(g, t): Ux \mapsto Ut^{-1}xg$, with the \mathbb{G}_{gr} acting through the character $\mathbf{G}_m \xrightarrow{(1, e^{-2\rho})} G \times T$. Now regard X as a G -space with the same action of \mathbb{G}_{gr} . Then $\text{Bun}_G^X \rightarrow \text{Bun}_G$, defined in [BZSV23, §10.2], can be identified with a $\text{Bun}_N^{\mathcal{F}_T}$ such that $\mathcal{F}_T^{\alpha_i} \cong \Omega_C$ for each i . Consider the composite map

$$\text{Bun}_G^X \xrightarrow{\text{ev}} \prod_{i \in \mathcal{I}} \underline{\mathbb{H}^1(C, \Omega)} \xrightarrow{\text{add}} \underline{\mathbb{H}^1(C, \Omega)} \cong \mathbf{A}^1. \quad (5.1.1)$$

⁷The two actually differ by the equivalence of removing the unit, cf. [FG12, Remark 6.4.6]. We are using the version of Francis-Gaitsgory.

⁸The formulation below is the one of [Ho17, Proposition 4.1.11].

Example 5.1.1. For $G = \mathrm{GL}_1$, the classical truncation of the stack Bun_G^X has R -points the groupoid of pairs

$$(\mathcal{L} \in \mathrm{Bun}_G(R), t: \mathcal{O}_C \boxtimes R \rightarrow \mathcal{L}).$$

Example 5.1.2. For $G = \mathrm{GL}_2$, the classical truncation of the stack Bun_G^X has R -points the groupoid of extensions

$$\Omega_C^{1/2} \boxtimes R \rightarrow \mathcal{F} \rightarrow \Omega_C^{-1/2} \boxtimes R.$$

The map $\mathrm{ev}: \mathrm{Bun}_G^X \rightarrow \mathbf{A}^1$ sends such an extension to its class in $\mathrm{H}^1(C_R, \Omega_C \boxtimes R) \cong R$.

Recall the Artin-Schreier sheaf on \mathbf{A}^1 from §2.3.2. We let $\Psi \in \mathrm{Shv}(\mathrm{Bun}_G^X)$ be the $*$ -pullback of the Artin-Schreier sheaf along (5.1.1). The (*automorphic*) *Whittaker sheaf* $\mathscr{W} \in \mathrm{Shv}(\mathrm{Bun}_G)$ is then the $!$ -pushforward⁹ of Ψ along the natural map $\mathrm{Bun}_G^X \rightarrow \mathrm{Bun}_G$.

Up to twist, the Whittaker sheaf corresponds to the dualizing sheaf of $\mathrm{Loc}_{\check{G}}$ under \mathbb{L}_G . More precisely, we normalize the Geometric Langlands Equivalence as in [BZSV23, §12.2.1] so that for ρ the half-sum of positive roots of G and $\check{\rho}$ the half-sum of positive roots for \check{G} , we have

$$\mathrm{Shv}(\mathrm{Bun}_G) \ni \mathscr{W} \langle (g-1)(\dim N - \langle 2\rho, 2\check{\rho} \rangle) \rangle \xrightarrow{\mathbb{L}_G} \omega_{\mathrm{Loc}_{\check{G}}} \langle -(g-1) \dim G \rangle \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}). \quad (5.1.2)$$

Note that $\dim \mathrm{Bun}_G = (g-1) \dim G$.

5.2. Hecke compatibility. We summarize [Gai15, §4], which formulates the ‘‘Hecke compatibility’’ property of the Geometric Langlands Equivalence \mathbb{L}_G .

5.2.1. *Spectral localization.* We will define a certain *spectral localization* functor

$$\mathrm{Loc}^{\mathrm{spec}}: \mathrm{QCoh}((B\check{G})_{\mathrm{Ran} C}) \rightarrow \mathrm{QCoh}(\mathrm{Loc}_{\check{G}})$$

following [Gai15, §4.3]. Recall from §4.5.3 the ‘‘evaluation map’’ $\mathrm{ev}: \mathrm{Ran}(C)_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}} \rightarrow (B\check{G})_{\mathrm{Ran} C}$. The diagram

$$\begin{array}{ccc} \mathrm{Ran}(C)_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}} & \xrightarrow{\mathrm{ev}} & (B\check{G})_{\mathrm{Ran} C} \\ \downarrow \mathrm{pr}_2 & & \\ \mathrm{Loc}_{\check{G}} & & \end{array}$$

defines the functor

$$\mathrm{Loc}^{\mathrm{spec}} = \mathrm{pr}_{2*} \mathrm{ev}^!: \mathrm{QCoh}((B\check{G})_{\mathrm{Ran} C}) \rightarrow \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}).$$

5.2.2. *Hecke operators.* The Ran version of the Hecke stack, denoted $\mathrm{Hk}_{G, \mathrm{Ran} C}$, has R -points the groupoid of tuples $(\underline{c}, \mathcal{F}_1, \mathcal{F}_2, \beta)$ where $\underline{c} \in \mathrm{Ran} C(R)$, $\mathcal{F}_i \in \mathrm{Bun}_G(R)$, and β is an isomorphism $\mathcal{F}_1|_{C_R \setminus \underline{c}} \cong \mathcal{F}_2|_{C_R \setminus \underline{c}}$. There is a correspondence diagram

$$\begin{array}{ccc} & \mathrm{Hk}_{G, \mathrm{Ran} C} & \\ h_1 \swarrow & & \searrow \mathrm{pr} \times h_2 \\ \mathrm{Bun}_G & & \mathrm{Ran} C \times \mathrm{Bun}_G \end{array} \quad (5.2.1)$$

Convolution defines a map

$$\mathrm{Hk}_{G, \mathrm{Ran} C} \times_{h_2, \mathrm{Bun}_G, h_1} \mathrm{Hk}_{G, \mathrm{Ran} C} \rightarrow \mathrm{Hk}_{G, \mathrm{Ran} C}. \quad (5.2.2)$$

Pull-push via the correspondence diagram

$$\begin{array}{ccc} \mathrm{Hk}_{G, \mathrm{Ran} C} \times_{h_2, \mathrm{Bun}_G, h_1} \mathrm{Hk}_{G, \mathrm{Ran} C} & \longrightarrow & \mathrm{Hk}_{G, \mathrm{Ran} C} \times \mathrm{Hk}_{G, \mathrm{Ran} C} \\ \downarrow (5.2.2) & & \\ \mathrm{Hk}_{G, \mathrm{Ran} C} \times_{\mathrm{Ran} C} (\mathrm{Ran} C \times \mathrm{Ran} C) & & \\ \downarrow \mathrm{Id} \times \mathrm{union} & & \\ \mathrm{Hk}_{G, \mathrm{Ran} C} & & \end{array}$$

defines a monoidal structure on $\mathrm{Shv}(\mathrm{Hk}_{G, \mathrm{Ran} C})$.

⁹In the de Rham case, note that this is defined because Ψ is holonomic [Gai15, p.47].

The correspondence diagram (5.2.1) defines an action of the monoidal category $\mathrm{Shv}(\mathrm{Hk}_{G, \mathrm{Ran} C})$ on $\mathrm{Shv}(\mathrm{Bun}_G)$. We refer to this as the ‘‘action of Hecke operators’’.

5.2.3. *Geometric Satake.* The Geometric Satake equivalence [MV07] induces a functor

$$\mathrm{QCoh}((\check{B}\check{G})_{\mathrm{Ran} C}) \xrightarrow{\mathrm{Sat}} \mathrm{Shv}(\mathrm{Hk}_{G, \mathrm{Ran} C}). \quad (5.2.3)$$

The *Hecke compatibility property* of the Geometric Langlands equivalences stipulates that \mathbb{L}_G intertwines the action of $\mathrm{QCoh}((\check{B}\check{G})_{\mathrm{Ran} C})$ on $\mathrm{Dmod}(\mathrm{Bun}_G)$ (via (5.2.3) and Hecke operators) and on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ (via spectral localization).

5.2.4. *Graded version.* By the discussion in §4.3, for a Λ^+ -graded object in $\mathrm{QCoh}((\check{B}\check{G})_{\mathrm{Ran} C})$, there is an alternative formulation of spectral localization in terms of graded configuration space $\mathrm{Div}^{\Lambda^+} C$ instead of $\mathrm{Ran} C$.

There is a parallel story for the Hecke action, using $\mathrm{Hk}_{G, \mathrm{Div}^{\Lambda^+} C}$ instead of $\mathrm{Hk}_{G, \mathrm{Ran} C}$. The stack $\mathrm{Hk}_{G, \mathrm{Div}^{\Lambda^+} C}$ has R -points the groupoid of tuples $(D, \mathcal{F}_1, \mathcal{F}_2, \beta)$ where $D \in \mathrm{Div}^{\Lambda^+} C(R)$, $\mathcal{F}_i \in \mathrm{Bun}_G(R)$, and β is an isomorphism $\mathcal{F}_1|_{C_R \setminus D} \cong \mathcal{F}_2|_{C_R \setminus D}$. The graded version of (5.2.3) is a functor

$$\mathrm{QCoh}((\check{B}\check{G})_{\mathrm{Div}^{\Lambda^+} C}) \xrightarrow{\mathrm{Sat}} \mathrm{Shv}(\mathrm{Hk}_{G, \mathrm{Div}^{\Lambda^+} C}). \quad (5.2.4)$$

Hecke compatibility then says that \mathbb{L}_G intertwines the action of $\mathrm{QCoh}((\check{B}\check{G})_{\mathrm{Div}^{\Lambda^+} C})$ on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$, via spectral localization and tensoring, with its action on $\mathrm{Dmod}(\mathrm{Bun}_G)$ via (5.2.4) and Hecke operators.

Remark 5.2.1. By definition, the spectral localization functor $\mathrm{Loc}^{\mathrm{spec}}$ takes a factorization (co)algebra in $\mathrm{QCoh}((\check{B}\check{G})_{\mathrm{Div}^{\Lambda^+} C})$ to its relative factorization homology over $\mathrm{Loc}_{\check{G}}$. Since the factorization homology takes the unit to the unit (cf. [FG12, Remark 6.4.6]), it will be convenient to formally define factorization homology on *unital augmented* (co)algebras by taking the direct sum with a copy of the unit (see §4.1).

Example 5.2.2 (Hecke compatibility for the standard representation). Let $G = \mathrm{GL}_r$, so $\check{G} = \mathrm{GL}_r$. We denote by $\mathrm{Std} \in \mathrm{Rep}(\check{G})$ the standard representation. Consider

$$\overline{\mathrm{Sym}}^\bullet(\mathrm{Std}) := \bigoplus_{n>0} \mathrm{Sym}^n(\mathrm{Std}) \in \mathrm{Rep}(\check{G}), \quad (5.2.5)$$

the free (non-unital) graded cocommutative coalgebra in $\mathrm{Rep}(\check{G})$ generated by Std in graded degree 1. There is a corresponding (under Theorem 4.3.4) cocommutative factorization coalgebra

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std})) = \left\{ \mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std})) \in \mathrm{QCoh}((\check{B}\check{G})_{C^{(n)}}) \right\}_{n>0}. \quad (5.2.6)$$

By Example 4.3.5, it is given explicitly by

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std}))_n \cong \pi_n!(\mathrm{Std}^{\boxtimes n})_{\Sigma_n} \quad (5.2.7)$$

where $\pi_n: C^n \rightarrow C^{(n)}$ is the natural quotient and Σ_n is the symmetric group on n elements.

Then the universally twisted (cf. §4.5.3) version

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std}))^{\mathrm{univ}} =: \{ \mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std}))_n^{\mathrm{univ}} \in C_{\mathrm{dR}}^{(n)} \times \mathrm{Loc}_{\check{G}} \}_{n>0} \quad (5.2.8)$$

is described as follows. We have a universal \check{G} -local system on $C \times \mathrm{Loc}_{\check{G}}$, and composing it with the standard representation gives $L^{\mathrm{univ}} \in \mathrm{QCoh}(C_{\mathrm{dR}} \times \mathrm{Loc}_{\check{G}})$. Taking the n -fold exterior product of L^{univ} in the C_{dR} factor gives $L_n^{\mathrm{univ}} \in \mathrm{QCoh}(C_{\mathrm{dR}}^n \times \mathrm{Loc}_{\check{G}})$. Then for $\pi_n: C^n \rightarrow C^{(n)}$ the obvious projection, we have

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std}))_n^{\mathrm{univ}} \cong (\pi_n! L_n^{\mathrm{univ}})_{\Sigma_n} \in \mathrm{QCoh}(C_{\mathrm{dR}}^{(n)} \times \mathrm{Loc}_{\check{G}}). \quad (5.2.9)$$

Note that under the Geometric Satake equivalence for GL_r , each $\mathrm{Sym}^d(\mathrm{Std}) \in \mathrm{Rep}(\check{G})$ corresponds to a shift of the *constant sheaf* on its support in Hk_G for every d . (This is well-known; the key point is that each non-zero weight space of $\mathrm{Sym}^d(\mathrm{Std})$ is one-dimensional.) Consequently, L_n^{univ} corresponds to the constant sheaf on Hk_{G, C^n} normalized to be relatively perverse (in the sense of [HS23]) over C^n , namely $k\langle n(r-1) \rangle$. Hence (5.2.4) takes

$$\left\{ \mathrm{Fact}(\overline{\mathrm{Sym}}^\bullet(\mathrm{Std}))_n \in \mathrm{QCoh}((\check{B}\check{G})_{C^{(n)}}) \right\}_{n>0} \mapsto \left\{ k\langle n(r-1) \rangle \in \mathrm{Shv}(\mathrm{Hk}_{G, C^{(n)}}) \right\}_{n>0}. \quad (5.2.10)$$

Now let $\mathrm{Sym}^\bullet(\mathrm{Std}) \in \mathrm{Rep}(\check{G})$ be the *unital* cocommutative coalgebra $k \oplus \overline{\mathrm{Sym}}^\bullet(\mathrm{Std})$. Then we may formally set

$$\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std})) = \{\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))_n \in \mathrm{QCoh}((B\check{G})_{C^{(n)}})\}_{n \geq 0} \quad (5.2.11)$$

to be

$$\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))_n := \begin{cases} \pi_n!(\mathrm{Std}^{\boxtimes n})_{\Sigma_n} \in \mathrm{QCoh}((B\check{G})_{C^{(n)}}) & n > 0, \\ \mathcal{O}_{\mathrm{pt}} \in \mathrm{QCoh}(\mathrm{pt}) & n = 0. \end{cases} \quad (5.2.12)$$

We also formally define the universally twisted version

$$\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))^{\mathrm{univ}} = \{\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))_n^{\mathrm{univ}} \in \mathrm{QCoh}(C_{\mathrm{dR}}^{(n)} \times \mathrm{Loc}_{\check{G}})\}_{n \geq 0} \quad (5.2.13)$$

to be

$$\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))_n^{\mathrm{univ}} = \begin{cases} (\pi_n! L_n^{\mathrm{univ}})_{\Sigma_n} \in \mathrm{QCoh}(C_{\mathrm{dR}}^{(n)} \times \mathrm{Loc}_{\check{G}}) & n > 0, \\ \mathcal{O}_{\mathrm{Loc}_{\check{G}}} \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}) & n = 0. \end{cases} \quad (5.2.14)$$

By definition, we have

$$\mathrm{Loc}^{\mathrm{Spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) := \mathrm{pr}_{2*} \left(\prod_{n \geq 0} \mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))_n^{\mathrm{univ}} \right) \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}). \quad (5.2.15)$$

Let $\mathrm{Hk}_{G,C^{(n)}}^{\mathrm{Std}}$ be the stack of tuples $(D, \mathcal{F}_1, \mathcal{F}_2, \beta)$ where $\beta: \mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ is an upper modification such that $\mathrm{coker}(\beta)$ is a line bundle over D . Let

$$\mathrm{Hk}_{G, \mathrm{Sym}^\bullet C} := \bigcup_{n \geq 0} \mathrm{Hk}_{G, C^{(n)}}^{\mathrm{Std}}. \quad (5.2.16)$$

The object of $\mathrm{Shv}(\mathrm{Hk}_{G, \mathrm{Sym}^\bullet C})$ corresponding to (5.2.12) under the Geometric Satake equivalence is

$$\mathrm{Sat}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))_n) \cong \begin{cases} k\langle n(r-1) \rangle \in \mathrm{Shv}(\mathrm{Hk}_{G, C^{(n)}}^{\mathrm{Std}}) & n > 0, \\ k \in \mathrm{Shv}(\mathrm{Hk}_{G, C^{(0)}} \cong \mathrm{Bun}_G) & n = 0. \end{cases} \quad (5.2.17)$$

Hecke compatibility then stipulates that the tensoring action of $\mathrm{Loc}^{\mathrm{Spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}})$ on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ is intertwined under \mathbb{L}_G with the action of the kernel sheaf

$$\prod_{n \geq 0} k\langle n(r-1) \rangle \in \mathrm{Shv}\left(\bigcup_{n \geq 0} \mathrm{Hk}_{G, C^{(n)}}^{\mathrm{Std}}\right) \quad (5.2.18)$$

on $\mathrm{Dmod}(\mathrm{Bun}_G)$ via convolving on the correspondence

$$\begin{array}{ccc} & \mathrm{Hk}_{G, \mathrm{Sym}^\bullet C} := \bigcup_{n \geq 0} \mathrm{Hk}_{G, C^{(n)}}^{\mathrm{Std}} & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{array} \quad (5.2.19)$$

5.3. Eisenstein compatibility. Let $P \subset G$ be a standard parabolic subgroup with Levi quotient M . This induces a standard parabolic subgroup $\check{P} \subset \check{G}$ with Levi quotient \check{M} . The Eisenstein compatibility property of \mathbb{L}_G is discussed in [Gai15, §6]. We will briefly summarize it.

With reference to the correspondence diagram

$$\begin{array}{ccc} & \mathrm{Bun}_P & \\ q \swarrow & & \searrow p \\ \mathrm{Bun}_M & & \mathrm{Bun}_G \end{array}$$

the *Eisenstein functor* [Gai15, §6.3]

$$\mathrm{Eis}_P: \mathrm{Shv}(\mathrm{Bun}_M) \rightarrow \mathrm{Shv}(\mathrm{Bun}_G)$$

is defined to be $\mathrm{Eis}_P := p!q^*$.

With reference to the correspondence diagram

$$\begin{array}{ccc} & \text{Loc}_{\check{P}} & \\ \check{q} \swarrow & & \searrow \check{p} \\ \text{Loc}_{\check{M}} & & \text{Loc}_{\check{G}} \end{array}$$

the spectral Eisenstein functor

$$\text{Eis}_{\check{P}}^{\text{spec}} : \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{M}}) \rightarrow \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$$

is defined to be $\text{Eis}_{\check{P}}^{\text{spec}} := \check{p}_* \circ \check{q}^!$. Note that our normalization is slightly different from [Gai15, §6.4], in that we use $\check{q}^!$ instead of \check{q}^* ; the two differ by tensoring by an invertible sheaf (namely, a shift of the determinant of $\mathbf{T}_{\check{q}}$) since \check{q} is quasi-smooth.

Then the diagram

$$\begin{array}{ccc} \text{Dmod}(\text{Bun}_M) & \xrightarrow[\sim]{\mathbb{L}_M} & \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{M}}) \\ \downarrow \text{Eis}_{\check{P}} & & \downarrow \text{Eis}_{\check{P}}^{\text{spec}} \\ \text{Dmod}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}) \end{array}$$

commutes up to tensoring by an explicit invertible sheaf in either column; the details are pinned down in [Gai15, §6.4.8].

Notation 5.3.1. Objects which do not lie in $\text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$ arise naturally (an example that will come up for us is the IndCoh -pushforward of \mathcal{O}_{pt} along the map $\text{pt} \rightarrow \text{Loc}_{\check{G}}$ corresponding to the trivial local system). There is a co-localization functor $\text{IndCoh}(\text{Loc}_{\check{G}}) \rightarrow \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$, which is right adjoint to the natural inclusion $\text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}) \hookrightarrow \text{IndCoh}(\text{Loc}_{\check{G}})$. By convention, whenever we write $\text{Eis}^{\text{spec}}(\dots) \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$, we are automatically applying this co-localization functor (if necessary). This has the same effect as applying the co-localization to the input object of $\text{IndCoh}(\text{Loc}_{\check{M}})$.

Part 2. Examples

6. TATE PERIOD

6.1. Setup. We study an ansatz that we call the ‘‘Tate period’’, as it corresponds to the integral representations in Tate’s thesis [Tat67] for the L -functions attached to Hecke characters.

6.1.1. Automorphic side. Let $X = \mathbf{A}^1$, with $G = \text{GL}_1$ acting by the standard representation.

6.1.2. Spectral side. Let $\check{X} = \mathbf{A}^1$, with $\check{G} = \text{GL}_1$ acting by the standard representation.

6.1.3. \mathbb{G}_{gr} -action. According to [BZSV23, Example 12.6.7], the ‘‘neutral’’ action of \mathbb{G}_{gr} is the standard scaling action on both X and \check{X} , so the ‘‘un-normalized’’ action of \mathbb{G}_{gr} is trivial on both X and \check{X} (cf. [BZSV23, Example 12.6.7]). Below we will take the un-normalized action of \mathbb{G}_{gr} .

6.1.4. Automorphic period. By definition, Bun_G^X is the (derived) mapping stack

$$\text{Bun}_G^X := \text{Map}(C, \mathbf{A}^1/\mathbf{G}_m)$$

with R -points the groupoid of pairs (\mathcal{L}, s) where:

- \mathcal{L} is a line bundle on C_R .
- s is a section of \mathcal{L} on C_R .

Remark 6.1.1. The derived structure plays no role until we study the functional equation in §6.5, so we will pass to the classical truncation until then.

The map $\pi^{\text{aut}} : \text{Bun}_G^X \rightarrow \text{Bun}_G$ forgets the datum of s . The automorphic period sheaf is

$$\mathcal{P}_X := \pi_1^{\text{aut}}(k_{\text{Bun}_G^X}) \in \text{Shv}(\text{Bun}_G).$$

In the language of [BZSV23], this is the un-normalized period sheaf for the un-normalized action of \mathbb{G}_{gr} .

6.1.5. *Spectral period.* By definition, $\mathrm{Loc}_{\tilde{G}}$ is the derived mapping stack

$$\mathrm{Loc}_{\tilde{G}}^{\tilde{X}} := \mathrm{Map}(C_{\mathrm{dR}}, \mathbf{A}^1/\mathbf{G}_m)$$

is the derived stack with R -points the groupoid of pairs (\mathcal{L}, s) where:

- $L \in \mathrm{Loc}_1(R)$ is a line bundle on C_R with a flat connection along C (informally speaking, an R -family of rank 1 local systems on C).
- $s \in \mathrm{Sect}(C_{\mathrm{dR}}, L)(R)$, or informally speaking a flat section of L .

The map $\pi_*^{\mathrm{spec}}: \mathrm{Loc}_{\tilde{G}}^{\tilde{X}} \rightarrow \mathrm{Loc}_{\tilde{G}}$ forgets the datum of s . The (co-localized) spectral period sheaf is

$$\mathcal{L}_{\tilde{X}} := (\pi_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_{\tilde{G}}^{\tilde{X}}})^{\vee} \in \mathrm{QCoh}(\mathrm{Loc}_{\tilde{G}}).$$

Note that $\pi_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_{\tilde{G}}^{\tilde{X}}} = (\pi_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_{\tilde{G}}^{\tilde{X}}})^{\vee}$ in this case, since the action of \mathbb{G}_{gr} is trivial. In the language of [BZSV23], this is the un-normalized \mathcal{L} -sheaf for the un-normalized action of \mathbb{G}_{gr} .

6.1.6. *Duality.* The main theorem of this section is the following:

Theorem 6.1.2 (Geometric Langlands duality for the Tate period). *The Geometric Langlands equivalence for $G = \mathrm{GL}_1$ takes*

$$\mathrm{Dmod}(\mathrm{Bun}_G) \ni \mathcal{P}_{X(g-1)} \xrightarrow{\mathbb{L}_G} \mathcal{L}_{\tilde{X}}^{\vee} \in \mathrm{IndCoh}(\mathrm{Loc}_{\tilde{G}}).$$

It is explained in [BZSV23, §12.2.3] that Theorem 6.1.2 is equivalent to [BZSV23, Conjecture 12.1.1] for the neutral action of \mathbb{G}_{gr} . (The duality involution in *loc. cit.* disappears under the translation from right actions to left actions – see [CV24, Appendix A].)

6.2. **Spectral side.** Let $Z \hookrightarrow \mathrm{Loc}_{\tilde{G}}^{\tilde{X}}$ be the zero section of π^{spec} and U be the open complement of Z . Note that U is isomorphic to $\mathrm{Map}(C_{\mathrm{dR}}, \mathbf{G}_m/\mathbf{G}_m) = \mathrm{pt}$, while Z is isomorphic to $\mathrm{Loc}_{\tilde{G}}$. We consider the exact triangle in $\mathrm{QCoh}(\mathrm{Loc}_{\tilde{G}})$ coming from filtering by sections with set-theoretic support in Z :

$$(\pi^{\mathrm{spec}})^{\wedge}_{Z*}(\omega_{\mathrm{Loc}_{\tilde{G}}^{\tilde{X}}}) \rightarrow \pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\tilde{G}}^{\tilde{X}}}) \rightarrow \pi_*^{\mathrm{spec}}|_U(\omega_{\mathrm{Loc}_{\tilde{G}}^{\tilde{X}}}|_U) \quad (6.2.1)$$

where we are using the notational conventions of §2.5.2.

6.2.1. *Open stratum.* We analyze the rightmost term of (6.2.1). It is evidently the direct image along $\mathrm{pt} \xrightarrow{\mathrm{triv}} \mathrm{Loc}_1$ at the trivial local system, i.e., the skyscraper sheaf $\mathcal{O}_{\mathrm{triv}}$ of the trivial local system.

6.2.2. *Closed stratum.* Before we analyze the leftmost term of (6.2.1), we establish some generalities.

Recall that an affine derived scheme S is n -coconnective if its homotopy groups vanish in degrees greater than n , and *eventually coconnective* if it is n -coconnective for some n [GR17a, Chapter 2, §1.2]. The definition of eventually coconnective derived stacks is a bit subtle: see [GR17a, Chapter 2, §2.6]. The significance for us is that if a derived stack S is eventually coconnective, then $\mathcal{O}_S \in \mathrm{Coh}(S)$, so $\mathrm{Perf}(S) \subset \mathrm{Coh}(S)$, which ind-extends to a natural embedding $\Xi: \mathrm{QCoh}(S) \hookrightarrow \mathrm{IndCoh}(S)$. In particular, for a finite type map of derived stacks $f: Y \rightarrow S$, the *relative dualizing complex* $\omega_{Y/S} := f^!(\mathcal{O}_S) \in \mathrm{IndCoh}(Y)$ is defined.

Lemma 6.2.1. *Let S be a locally finite type derived stack over a field and let $f: Y_0 \hookrightarrow Y$ be a closed embedding of finite type derived stacks over S . Let $\mathcal{Y}_0 := Y_{Y_0}^{\wedge}$ be the formal completion of Y along Y_0 and $\pi: \mathcal{Y} \rightarrow S$ be the restriction of π . Then for $\pi: Y \rightarrow S$ the structure map to S , we have isomorphisms*

$$\pi_{\mathcal{Y}_0*}^{\wedge}(\omega_Y) \cong \pi_*(\omega_{Y_0}) \in \mathrm{IndCoh}(S).$$

If S is eventually coconnective, so that $\omega_{Y_0/S} \in \mathrm{IndCoh}(Y)$ is defined, then we also have an isomorphism

$$\pi_{\mathcal{Y}_0*}^{\wedge}(\omega_{Y/S}) \cong \pi_*(\omega_{Y_0/S}) \in \mathrm{IndCoh}(S).$$

Proof. By definition, $\pi_{\mathcal{Y}_0*}^{\wedge}(\omega_Y)$ is the composition of π_* with the counit $\mathrm{IndCoh}(Y) \twoheadrightarrow \mathrm{IndCoh}_{Y_0}(Y) \hookrightarrow \mathrm{IndCoh}(Y)$. Letting $\hat{i}: \mathcal{Y} \rightarrow Y$ be the map induced by formal completion, the equivalence $\mathrm{IndCoh}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{IndCoh}_{Y_0}(Y)$ is implemented by the adjunction $(\hat{i}_*, \hat{i}^!)$, cf. §4.4.1. Under this equivalence, we have

$$\pi_{\mathcal{Y}_0*}^{\wedge}(\omega_Y) \cong \pi_* \hat{i}_* \hat{i}^!(\omega_Y) \cong \pi_*(\omega_Y) \in \mathrm{IndCoh}(S)$$

since $\hat{i}^! \omega_Y \cong \omega_Y$, and similarly for the relative dualizing complex if S is eventually coconnective. \square

We will use Proposition 4.5.2, with $Y := \tilde{X}$ and $y := 0 \hookrightarrow \mathbf{A}^1 = Y$, to calculate $(\pi^{\text{spec}})_{Z*}^{\wedge}(\omega_{\text{Loc}_{\tilde{X}}})$. For this we first need to compute the unital cocommutative coalgebra $\Gamma_0(\tilde{X}; \omega_{\tilde{X}})$. It will be convenient to establish a more general statement for future use.

Lemma 6.2.2. *Let S be an eventually coconnective locally finite type derived stack over k (hence of characteristic 0), \tilde{G} be a reductive group, and let $\pi: E \rightarrow S$ be a \tilde{G} -equivariant derived vector bundle associated to a perfect complex $\mathcal{E} \in \text{Perf}(S/\tilde{G})$ with tor-amplitude in $[0, \infty)$, where \tilde{G} acts trivially on S . We view $S \hookrightarrow E$ as the zero section. Equipping E with the scalar \mathbf{G}_m -action, we have a natural isomorphism*

$$\pi_{S*}^{\wedge}(\omega_{E/S}) \cong \text{Sym}^{\bullet}(\mathcal{E}) \in \text{QCoh}(S/\tilde{G})$$

as \mathbf{G}_m -graded unital cocommutative coalgebras. Here $\text{Sym}^{\bullet}(\mathcal{E}) := \bigoplus_{n \geq 0} \text{Sym}^n(\mathcal{E})$ is the free graded \tilde{G} -equivariant unital cocommutative coalgebra over S generated by \mathcal{E} (placed in graded degree 1).

Remark 6.2.3. Contrast the statement of Lemma 6.2.2 with the tautological statement

$$\pi_*(\mathcal{O}_E) \cong \text{Sym}^{\bullet}(\mathcal{E}^{\vee}) \quad (6.2.2)$$

the free graded unital commutative algebra in $\text{Rep}(\tilde{G})$ generated by \mathcal{E}^{\vee} (placed in graded degree -1), which is the dual \mathbf{G}_m -graded unital commutative algebra. The appearance of this duality is somehow crucial.

Proof. Let $z: S \hookrightarrow E$ be the zero-section. Let

$$\begin{array}{ccccccc} S = S^{(0)} & \hookrightarrow & S^{(1)} & \hookrightarrow & \dots & \hookrightarrow & E_S^{\wedge} \hookrightarrow E \\ \downarrow \pi^{(0)} & & \downarrow \pi^{(1)} & & \downarrow \pi^{(2)} & & \downarrow \pi \\ S & \longleftarrow & S & \longleftarrow & S & \longleftarrow & S \longleftarrow S \end{array}$$

be the system of derived infinitesimal neighborhoods of $z: S \hookrightarrow E$ as in (4.4.1), with $E_S^{\wedge} := \text{colim}_n S^{(n)}$. Then by Lemma 6.2.1 we have $\pi_{S*}^{\wedge}(\omega_{E/S}) \cong \pi_*(\omega_{E_S^{\wedge}/S})$, and by definition of ind-coherent sheaves on ind-schemes we have (cf. §4.4.1)

$$\pi_*(\omega_{E_S^{\wedge}/S}) = \text{colim}_n \pi_*^{(n)}(\omega_{S^{(n)}/S}) \in \text{IndCoh}(S). \quad (6.2.3)$$

Since each $\pi^{(n)}$ is proper, Grothendieck duality applies to give a trace map $\pi_*^{(n)}(\omega_{S^{(n)}/S}) \rightarrow \mathcal{O}_S$. Taking the colimit in n , we get a trace map

$$\pi_{S*}^{\wedge}(\omega_{E_S^{\wedge}/S}) \rightarrow \mathcal{O}_S. \quad (6.2.4)$$

This induces a pairing

$$\pi_{S*}^{\wedge}(\omega_{E_S^{\wedge}/S}) \otimes \pi_*(\mathcal{O}_E) \rightarrow \pi_{S*}^{\wedge}(\omega_{E_S^{\wedge}/S}) \xrightarrow{(6.2.4)} \mathcal{O}_S, \quad (6.2.5)$$

which we claim is perfect in each degree of the grading. This can be checked locally on S . Affine-locally on S we may find a presentation for \mathcal{E} as a complex of finite projective modules, which necessarily splits, so we may assume that $\mathcal{E} \cong \bigoplus_{i \geq 0} \mathcal{E}^i[-i]$ where each \mathcal{E}^i is a locally free coherent sheaf, which induces a splitting $E \cong \bigoplus_{i \geq 0} E^i[-i]$. This then reduces to the case where E is concentrated in a single degree $i \geq 0$.

If $i > 0$, then $\pi: E \rightarrow S$ is proper, as the classical truncation of π is an isomorphism, and the claim follows directly from Grothendieck duality. We therefore reduce to the case $i = 0$, where E is a classical vector bundle over S . Returning to the situation of (6.2.3), since each $\pi^{(n)}$ is proper, Grothendieck duality applies to give a natural isomorphism

$$\pi_*^{(n)}(\omega_{S^{(n)}/S}) \cong (\pi_*^{(n)} \mathcal{O}_{S^{(n)}})^{\vee} \in \text{IndCoh}(S).$$

Since z is a regular embedding, $\pi_*^{(n)} \mathcal{O}_{S^{(n)}}$ is the quotient of $\text{Sym}^{\bullet}(\mathcal{E}^{\vee})$ by the n th power of its augmentation ideal. Dualizing this and taking the colimit over n completes the proof that (6.2.5) is a perfect pairing.

Finally we observe that the commutative coalgebra structure on $\pi_{S*}^{\wedge}(\omega_{E_S^{\wedge}/S})$ is graded dual to the commutative algebra structure on $\pi_*(\mathcal{O}_E)$ under (6.2.5), so we conclude from (6.2.2). \square

Corollary 6.2.4. *The cocommutative coalgebra $\Gamma_0(\tilde{X}; \omega_{\tilde{X}}) \in \text{Rep}(\text{GL}_1)$ identifies with $\text{Sym}^{\bullet}(\text{Std})$, the free unital cocommutative coalgebra in $\text{Rep}(\text{GL}_1)$ on Std placed in graded degree 1.*

Proof. Apply Lemma 6.2.2 with $\tilde{G} = \text{GL}_1$, $S = \text{pt}$, $X = \tilde{X}$. \square

Proposition 6.2.5. *Let $\text{Fact}(\text{Sym}^\bullet(\text{Std}))$ be the graded unital cocommutative factorization coalgebra (5.2.12). Then we have an isomorphism*

$$(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}}) \cong \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet(\text{Std}))) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}).$$

Proof. Applying Proposition 4.5.2 with y being the origin inside the \check{G} -scheme Std , placed in graded degree 1 (corresponding to the standard scaling action of \mathbb{G}_{gr}), we obtain an isomorphism

$$\overline{(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}})} \cong (\text{pr}_{2*} \text{Fact}(\overline{\Gamma}_0(\check{X}; \omega_{\check{X}}))^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}). \quad (6.2.6)$$

In Corollary 6.2.4 we saw that $\overline{\Gamma}_0(\check{X}; \omega_{\check{X}}) \cong \overline{\text{Sym}}^\bullet(\text{Std}) \in \text{Rep}(\check{G})^{\mathbf{Z}_{>0}}$. Inserting this into (6.2.6) yields an isomorphism

$$\overline{(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}})} \cong (\text{pr}_{2*} \text{Fact}(\overline{\text{Sym}}^\bullet(\text{Std}))^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}). \quad (6.2.7)$$

Recall from §4.1 that we may pass between non-unital cocommutative coalgebras and augmented unital graded cocommutative coalgebras by taking the direct sum with the unit. Applying this to (6.2.7), we obtain an isomorphism

$$(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}}) \cong (\text{pr}_{2*} \text{Fact}(\text{Sym}^\bullet(\text{Std}))^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}) \quad (6.2.8)$$

where the meaning of $\text{Fact}(\text{Sym}^\bullet(\text{Std}))^{\text{univ}}$ is as in (5.2.14). We conclude by using that

$$\text{pr}_{2*} \text{Fact}(\text{Sym}^\bullet(\text{Std}))^{\text{univ}} \cong \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet(\text{Std}))) \in \text{QCoh}(\text{Loc}_{\check{G}})$$

by definition (cf. (5.2.15)). □

6.2.3. Extension class. Putting together §6.2.1 and §6.2.2, we have produced an exact triangle in $\text{IndCoh}(\text{Loc}_{\check{G}})$,

$$\text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet(\text{Std}))) \otimes \omega_{\text{Loc}_{\check{G}}} \rightarrow \pi_{\text{spec}*}(\omega_{\text{Loc}_{\check{G}}^{\check{X}}}) \rightarrow \mathcal{O}_{\text{triv}}. \quad (6.2.9)$$

Next we want to understand the connecting homomorphism

$$\mathcal{O}_{\text{triv}} \rightarrow \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet(\text{Std}))) \otimes \omega_{\text{Loc}_{\check{G}}}[1] \cong \pi_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}})[1].$$

Let us factor $\text{triv}: \text{pt} \rightarrow \text{Loc}_{\check{G}}$ as a composition

$$\begin{array}{ccc} \text{pt} & & \\ q \downarrow & \searrow^{\text{triv}} & \\ \mathbf{BG}_m & \xrightarrow{\iota} & \text{Loc}_{\check{G}} \end{array}$$

Since ι is a closed embedding, we have an adjunction $(\iota_*, \iota^!)$, hence

$$\begin{aligned} \text{Hom}_{\text{Loc}_{\check{G}}}(\mathcal{O}_{\text{triv}}, (\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}})[1]) &\cong \text{Hom}_{\text{Loc}_{\check{G}}}(\iota_* q_* \mathcal{O}_{\text{pt}}, (\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}})[1]) \\ &\cong \text{Hom}_{\mathbf{BG}_m}(q_* \mathcal{O}_{\text{pt}}, \iota^!(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}^{\check{X}}})[1]). \end{aligned} \quad (6.2.10)$$

We have a derived Cartesian square

$$\begin{array}{ccc} V & \xrightarrow{\iota} & \text{Loc}_{\check{G}}^{\check{X}} \\ \downarrow \pi & & \downarrow \pi^{\text{spec}} \\ \mathbf{BG}_m & \xrightarrow{\iota} & \text{Loc}_{\check{G}} \end{array} \quad (6.2.11)$$

where $V := \text{Spec}(\text{Sym}^\bullet(\Gamma_{\text{dR}}(C; k)^\vee))$ is the derived vector bundle associated to the perfect complex $\Gamma_{\text{dR}}(C; k)$ over k . (In fact, the argument of [FYZ21, Proposition 5.34] shows that $\text{Loc}_{\check{G}}^{\check{X}}$ is the derived vector bundle over

$\mathrm{Loc}_{\check{G}}$ associated to the perfect complex $L \mapsto \Gamma_{\mathrm{dR}}(C; L)$.) We may further base change along its zero-section $Z \hookrightarrow \mathrm{Loc}_{\check{G}}^{\check{X}}$ as well as the formal completion $\mathcal{Z} \rightarrow \mathrm{Loc}_{\check{G}}^{\check{X}}$ thereof, giving a tower of derived Cartesian squares

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\iota} & \mathcal{Z} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\iota} & \mathrm{Loc}_{\check{G}}^{\check{X}} \\ \downarrow \pi & & \downarrow \pi^{\mathrm{spec}} \\ \mathbf{BG}_m & \xrightarrow{\iota} & \mathrm{Loc}_{\check{G}} \end{array} \quad (6.2.12)$$

We abuse notation by calling all the horizontal maps ι . In particular, from the outer derived Cartesian square

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\iota} & \mathcal{Z} \\ \downarrow \pi & & \downarrow \pi^{\mathrm{spec}} \\ \mathbf{BG}_m & \xrightarrow{\iota} & \mathrm{Loc}_{\check{G}} \end{array} \quad (6.2.13)$$

we get a base change natural isomorphism

$$\iota^! \pi_*^{\mathrm{spec}}(\omega_{\mathcal{Z}}) \cong \pi_* \iota^!(\omega_{\mathcal{Z}}) \cong \pi_*(\omega_{\mathcal{V}}) \in \mathrm{QCoh}(\mathbf{BG}_m). \quad (6.2.14)$$

Inserting this into (6.2.10) and recalling from Lemma 6.2.1 that $\pi_*^{\mathrm{spec}}(\omega_{\mathcal{Z}}) \cong (\pi^{\mathrm{spec}})_{Z^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Loc}_{\check{G}}}(\mathcal{O}_{\mathrm{triv}}, (\pi^{\mathrm{spec}})_{Z^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})[1]) \cong \mathrm{Hom}_{\mathbf{BG}_m}(q_* \mathcal{O}_{\mathrm{pt}}, \pi_* \omega_{\mathcal{V}}[1]). \quad (6.2.15)$$

Recall that the functor q^* induces $\mathrm{QCoh}(\mathbf{BG}_m) \xrightarrow{\sim} \mathrm{Rep}(\mathbf{G}_m)$. Note that $q^! \cong q^*[1]$ since q is a \mathbf{G}_m -torsor (hence smooth of relative dimension 1, with trivial relative dualizing sheaf), so by base change and Lemma 6.2.2 we have

$$q^* \pi_*(\omega_{\mathcal{V}}) \cong \mathrm{Sym}^{\bullet}(\Gamma_{\mathrm{dR}}(C; k))[-1]$$

where the \mathbf{G}_m -action is via the natural scaling action on $\Gamma_{\mathrm{dR}}(C, k)$. Putting this into (6.2.15) gives

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Loc}_{\check{G}}}(\mathcal{O}_{\mathrm{triv}}, (\pi^{\mathrm{spec}})_{Z^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})[1]) &\cong \mathrm{Hom}_{\mathbf{BG}_m}(q_* \mathcal{O}_{\mathrm{pt}}, \pi_* \omega_{\mathcal{V}}[1]) \\ &\cong \mathrm{Hom}_{\mathbf{G}_m}(\mathcal{O}(\mathbf{G}_m), \mathrm{Sym}^{\bullet}(\Gamma_{\mathrm{dR}}(C; k))[-1][1]) \\ &\cong \prod_{n \geq 0} \mathrm{H}_{\mathrm{dR}}^0(C; k). \end{aligned} \quad (6.2.16)$$

Hence the isomorphism class of the underlying lying object $\pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}) \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ is determined by the extension class $[\pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})]$ of (6.2.1) in each of the 1-dimensional spaces $\mathrm{H}_{\mathrm{dR}}^0(C; k)$ indexed by $n \geq 0$ in (6.2.16), modulo the action of k^{\times} . In other words, we have pinned down the isomorphism class of $\pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}) \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ once we know whether or not the extension (6.2.1) is split for each n . This is addressed by the following Lemma.

Lemma 6.2.6. *For each $n \geq 0$, the extension (6.2.1) maps to the class of $1 \in \mathrm{H}_{\mathrm{dR}}^0(C; k)$ in (6.2.16).*

Proof. The statement can be proved after base change along $\iota: \mathbf{BG}_m \rightarrow \mathrm{Loc}_{\check{G}}$. Examining the computation of extensions, we see that it also suffices to prove for the classical truncation of V_{cl} of \mathcal{V} (since the derived part only contributes in higher degrees), which is $\mathbf{A}^1/\mathbf{G}_m$ with the standard action of \mathbf{G}_m . Its relative dualizing sheaf over \mathbf{BG}_m is $\omega_{V_{\mathrm{cl}}} \cong \mathcal{O}_{V_{\mathrm{cl}}}[1]$. Then the statement is that in the exact triangle

$$\pi_{0*}^{\wedge}(\omega_{V_{\mathrm{cl}}}) \rightarrow \pi_*(\omega_{V_{\mathrm{cl}}}) \rightarrow \mathcal{O}_{\mathrm{triv}} \in \mathrm{QCoh}(\mathbf{BG}_m)$$

the map $\mathrm{R}^0 \pi_*(\mathcal{O}_{\mathrm{triv}}) \rightarrow \mathrm{R}^1 \pi_{0*}^{\wedge}(\omega_{V_{\mathrm{cl}}})$ is an isomorphism in the n graded component for each $n \geq 0$; and in fact is the identity map with respect to the canonical identification $\mathrm{R}^1 \pi_{0*}^{\wedge}(\omega_{V_{\mathrm{cl}}}) \cong k$. Applying $q^!$ and using base change, this reduces to the explicit computation of the exact triangle

$$\Gamma_0(\mathbf{A}^1; \omega_{\mathbf{A}^1}) \rightarrow \Gamma(\mathbf{A}^1; \omega_{\mathbf{A}^1}) \rightarrow \Gamma(\mathbf{G}_m; \omega_{\mathbf{G}_m}).$$

Since \mathbf{A}^1 and \mathbf{G}_m are smooth, we have compatible identifications $\omega_{\mathbf{A}^1} \cong \mathcal{O}_{\mathbf{A}^1}[1]$ and $\omega_{\mathbf{G}_m} \cong \mathcal{O}_{\mathbf{G}_m}[1]$. The assertion that the map is an isomorphism clear from the explicit identification of $\Gamma(\mathbf{A}^1; \mathcal{O}_{\mathbf{A}^1}) \rightarrow \Gamma(\mathbf{G}_m; \mathcal{O}_{\mathbf{G}_m})$, which also provides the canonical identification $R^1\Gamma_0(\mathbf{A}^1; \omega_{\mathbf{A}^1}) \cong \text{Std}$, and then the statement that the extension class is the identity is tautological. \square

6.3. Automorphic side. We let $i: Z \hookrightarrow \text{Bun}_G^X$ be the zero section and $j: U \hookrightarrow \text{Bun}_G^X$ its open complement.

We filter $\pi_!^{\text{aut}}(\underline{k}_{\text{Bun}_G^X})$ using the open-closed exact triangle in $\text{Shv}(\text{Bun}_G^X)$,

$$j_! \underline{k}_U \rightarrow \underline{k}_{\text{Bun}_G^X} \rightarrow i_* \underline{k}_Z, \quad (6.3.1)$$

which after applying $\pi_!^{\text{aut}}$ gives an exact triangle in $\text{Shv}(\text{Bun}_G)$,

$$\pi_!^{\text{aut}} j_!(\underline{k}_U) \rightarrow \pi_!^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \pi_!^{\text{aut}} i_*(\underline{k}_Z). \quad (6.3.2)$$

6.3.1. Closed stratum. Since the π^{aut} restricts to an isomorphism $Z \xrightarrow{\sim} \text{Bun}_G$, we have

$$\pi_!^{\text{aut}}(\underline{k}_Z) \cong \underline{k}_{\text{Bun}_G} \in \text{Shv}(\text{Bun}_G). \quad (6.3.3)$$

6.3.2. Open stratum. Recall that for $G = \text{GL}_1$, we have $\text{Bun}_G = \coprod_{d \in \mathbf{Z}} \text{Bun}_G^d$ where Bun_G^d is the connected component parametrizing line bundles of degree d .

The intersection $U \cap \text{Bun}_G^d$, which parametrizes degree d line bundles plus a non-zero section, is isomorphic to $C^{(d)}$, by the map sending a section to its divisor. Hence we have

$$U \cong \text{Sym}^\bullet C = \coprod_{d \geq 0} C^{(d)}.$$

The restriction $\pi^{\text{aut}}|_U: U \rightarrow \text{Bun}_G$ is a version of the Abel-Jacobi map: it sends a divisor D to the line bundle $\mathcal{O}(D)$. Therefore we denote it $\text{AJ}: U \rightarrow \text{Bun}_G$, and we write AJ^d for the restriction to the connected component $C^{(d)} \subset U$, which lands in Bun_G^d . Hence we have

$$\pi_!^{\text{aut}}(\underline{k}_U) \cong \text{AJ}_!(\underline{k}_U) \cong \text{AJ}_!(\underline{k}_{\text{Sym}^\bullet C}).$$

6.3.3. Extension class. We have just produced an exact triangle in $\text{Shv}(\text{Bun}_G)$:

$$\text{AJ}_!(\underline{k}_{\text{Sym}^\bullet C}) \rightarrow \pi_!^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \underline{k}_{\text{Bun}_G}. \quad (6.3.4)$$

To analyze the extension class of (6.3.2), we will use a trick that was explained to us by Akshay Venkatesh. Consider the exact triangle in $\text{Shv}(\text{Bun}_G^X)$,

$$i_! i^!(\underline{k}_{\text{Bun}_G^X}) \rightarrow \underline{k}_{\text{Bun}_G^X} \rightarrow j_* j^*(\underline{k}_{\text{Bun}_G^X}). \quad (6.3.5)$$

Now apply π_*^{aut} to (6.3.5). Since i is a closed embedding, we have $i_! = i_*$, so we get an exact triangle in $\text{Shv}(\text{Bun}_G)$,

$$\pi_*^{\text{aut}} i_* i^!(\underline{k}_{\text{Bun}_G^X}) \rightarrow \pi_*^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \pi_*^{\text{aut}} j_* j^*(\underline{k}_{\text{Bun}_G^X}). \quad (6.3.6)$$

Under the identification $\pi^{\text{aut}}: Z \cong \text{Bun}_G$, we have $\pi^{\text{aut}} \circ i = \text{Id}$. There is a \mathbf{G}_m -action on Bun_G^X contracting the fibers to the zero section, which by the Contraction Principle [DG15, Theorem C.5.3] implies that the natural transformation $i^! \rightarrow \pi_!^{\text{aut}}$ is an isomorphism of functors from \mathbf{G}_m -monodromic sheaves on Bun_G^X to sheaves on Bun_G . Putting this into (6.3.6) gives the exact triangle

$$\pi_!^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \pi_*^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \pi_*^{\text{aut}} j_* j^*(\underline{k}_{\text{Bun}_G^X}). \quad (6.3.7)$$

Now, again by the Contraction Principle the natural transformation $\pi_*^{\text{aut}} \rightarrow i_*$ is an isomorphism of functors from \mathbf{G}_m -monodromic sheaves on Bun_G^X to sheaves on Bun_G , giving in particular an isomorphism $\pi_*^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \cong \underline{k}_{\text{Bun}_G}$. Finally, observe that

$$\pi_*^{\text{aut}} \circ j_* \cong (\pi^{\text{aut}} \circ j)_* = \text{AJ}_*$$

which transforms (6.3.7) into the exact triangle

$$\pi_!^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \underline{k}_{\text{Bun}_G} \rightarrow \text{AJ}_*(j^* \underline{k}_{\text{Bun}_G^X}). \quad (6.3.8)$$

Note that we have a natural isomorphism $\mathrm{AJ}_* \cong \mathrm{AJ}_! [1]$ because AJ is the composition of a \mathbf{G}_m -torsor and a projective space bundle. Unraveling the construction, we have a commutative diagram

$$\begin{array}{ccc} \underline{k}_{\mathrm{Bun}_G} & \xrightarrow{(6.3.4)} & \mathrm{AJ}_!(j^* \underline{k}_{\mathrm{Bun}_G^X})[1] \\ \parallel & & \downarrow \sim \\ \underline{k}_{\mathrm{Bun}_G} & \xrightarrow{(6.3.8)} & \mathrm{AJ}_*(j^* \underline{k}_{\mathrm{Bun}_G^X}) \end{array}$$

where the top row is the connecting homomorphism for (6.3.4), which therefore lies in the space

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Bun}_G}(\underline{k}_{\mathrm{Bun}_G}, \mathrm{AJ}_*(\underline{k}_{\mathrm{Sym}^\bullet C})) &\cong \mathrm{Hom}_{\mathrm{Sym}^\bullet C}(\underline{k}_{\mathrm{Sym}^\bullet C}, \underline{k}_{\mathrm{Sym}^\bullet C}) \\ &\cong \prod_{d \geq 0} \mathrm{Hom}_{\mathrm{Sym}^d C}(\underline{k}_{\mathrm{Sym}^d C}, \underline{k}_{\mathrm{Sym}^d C}) \cong \prod_{d \geq 0} k. \end{aligned} \quad (6.3.9)$$

Lemma 6.3.1. *For each $d \geq 0$, under our identifications the connecting map for (6.3.4) is adjoint to the identity map in $\mathrm{Hom}_{\mathrm{Sym}^d C}(\underline{k}_{\mathrm{Sym}^d C}, \underline{k}_{\mathrm{Sym}^d C})$ in (6.3.9).*

Proof. The map is given by a scalar for each d . To compute the scalar, we may base change to a point $\mathcal{L} \in \mathrm{Bun}_G(\mathbf{F})$. Let

$$\begin{array}{ccc} V & \xrightarrow{l} & \mathrm{Bun}_G^X \\ \downarrow \pi & & \downarrow \pi^{\mathrm{aut}} \\ \mathcal{L} & \xrightarrow{l} & \mathrm{Bun}_G \end{array}$$

Note that $l^* \pi^{\mathrm{aut}} \cong \pi_* l^*$ on \mathbf{G}_m -monodromic sheaves, by hyperbolic localization. Here V is the derived vector bundle associated to $\Gamma(C; \mathcal{L})$. Write $i: 0 \hookrightarrow V$ for the origin and j for its open complement. The map $\pi_*^{\mathrm{aut}}(\underline{k}_{\mathrm{Bun}_G^X}) \rightarrow \mathrm{AJ}_*(j^* \underline{k}_{\mathrm{Bun}_G^X})$ from (6.3.7) pulls back under l^* to the map given by the unit of j in the left side of

$$\mathrm{Hom}(\pi_* \underline{k}_V, \pi_* j_* j^* \underline{k}_V) \cong \mathrm{Hom}(j^* \pi^* \pi_* \underline{k}_V, j^* \underline{k}_V),$$

which transports to the counit of π on the right side, which is evidently the identity upon identifying $\pi^* \pi_* \underline{k}_V = \underline{k}_V$. The hyperbolic localization is implemented by the map $\pi_* \underline{k}_V \rightarrow \pi_* i_* i^* \underline{k}_V$, and under the identifications $\pi_* \underline{k}_V = \underline{k}_L$ and $i^* \underline{k}_V = \underline{k}_L$, it is the identity map. Hence we are done. \square

6.4. Comparison. We will now complete the proof of Theorem 6.1.2. We note that in this case $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\tilde{G}}) \cong \mathrm{QCoh}(\mathrm{Loc}_{\tilde{G}})$ because \tilde{G} is a torus.

We first collect some normalizations. The Whittaker compatibility (5.1.2) specializes in this case to the statement that for the trivial bundle $\mathrm{triv} \in \mathrm{Bun}_G$, we have

$$\mathrm{Dmod}(\mathrm{Bun}_1) \ni \delta_{\mathrm{triv}} \xrightarrow{\mathbb{L}_G} \omega_{\mathrm{Loc}_1} \langle -(g-1) \rangle \in \mathrm{QCoh}(\mathrm{Loc}_1). \quad (6.4.1)$$

On the other hand, as $\dim \mathrm{Bun}_1 = (g-1)$, we have for $\mathrm{triv} \in \mathrm{Loc}_1$ the trivial local system,

$$\mathrm{Dmod}(\mathrm{Bun}_1) \ni \underline{k} \langle g-1 \rangle \xrightarrow{\mathbb{L}_G} \mathcal{O}_{\mathrm{triv}} \in \mathrm{QCoh}(\mathrm{Loc}_1). \quad (6.4.2)$$

Lemma 6.4.1. *We have*

$$\mathrm{Dmod}(\mathrm{Bun}_1) \ni \mathrm{AJ}_!(\underline{k}_{\mathrm{Sym}^\bullet C}) \langle g-1 \rangle \xrightarrow{\mathbb{L}_G} \mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) \otimes \omega_{\mathrm{Loc}_{\tilde{G}}} \in \mathrm{QCoh}(\mathrm{Loc}_1).$$

Proof. By the special case of Hecke compatibility of \mathbb{L}_G in Example 5.2.2, \mathbb{L}_G intertwines the tensoring action of $\mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) \in \mathrm{QCoh}(\mathrm{Loc}_{\tilde{G}})$ with the Hecke convolution action of $\prod_{n \geq 0} \underline{k}_{\mathrm{HK}_{G,C(n)}^{\mathrm{Std}}} \langle n(r-1) \rangle \in \mathrm{Shv}(\bigcup_{n \geq 0} \mathrm{Hk}_{G,C(n)})$ on $\mathrm{Dmod}(\mathrm{Bun}_G)$. In this case, note that $n(r-1) = 0$ because $r = 1$, and $\mathrm{Hk}_{G,C(n)}^{\mathrm{Std}}$ can be identified with the stack of tuples $(D, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_2 \cong \mathcal{L}_1(D))$. Applying this statement to (6.4.1), we deduce that \mathbb{L}_G takes

$$\mathrm{Dmod}(\mathrm{Bun}_G) \ni \left(\prod_{n \geq 0} \underline{k}_{\mathrm{HK}_{G,C(n)}^{\mathrm{Std}}} \right) \star \delta_{\mathrm{triv}} \xrightarrow{\mathbb{L}_G} \mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) \otimes \omega_{\mathrm{Loc}_{\tilde{G}}} \langle -(g-1) \rangle \in \mathrm{QCoh}(\mathrm{Loc}_{\tilde{G}}). \quad (6.4.3)$$

Finally, unraveling the definition of the Hecke convolution reveals that

$$\underline{k}_{\mathrm{Hk}_{G,C^{(n)}}^{\mathrm{Std}}} \star \delta_{\mathrm{triv}} \cong \mathrm{AJ}_!^n(\underline{k}_{C^{(n)}})$$

so that the left side of (6.4.3) is identified with $\mathrm{AJ}_!(\underline{k}_{\mathrm{Sym}^\bullet C})$. Inserting this into (6.4.3) and twisting by $\langle g-1 \rangle$ completes the proof. \square

Inspecting (6.2.9) and (6.3.4), and using Proposition 6.2.5, we have a diagram where \mathbb{L}_G intertwines the indicated objects:

$$\begin{array}{ccccc} \mathrm{AJ}_!(\underline{k}_{\mathrm{Sym}^\bullet C})^{\langle g-1 \rangle} & \longrightarrow & \pi_!^{\mathrm{aut}}(\underline{k}_{\mathrm{Bun}_G^X})^{\langle g-1 \rangle} & \rightarrow & \underline{k}_{\mathrm{Bun}_G}^{\langle g-1 \rangle} & \in \mathrm{Dmod}(\mathrm{Bun}_G) \\ \text{Lemma 6.4.1} \downarrow \mathbb{L}_G & & \downarrow \mathbb{L}_G? & & \downarrow \mathbb{L}_G & \\ \mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) \otimes \omega_{\mathrm{Loc}_{\tilde{G}}} & \longrightarrow & \pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\tilde{G}}^X}) & \longrightarrow & \mathcal{O}_{\mathrm{triv}} & \in \mathrm{QCoh}(\mathrm{Loc}_{\tilde{G}}) \end{array} \quad (6.4.4)$$

To verify that \mathbb{L}_G takes $\pi_!^{\mathrm{aut}}(\underline{k}_{\mathrm{Bun}_G^X})^{\langle g-1 \rangle}$ to $\pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\tilde{G}}^X})$, it therefore suffices to verify that \mathbb{L}_G takes the connecting map $\underline{k}_{\mathrm{Bun}_G}^{\langle g-1 \rangle} \rightarrow \mathrm{AJ}_!(\underline{k}_{\mathrm{Sym}^\bullet C})^{\langle g-1 \rangle}[1]$ to the connecting map $\mathcal{O}_{\mathrm{triv}} \rightarrow \mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet(\mathrm{Std}))) \otimes \omega_{\mathrm{Loc}_{\tilde{G}}}[1]$. The functor \mathbb{L}_G is the identity on the extension groups in question with respect to the identifications (6.2.16) and (6.3.9), and then we conclude by using Lemma 6.2.6 and Lemma 6.3.1 to identify the connecting maps on each side. \square

6.5. Functional equation. We explain a categorification of the *functional equation* for Tate's zeta functions, which compares the automorphic periods for the standard action of \mathbf{G}_m on \mathbf{A}^1 and its inverse. To emphasize the distinction, we write $X := \mathrm{Std}$ for \mathbf{A}^1 with the standard scaling action of $G = \mathbf{G}_m$, and $X' := \mathrm{Std}^\vee$ for \mathbf{A}^1 with the inverse of the standard action. We equip X and X' with the standard scaling \mathbb{G}_{gr} -action (i.e., the neutral actions).

6.5.1. Normalized period sheaf for X . Let $\mathcal{E} := \underline{\mathrm{RHom}}(\mathcal{O}_C, \mathcal{L}^{\mathrm{univ}} \otimes \Omega_C^{1/2})$ be the perfect complex on Bun_1 , whose pullback to $\mathcal{L} \in \mathrm{Bun}_1(R)$ is $\Gamma(C_R; \mathcal{L})$ viewed as an animated R -module. Let $\pi: E \rightarrow \mathrm{Bun}_1$ be the associated derived vector bundle. We define the derived stack Bun_G^X as in [BZSV23, (10.6)] but taking the *derived* fibered product and forming mapping spaces as *derived stacks*:

$$\begin{array}{ccc} \mathrm{Bun}_G^X & \longrightarrow & \mathrm{Map}(C, \frac{X}{G \times \mathbb{G}_{gr}}) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G & \xrightarrow{\mathrm{Id} \boxtimes \Omega_C^{1/2}} & \mathrm{Bun}_{G \times \mathbb{G}_{gr}} \end{array}$$

By [FYZ21, Proposition 5.34], Bun_G^X is canonically identified as a derived stack with E , in a manner intertwining the projection π with $\pi^{\mathrm{aut}}: \mathrm{Bun}_G^X \rightarrow \mathrm{Bun}_G$. Hence the un-normalized period sheaf for X is $\mathcal{P}_X \cong \pi_! \underline{k}_E$. To normalize \mathcal{P}_X , we note that $\eta_X: G \rightarrow \mathbf{G}_m$ is the standard character (i.e., the identity). We pull back the degree function $\mathrm{deg}: \mathrm{Bun}_{\mathbf{G}_m} \rightarrow \mathbf{Z}$ to Bun_G via η_X , so it assigns to $\mathcal{L} \in \mathrm{Bun}_G(R)$ the integer $\mathrm{deg} \mathcal{L}$. Then the *normalized* period sheaf [BZSV23, (10.12)] is

$$\mathcal{P}_X^{\mathrm{norm}} \cong \pi_! \underline{k}_E^{\langle \mathrm{deg} + (g-1) \rangle} \in \mathrm{Shv}(\mathrm{Bun}_G). \quad (6.5.1)$$

6.5.2. Normalized period sheaf for X' . Let $\mathcal{E}' := \underline{\mathrm{RHom}}(\mathcal{O}_C, (\mathcal{L}^{\mathrm{univ}})^\vee \otimes \Omega_C^{1/2})$ be the perfect complex on Bun_1 , which over $\mathcal{L} \in \mathrm{Bun}_1(R)$ is $\Gamma(C_R; \mathcal{L}^\vee \otimes \Omega_C^{1/2})$. Let $\pi': E' \rightarrow \mathrm{Bun}_1$ be the associated derived vector bundle. Again, the derived stack $\mathrm{Bun}_G^{X'}$ is canonically identified with E' , in a manner intertwining the projection π' with $\pi^{\mathrm{aut}}: \mathrm{Bun}_G^{X'} \rightarrow \mathrm{Bun}_G$. Hence the un-normalized period sheaf for X is $\mathcal{P}_{X'} \cong \pi_! \underline{k}_{E'}$. Now, in this case $\eta_{X'}: G \rightarrow \mathbf{G}_m$ is the *inverse* of the standard character. Therefore, the degree function $\mathrm{deg}: \mathrm{Bun}_{\mathbf{G}_m} \rightarrow \mathbf{Z}$ pulls back to $-\mathrm{deg}: \mathrm{Bun}_G \rightarrow \mathbf{Z}$ under $\eta_{X'}$. Hence the *normalized* period sheaf is

$$\mathcal{P}_{X'}^{\mathrm{norm}} \cong \pi_! \underline{k}_{E'}^{\langle -\mathrm{deg} + (g-1) \rangle} \in \mathrm{Shv}(\mathrm{Bun}_G). \quad (6.5.2)$$

6.5.3. *Categorification of the functional equation.* The following result is the categorification of the functional equation for the L -functions attached to Hecke characters of GL_1 .

Proposition 6.5.1. *With the notation above, there is a canonical isomorphism*

$$\mathcal{P}_X^{\mathrm{norm}} \cong \mathcal{P}_{X'}^{\mathrm{norm}} \in \mathrm{Shv}(\mathrm{Bun}_1).$$

Proof. Let $r_E := \mathrm{rank}(E)$ be the virtual rank of E over Bun_1 . Note that $r_E = \mathrm{deg}$ by Riemann-Roch. Regarding Bun_1 as the total space of the zero vector bundle over itself, we have from §3.1.5 natural isomorphisms

$$\pi_! k_E = \mathrm{FT}_0(\pi_! k_E) \cong z_{E^\vee}^* \mathrm{FT}_E(k_E)[-r_E] \cong z_{E^\vee}^*(\delta_{E^\vee})\langle -2r_E \rangle \quad (6.5.3)$$

where z_{E^\vee} is the zero-section of E^\vee . Note that by Serre duality provides an isomorphism of perfect complexes over R ,

$$\Gamma(C_R; \mathcal{L} \otimes \Omega_C^{1/2})^\vee \cong \Gamma(C_R; \mathcal{L}^\vee \otimes \Omega_C^{1/2}[1])$$

so that E is dual to $E'[1]$. Hence we have a derived Cartesian square

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & \mathrm{Bun}_1 \\ \downarrow \pi' & & \downarrow z_{E^\vee} \\ \mathrm{Bun}_1 & \xrightarrow{z_{E^\vee}} & E^\vee \end{array} \quad (6.5.4)$$

Proper base change applied to (6.5.4) gives a natural isomorphism

$$z_{E^\vee}^*(\delta_{E^\vee}) = z_{E^\vee}^* z_{E^\vee!}(k_{\mathrm{Bun}_1}) \cong \pi_1'(\pi')^*(k_{\mathrm{Bun}_1}) \cong \pi_1'(k_{E'}) \in \mathrm{Shv}(\mathrm{Bun}_1), \quad (6.5.5)$$

and the result follows from combining (6.5.3) and (6.5.5), and then comparing them to (6.5.1) and (6.5.2). \square

Remark 6.5.2. On quasicompact open substacks of Bun_1 , an argument for Proposition 6.5.1 without using derived Fourier analysis appears in [BG02, Lemma 7.3.6] and is credited there to Drinfeld. Drinfeld's argument is based on an ingenious construction, which we invite the reader to admire. Nevertheless, we feel that the perspective of the derived Fourier transform puts the statement in its proper general context. In fact, trying to understand the functional equation is what led the authors to the derived Fourier transform, the details of which were then written up in [FYZ23].

7. HECKE PERIOD

7.1. **Setup.** We study an ansatz that we call the ‘‘Hecke period’’, as it corresponds to Hecke’s construction [Hec37] of L -functions for modular forms (i.e., automorphic forms on GL_2) by Mellin transform.

7.1.1. *Automorphic side:* Let $A := \begin{pmatrix} 1 & \\ & * \end{pmatrix} \cong \mathbf{G}_m \hookrightarrow \mathrm{GL}_2$ and $X = \mathrm{GL}_2/A$, with $G = \mathrm{GL}_2$ acting by left translation.

7.1.2. *Spectral side:* Let $\check{X} = \mathbf{A}^2$ with $\check{G} = \mathrm{GL}_2$ acting by the standard representation.

7.1.3. \mathbb{G}_{gr} -action. According to [BZSV23, Example 12.6.7], the neutral \mathbb{G}_{gr} -action on X is trivial, while the neutral \mathbb{G}_{gr} -action on \check{X} is the standard scaling action. The un-normalized \mathbb{G}_{gr} -action on X is via the *inverse* of the standard character into the center $\mathbf{G}_m \cong Z(\mathrm{GL}_2)$, while the un-normalized \mathbb{G}_{gr} -action on \check{X} is still the standard scaling action. Below we will take the un-normalized actions of \mathbb{G}_{gr} .

7.1.4. *Automorphic period sheaf.* Examining the recipe of [BZSV23, §10.2], we see that Bun_G^X can be identified with Bun_1 in such a way that the map $\pi^{\mathrm{aut}}: \mathrm{Bun}_G^X \rightarrow \mathrm{Bun}_G$ is identified with the map sending

$$\mathcal{L} \mapsto (\mathcal{L} \otimes \Omega_C^{1/2}) \oplus \Omega_C^{1/2}$$

where $\Omega_C^{1/2}$ is the fixed spin structure (§2.4.1). The automorphic period sheaf is

$$\mathcal{P}_X := \pi_1^{\mathrm{aut}}(k_{\mathrm{Bun}_G^X}) \in \mathrm{Shv}(\mathrm{Bun}_G).$$

In the language of [BZSV23], this is the un-normalized period sheaf for the un-normalized action of \mathbb{G}_{gr} .

7.1.5. *Spectral period sheaf.* By definition, $\mathrm{Loc}_{\check{G}}^{\check{X}}$ is the derived mapping stack

$$\mathrm{Loc}_{\check{G}}^{\check{X}} = \mathrm{Map}(C_{\mathrm{dR}}, \mathbf{A}^2 / \mathrm{GL}_2)$$

with R -points the groupoid of pairs (E, s) , where:

- $E \in \mathrm{Loc}_2(R)$ is a rank 2 vector bundle on C_R with a flat connection along C ,
- $s \in \mathrm{Sect}(C_{\mathrm{dR}}, E)(R)$, or informally speaking an R -family of flat sections of E .

The map $\pi_*^{\mathrm{spec}} : \mathrm{Loc}_{\check{G}}^{\check{X}} \rightarrow \mathrm{Loc}_{\check{G}}$ forgets the datum of s . The (co-localized) spectral period sheaf is

$$\mathcal{L}_{\check{X}} \cong (\pi_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})^{\vee} := \bigoplus_n (\pi_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})_n \langle n \rangle \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$$

where $(\pi_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}})_n$ is the n th graded piece for the \mathbb{G}_{gr} -action. In the language of [BZSV23], this is the un-normalized \mathcal{L} -sheaf for the un-normalized action of \mathbb{G}_{gr} .

7.1.6. *Duality.* In this case, the projection of [BZSV23, Conjecture 12.1.1] to the formulation of Geometric Langlands in [AG15] is equivalent to

$$\mathrm{Dmod}(\mathrm{Bun}_G) \ni \mathcal{P}_{X \langle 5(g-1) \rangle} \xrightarrow{\mathbb{L}_G} \mathcal{L}_{\check{X}} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}).$$

Indeed, [BZSV23, Proposition 12.6.4] says that [BZSV23, Conjecture 12.1.1] is equivalent to $\mathcal{P}_{X \langle r \rangle} \xrightarrow{\mathbb{L}_G} \mathcal{L}_{\check{X}}^d$ where

$$r = \beta_{X'} + \beta_{\check{X}'} - (g-1)\tau.$$

where the $(-)'$ refers to un-normalizing \mathbb{G}_{gr} -action. The duality involution disappears under the translation from right actions to left actions (cf. [CV24, Appendix A]). Here $\tau = \langle \eta, \check{\eta} \rangle = 0$ where η is the character by which G scales the volume form on X ; it vanishes because GL_2 and A are unimodular, so GL_2/A admits a left GL_2 -invariant volume form. Then $\beta_{X'} = (g-1)(\dim G - \dim X' + \gamma_{X'})$, but $\gamma_{X'} = 0$ since the action of \mathbf{G}_m is through G . Also $\beta_{\check{X}'} = (g-1)(\dim \check{G} - \dim \check{X}' + \gamma_{\check{X}'})$ and $\gamma_{\check{X}'} = 2$ since the \mathbb{G}_{gr} -action scales the volume form by the square of the standard character. All in all, we see that

$$r = (g-1)(4-3) + (g-1)(4-2+2) = 5(g-1).$$

7.2. **Spectral side.** We let $Z \hookrightarrow \mathrm{Loc}_{\check{G}}^{\check{X}}$ be the zero section and U be the open complement of Z . We consider the exact triangle in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ coming from filtering by sections with set-theoretic support in Z :

$$(\pi_*^{\mathrm{spec}})_{Z^*}^{\wedge}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}) \rightarrow \pi_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}) \rightarrow \pi_*^{\mathrm{spec}}|_U(\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}|_U) \quad (7.2.1)$$

where we have implicitly co-localized to the category of sheaves with nilpotent singular support (§2.5.2).

7.2.1. *Open stratum.* We analyze the rightmost term of (7.2.1). Since a flat section of a family of local systems is non-zero at one point if and only if it is non-zero at every point, U is isomorphic to the derived mapping stack $\mathrm{Map}(C_{\mathrm{dR}}, (\mathbf{A}^2 \setminus 0) / \mathrm{GL}_2)$ over $\mathrm{Loc}_{\check{G}}$. The action of GL_2 on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ identifies $\mathbf{A}^2 \setminus 0$ with $\mathrm{GL}_2 / \mathrm{Mir}_2$, where

$$\mathrm{Mir}_2 := \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \hookrightarrow \mathrm{GL}_2$$

is the mirabolic subgroup. This induces a presentation of U as the derived mapping stack

$$U \cong \mathrm{Map}(C_{\mathrm{dR}}, \mathrm{B} \mathrm{Mir}_2)$$

such that the map $\pi_*^{\mathrm{spec}}|_U : U \rightarrow \mathrm{Loc}_{\check{G}}$ is the one induced by the inclusion $\mathrm{Mir}_2 \hookrightarrow \mathrm{GL}_2$.

Therefore, with respect to the diagram

$$\begin{array}{ccc} & \mathrm{Loc}_{\mathrm{Mir}_2} & \\ q_{\mathrm{Mir}} \swarrow & & \searrow p_{\mathrm{Mir}} \\ \mathrm{Loc}_1 & & \mathrm{Loc}_2 \end{array}$$

we have

$$\pi_*^{\mathrm{spec}}|_U(\omega_U) \cong p_{\mathrm{Mir}*} q_{\mathrm{Mir}}^!(\omega_{\mathrm{Loc}_1}) \in \mathrm{IndCoh}(\mathrm{Loc}_2).$$

In order to fit the formulation of Eisenstein compatibility in §5.3, we rewrite this in terms of usual spectral Eisenstein functors (for parabolic subgroups). Let $\check{B} \subset \check{G}$ be the standard Borel subgroup containing Mir_2 , and T its Levi quotient. We consider the correspondence diagram

$$\begin{array}{ccc} & \text{Loc}_{\check{B}} & \\ q \swarrow & & \searrow p \\ \text{Loc}_{\check{T}} & & \text{Loc}_{\check{G}} \end{array}$$

We then have a commutative diagram

$$\begin{array}{ccccc} & & \text{Loc}_{\text{Mir}_2} & & \\ & q_{\text{Mir}} \swarrow & \downarrow f & \searrow p_{\text{Mir}} & \\ \text{Loc}_1 & & \text{Loc}_{\check{B}} & & \\ \downarrow g & \swarrow q & & \searrow p & \\ \text{Loc}_{\check{T}} & & & & \text{Loc}_{\check{G}} \end{array}$$

where the upper left square is Cartesian. By base change for that Cartesian square, we get isomorphisms

$$p_{\text{Mir}*}(\omega_U) \cong p_{\text{Mir}*}q_{\text{Mir}}^!(\omega_{\text{Loc}_1}) \cong p_*q^!g_*(\omega_{\text{Loc}_1}) \in \text{IndCoh}(\text{Loc}_{\check{G}}). \quad (7.2.2)$$

Now, g can be identified with the map

$$\text{pt} \times \text{Loc}_1 \xrightarrow{\text{triv} \times \text{Id}} \text{Loc}_1 \times \text{Loc}_1$$

so that

$$g_*(\omega_{\text{Loc}_1}) \cong \mathcal{O}_{\text{triv}} \boxtimes \omega_{\text{Loc}_1} \in \text{IndCoh}(\text{Loc}_1 \times \text{Loc}_1).$$

Inserting this into (7.2.2), and co-localizing to the category with nilpotent singular support, we have established:

Proposition 7.2.1. *We have an isomorphism (using the notational convention of §2.5.2)*

$$\pi_{\text{spec}}|_{U*}(\omega_U) \cong \text{Eis}_{\check{B}}^{\text{spec}}(\mathcal{O}_{\text{triv}} \boxtimes \omega_{\text{Loc}_1}) \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}).$$

7.2.2. Closed stratum. Next we analyze the leftmost term in (7.2.1), with a similar approach to that of §6.2.2. The origin $0 \hookrightarrow \text{Std} = \check{X}$ is the only \check{G} -fixed point. Let \mathcal{V} be its formal completion in \check{X} . By Lemma 6.2.1, we have an isomorphism of unital cocommutative coalgebras

$$\Gamma_0(\check{X}; \omega_{\check{X}}) \cong \Gamma(\mathcal{V}; \omega_{\mathcal{V}}) \in \text{Rep}(\check{G} \times \mathbb{G}_{gr}). \quad (7.2.3)$$

The \mathbb{G}_{gr} -action equips $\Gamma_0(\check{X}; \omega_{\check{X}})$ with a non-negative grading.

Lemma 7.2.2. *As a commutative coalgebra in $\text{Rep}(\check{G})^{\mathbb{Z} > 0}$, $\Gamma_0(\check{X}; \omega_{\check{X}})$ identifies with $\text{Sym}^\bullet(\text{Std})$ where Std is the standard representation of \check{G} in graded degree 1.*

Proof. This follows from Lemma 6.2.2. □

This gives the following description of the leftmost term in (7.2.1).

Proposition 7.2.3. *Let $\text{Fact}(\text{Sym}^\bullet(\text{Std}))$ be the graded unital cocommutative factorization coalgebra (5.2.12). Then we have an isomorphism*

$$(\pi^{\text{spec}})_{Z*}^\wedge(\omega_{\text{Loc}_{\check{X}}}) \cong \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std})) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}).$$

Proof. Applying Proposition 4.5.2 to the origin inside Std viewed as a \check{G} -scheme, equipped with the standard scaling action of \mathbb{G}_{gr} , we obtain an isomorphism

$$(\pi^{\text{spec}})_{Z*}^\wedge(\omega_{\text{Loc}_{\check{X}}}) \cong \text{pr}_{2*}(\text{Fact}(\bar{\Gamma}_0(\check{X}; \omega_{\check{X}}))^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}). \quad (7.2.4)$$

In (7.2.3) and Lemma 7.2.2 we saw that $\bar{\Gamma}_0(\check{X}; \omega_{\check{X}}) \cong \overline{\text{Sym}^\bullet \text{Std}} \in \text{Rep}(\check{G})^{\mathbb{Z} > 0}$. Inserting this into (7.2.4) yields an isomorphism

$$(\pi^{\text{spec}})_{Z*}^\wedge(\omega_{\text{Loc}_{\check{X}}}) \cong \text{pr}_{2*}(\text{Fact}(\overline{\text{Sym}^\bullet \text{Std}})^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}) \quad (7.2.5)$$

where the meaning of $\text{Fact}(\overline{\text{Sym}}^\bullet \text{Std})^{\text{univ}}$ is as in Example 5.2.2. The rest follows from adding a copy of the unit, as in the proof of Proposition 6.2.5. \square

7.2.3. *Summary.* Putting together §7.2.1 and §7.2.1, we have produced an exact triangle in $\text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$:

$$\text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std})) \otimes \omega_{\text{Loc}_{\check{G}}} \rightarrow \pi_*^{\text{spec}}(\omega_{\text{Loc}_{\check{X}}}^{\text{spec}}) \rightarrow \text{Eis}_{\check{B}}^{\text{spec}}(\mathcal{O}_{\text{triv}} \boxtimes \omega_{\text{Loc}_1}) \quad (7.2.6)$$

where we are invoking the notational convention of §2.5.2.

7.3. **Automorphic side.** Twisting by Ω_C , we may identify $\pi^{\text{aut}}: \text{Bun}_G^X \rightarrow \text{Bun}_2$ with the map $\text{Bun}_1 \rightarrow \text{Bun}_2$ sending $\mathcal{L} \mapsto (\mathcal{L} \otimes \Omega_C^{-1/2}) \oplus \Omega_C^{1/2}$. We will write $\pi^{\text{aut}}: \text{Bun}_1 \rightarrow \text{Bun}_2$ when using the latter perspective.

We seek a filtration of \mathcal{P}_X corresponding to the filtration of $\mathcal{L}_{\check{X}}$ in (8.2.9), which came from the stratification of \check{X} by \check{G} -orbits. But note that unlike in the example of the Tate period, here there is no obvious “open-closed” decomposition of $\pi_1^{\text{aut}}(\underline{k}_{\text{Bun}_1})$ coming from a stratification of X to match the stratification of \check{X} , since the G -action on X is homogeneous. (However, see §7.5 for matching stratifications in the hyperspherical varieties.) We will instead look for a decomposition in the *Fourier dual space*.

7.3.1. *Unfolding.* On Bun_1 there is a perfect complex $\text{RHom}(\mathcal{L}^{\text{univ}}, \Omega_C[1])$ whose pullback to $\mathcal{L} \in \text{Bun}_1(R)$, for any animated \mathbf{F} -algebra R , is $\text{RHom}_{C_R}(\mathcal{L}, \Omega_C \boxtimes R[1])$ regarded as an animated R -module. We denote its total space by $E \rightarrow \text{Bun}_1$. By Serre duality, the dual derived vector bundle $E^\vee \rightarrow \text{Bun}_1$ can be identified with the total space of the perfect complex $\text{RHom}(\mathcal{O}_C, \mathcal{L}^{\text{univ}})$, whose pullback to $\mathcal{L} \in \text{Bun}_1(R)$ is $\text{RHom}_{C_R}(\mathcal{O} \boxtimes R, \mathcal{L})$ regarded as an animated R -module.

In fact, E is a classical stack, with R -points the groupoid of extensions

$$\Omega_C \boxtimes R \rightarrow \mathcal{F} \rightarrow \mathcal{L} \quad (7.3.1)$$

where $\mathcal{L} \in \text{Bun}_1(R)$. The zero-section $\text{Bun}_1 \rightarrow E$ sends $\mathcal{L} \in \text{Bun}_1(R)$ to the split extension $\mathcal{L} \oplus (\Omega_C \boxtimes R)$.

Let deg be the locally constant function on Bun_1 assigning to a family of line bundles the fiberwise degree. The virtual rank of E as a derived vector bundle over Bun_1 is

$$r_E := \chi(\text{RHom}(\mathcal{L}^{\text{univ}}, \Omega_C[1])) = 1 - g + \text{deg}.$$

Hence by Example 3.1.3 we have an isomorphism

$$\delta_E \cong \text{FT}_{E^\vee}(k_{E^\vee}r_E) \in \text{Shv}(E). \quad (7.3.2)$$

We will then analyze the decomposition of δ_E coming from the the stratification of E^\vee into its zero section and the complement. To unravel what this means, consider the commutative diagram

$$\begin{array}{ccccc} & & E \times_{\text{Bun}_1} E^\vee & & \\ & \swarrow \text{pr}_1 & & \searrow \text{pr}_2 & \\ & E & & E^\vee & \\ & \swarrow f & & \swarrow & \\ \text{Bun}_2 & & & & \text{Bun}_1 \end{array} \quad (7.3.3)$$

where f sends (7.3.1) to $\mathcal{F} \otimes \Omega_C^{-1/2} \in \text{Bun}_2(R)$, and the middle square is derived Cartesian. The classical truncation $(E \times_{\text{Bun}_1} E^\vee)_{\text{cl}}$ has R -points the groupoid of diagrams

$$\left\{ \begin{array}{c} \Omega_C \boxtimes R \\ \downarrow \\ \mathcal{F} \\ \downarrow \\ \mathcal{O}_C \boxtimes R \xrightarrow{s} \mathcal{L} \end{array} \right\} \quad (7.3.4)$$

where

- \mathcal{F} is a rank 2 vector bundle on C_R ,
- \mathcal{L} is a line bundle on C_R , and

- $s: \mathcal{O}_C \boxtimes R \rightarrow \mathcal{L}$ is any section (possibly zero).

The map

$$\text{ev}: E \times_{\text{Bun}_1} E^\vee \rightarrow \mathbf{A}^1$$

assigns to a diagram (7.3.4) the extension class of the pullback of \mathcal{F} along s , which is an element of $H^1(C_R, \Omega^1 \boxtimes R) \cong R$.

Lemma 7.3.1. *Recall that \mathcal{A} is the Artin-Schreier sheaf on \mathbf{A}^1 . With the notation above, we have an isomorphism*

$$\pi_1^{\text{aut}}(\underline{k}_{\text{Bun}_1}) = f_! \delta_E \cong f_! \text{pr}_{1!}(\text{ev}^* \mathcal{A}(2r_E)) \in \text{Shv}(\text{Bun}_2).$$

Proof. This follows from (7.3.2) and the commutative diagram (7.3.3) by writing out the definition of FT. \square

Consider the derived fibered product

$$\begin{array}{ccc} Z & \xleftarrow{i} & E \times_{\text{Bun}_1} E^\vee \\ \downarrow & & \downarrow \text{pr}_1 \\ \text{Bun}_1 & \xrightarrow{0} & E \end{array}$$

Informally speaking, $i: Z \hookrightarrow E \times_{\text{Bun}_1} E^\vee$ is the closed substack where the section s is zero. Let $j: U \rightarrow E \times_{\text{Bun}_1} E^\vee$ be its open complement. Then we have an exact triangle in $\text{Shv}(E \times_{\text{Bun}_1} E^\vee)$,

$$j_! j^*(\text{ev}^* \mathcal{A}) \rightarrow \text{ev}^* \mathcal{A} \rightarrow i_* i^*(\text{ev}^* \mathcal{A}).$$

Then applying $f_! \text{pr}_{1!}$ and using Lemma 7.3.1 gives an exact triangle in $\text{Shv}(\text{Bun}_2)$:

$$f_! \text{pr}_{1!} j_! j^*(\text{ev}^* \mathcal{A})(2r_E) \rightarrow \pi_1^{\text{aut}}(\underline{k}_{\text{Bun}_1}) \rightarrow f_! \text{pr}_{1!} i_* i^*(\text{ev}^* \mathcal{A})(2r_E). \quad (7.3.5)$$

Henceforth we will have no further need to refer to the derived structure on the objects involved, so we pass to classical truncations.

7.3.2. Closed stratum. We analyze the rightmost term of (7.3.5).

The closed stratum $Z \hookrightarrow E \times_{\text{Bun}_1} E$ may be identified with the (classical) stack $\text{Bun}_{\text{Mir}_2}^\Omega$, whose R -points form the groupoid of

- \mathcal{L} , a line bundle on $C \times R$.
- An extension

$$\Omega_C \boxtimes R \rightarrow \mathcal{F} \rightarrow \mathcal{L}. \quad (7.3.6)$$

The stack $\text{Bun}_{\text{Mir}_2}^\Omega$ is a twisted version of the stack of torsors for the mirabolic group $\text{Mir}_2 \hookrightarrow \text{GL}_2$.

Since $\text{ev}|_Z$ is identified with projection to $0 \times \text{Bun}_1 \hookrightarrow \mathbf{A}^1 \times \text{Bun}_1$, the $*$ -restriction of $\text{ev}^* \mathcal{A}$ to Z is identified with the constant sheaf $\underline{k}_{\text{Bun}_{\text{Mir}_2}^\Omega}$. Let $p_{\text{Mir}} = f \circ i: \text{Bun}_{\text{Mir}_2}^\Omega \rightarrow \text{Bun}_2$ be the map sending (7.3.6) to $\mathcal{F} \otimes \Omega_C^{-1/2}$. So the rightmost term in (7.3.5) is isomorphic to $p_{\text{Mir}!}(\underline{k}_{\text{Bun}_{\text{Mir}_2}^\Omega})(2r_E)$. We rewrite this in terms of the Eisenstein functor. Let B be the standard parabolic of GL_2 containing Mir_2 . Then we have a commutative diagram

$$\begin{array}{ccccc} & & \text{Bun}_{\text{Mir}_2}^\Omega & & \\ & q_{\text{Mir}} \swarrow & \downarrow g' & \searrow p_{\text{Mir}} & \\ \text{Bun}_1 & & \text{Bun}_B & & \text{Bun}_2 \\ \downarrow g & \swarrow q & & \searrow p & \\ \text{Bun}_1 \times \text{Bun}_1 & & & & \end{array}$$

in which the left square is Cartesian, and g is the map $\mathcal{L} \mapsto (\mathcal{L} \otimes \Omega_C^{-1/2}, \Omega_C^{1/2})$. By proper base change, we have

$$p_{\text{Mir}!}(\underline{k}_{\text{Bun}_{\text{Mir}_2}^\Omega}) \cong p_{\text{Mir}!} q_{\text{Mir}}^*(\underline{k}_{\text{Bun}_1}) \cong p_! q^* g_!(\underline{k}_{\text{Bun}_1}) = \text{Eis}_B(\underline{k}_{\text{Bun}_1} \boxtimes \delta_{\text{spin}}) \in \text{Shv}(\text{Bun}_G)$$

where δ_{spin} is the delta sheaf at $\Omega_C^{1/2} \in \text{Bun}_1$, meaning $\text{spin}_!(\underline{k}_{\text{pt}})$ for the map $\text{spin}: \text{pt} \rightarrow \text{Bun}_1$ corresponding to the chosen spin structure $\Omega_C^{1/2}$. In conclusion, we have identified the term $f_! i_* i^*(\text{ev}^* \mathcal{A})$ from (7.3.5) with

$$p_{\text{Mir}!}(\underline{k}_{\text{Bun}_{\text{Mir}_2}^\Omega}) \cong \text{Eis}_B(\underline{k}_{\text{Bun}_1} \boxtimes \delta_{\text{spin}}) \in \text{Shv}(\text{Bun}_G).$$

7.3.3. *Drinfeld's compactifications.* In preparation for analyzing the open stratum, we take a digression to recall Drinfeld's compactifications. Let G be a reductive group. Choose a Borel subgroup $B \subset G$ and let $N \subset B$ be the unipotent radical. Let $T = B/N$ and $\mathcal{F}_T \in \text{Bun}_T(\mathbf{F})$.

The map $u: \text{Bun}_N^{\mathcal{F}_T} \rightarrow \text{Bun}_G$ is not proper, even after quotienting out by the T -action on the source. Drinfeld constructed a stack $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ fitting into a commutative diagram

$$\begin{array}{ccc} \text{Bun}_N^{\mathcal{F}_T} & \hookrightarrow & \overline{\text{Bun}}_N^{\mathcal{F}_T} \\ & \searrow u & \downarrow \bar{u} \\ & & \text{Bun}_G \end{array}$$

such that the induced map $\overline{\text{Bun}}_N^{\mathcal{F}_T}/T \rightarrow \text{Bun}_G$ is proper; this is explained in [FGV01, §2.2]. To explain its functor of points, we recall that the groupoid $\text{Bun}_N^{\mathcal{F}_T}(R)$ admits a description as classifying pairs (\mathcal{F}_G, κ) where \mathcal{F}_G is a G -torsor on C_R and κ is a system of (saturated) vector bundle embeddings

$$\kappa^{\check{\lambda}}: \mathcal{F}_T^{\check{\lambda}} \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}, \text{ for all } \check{\lambda} \in X^*(T),$$

satisfying the Plücker relations, where $\mathcal{F}_T^{\check{\lambda}}$ is the line bundle on C_R obtained from \mathcal{F}_T via the homomorphism $\check{\lambda}: T \rightarrow \mathbf{G}_m$; and $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$ is the vector bundle on C_R obtained by twisting irreducible representation of G with highest weight $\check{\lambda}$ by \mathcal{F}_G . Then $\overline{\text{Bun}}_N^{\mathcal{F}_T}(R)$ parametrizes tuples (\mathcal{F}_G, κ) defined similarly, except that the maps $\kappa^{\check{\lambda}}$ are only required to be (not necessarily saturated) embeddings of coherent sheaves fiberwise over $\text{Spec } R$. The maps u and \bar{u} are both described by $(\mathcal{F}_G, \kappa) \mapsto \mathcal{F}_G$.

Example 7.3.2. We explain the picture for $G = \text{GL}_2$, which is the only case we will use in this paper. In this case, \mathcal{F}_T corresponds to a pair of line bundles $\mathcal{L}_1, \mathcal{L}_2$ on C . The stack $\text{Bun}_N^{\mathcal{F}_T}$ has R -points the groupoid of extensions

$$\mathcal{L}_1 \boxtimes R \xrightarrow{\kappa_1} \mathcal{F} \xrightarrow{\kappa_2} \mathcal{L}_2 \boxtimes R$$

of vector bundles on C_R . The stack $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ has R -points the groupoid of triples $(\mathcal{F}, \kappa_1, \kappa_2)$ where

- \mathcal{F} is a rank 2 vector bundle on C_R ,
- $\kappa_1: \mathcal{L}_1 \boxtimes R \hookrightarrow \mathcal{F}$ is an injection of coherent sheaves fiberwise on $\text{Spec } R$,
- $\kappa_2: \mathcal{F} \rightarrow \mathcal{L}_2 \boxtimes R$ is a map of coherent sheaves which is non-zero fiberwise on $\text{Spec } R$, and
- the composition $\kappa_2 \circ \kappa_1$ is the zero map.

Let $\Lambda := X_*(T)_{\geq 0}$. There is a stratification of $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ indexed by Λ -colored divisors $D = \sum_i \nu_i \cdot x_i \in \text{Div}^\Lambda C$; recall this means that $\nu_i \geq 0$ for each i .¹⁰ The stratum $\overline{\text{Bun}}_N^{\mathcal{F}_T}(D)$ indexed by D is characterized by the property that an \mathbf{F} -point $(\mathcal{F}_G, \kappa) \in \overline{\text{Bun}}_N^{\mathcal{F}_T}$ lies in $\overline{\text{Bun}}_N^{\mathcal{F}_T}(D)$ precisely when

$$\text{the saturation of } \kappa^{\check{\lambda}}: \mathcal{F}_T^{\check{\lambda}} \rightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \text{ is } \mathcal{F}_T^{\check{\lambda}} \left(\sum_i \langle \nu_i, \check{\lambda} \rangle \cdot x_i \right) \text{ for each } \check{\lambda} \in X^*(T).$$

In particular, the stratum indexed by $D = 0$ is the open substack $\text{Bun}_N^{\mathcal{F}_T} \hookrightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}$.

Example 7.3.3. We explain the picture for $G = \text{GL}_2$ in terms of Example 7.3.2. In this case, the stratification is indexed by a pair of effective divisors D_1, D_2 on C . The stratum corresponding to (D_1, D_2) has \mathbf{F} -points the diagrams

$$\mathcal{L}_1 \xrightarrow{\kappa_1} \mathcal{F} \xrightarrow{\kappa_2} \mathcal{L}_2$$

such that the saturation of κ_1 is $\mathcal{L}_1(D_1) \subset \mathcal{F}$, and the image of κ_2 is $\mathcal{L}_2(-D_2) \subset \mathcal{L}_2$.

¹⁰We have reverse the conventions on positivity/negative from [FGV01] for consistency with our discussion of graded configuration space.

7.3.4. *Open stratum.* We will analyze the leftmost term in (7.3.5).

For $d \geq 0$, we abbreviate $\mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}$ for the Hecke stack $\mathrm{Hk}_{\mathrm{GL}_2,C^{(d)}}^{\mathrm{Std}}$ from (5.2.16), and

$$\mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}} := \bigcup_{d \geq 0} \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}$$

for the stack $\mathrm{Hk}_{\mathrm{GL}_2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}}$ from (5.2.16). We have a correspondence diagram

$$\begin{array}{ccccc} & & \mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}} & & \\ & \swarrow h_l & & \searrow h_r & \\ \mathrm{Bun}_2 & & & & \mathrm{Bun}_2 \\ & & \searrow h_r \times \mathrm{pr} & & \\ & & \mathrm{Bun}_2 \times \mathrm{Sym}^\bullet C & \longrightarrow & \mathrm{Bun}_2 \end{array}$$

where for a modification $(\mathcal{F} \hookrightarrow \mathcal{F}') \in \mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}}(R)$, we define

$$h_l(\mathcal{F} \hookrightarrow \mathcal{F}') := \mathcal{F} \in \mathrm{Bun}_2(R)$$

$$h_r(\mathcal{F} \hookrightarrow \mathcal{F}') := \mathcal{F}' \in \mathrm{Bun}_2(R)$$

and $\mathrm{pr}(\mathcal{F} \hookrightarrow \mathcal{F}')$ is the support divisor of $(\mathcal{F}'/\mathcal{F})$. We write

$$\mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}} \star(-) := \sum_{d \geq 0} \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}} \star(-) \quad (7.3.7)$$

for the endofunctor of $\mathrm{Shv}(\mathrm{Bun}_G)$ induced by the convolving with the kernel sheaf

$$\prod_{d \geq 0} \underline{k}^{(d)} \in \mathrm{Shv}\left(\bigcup_{d \geq 0} \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}\right) \quad (7.3.8)$$

from (5.2.18).

Remark 7.3.4 (Satake sheaves). In principle one should use the IC complexes coming from the Geometric Satake equivalence as kernel sheaves for Hecke convolution. Recall from Example 5.2.2 that the definition (7.3.7) is justified by the fact that the Satake sheaf associated to $\mathrm{Sym}^d(\mathrm{Std}) \in \mathrm{Rep}(\mathrm{GL}_2)$ is the (shifted and twisted) *constant sheaf* $\underline{k}^{(d)}$ on the corresponding Schubert variety.

Recall that $\mathscr{W} \in \mathrm{Shv}(\mathrm{Bun}_2)$ is the Whittaker sheaf (cf. §5.1). We write $\langle \deg \rangle: \mathrm{Shv}(\mathrm{Bun}_2) \rightarrow \mathrm{Shv}(\mathrm{Bun}_2)$ for the endofunctor given by $\langle d \rangle$ on $\mathrm{Shv}(\mathrm{Bun}_2^d)$, where Bun_2^d is the connected component parametrizing bundles with degree d .

Proposition 7.3.5. *With notation as in §7.3.1, we have an isomorphism*

$$\mathrm{Hk}_{2,\mathrm{Sym}^\bullet(C)}^{\mathrm{Std}} \star \mathscr{W} \cong f_! \mathrm{pr}_1! j_! j^*(\mathrm{ev}^* \mathscr{A}) \langle \deg \rangle \in \mathrm{Shv}(\mathrm{Bun}_G).$$

Proof. Let $\mathcal{F}_T := \Omega_C^{1/2} \oplus \Omega_C^{-1/2}$. Then, with notation as in §7.3.3, $\mathrm{Bun}_N^{\mathcal{F}_T}$ classifies families of extensions

$$0 \rightarrow \Omega_C^{1/2} \boxtimes R \rightarrow \mathcal{E} \rightarrow \Omega_C^{-1/2} \boxtimes R \rightarrow 0.$$

Recall that the Whittaker sheaf \mathscr{W} is obtained by pulling back the Artin-Schreier sheaf \mathscr{A} via the map $\mathrm{Bun}_N^{\mathcal{F}_T} \rightarrow \mathbf{A}^1$, obtaining $\Psi \in \mathrm{Shv}(\mathrm{Bun}_N^{\mathcal{F}_T})$, and then $!$ -pushing forward to Bun_2 . There is a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Bun}_N^{\mathcal{F}_T} \times_{\mathrm{Bun}_2} \mathrm{Hk}_{2,\mathrm{Sym}^\bullet(C)}^{\mathrm{Std}} & & \\ & \swarrow h_l & \downarrow u & \searrow h & \\ \mathrm{Bun}_N^{\mathcal{F}_T} & & \mathrm{Hk}_{2,\mathrm{Sym}^\bullet(C)}^{\mathrm{Std}} & & \overline{\mathrm{Bun}}_N^{\mathcal{F}_T} \\ \downarrow u & \swarrow h_l & & \searrow h_r & \downarrow \bar{u} \\ \mathrm{Bun}_2 & & & & \mathrm{Bun}_2 \end{array} \quad (7.3.9)$$

where the left square is Cartesian. To describe the dashed arrow h , note that

$$\mathrm{Bun}_N^{\mathcal{F}_T} \times_{\mathrm{Bun}_2} \mathrm{Hk}_{2,\mathrm{Sym}^\bullet(C)}^{\mathrm{Std}} = \bigcup_{d \geq 0} \mathrm{Bun}_N^{\mathcal{F}_T} \times_{\mathrm{Bun}_2} \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}$$

has R -points the groupoid of diagrams of the form

$$\begin{array}{ccc}
 \Omega_C^{1/2} \boxtimes R & & \\
 \downarrow & & \\
 \mathcal{E} & \hookrightarrow & \mathcal{E}' \\
 \downarrow & & \\
 \Omega_C^{-1/2} \boxtimes R & &
 \end{array} \tag{7.3.10}$$

where the cokernel of $\mathcal{E} \hookrightarrow \mathcal{E}'$ is a line bundle over some effective divisor $D \in C^{(d)}(R)$. The dashed arrow h in (7.3.9) sends (7.3.10) to the inclusion of coherent sheaves $(\Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{E}') \in \overline{\text{Bun}}_N^{\mathcal{F}_T}(R)$ in terms of the description in Example 7.3.2.

There is an open embedding $\bar{j}: U \rightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T}$ sending (7.3.4) to the inclusion of coherent sheaves $\Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{F} \otimes \Omega_C^{-1/2}$. We claim that

$$h_1 h_1^*(\Psi) \cong \bar{j}_! \text{ev}^*(\mathcal{A}) \in \text{Shv}(\overline{\text{Bun}}_N^{\mathcal{F}_T}). \tag{7.3.11}$$

We will deduce this from the Geometric Casselman-Shalika formula of Frenkel-Gaitsgory-Vilonen [FGV01, Theorem 4], which is a similar statement but for Hecke modification at a ‘‘fixed leg’’; thus our claim (7.3.11) can be viewed as a form of Geometric Casselman-Shalika ‘‘with moving legs’’. We first produce a map

$$\text{ev}^* \mathcal{A} \rightarrow \bar{j}^* h_1 h_1^*(\Psi) \in \text{Shv}(U). \tag{7.3.12}$$

Indeed, the fibered product U' in the Cartesian diagram

$$\begin{array}{ccc}
 U' & \xrightarrow{h'} & U \\
 \downarrow \bar{j}' & & \downarrow \bar{j} \\
 \text{Bun}_N^{\mathcal{F}_T} \times_{\text{Bun}_2} \text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std}} & \xrightarrow{h} & \overline{\text{Bun}}_N^{\mathcal{F}_T}
 \end{array} \tag{7.3.13}$$

parametrizes divisors $D \in \text{Sym}^\bullet(C)(R)$ along with commutative diagrams

$$\begin{array}{ccc}
 \Omega_C^{1/2} \boxtimes R & \xlongequal{\quad} & \Omega_C^{1/2} \boxtimes R \\
 \downarrow & & \downarrow \\
 \mathcal{E} & \hookrightarrow & \mathcal{E}' \\
 \downarrow & & \downarrow \\
 \Omega_C^{-1/2} \boxtimes R & \hookrightarrow & \Omega_C^{-1/2} \otimes_{\mathcal{O}_C} \mathcal{L}_2
 \end{array} \tag{7.3.14}$$

where $\text{coker}(\Omega_C^{-1/2} \boxtimes R \rightarrow \Omega_C^{-1/2} \otimes_{\mathcal{O}_C} \mathcal{L}_2) \cong \text{coker}(\mathcal{E} \hookrightarrow \mathcal{E}')$ is a line bundle over D . The map $h': U' \rightarrow U$ projects to the data of D along with the right column. But this data is enough to reconstruct the entire diagram, as the left column must then be the pullback of the right column along the composition

$$\Omega_C^{-1/2} \boxtimes R \cong \Omega_C^{-1/2} \otimes_{\mathcal{O}_C} \mathcal{L}_2(-D) \hookrightarrow \Omega_C^{-1/2} \otimes_{\mathcal{O}_C} \mathcal{L}_2.$$

Hence the map $h': U' \xrightarrow{\sim} U$ is an isomorphism. This induces isomorphisms

$$\text{ev}^* \mathcal{A} \rightarrow h_1'(\bar{j}')^* h_1^*(\Psi) \cong \bar{j}^* h_1 h_1^*(\Psi) \in \text{Shv}(U), \tag{7.3.15}$$

giving the claimed map (7.3.12).

As \bar{j} is an open embedding, adjunction from (7.3.15) gives a map

$$\bar{j}_! \text{ev}^* \mathcal{A} \rightarrow h_1 h_1^*(\Psi) \in \text{Shv}(\overline{\text{Bun}}_N^{\mathcal{F}_T}) \tag{7.3.16}$$

which we want to show is an isomorphism. This can be checked at stalks, so we may in particular restrict to the fiber over $D \in C^{(d)} \subset \text{Sym}^\bullet C$ with respect to the map $\overline{\text{Bun}}_N^{\mathcal{F}_T} \rightarrow \text{Sym}^\bullet C$. Now this is a special case of [FGV01, Theorem 4] (applied iteratively to the different points in the support of D), which says that

the Hecke operator attached to $V \in \text{Rep}(G)$ acting on Ψ is the *clean* extension of the stratum indexed by the highest weight. Indeed, the highest weight of $\text{Sym}^d(\text{Std})$ is $(d, 0)$, and the stratum indexed by $(d, 0)$ in $\overline{\text{Bun}}_N^{\mathcal{F}_T}$ is the one where $\Omega_C^{1/2} \boxtimes R \rightarrow \mathcal{E}'$ has no zeros, or in other words is already saturated, so that $\text{coker}(\Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{E}')$ is a locally free of rank d over D . This is precisely the fiber of U over $D \in C^{(d)}$, which then establishes (7.3.11) by invoking the Geometric Casselman-Shalika formula, noting Remark 7.3.4 that the perverse sheaf associated to V by the Geometric Satake equivalence is the shift and twist of the constant sheaf that we are using.

Apply $\bar{u}: \overline{\text{Bun}}_N^{\mathcal{F}_T} \rightarrow \text{Bun}_2$ to both sides of (7.3.11) gives an isomorphism

$$\bar{u}_! h_! h_l^*(\Psi) \cong \bar{u}_! \bar{j}_! \bar{j}_1^*(\text{ev}^* \mathcal{A}). \quad (7.3.17)$$

From the commutativity of the diagram (7.3.9), the left side of (7.3.17) identifies with $\text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std}} \langle -\text{deg} \rangle \star \mathcal{W} \in \text{Shv}(\text{Bun}_G)$ since the kernel sheaf (7.3.8) is the constant sheaf shifted by $\langle d \rangle$ on the component over $C^{(d)}$. On the other hand, we see by inspection $\bar{u} \circ \bar{j} = f \circ \text{pr}_1 \circ j$, so the right side of (7.3.17) identifies with $f_! j_! j^*(\text{ev}^* \mathcal{A}) \in \text{Shv}(\text{Bun}_G)$. This completes the proof. \square

7.3.5. *Summary.* Putting together §7.3.2 and §7.3.4 we have produced an exact triangle in $\text{Shv}(\text{Bun}_G)$,

$$(\text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std}}) \star \mathcal{W} \langle 2r_E - \text{deg} \rangle \rightarrow \pi_{\text{aut}!}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \text{Eis}_B(\underline{k}_{\text{Bun}_1} \boxtimes \delta_{\text{spin}}) \langle 2r_E \rangle \quad (7.3.18)$$

7.4. **Comparison.** We compare the extensions (7.2.6) and (7.3.18) for the L -sheaf and period sheaf.

7.4.1. *Rightmost terms.* First we examine the terms on the right. By (6.4.1) and (6.4.2), we have

$$\text{Dmod}(\text{Bun}_1 \times \text{Bun}_1) \ni (\underline{k}_{\text{Bun}_1} \boxtimes \delta_{\text{triv}}) \langle 2g-2 \rangle \xrightarrow{\mathbb{L}_{\text{GL}_1 \times \text{GL}_1}} \mathcal{O}_{\text{triv}} \boxtimes \omega_{\text{Loc}_1} \in \text{QCoh}(\text{Loc}_1 \times \text{Loc}_1).$$

Note that we are using $\text{QCoh}(\text{Loc}_1 \times \text{Loc}_1) = \text{IndCoh}_{\text{Nilp}}(\text{Loc}_1 \times \text{Loc}_1)$ and the notational convention of §2.5.2. In this case Eisenstein compatibility (§5.3) says that the diagram

$$\begin{array}{ccc} \text{Dmod}(\text{Bun}_1 \times \text{Bun}_1) & \xrightarrow[\sim]{\mathbb{L}_{\text{GL}_1 \times \text{GL}_1}} & \text{QCoh}(\text{Loc}_1 \times \text{Loc}_1) \\ \downarrow \text{Eis}_B & & \downarrow \text{Eis}_B^{\text{spec}} \\ \text{Dmod}(\text{Bun}_2) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{Loc}_2) \end{array}$$

commutes after translating by $(\Omega_C^{1/2}, \Omega_C^{-1/2})$ on the source and twisting by $\langle g-1+\text{deg} \rangle$, so we deduce that

$$\text{Dmod}(\text{Bun}_2) \ni \text{Eis}_B(\underline{k}_{\text{Bun}_1} \boxtimes \delta_{\text{spin}}) \langle 3g-3+\text{deg} \rangle \xrightarrow{\mathbb{L}_G} \text{Eis}_B^{\text{spec}}(\mathcal{O}_{\text{triv}} \boxtimes \omega_{\text{Loc}_1}) \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_2). \quad (7.4.1)$$

7.4.2. *Leftmost terms.* Next we examine the terms on the left.

Lemma 7.4.1. *With notation as in Proposition 7.2.3 and Proposition 7.3.5, we have*

$$\text{Dmod}(\text{Bun}_G) \ni \text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std}} \star \mathcal{W} \langle 3(g-1) \rangle \xrightarrow{\mathbb{L}_G} \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std})) \otimes \omega_{\text{Loc}_G} \in \text{QCoh}(\text{Loc}_G).$$

Proof. The Whittaker normalization (5.1.2) in this case reads

$$\text{Dmod}(\text{Bun}_G) \ni \mathcal{W} \langle -(g-1) \rangle \xrightarrow{\mathbb{L}_G} \omega_{\text{Loc}_G} \langle -(g-1)4 \rangle \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_G). \quad (7.4.2)$$

By the special case of Hecke compatibility of \mathbb{L}_G in Example 5.2.2, \mathbb{L}_G intertwines the tensoring action of

$$\text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std})) \in \text{QCoh}(\text{Loc}_G) \text{ on } \text{IndCoh}_{\text{Nilp}}(\text{Loc}_G)$$

with the Hecke convolution action of

$$\prod_{n \geq 0} \underline{k}_{\text{Hk}_{G, C^{(n)}}^{\text{Std}}} \langle n \rangle \in \text{Shv}(\text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std}}) \text{ on } \text{Dmod}(\text{Bun}_G).$$

Acting by these operators on either side of (7.4.2) and twisting by $\langle 4(g-1) \rangle$ yields the result. \square

7.4.3. *Conclusion.* Twisting (7.3.18) by $\langle 5(g-1) \rangle$, we get an exact triangle in $\mathrm{Shv}(\mathrm{Bun}_G)$:

$$(\mathrm{Hk}_{2, \mathrm{Sym}^\bullet C}^{\mathrm{Std}}) \star \mathscr{W} \langle 3(g-1) + \mathrm{deg} \rangle \rightarrow \pi_{\mathrm{aut}}!(\underline{k}_{\mathrm{Bun}_G^X} \langle 5(g-1) \rangle) \rightarrow \mathrm{Eis}_B(\underline{k}_{\mathrm{Bun}_1} \boxtimes \delta_{\mathrm{spin}}) \langle 2(g-1) + 2 \mathrm{deg} \rangle$$

where we used that $2r_E + 5(g-1) = 2(1-g + \mathrm{deg}) + 5(g-1) = 3(g-1) + 2 \mathrm{deg}$. By inspection, the twist $\langle \mathrm{deg} \rangle$ on the flanking terms correspond to the shearing operation $(-)^{\mathcal{J}}$ on $\mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet \mathrm{Std}))$ and $\mathrm{Eis}_B^{\mathrm{spec}}(\mathcal{O}_{\mathrm{triv}} \boxtimes \omega_{\mathrm{Loc}_1}) \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_G)$ for their respective \mathbf{G}_m -actions (coming from Proposition 7.2.3 in the former case), under Lemma 7.4.1 and (7.4.1). Therefore, we have proved the following theorem.

Theorem 7.4.2. *The Geometric Langlands equivalence $\mathbb{L}_G: \mathrm{Dmod}(\mathrm{Bun}_2) \rightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_2)$ intertwines the indicated objects in the diagram below, where each row is exact*

$$\begin{array}{ccccc} \mathrm{Hk}_{2, \mathrm{Sym}^\bullet C}^{\mathrm{Std}} \star \mathscr{W} \langle 3(g-1) + \mathrm{deg} \rangle & \rightarrow & \mathcal{P}_X \langle 5(g-1) \rangle & \rightarrow & \mathrm{Eis}_B(\delta_{\mathrm{spin}} \boxtimes \underline{k}_{\mathrm{Bun}_1}) \langle 3(g-1) + 2 \mathrm{deg} \rangle \\ \text{Lemma 7.4.1} \downarrow \mathbb{L}_G & & \downarrow \mathbb{L}_G? & & (7.4.1) \downarrow \mathbb{L}_G \\ \mathrm{Loc}^{\mathrm{spec}}(\mathrm{Fact}(\mathrm{Sym}^\bullet \mathrm{Std}))^{\mathcal{J}} & \longrightarrow & \mathcal{L}_{\check{X}} & \longrightarrow & \mathrm{Eis}_B^{\mathrm{spec}}(\mathcal{O}_{\mathrm{triv}} \boxtimes \omega_{\mathrm{Loc}_1})^{\mathcal{J}} \end{array} \quad (7.4.3)$$

7.5. **Microlocal interpretation.** Here we give an explanation for the unfolding decomposition (7.3.5) in terms of the geometry of T^*X . The fact that the ultimately was seen in the Fourier dual space suggests that in general one should look for decompositions in the *microlocal* geometry of X . Indeed, we shall see a decomposition of its hyperspherical variety T^*X analogous to (1.2.4) that reflects the decomposition of periods. We thank David Ben-Zvi and Akshay Venkatesh for explaining this to us.

Let $\mathfrak{a} := \mathrm{Lie} A$ and $\mathfrak{n} := \mathrm{Lie} N$; note that $\mathrm{Lie} \mathrm{Mir}_2 = \mathfrak{a} \oplus \mathfrak{n}$. With respect to the moment maps $\mu_A: T^*G \rightarrow \mathfrak{a}^*$ and $\mu_{\mathrm{Mir}}: T^*G \rightarrow \mathfrak{a}^* \oplus \mathfrak{n}^*$, we have the following decomposition of the hyperspherical variety T^*X :

$$T^*X \cong \frac{\mu_A^{-1}(0)}{A} = \frac{\mu_{\mathrm{Mir}}^{-1}(0 \times \mathfrak{n}^*)}{A} = \left(\frac{\mu_{\mathrm{Mir}}^{-1}(0 \times 0)}{A} \right) \sqcup \left(\frac{\mu_{\mathrm{Mir}}^{-1}(0 \times (\mathfrak{n}^* \setminus 0))}{A} \right). \quad (7.5.1)$$

(Note that μ^{-1} should be formed in the derived sense a priori, although it does not make a difference in this example.) Now observe that

$$\frac{\mu_{\mathrm{Mir}}^{-1}(0 \times (\mathfrak{n}^* \setminus 0))}{A} \cong T^*(G/N, \psi)$$

is isomorphic to the hyperspherical variety of the Whittaker period (using that A acts freely on $\mathfrak{n}^* \setminus 0$)¹¹, while $\mu_{\mathrm{Mir}}^{-1}(0 \times 0)/A$ can be thought of as the embedding in T^*X of the Lagrangian correspondence

$$T^*X \longleftarrow X \times_{X/N} T^*(X/N) \longrightarrow T^*(X/N) = T^*(G/\mathrm{Mir}_2)$$

associated to the map $X \twoheadrightarrow X/N$. Thus, (7.5.1) realizes a decomposition of T^*X into $T^*(G/N, \psi)$ and $T^*(G/\mathrm{Mir}_2)$ in the sense of G -Hamiltonian spaces. These terms correspond to the decomposition of $\pi_1^{\mathrm{aut}}(\underline{k}_{\mathrm{Bun}_G^X})$ in (7.3.18).

8. SINGULAR RANKIN-SELBERG PERIOD

8.1. **Setup.** In this example we will study a duality which is closely related to the *Rankin-Selberg* period. In the Rankin-Selberg period for GL_2 , $X^{\mathrm{RS}} = \mathrm{GL}_2 \times \mathbf{A}^2$ as a spherical variety for $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ and $\check{X}^{\mathrm{RS}} = \mathrm{Std} \otimes \mathrm{Std}$. We shall pass to the ‘‘rank one’’ locus inside in \check{X}^{RS} , which corresponds to passing to a certain open subset inside X^{RS} .

The Rankin-Selberg period is a fundamental construction, which is also the starting point for other approaches to automorphic periods such as the Relative Trace Formula. Our particular interest in passing from \check{X}^{RS} to \check{X} owes to the latter being *singular*. Singular spherical varieties lie beyond the scope of [BZSV23] but encompass many useful examples, so we would like a theory that covers them. Our analysis here suggests that [BZSV23, Conjecture 12.1.1] extends well to the singular case, with modifications to the normalizations.

8.1.1. *Automorphic side.* Let $X = (\mathrm{GL}_2 \times \mathrm{GL}_2) \times^{\mathrm{GL}_2} (\mathbf{A}^2 \setminus 0)$, where the quotient by GL_2 is with respect to its diagonal embedding in $\mathrm{GL}_2 \times \mathrm{GL}_2$, and $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ acting by left translation on the left factor. (In other words, X is the induction of $\mathbf{A}^2 \setminus 0$ along the diagonal subgroup $\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_2$.)

¹¹This is a reflection of the equivalence between the Whittaker model and the Kirillov model.

8.1.2. *Spectral side.* Let $\check{X} = (\text{Std} \otimes \text{Std})^{\text{rank} \leq 1}$ (the locus of rank ≤ 1 tensors) and $\check{G} = \text{GL}_2 \times \text{GL}_2$, acting factorwise through the standard representation.

8.1.3. \mathbb{G}_{gr} -action. We will take action of \mathbb{G}_{gr} on X induced by the inverse standard scaling action of \mathbb{G}_{gr} on $\mathbf{A}^2 \setminus 0$, and the trivial action of \mathbb{G}_{gr} on \check{X} , which we propose to think of as the un-normalized \mathbb{G}_{gr} -actions.

This is justified as follows: X appears an open subvariety of the Rankin-Selberg period $X^{\text{RS}} := (\text{GL}_2 \times \text{GL}_2) \times^{\text{GL}_2} \mathbf{A}^2$, and \check{X} appears as a closed subvariety of the dual $\check{X}^{\text{RS}} := \text{Std} \otimes \text{Std}$. As X^{RS} and \check{X}^{RS} are both smooth and affine, they fall under the scope of [BZSV23], wherein the neutral action of \mathbb{G}_{gr} on X^{RS} is induced by the scaling action on \mathbf{A}^2 , while the neutral action of \mathbb{G}_{gr} on \check{X}^{RS} is also the scaling action. Then the un-normalized action of \mathbb{G}_{gr} on X^{RS} is induced by the inverse standard scaling on \mathbf{A}^2 , while the un-normalized action of \mathbb{G}_{gr} on \check{X}^{RS} is trivial. Passing to the subvarieties $X \hookrightarrow X^{\text{RS}}$ and $\check{X} \hookrightarrow \check{X}^{\text{RS}}$ then results in the actions of the preceding paragraph.

Remark 8.1.1. Let us compare this to the outcome of naively using the recipe of [BZSV23]. On X , the naive neutral action of \mathbb{G}_{gr} should be induced by the scaling action on $\mathbf{A}^2 \setminus 0$, and the naive neutral action of \mathbb{G}_{gr} on \check{X} should be the standard scaling action (we use the adjective ‘naive’ because strictly speaking X and \check{X} fall outside the scope of [BZSV23], so we are performing some naive extrapolation from their formulas). Writing $X \cong \text{GL}_2 \times (\mathbf{A}^2 \setminus 0)$ and taking Haar measure on GL_2 and Lebesgue measure on $\mathbf{A}^2 \setminus 0$, we have $\eta = (1, \det): \text{GL}_2 \times \text{GL}_2 \rightarrow \mathbf{G}_m$. Writing $(\check{X})^{\text{smooth}} = (\mathbf{A}^2 \setminus 0) \times^{\mathbf{G}_m} (\mathbf{A}^2 \setminus 0)$ and descending Lebesgue measure, we get $\tilde{\eta} = (\det, \det): \text{GL}_2 \times \text{GL}_2 \rightarrow \mathbf{G}_m$. Hence the ‘naive un-normalized’ action of \mathbb{G}_{gr} on X is trivial, and the ‘naive un-normalized’ action of \mathbb{G}_{gr} on \check{X} is also trivial.

8.1.4. *Automorphic period.* We define the mirabolic subgroup

$$\text{Mir}_2 := \left\{ \begin{pmatrix} 1 & * \\ & * \end{pmatrix} \right\} \subset \text{GL}_2.$$

Then $\mathbf{A}^2 \setminus 0 \cong \text{GL}_2 / \text{Mir}_2$ as a GL_2 -space, so $X = (\text{GL}_2 \times \text{GL}_2) \times^{\text{GL}_2} (\text{GL}_2 / \text{Mir}_2)$ as a $G = \text{GL}_2 \times \text{GL}_2$ -space. Thus $X/G \cong \text{B Mir}_2$. Examining the recipe of [BZSV23, §10.2], we see that Bun_G^X can be identified with the space $\text{Bun}_{\text{Mir}_2}^\Omega$ with R -points the groupoid of extensions

$$\Omega_C^{1/2} \boxtimes R \rightarrow \mathcal{F} \rightarrow \mathcal{L} \boxtimes \Omega_C^{1/2} \quad (8.1.1)$$

where $\mathcal{L} \in \text{Bun}_1(R)$ and $\Omega_C^{1/2}$ is the fixed spin structure (cf. §2.4.1), in such a way that the map $\pi^{\text{aut}}: \text{Bun}_G^X \rightarrow \text{Bun}_G$ is identified with the map sending (8.1.1) to $(\mathcal{F}, \mathcal{F}) \in \text{Bun}_2 \times \text{Bun}_2(R)$. The automorphic period sheaf is

$$\mathcal{P}_X := \pi_1^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \in \text{Shv}(\text{Bun}_G).$$

In the language of [BZSV23], this is the un-normalized period sheaf for the un-normalized action of \mathbb{G}_{gr} .

8.1.5. *Spectral period.* By definition, $\text{Loc}_G^{\check{X}}$ is the derived mapping stack

$$\text{Loc}_G^{\check{X}} = \text{Map}(C_{\text{dR}}, \check{X}/\check{G})$$

with R -points the groupoid of triples (F_1, F_2, s) , where:

- $F_1, F_2 \in \text{Loc}_2(R)$ are rank 2 vector bundles on C_R with flat connections along C .
- $s \in \text{Sect}(C_{\text{dR}}, (F_1 \otimes F_2)^{\text{rank} \leq 1})(R)$, or informally speaking a flat section of the flat bundle $F_1 \otimes F_2$ landing pointwise in the locus of tensors of rank ≤ 1 .

The (co-localized) spectral period sheaf is

$$\mathcal{L}_{\check{X}} \cong (\pi_*^{\text{spec}} \omega_{\text{Loc}_G^{\check{X}}})^{\mathcal{J}} \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_G^{\check{X}}).$$

(Note that $\pi_*^{\text{spec}} \omega_{\text{Loc}_G^{\check{X}}} = (\pi_*^{\text{spec}} \omega_{\text{Loc}_G^{\check{X}}})^{\mathcal{J}}$ in this case, since the action of \mathbb{G}_{gr} is trivial.) In the language of [BZSV23], this is the un-normalized \mathcal{L} -sheaf for the un-normalized action of \mathbb{G}_{gr} .

8.1.6. *Duality.* Since \check{X} is singular, [BZSV23, Conjecture 12.1.1] does not formally apply to this case. However, we propose that its statement should be extended as follows.

Conjecture 8.1.2. *Under the Geometric Langlands equivalence for $G = \text{GL}_2 \times \text{GL}_2$, we have*

$$\text{Dmod}(\text{Bun}_G) \ni \mathcal{P}_X \langle 8(g-1) \rangle \xrightarrow{\text{L}_G} \mathcal{L}_{\check{X}} \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_G^{\check{X}}).$$

8.2. **Spectral side.** We let $Z \hookrightarrow \text{Loc}_{\check{G}}^{\check{X}}$ be the zero section and U be the open complement of Z . We consider the exact triangle coming from filtering by sections with set-theoretic support in Z :

$$(\pi^{\text{spec}})_{Z*}^{\wedge}(\omega_{\text{Loc}_{\check{G}}^{\check{X}}}) \rightarrow \pi_*^{\text{spec}}(\omega_{\text{Loc}_{\check{G}}^{\check{X}}}) \rightarrow (\pi^{\text{spec}}|_U)_*\omega_U. \quad (8.2.1)$$

8.2.1. *Open stratum.* We analyze the rightmost term in (8.2.1), which corresponds to the open substack $U \subset \text{Loc}_{\check{G}}^{\check{X}}$ where the flat section s has rank *exactly* equal to 1. The locus of rank 1 vectors in the representation $\text{Std} \otimes \text{Std}$ of $\text{GL}_2 \times \text{GL}_2$ is homogeneous. Taking $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the basepoint, the stabilizer is generated by the subgroups $\text{Mir}_2 \times \text{Mir}_2$ and A in \check{G} , where

$$\text{Mir}_2 = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \subset \text{GL}_2, \quad \text{and} \quad A = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix}, \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} \right\} \subset \text{GL}_2 \times \text{GL}_2.$$

This choice of basepoint induces a presentation

$$(\mathbf{A}^2 \otimes \mathbf{A}^2)^{\text{rank}=1} \cong (\text{GL}_2 / \text{Mir}_2) \times^A (\text{GL}_2 / \text{Mir}_2)$$

identifying U with the derived mapping stack $\text{Map}(C_{\text{dR}}, B\check{H}) = \text{Loc}_{\check{H}}$ for the subgroup

$$\check{H} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} a^{-1} & b' \\ 0 & d' \end{pmatrix} \right\} \subset \text{GL}_2 \times \text{GL}_2 = \check{G} \quad (8.2.2)$$

and the map $\text{Loc}_{\check{G}}^{\check{X}} \rightarrow \text{Loc}_{\check{G}}$ is the one induced by the homomorphism $\check{H} \hookrightarrow \check{G}$.

We will rewrite the contribution from this orbit in terms of Eisenstein series. Let \check{B} be the standard parabolic in GL_2 and \check{T} its Levi quotient. Then we have a commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & \swarrow q' & & \searrow g' & \\ \text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1 & & & & \text{Loc}_{\check{B}} \times \text{Loc}_{\check{B}} \\ & \searrow g & & \swarrow q & \\ & & \text{Loc}_{\check{T}} \times \text{Loc}_{\check{T}} & & \\ & & & & \searrow p \\ & & & & \text{Loc}_{\check{G}} \end{array} \quad (8.2.3)$$

where the left square is derived Cartesian. Here:

- The map q is induced by the Levi quotient homomorphism $\check{B} \times \check{B} \rightarrow \check{T} \times \check{T}$; in coordinates, it sends

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \mapsto (a, d) \times (a', d').$$

- The map q' is induced by the group homomorphism $\check{H} \rightarrow A \times \mathbf{G}_m \times \mathbf{G}_m$ sending (8.2.2) to $a \times (d, d') \in A \times \mathbf{G}_m \times \mathbf{G}_m$.
- The map g is induced by the group homomorphism $A \times \mathbf{G}_m \times \mathbf{G}_m \rightarrow \check{T} \times \check{T}$ which sends

$$A \times \mathbf{G}_m \times \mathbf{G}_m \ni (a, d, d') \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \times \begin{pmatrix} a^{-1} & 0 \\ 0 & d' \end{pmatrix} \in \check{T} \times \check{T}.$$

- The map g' is induced by the inclusion of \check{H} in $\check{B} \times \check{B}$.

By base change applied to the commutative diagram (8.2.3), we have isomorphisms

$$g'_*(\omega_U) \cong g'_*(q')^!(\omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1}) \cong q'^!g_*(\omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1}) \in \text{IndCoh}(\text{Loc}_{\check{B}} \times \text{Loc}_{\check{B}}). \quad (8.2.4)$$

Then applying p_* to (8.2.4) and co-localizing, we obtain an isomorphism

$$(\pi^{\text{spec}}|_U)_*\omega_U \cong p_*q'^!g_*(\omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1}) = \text{Eis}_{\check{B} \times \check{B}}^{\text{spec}}(g_*\omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1}) \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}). \quad (8.2.5)$$

8.2.2. *Closed stratum.* We begin with a general construction that amalgamates two graded (co)commutative (co)algebras.

Construction 8.2.1. Let Λ be a finite rank free abelian monoid.

(1) Suppose $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ and $B = \bigoplus_{\lambda \in \Lambda} B_\lambda$ are Λ -graded commutative algebras in a symmetric monoidal category (\mathbf{C}, \otimes) . Then

$$A \overset{\Delta}{\otimes} B := \bigoplus_{\lambda \in \Lambda} A_\lambda \otimes B_\lambda$$

has a Λ -graded commutative algebra structure, with multiplication being the composition

$$\begin{aligned} (A \overset{\Delta}{\otimes} B) \otimes (A \overset{\Delta}{\otimes} B) &= \left(\bigoplus_{i \in \Lambda} A_i \otimes B_i \right) \otimes \left(\bigoplus_{j \in \Lambda} A_j \otimes B_j \right) \\ &\cong \bigoplus_{i, j \in \Lambda} ((A_i \otimes B_i) \otimes (A_j \otimes B_j)) \\ &\hookrightarrow \bigoplus_{\lambda \in \Lambda} \left(\left(\bigoplus_{i+j=\lambda} A_i \otimes A_j \right) \otimes \left(\bigoplus_{i'+j'=\lambda} B_{i'} \otimes B_{j'} \right) \right) \\ &\rightarrow \bigoplus_{\lambda \in \Lambda} A_\lambda \otimes B_\lambda = A \overset{\Delta}{\otimes} B \end{aligned}$$

where the third line includes into the summands indexed by $i = i', j = j'$ and the fourth line uses the multiplications on A, B .

(2) Similarly, if A and B are Λ -graded cocommutative coalgebras in (\mathbf{C}, \otimes) , then

$$A \overset{\Delta}{\otimes} B := \bigoplus_{\lambda \in \Lambda} A_\lambda \otimes B_\lambda$$

has a Λ -graded cocommutative coalgebra structure, by the dual formulas. If A_λ, B_λ are dualizable in \mathbf{C} for all λ , then the graded duals $A^\vee := \bigoplus_{\lambda \in \Lambda} A_\lambda^\vee$ and $B^\vee := \bigoplus_{\lambda \in \Lambda} B_\lambda^\vee$ are Λ -graded commutative algebras, and $(A \overset{\Delta}{\otimes} B)^\vee \cong (A^\vee \overset{\Delta}{\otimes} B^\vee)$.

Next we analyze the leftmost term of (8.2.1). We may view $Z \hookrightarrow \text{Loc}_{\check{G}}^{\check{X}}$ as the zero section of $\pi^{\text{spec}}: \text{Loc}_{\check{G}}^{\check{X}} \rightarrow \text{Loc}_{\check{G}}$.

We compute the coalgebra of distributions on X supported at 0. Note that unlike in the Tate and Hecke periods considered in the previous sections, \check{X} is not a vector space, so Lemma 6.2.2 does not apply.

Lemma 8.2.2. *As a cocommutative coalgebra in $(\text{Rep}(\text{GL}_2) \otimes \text{Rep}(\text{GL}_2))^{\mathbf{Z}_{\geq 0}}$, we have*

$$\Gamma_0(\check{X}; \omega_{\check{X}}) \cong \text{Sym}^\bullet(\text{Std}) \overset{\Delta}{\otimes} \text{Sym}^\bullet(\text{Std})$$

where $\text{Sym}^\bullet(\text{Std}) \in \text{Rep}(\text{GL}_2)^{\mathbf{Z}_{\geq 0}}$ is the free $\mathbf{Z}_{\geq 0}$ -graded cocommutative coalgebra on the standard representation placed in graded degree 1.

Proof. The variety \check{X} is a quadric cone in \mathbf{A}^4 . It admits a smooth resolution by $\check{Y} := [\mathbf{A}^2 \times \mathbf{A}^2 \setminus (0, 0)]/\mathbf{G}_m$ via the map

$$\check{Y} \ni (y_1, y_2) \xrightarrow{r} y_1 \otimes y_2$$

where \mathbf{G}_m acts via the standard scaling on the first factor and its inverse on the second factor. The map $r: \check{Y} \rightarrow \check{X}$ is in fact a rational resolution (quadric cone singularities are rational) so the natural map $r_* \omega_{\check{Y}} \rightarrow \omega_{\check{X}}$ is an isomorphism. Furthermore, the resolution r restricts to an isomorphism on the complement of $0 \in \check{X}$. The exceptional fiber $W = r^{-1}(0)$ is the disjoint union of $\mathbf{P}^1 \times 0$ and $0 \times \mathbf{P}^1$. By adjunction we have a commutative diagram of exact triangles in $\text{IndCoh}(\check{X})$,

$$\begin{array}{ccccc} r_{W*}(\omega_{\check{Y}}) & \longrightarrow & r_* \omega_{\check{Y}} & \longrightarrow & r|_{(\check{Y} \setminus W)*}(\omega_{\check{Y} \setminus W}) \\ \downarrow \sim & & \downarrow \sim & & \parallel \\ \omega_{\check{Y}} & \longrightarrow & \omega_{\check{X}} & \longrightarrow & \omega_{\check{X} \setminus 0} \end{array}$$

where \mathcal{V} is the derived formal completion of \check{X} at 0. Hence we obtain that the map $\Gamma_W(\check{Y}; \omega_{\check{Y}}) \rightarrow \Gamma_0(\check{X}; \omega_{\check{X}})$ is an isomorphism of cocommutative coalgebras. It therefore suffices to compute $\Gamma_W(\check{Y}; \omega_{\check{Y}})$ as a \check{G} -equivariant graded cocommutative coalgebra. To do this we use (4.4.1); note that $W \hookrightarrow \check{Y}$ is a regular embedding (unlike $0 \hookrightarrow \check{X}$), so that its system of derived infinitesimal thickenings coincides with the classical such notion (whereas the analogous picture for $0 \hookrightarrow \check{X}$ is in principle complicated, by Remark 4.4.1).

First we compute the (underived) global functions $\mathcal{O}(\check{Y})$ as a commutative algebra in $\text{Rep}(\check{G})$. We have

$$\begin{aligned} \mathcal{O}(\check{Y}) &\cong \mathcal{O}([\text{Std} \times \text{Std} \setminus (0, 0)])^{\mathbf{G}_m} \\ &\cong (\text{Sym}^\bullet(\text{Std}^\vee) \otimes \text{Sym}^\bullet(\text{Std}^\vee))^{\mathbf{G}_m} \\ &\cong \bigoplus_{n \geq 0} \text{Sym}^n(\text{Std}^\vee) \boxtimes \text{Sym}^n(\text{Std}^\vee) \end{aligned} \quad (8.2.6)$$

where in the last isomorphism we note that $\text{Sym}^n(\text{Std}^\vee) \boxtimes 1$ has weight n for the \mathbf{G}_m -action and $1 \boxtimes \text{Sym}^n(\text{Std}^\vee)$ has weight $-n$ for the \mathbf{G}_m -action.

Now we turn to calculating the cocommutative coalgebra $\Gamma_W(\check{Y}; \omega_{\check{Y}}) \in \text{Rep}(\text{GL}_2) \otimes \text{Rep}(\text{GL}_2)$. Let

$$W = W^{(0)} \hookrightarrow W^{(1)} \hookrightarrow \dots \hookrightarrow \mathcal{W} \hookrightarrow \check{Y}$$

be the system of formal neighborhoods of W in \check{Y} as in (4.4.1), with colimit the formal completion $\mathcal{W} := Y_{\check{W}}^\wedge$. According to Lemma 6.2.1 we have

$$\Gamma_W(\check{Y}; \omega_{\check{Y}}) \cong \Gamma(\mathcal{W}; \omega_{\mathcal{W}}) \in \text{Rep}(\text{GL}_2) \otimes \text{Rep}(\text{GL}_2),$$

which in turn can be presented as

$$\Gamma(\mathcal{W}; \omega_{\mathcal{W}}) \cong \text{colim}_n \Gamma(W^{(n)}; \omega_{W^{(n)}}).$$

By Grothendieck duality, $\Gamma(W^{(n)}; \omega_{W^{(n)}})$ is the dual cocommutative coalgebra to the commutative algebra $\Gamma(W^{(n)}; \mathcal{O}_{W^{(n)}})$. Since $W^{(n)}$ is the n th order neighborhood of $W \cong \mathbf{P}^1 \amalg \mathbf{P}^1$, it has no higher cohomology. Since $W \hookrightarrow \check{Y}$ is a regular embedding, $\mathcal{O}_{W^{(n)}}$ is the quotient of (8.2.6) by the n th power of the ideal of W . Dualizing and taking the colimit over n gives the dual cocommutative coalgebra to (8.2.6), completing the proof. \square

Proposition 8.2.3. *Viewing $\text{Sym}^\bullet(\text{Std}) \overset{\Delta}{\otimes} \text{Sym}^\bullet(\text{Std})$ as a cocommutative coalgebra in $\text{Rep}(\check{G})^{\mathbf{Z}_{\geq 0}}$ with Std in grading degree 1, let $\text{Fact}(\text{Sym}^\bullet \text{Std} \overset{\Delta}{\otimes} \text{Sym}^\bullet \text{Std}) \in \text{QCoh}((B\check{G})_{\text{Sym}^\bullet C})$ be the associated factorization algebra, augmented by the unit analogously to (5.2.12). Then we have an isomorphism*

$$(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}}^{\check{X}}) \cong \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std} \overset{\Delta}{\otimes} \text{Sym}^\bullet \text{Std})) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}).$$

Proof. Applying Proposition 4.5.2 to the origin inside the \check{G} -scheme \check{X} , with grading coming from the standard scaling action of \mathbb{G}_{gr} , we obtain an isomorphism

$$\overline{(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}}^{\check{X}})} \cong \text{pr}_{2*}(\text{Fact}(\bar{\Gamma}_0(\check{X}; \omega))^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}). \quad (8.2.7)$$

From Lemma 8.2.2 we have

$$\bar{\Gamma}_0(\check{X}; \omega) \cong \overline{\text{Sym}^\bullet(\text{Std})} \overset{\Delta}{\otimes} \overline{\text{Sym}^\bullet(\text{Std})} \in \text{Rep}(\check{G})^{\mathbf{Z}_{> 0}}.$$

Inserting this into (8.2.7) yields an isomorphism

$$\overline{(\pi^{\text{spec}})_{Z^*}^\wedge(\omega_{\text{Loc}_{\check{G}}}^{\check{X}})} \cong \text{pr}_{2*}(\text{Fact}(\overline{\text{Sym}^\bullet(\text{Std})} \overset{\Delta}{\otimes} \overline{\text{Sym}^\bullet(\text{Std})})^{\text{univ}}) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}) \quad (8.2.8)$$

where $\text{Fact}(\overline{\text{Sym}^\bullet(\text{Std})} \overset{\Delta}{\otimes} \overline{\text{Sym}^\bullet(\text{Std})})$ is the universally twisted relative factorization algebra over $\text{Loc}_{\check{G}}$, analogous to (5.2.14). The rest follows from adding a copy of the unit, as in the proof of Proposition 6.2.5. \square

8.2.3. Summary. We have produced an exact triangle in $\text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$:

$$\text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std} \overset{\Delta}{\otimes} \text{Sym}^\bullet \text{Std})) \otimes \omega_{\text{Loc}_{\check{G}}} \rightarrow \pi_*^{\text{spec}}(\omega_{\text{Loc}_{\check{G}}}^{\check{X}}) \rightarrow \text{Eis}_{\check{B}}^{\text{spec}}(g_* \omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1}). \quad (8.2.9)$$

8.3. Automorphic side. We seek a filtration of \mathcal{P}_X corresponding to the filtration of $\mathcal{L}_{\check{X}}$ in (8.2.9), which came from the stratification of \check{X} by \check{G} -orbits. But, as in the case of the Hecke period, X is a homogeneous G -variety, so the stratification of X by G -orbits is trivial. Therefore, we will again look for a stratification in the Fourier dual space.

8.3.1. Unfolding. We may identify $\mathrm{Bun}_{\mathrm{Mir}_2}^\Omega$ with the derived vector bundle over Bun_1 , associated to the perfect complex $\mathrm{RHom}(\mathcal{L}^{\mathrm{univ}}, \Omega_C[1])$ whose pullback to $\mathcal{L} \in \mathrm{Bun}_1(R)$ is $\mathrm{RHom}_{C_R}(\mathcal{L}, \Omega_C \boxtimes R[1])$ as an animated R -module. We abbreviate $E := \mathrm{Bun}_{\mathrm{Mir}_2}^\Omega$ viewed in this way. Then there is a factorization of π^{aut} as

$$E \begin{array}{c} \xrightarrow{\Delta} \\ \xrightarrow{\pi^{\mathrm{aut}}} \\ \xrightarrow{f} \end{array} E \times_{\mathrm{Bun}_1} E \longrightarrow \mathrm{Bun}_2 \times \mathrm{Bun}_2 \quad (8.3.1)$$

where f sends

$$E \times_{\mathrm{Bun}_1} E(R) \ni \left\{ \begin{array}{c} \Omega_C \boxtimes R \rightarrow \mathcal{F}_1 \rightarrow \mathcal{L} \\ \Omega_C \boxtimes R \rightarrow \mathcal{F}_2 \rightarrow \mathcal{L} \end{array} \right\} \mapsto \left\{ \begin{array}{c} \mathcal{F}_1 \otimes_{\mathcal{O}_C} \Omega_C^{-1/2} \\ \mathcal{F}_2 \otimes_{\mathcal{O}_C} \Omega_C^{-1/2} \end{array} \right\} \in \mathrm{Bun}_2 \times \mathrm{Bun}_2(R). \quad (8.3.2)$$

This induces an isomorphism

$$\pi_1^{\mathrm{aut}}(\underline{k}_E) \cong f_!(\Delta_! \underline{k}_E) \in \mathrm{Shv}(\mathrm{Bun}_G). \quad (8.3.3)$$

From §3.1.5 we have a natural isomorphism

$$\Delta_! \mathrm{FT}_{E^\vee} \cong \mathrm{FT}_{E^\vee \times_{\mathrm{Bun}_1} E^\vee}(\Delta^\vee)^*r_E \quad (8.3.4)$$

where $r_E := \mathrm{rank}(E) = 1 - g + \deg \mathcal{L}^{\mathrm{univ}}$ is the virtual rank of E ; and $\Delta^\vee: E^\vee \times_{\mathrm{Bun}_1} E^\vee \rightarrow E^\vee$ is dual map of Δ , which can be identified with the addition on E^\vee . In particular, applying (8.3.4) to the δ -sheaf on E^\vee , and using that $\mathrm{FT}_{E^\vee}(\delta_{E^\vee}) \cong \underline{k}_E[r_E]$ (cf. Example 3.1.3), we obtain an isomorphism

$$\pi_1^{\mathrm{aut}}(\underline{k}_E) \cong f_! \mathrm{FT}_{E^\vee \times_{\mathrm{Bun}_1} E^\vee}((\Delta^\vee)^* \delta_{E^\vee})(r_E) \in \mathrm{Shv}(\mathrm{Bun}_G). \quad (8.3.5)$$

We have a derived Cartesian square

$$\begin{array}{ccc} E^\vee & \xrightarrow{z'} & E^\vee \times_{\mathrm{Bun}_1} E^\vee \\ \downarrow & & \downarrow \Delta^\vee \\ \mathrm{Bun}_1 & \xrightarrow{z} & E^\vee \end{array} \quad (8.3.6)$$

where the map z is the zero section, and z' is the anti-diagonal embedding $e \mapsto (e, -e)$. Applying proper base change, we obtain an isomorphism

$$(\Delta^\vee)^*(\delta_{E^\vee}) = (\Delta^\vee)^* z'_!(\underline{k}_{\mathrm{Bun}_1}) \cong z'_!(\underline{k}_{E^\vee}) \in \mathrm{Shv}(E^\vee \times_{\mathrm{Bun}_1} E^\vee). \quad (8.3.7)$$

Putting this into (8.3.5) gives an isomorphism

$$\pi_1^{\mathrm{aut}}(\underline{k}_E) \cong f_! \mathrm{FT}_{E^\vee \times_{\mathrm{Bun}_1} E^\vee}(z'_! \underline{k}_{E^\vee})(r_E) \in \mathrm{Shv}(\mathrm{Bun}_G). \quad (8.3.8)$$

Now we unravel the meaning of the right side of (8.3.8). Consider the commutative diagram

$$\begin{array}{ccccc} & & E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee & & \\ & & \downarrow z'' & \searrow \mathrm{pr}'_r & \\ & & E^\vee & & \\ & & \downarrow z' & & \\ E & \xrightarrow{\Delta} & E \times_{\mathrm{Bun}_1} E & \xrightarrow{f} & \mathrm{Bun}_2 \times \mathrm{Bun}_2 \\ \downarrow \Delta & & \downarrow \mathrm{pr}_l & & \downarrow \Delta^\vee \\ E \times_{\mathrm{Bun}_1} E & \xrightarrow{\mathrm{pr}_l} & E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee \times_{\mathrm{Bun}_1} E^\vee & \xrightarrow{\mathrm{pr}_r} & E^\vee \times_{\mathrm{Bun}_1} E^\vee \\ & & \downarrow \mathrm{pr}_r & & \downarrow \Delta^\vee \\ & & E^\vee & & E^\vee \\ & & \downarrow \Delta^\vee & & \\ & & \mathrm{Bun}_1 & & \end{array} \quad (8.3.9)$$

where the parallelograms are derived Cartesian. By base change and the projection formula, we have

$$\begin{aligned} \mathrm{FT}_{E^\vee \times_{\mathrm{Bun}_1} E^\vee}(z'_! k_{E^\vee}) &= \mathrm{pr}_{l!}(\mathrm{ev}^* \mathcal{A} \otimes \mathrm{pr}_r^* z'_! k_{E^\vee})[2r_E] \\ &\cong \mathrm{pr}_{l!}(\mathrm{ev}^* \mathcal{A} \otimes z'_! \mathrm{pr}_r^*(k_{E^\vee}))[2r_E] \\ &\cong \mathrm{pr}_{l!}(z'_!(z''^* \mathrm{ev}^* \mathcal{A}))[2r_E]. \end{aligned} \quad (8.3.10)$$

Explicitly, $E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee$ is a derived Artin stack whose classical truncation has R -points the groupoid of diagrams

$$\left\{ \begin{array}{ccc} & \Omega_C \boxtimes R & \Omega_C \boxtimes R \\ & \downarrow & \downarrow \\ & \mathcal{F}_1 & \mathcal{F}_2 \\ & \downarrow & \downarrow \\ \mathcal{O}_C \boxtimes R & \xrightarrow{s} \mathcal{L} & \mathcal{O}_C \boxtimes R \xrightarrow{-s} \mathcal{L} \end{array} \right\} \quad (8.3.11)$$

where

- $\mathcal{F}_1, \mathcal{F}_2$ are rank 2 vector bundles on C_R ,
- \mathcal{L} is a line bundle on C_R ,
- The columns are extensions of \mathcal{L} by $\Omega_C \boxtimes R$, and
- $s: \mathcal{O}_C \boxtimes R \rightarrow \mathcal{L}$ is any global section of \mathcal{L} (possibly zero).

The map

$$\Pi := f \circ \mathrm{pr}_l \circ z'' : E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee \rightarrow \mathrm{Bun}_G \quad (8.3.12)$$

sends (8.3.11) to the underlying bundles $(\mathcal{F}_1 \otimes_{\mathcal{O}_C} \Omega_C^{-1/2}, \mathcal{F}_2 \otimes_{\mathcal{O}_C} \Omega_C^{-1/2}) \in \mathrm{Bun}_2 \times \mathrm{Bun}_2(R)$. The map $\mathrm{ev} \circ z''$ sends (8.3.11) to the sum of the extension classes in $\mathrm{Ext}^1(\mathcal{O}_C \boxtimes R, \Omega_C \boxtimes R) \cong R$ obtained by pulling back the extensions in the columns via the horizontal maps s and $-s$ respectively. We abuse notation by abbreviating

$$\mathrm{ev}^* \mathcal{A} := z''^* \mathrm{ev}^* \mathcal{A} \in \mathrm{Shv}(E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee). \quad (8.3.13)$$

Putting together (8.3.8) and (8.3.10), we have established:

Proposition 8.3.1. *We have an isomorphism*

$$\pi_1^{\mathrm{aut}}(k_{\mathrm{Bun}_G^X}) \cong \Pi_! \mathrm{ev}^* \mathcal{A} \langle 2r_E \rangle \in \mathrm{Shv}(\mathrm{Bun}_G).$$

Now we contemplate the decomposition coming from the stratification by the vanishing of s . Let $i: Z \hookrightarrow E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee$ be the closed substack where $s = 0$, which may be viewed as the pullback of the zero section of E^\vee under pr_r . Let $j: U \hookrightarrow E \times_{\mathrm{Bun}_1} E \times_{\mathrm{Bun}_1} E^\vee$ be its open complement. Then, using Proposition 8.3.1, we have the following exact triangle in $\mathrm{Shv}(\mathrm{Bun}_G)$:

$$\Pi_! j_! j^*(\mathrm{ev}^* \mathcal{A}) \langle 2r_E \rangle \rightarrow \pi_1^{\mathrm{aut}}(k_{\mathrm{Bun}_G^X}) \rightarrow \Pi_! i_* i^*(\mathrm{ev}^* \mathcal{A}) \langle 2r_E \rangle. \quad (8.3.14)$$

Henceforth we will have no further need of the derived structure on the objects involved, so we pass to classical truncations.

8.3.2. Closed stratum. We analyze the rightmost term of (8.3.14). Note that $Z \cong E \times_{\mathrm{Bun}_1} E$ and $i^* \mathrm{ev}^* \mathcal{A} \cong k_Z$. Let B be the standard Borel subgroup of GL_2 containing Mir_2 , and T its Levi quotient. We have an equality of maps

$$\Pi \circ i = p \circ g' : Z \rightarrow \mathrm{Bun}_G \quad (8.3.15)$$

referring to the diagram

$$\begin{array}{ccc} & \Pi & \\ & \curvearrowright & \\ E \times_{\mathrm{Bun}_1} E & \xrightarrow{g'} \mathrm{Bun}_B \times \mathrm{Bun}_B & \xrightarrow{p} \mathrm{Bun}_2 \times \mathrm{Bun}_2 \end{array} \quad (8.3.16)$$

where g' sends (8.3.11) to

$$(\Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{F}_1 \otimes \Omega_C^{-1/2}) \times (\Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{F}_2 \otimes \Omega_C^{-1/2}) \in \mathrm{Bun}_B(R) \times \mathrm{Bun}_B(R). \quad (8.3.17)$$

The map g' fits into a derived Cartesian square

$$\begin{array}{ccc} E \times_{\mathrm{Bun}_1} E & \xrightarrow{g'} & \mathrm{Bun}_B \times \mathrm{Bun}_B \\ \downarrow q' & & \downarrow q \\ \mathrm{Bun}_1 & \xrightarrow{g} & \mathrm{Bun}_T \times \mathrm{Bun}_T \end{array} \quad (8.3.18)$$

where

- The map q is induced by the Levi quotient $B \rightarrow T$.
- The map q' sends (8.3.11) to \mathcal{L} .
- The bottom map g sends

$$\mathcal{L} \in \mathrm{Bun}_1(R) \mapsto (\Omega_C^{1/2} \boxtimes R, \mathcal{L} \otimes_{\mathcal{O}_C} \Omega_C^{-1/2}) \times (\Omega_C^{1/2} \boxtimes R, \mathcal{L} \otimes_{\mathcal{O}_C} \Omega_C^{-1/2}) \in \mathrm{Bun}_T \times \mathrm{Bun}_T(R). \quad (8.3.19)$$

Hence by proper base change, we have natural isomorphisms

$$g'_!(\underline{k}_Z) \cong g'_!(q')^*(\underline{k}_{\mathrm{Bun}_1}) \cong q^*(g'_!(\underline{k}_{\mathrm{Bun}_1})) \in \mathrm{Shv}(\mathrm{Bun}_B \times \mathrm{Bun}_B). \quad (8.3.20)$$

Combining (8.3.20) with the identity of maps (8.3.15) and the observation $i^* \mathrm{ev}^* \mathcal{A} \cong \underline{k}_Z$, we have an isomorphism

$$\Pi i_* i^*(\mathrm{ev}^* \mathcal{A}) \cong p_1 g'_!(\underline{k}_Z) \cong p_1 q^*(g'_!(\underline{k}_{\mathrm{Bun}_1})) \cong \mathrm{Eis}_{B \times B}(g'_!(\underline{k}_{\mathrm{Bun}_1})). \quad (8.3.21)$$

8.3.3. *Open stratum.* We analyze the leftmost term in (8.3.11).

Twisting by $\Omega_C^{1/2}$, we may rewrite the R -points of U (recall that we replaced the original U by its classical truncation) as the groupoid of diagrams

$$\left\{ \begin{array}{ccc} & \Omega_C^{1/2} \boxtimes R & \Omega_C^{1/2} \boxtimes R \\ & \downarrow \mathcal{E}_1 & \downarrow \mathcal{E}_2 \\ & \downarrow & \downarrow \\ \Omega_C^{-1/2} \boxtimes R \xleftarrow{-s} \mathcal{L} \otimes \Omega_C^{-1/2} & & \Omega_C^{-1/2} \boxtimes R \xleftarrow{s} \mathcal{L} \otimes \Omega_C^{-1/2} \end{array} \right\} \quad (8.3.22)$$

where the columns are extensions of vector bundles and where s is required to be injective as a map of coherent sheaves fiberwise over $\mathrm{Spec} R$.

Let $\mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}$ and $\mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}}$ be as in §7.3.4. Write

$$\mathrm{Hk}_{G,C^{(d)}}^{\mathrm{Std} \otimes \Delta \mathrm{Std}} := \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}} \times_{C^{(d)}} \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}$$

which parametrizes pairs of modifications $(\mathcal{F}_1 \hookrightarrow \mathcal{F}'_1 \in \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}(R), \mathcal{F}_2 \hookrightarrow \mathcal{F}'_2 \in \mathrm{Hk}_{2,C^{(d)}}^{\mathrm{Std}}(R))$ such that

$$\mathrm{supp}(\mathrm{coker}(\mathcal{F}_1 \hookrightarrow \mathcal{F}'_1)) = \mathrm{supp}(\mathrm{coker}(\mathcal{F}_2 \hookrightarrow \mathcal{F}'_2)) \in C^{(d)}(R).$$

Write

$$\mathrm{Hk}_{G,\mathrm{Sym}^\bullet C}^{\mathrm{Std} \otimes \Delta \mathrm{Std}} := \bigcup_{d \geq 0} \mathrm{Hk}_{G,C^{(d)}}^{\mathrm{Std} \otimes \Delta \mathrm{Std}} \cong \mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}} \times_{\mathrm{Sym}^\bullet C} \mathrm{Hk}_{2,\mathrm{Sym}^\bullet C}^{\mathrm{Std}}.$$

We have a correspondence diagram

$$\begin{array}{ccccc} & & \mathrm{Hk}_{G,\mathrm{Sym}^\bullet C}^{\mathrm{Std} \otimes \Delta \mathrm{Std}} & & \\ & \swarrow h_l & & \searrow h_r & \\ \mathrm{Bun}_G & & & & \mathrm{Bun}_G \\ & & \searrow h_r \times \mathrm{pr} & & \nearrow \\ & & \mathrm{Bun}_G \times \mathrm{Sym}^\bullet C & \longrightarrow & \mathrm{Bun}_G \end{array} \quad (8.3.23)$$

where

$$\begin{aligned} h_l(\mathcal{F}_1 \hookrightarrow \mathcal{F}'_1, \mathcal{F}_2 \hookrightarrow \mathcal{F}'_2) &:= (\mathcal{F}_1, \mathcal{F}_2) \in \mathrm{Bun}_2 \times \mathrm{Bun}_2(R) \\ h_r(\mathcal{F}_1 \hookrightarrow \mathcal{F}'_1, \mathcal{F}_2 \hookrightarrow \mathcal{F}'_2) &:= (\mathcal{F}'_1, \mathcal{F}'_2) \in \mathrm{Bun}_2 \times \mathrm{Bun}_2(R) \\ \mathrm{pr}(\mathcal{F}_1 \hookrightarrow \mathcal{F}'_1, \mathcal{F}_2 \hookrightarrow \mathcal{F}'_2) &:= \mathrm{supp}(\mathrm{coker}(\mathcal{F}_i \hookrightarrow \mathcal{F}'_i)) \in \mathrm{Sym}^\bullet C(R). \end{aligned}$$

Consider the complex

$$\prod_{d \geq 0} k\langle 2d \rangle \in \text{Shv} \left(\bigcup_{d \geq 0} \text{Hk}_{G, C^{(d)}}^{\text{Std} \hat{\otimes} \text{Std}} \right) = \text{Shv}(\text{Hk}_{G, \text{Sym}^\bullet C}^{\text{Std} \hat{\otimes} \text{Std}}) \quad (8.3.24)$$

as a kernel sheaf on $\text{Hk}_{G, \text{Sym}^\bullet C}^{\text{Std} \hat{\otimes} \text{Std}}$. We write $\text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std} \hat{\otimes} \text{Std}} \star(-)$ for the endofunctor $\text{pr}_{r!}((8.3.24) \otimes \text{pr}_l^*(-))$ of $\text{Shv}(\text{Bun}_G)$ induced by (8.3.24).

Let N be the unipotent radical of B , so the Whittaker space for G is $G/(N \times N, \psi \times \psi)$. Let $\mathcal{F}_T := \Omega_C^{1/2} \oplus \Omega_C^{-1/2} \in \text{Bun}_T(\mathbf{F})$. Then, with notation as in §7.3.3, $\text{Bun}_N^{\mathcal{F}_T}(R)$ classifies extensions

$$0 \rightarrow \Omega_C^{1/2} \boxtimes R \rightarrow \mathcal{E} \rightarrow \Omega_C^{-1/2} \boxtimes R \rightarrow 0.$$

The Whittaker sheaf $\mathcal{W} \in \text{Shv}(\text{Bun}_G)$ is obtained by pulling back the Artin-Schreier sheaf \mathcal{A} via the map $\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T} \rightarrow \mathbf{A}^1$, to get $\Psi \in \text{Shv}(\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T})$, and then $!$ -pushing forward to Bun_G .

We write $\langle \text{deg} \rangle: \text{Shv}(\text{Bun}_G) \rightarrow \text{Shv}(\text{Bun}_G)$ for the endofunctor given by $\langle d_1 + d_2 \rangle$ on $\text{Shv}(\text{Bun}_G^{(d_1, d_2)})$, where Bun_G^d is the connected component parametrizing pairs of bundles $(\mathcal{F}_1, \mathcal{F}_2)$ with $\text{deg } \mathcal{F}_i = d_i$.

Proposition 8.3.2. *With notation as in §8.3.1, we have a natural isomorphism*

$$\text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \hat{\otimes} \text{Std}} \star \Psi \cong \Pi_{!} j_{!} j^*(\text{ev}^* \mathcal{A})_{\langle \text{deg} \rangle} \in \text{Shv}(\text{Bun}_G).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} & & (\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T}) \times_{\text{Bun}_G} \text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \hat{\otimes} \text{Std}} & & \\ & \swarrow h_l & \downarrow u & \dashrightarrow h & \\ (\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T}) & & \text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \hat{\otimes} \text{Std}} & & \overline{\text{Bun}}_N^{\mathcal{F}_T} \times \overline{\text{Bun}}_N^{\mathcal{F}_T} \\ & \swarrow h_l & \searrow h_r & & \downarrow \bar{u} \\ & & \text{Bun}_G & & \text{Bun}_G \end{array} \quad (8.3.25)$$

where:

- The left square is Cartesian.
- The dashed arrow h is as follows. Observe that $(\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T}) \times_{\text{Bun}_G} \text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \hat{\otimes} \text{Std}}$ parametrizes diagrams of the form

$$\begin{array}{ccc} \Omega_C^{1/2} \boxtimes R & & \Omega_C^{1/2} \boxtimes R \\ \downarrow & & \downarrow \\ \mathcal{E}_1 & \hookrightarrow & \mathcal{E}'_1 \\ \downarrow & & \downarrow \\ \Omega_C^{-1/2} \boxtimes R & & \Omega_C^{-1/2} \boxtimes R \end{array} \quad (8.3.26)$$

where $\text{coker}(\mathcal{E}_1 \hookrightarrow \mathcal{E}'_1)$ and $\text{coker}(\mathcal{E}_2 \hookrightarrow \mathcal{E}'_2)$ have the same divisor of supports, which is some effective divisor $D \in C^{(d)}(R)$. The dashed arrow h sends (8.3.26) to the pair of injections of coherent sheaves

$$(\Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{E}'_1, \Omega_C^{1/2} \boxtimes R \hookrightarrow \mathcal{E}'_2) \in \overline{\text{Bun}}_N^{\mathcal{F}_T} \times \overline{\text{Bun}}_N^{\mathcal{F}_T}(R).$$

There is an open embedding $\bar{j}: U \rightarrow \overline{\text{Bun}}_N^{\mathcal{F}_T} \times \overline{\text{Bun}}_N^{\mathcal{F}_T}$ sending (8.3.22) to the inclusions of coherent sheaves $(\Omega_C^{1/2} \boxtimes R \xrightarrow{s} \mathcal{E}'_1, \Omega_C^{1/2} \boxtimes R \xrightarrow{s} \mathcal{E}'_2)$. We claim that

$$h_! h_l^*(\Psi) \cong \bar{j}_!(\text{ev}^* \mathcal{A}) \in \text{Shv}(\overline{\text{Bun}}_N^{\mathcal{F}_T} \times \overline{\text{Bun}}_N^{\mathcal{F}_T}). \quad (8.3.27)$$

Consider the commutative diagram

$$\begin{array}{ccc}
& (\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T}) \times_{\text{Bun}_G} \text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std} \otimes \Delta \text{Std}} & \\
& \downarrow \iota & \\
& (\text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T}) \times_{\text{Bun}_G} (\text{Hk}_{2, \text{Sym}^\bullet C}^{\text{Std}} \times \text{Hk}_{2, \text{Sym}^\bullet C}^{\text{Std}}) & \\
\begin{array}{c} \curvearrowright h_i \\ \downarrow \\ \text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T} \end{array} & \xleftarrow{\tilde{h}_i} & \begin{array}{c} \text{Bun}_N^{\mathcal{F}_T} \times \text{Bun}_N^{\mathcal{F}_T} \\ \xrightarrow{\tilde{h}} \end{array}
\end{array} \tag{8.3.28}$$

where the map ι is the base change along the diagonal map $\text{Sym } C \xrightarrow{\Delta} \text{Sym } C \times \text{Sym } C$, which is the diagonal embedding $C^{(d)} \rightarrow C^{(d)} \times C^{(d)}$ on the d th component.

We will use the Geometric Casselman-Shalika formula [FGV01, Theorem 4] to analyze the action of the kernel sheaf

$$\underline{k}_{\langle \text{deg} \rangle} := \prod_{d \geq 0} \underline{k}_{\langle 2d \rangle} \in \text{Shv} \left(\text{Hk}_{2, \text{Sym}^\bullet C}^{\text{Std}} \times \text{Hk}_{2, \text{Sym}^\bullet C}^{\text{Std}} \right)$$

through the correspondence (8.3.28). The proof of Proposition 7.3.5 shows that $\tilde{h}_i \tilde{h}_i^*(\Psi \otimes \underline{k})$ is the clean extension from the stratum of $(\overline{\text{Bun}}_N^{\mathcal{F}_T} \times \overline{\text{Bun}}_N^{\mathcal{F}_T})$ parametrizing diagrams

$$\begin{array}{ccc}
\Omega_C^{1/2} \boxtimes R & \xlongequal{\quad} & \Omega_C^{1/2} \boxtimes R & & \Omega_C^{1/2} \boxtimes R & \xlongequal{\quad} & \Omega_C^{1/2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}_1 & \xrightarrow{\quad} & \mathcal{F}'_1 & & \mathcal{F}_2 & \xrightarrow{\quad} & \mathcal{F}'_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega_C^{-1/2} \boxtimes R & \xrightarrow[\neq 0]{s_1} & \mathcal{L}_1 \otimes \Omega_C^{-1/2} & & \Omega_C^{-1/2} \boxtimes R & \xrightarrow[\neq 0]{s_2} & \mathcal{L}_2 \otimes \Omega_C^{-1/2}
\end{array} \tag{8.3.29}$$

Therefore, base changing along the diagonal map $\iota: C^{(d)} \rightarrow C^{(d)} \times C^{(d)}$, we find that $h_i h_i^*(\Psi)$ is the clean extension along the locus where $\text{coker}(\mathcal{F}_1 \hookrightarrow \mathcal{F}'_1)$ has the same divisor of supports as $\text{coker}(\mathcal{F}_2 \hookrightarrow \mathcal{F}'_2)$. This is precisely U , which completes the proof of the claim (8.3.27).

Applying $\bar{u}: \overline{\text{Bun}}_N^{\mathcal{F}_T} \times \overline{\text{Bun}}_N^{\mathcal{F}_T} \rightarrow \text{Bun}_G$ to both sides of (8.3.27) gives an isomorphism

$$\bar{u}_i h_i h_i^*(\Psi) \cong \bar{u}_i \bar{j}_i \bar{j}_i^*(\text{ev}^* \mathcal{A}) \in \text{Shv}(\text{Bun}_G). \tag{8.3.30}$$

From the commutativity of the diagram (8.3.25), the left side of (8.3.30) identifies with $\text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \otimes \Delta \text{Std}} \star \mathcal{W} \langle -\text{deg} \rangle \in \text{Shv}(\text{Bun}_G)$ since the kernel sheaf (8.3.24) is the constant sheaf shifted by $\langle 2d \rangle$ on the component over $C^{(d)}$. On the other hand, by inspection $\bar{u} \circ \bar{j} = \Pi \circ j$, so the right side of (8.3.30) identifies with $\Pi_! j_! j^*(\text{ev}^* \mathcal{A}) \in \text{Shv}(\text{Bun}_G)$. This completes the proof. \square

8.3.4. *Summary.* In summary, combining §8.3.2 and §8.3.3 we have produced an exact triangle in $\text{Shv}(\text{Bun}_G)$,

$$\text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \otimes \Delta \text{Std}} \star \mathcal{W} \langle 2r_E - \text{deg} \rangle \rightarrow \pi_1^{\text{aut}}(\underline{k}_{\text{Bun}_G^X}) \rightarrow \text{Eis}_{B \times B}(g_! \underline{k}_{\text{Bun}_1}) \langle 2r_E \rangle. \tag{8.3.31}$$

8.4. **Comparison.** We compare (8.2.9) and (8.3.31) for the L -sheaf and period sheaf.

8.4.1. *Rightmost terms.* First we examine the terms on the right. On the spectral side, let us rename the object in question by $\text{Eis}_{\check{B} \times \check{B}}^{\text{spec}}(g_*^{\text{spec}} \omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1})$, as the same name g is regrettably used for a different map on the automorphic side. To spell out the meaning of $g_*^{\text{spec}}(\omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1})$, let us label the two factors of the maximal torus $\check{T} \times \check{T} \subset \check{G}$ as $\check{T}_1 \times \check{T}_2$, and then write $\check{T}_1 \cong \mathbf{G}_m^{(1)} \times \mathbf{G}_m^{(2)}$ and $\check{T}_2 \cong \mathbf{G}_m^{(3)} \times \mathbf{G}_m^{(4)}$. In these terms, the map

$$g^{\text{spec}}: \mathbf{G}_m \times \mathbf{G}_m \times \mathbf{G}_m \rightarrow \mathbf{G}_m^{(1)} \times \mathbf{G}_m^{(2)} \times \mathbf{G}_m^{(3)} \times \mathbf{G}_m^{(4)}$$

sends

$$(a, d, d') \mapsto (a, d, a^{-1}, d').$$

As the anti-diagonal embedding $\check{\mathbf{G}}_m \hookrightarrow \check{\mathbf{G}}_m \times \check{\mathbf{G}}_m$ is dual to the pullback along the map $\mathbf{G}_m \times \mathbf{G}_m \rightarrow \mathbf{G}_m$ given by $(x, y) \mapsto x/y$, we have using (6.4.2) that

$$\mathrm{Dmod}(\mathrm{Bun}_{T \times T}) \ni \Delta_{!}^{(13)}(\underline{k}_{\mathrm{Bun}_1}) \boxtimes \delta_{\mathrm{triv}}^{(2)} \boxtimes \delta_{\mathrm{triv}}^{(4)(g-1)} \xrightarrow{\mathbb{L}_{T \times T}} g_*^{\mathrm{spec}}(\omega_{\mathrm{Loc}_A \times \mathrm{Loc}_1 \times \mathrm{Loc}_1}) \in \mathrm{QCoh}(\mathrm{Loc}_{\check{T} \times \check{T}}) \quad (8.4.1)$$

where the superscripts on the RHS indicate the factors in which they lie, and $\Delta^{(13)}$ is the diagonal embedding of Bun_1 into the $\mathrm{Bun}_{\mathbf{G}_m^{(1)}} \times \mathrm{Bun}_{\mathbf{G}_m^{(3)}}$.

In this case Eisenstein compatibility (§5.3) says that the diagram

$$\begin{array}{ccc} \mathrm{Dmod}(\mathrm{Bun}_T \times \mathrm{Bun}_T) & \xrightarrow[\sim]{\mathbb{L}_{T \times T}} & \mathrm{QCoh}(\mathrm{Loc}_{\check{T}} \times \mathrm{Loc}_{\check{T}}) \\ \downarrow \mathrm{Eis}_{B \times B} & & \downarrow \mathrm{Eis}_{\check{B} \times \check{B}}^{\mathrm{spec}} \\ \mathrm{Dmod}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}) \end{array}$$

commutes after translating by $(\Omega_C^{1/2} \oplus \Omega_C^{-1/2}) \times (\Omega_C^{1/2} \oplus \Omega_C^{-1/2}) \in \mathrm{Bun}_{\check{T}} \times \mathrm{Bun}_{\check{T}}$ and twisting by $\langle 2(g-1) + \mathrm{deg} \rangle$. Combining this with (8.4.1), we deduce that

$$\mathrm{Dmod}(\mathrm{Bun}_G) \ni \mathrm{Eis}_{B \times B}(g! \underline{k}_{\mathrm{Bun}_1})_{(6(g-1) + \mathrm{deg})} \xrightarrow{\mathbb{L}_G} \mathrm{Eis}_{\check{B} \times \check{B}}^{\mathrm{spec}}(g_*^{\mathrm{spec}} \omega_{\mathrm{Loc}_A \times \mathrm{Loc}_1 \times \mathrm{Loc}_1}) \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}) \quad (8.4.2)$$

where we have used the notational convention of §2.5.2.

8.4.2. Leftmost terms. Next we examine the terms on the left. In preparation we observe here a binary operation on (co)commutative Λ^+ -graded factorization (co)algebras, which will be seen to be parallel to Construction 8.2.1.

Lemma/Construction 8.4.1. *Let Λ be a finite rank free abelian monoid with a chosen basis, as in §4.3.*

(1) *Let $\mathcal{A} = \{\mathcal{A}^\lambda\}_{\lambda \in \Lambda^+}$ and $\mathcal{B} = \{\mathcal{B}^\lambda\}_{\lambda \in \Lambda^+}$ be commutative Λ^+ -graded factorization algebras in $\mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C)$. Then*

$$\mathcal{A} \overset{\Delta}{\otimes} \mathcal{B} := \{\mathcal{A}^\lambda \overset{\dagger}{\otimes} \mathcal{B}^\lambda\}_{\lambda \in \Lambda^+} \in \mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C)$$

has a natural structure of commutative Λ^+ -graded factorization algebra.

(2) *Let $\mathcal{A} = \{\mathcal{A}^\lambda\}_{\lambda \in \Lambda^+}$ and $\mathcal{B} = \{\mathcal{B}^\lambda\}_{\lambda \in \Lambda^+}$ be cocommutative Λ^+ -graded factorization coalgebras in $\mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C)$. Then*

$$\mathcal{A} \overset{\Delta}{\otimes} \mathcal{B} = \{\mathcal{A}^\lambda \otimes \mathcal{B}^\lambda\}_{\lambda \in \Lambda^+} \in \mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C)$$

has a natural structure of cocommutative Λ^+ -graded factorization coalgebra.

Proof. (1) Define the factorization algebra structure (cf. (4.3.2))

$$(\mathcal{A}^{\lambda_1 + \lambda_2} \overset{\dagger}{\otimes} \mathcal{B}^{\lambda_1 + \lambda_2})|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\mathrm{disj}}} \cong (\mathcal{A}^{\lambda_1} \overset{\dagger}{\otimes} \mathcal{B}^{\lambda_1}) \boxtimes (\mathcal{A}^{\lambda_2} \overset{\dagger}{\otimes} \mathcal{B}^{\lambda_2})|_{(C^{(\lambda_1)} \times C^{(\lambda_2)})_{\mathrm{disj}}} \quad (8.4.3)$$

as the $\overset{\dagger}{\otimes}$ of the respective isomorphisms for \mathcal{A} and \mathcal{B} . Define the commutative factorization algebra structure (cf. (4.3.3))

$$(\mathcal{A}^{\lambda_1} \overset{\dagger}{\otimes} \mathcal{B}^{\lambda_1}) \boxtimes (\mathcal{A}^{\lambda_2} \overset{\dagger}{\otimes} \mathcal{B}^{\lambda_2}) \rightarrow \mathrm{add}_{\lambda_1, \lambda_2}^{\dagger}(\mathcal{A}^{\lambda_1 + \lambda_2} \overset{\dagger}{\otimes} \mathcal{B}^{\lambda_1 + \lambda_2}) \quad (8.4.4)$$

as the $\overset{\dagger}{\otimes}$ of the respective maps for \mathcal{A} and \mathcal{B} . It is straightforward to check that these define a commutative Λ^+ -graded factorization algebra.

(2) Similar, with dual formulas. \square

Lemma 8.4.2. (1) *Let $A, B \in \mathrm{ComAlg}(\mathrm{Shv}(C), \overset{\dagger}{\otimes})^{\Lambda^+}$. Let $\mathrm{Fact}(A), \mathrm{Fact}(B) \in \mathrm{ComAlg}^*(\mathrm{Shv}(\mathrm{Div}^{\Lambda^+} C))$ be the corresponding commutative factorization algebras (under Theorem 4.3.2). Then there is a natural isomorphism*

$$\mathrm{Fact}(A \overset{\Delta}{\otimes} B) \cong \mathrm{Fact}(A) \overset{\Delta}{\otimes} \mathrm{Fact}(B).$$

(2) Let $A, B \in \text{ComCoalg}(\text{Shv}_{\text{hol}}(C), \otimes)^{\Lambda^+}$. Let $\text{Fact}(A), \text{Fact}(B) \in \text{ComCoalg}^*(\text{Shv}_{\text{hol}}(\text{Div}^{\Lambda^+} C))$ be the corresponding cocommutative factorization coalgebras (under Theorem 4.3.4). Then there is a natural isomorphism

$$\text{Fact}(A \overset{\Delta}{\otimes} B) \cong \text{Fact}(A) \overset{\Delta}{\otimes} \text{Fact}(B).$$

Proof. (1) Since Fact is the inverse of $\Delta^!$ from Theorem 4.3.2, an equivalent statement is that for any commutative Λ^+ -graded factorization algebras $\mathcal{A}, \mathcal{B} \in \text{Shv}(\text{Div}^{\Lambda^+} C)$, there is a natural isomorphism

$$(\Delta^! \mathcal{A}) \overset{\Delta}{\otimes} (\Delta^! \mathcal{B}) \cong \Delta^!(\mathcal{A} \overset{\Delta}{\otimes} \mathcal{B}) \in \text{ComAlg}(\text{Shv}(C)^{\Lambda^+}, \overset{!}{\otimes})$$

where the “ $\overset{\Delta}{\otimes}$ ” on the left side to Construction 8.2.1 and the “ $\overset{\Delta}{\otimes}$ ” on the right side refers to Construction 8.4.1. This follows directly from the combinatorics of the definitions.

(2) Follows from dualizing (1). \square

Lemma 8.4.3. *With notation as in Proposition 8.2.3 and Proposition 8.3.2, we have*

$$\text{Dmod}(\text{Bun}_G) \ni \text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \overset{\Delta}{\otimes} \text{Std}} \star \mathscr{W}_{\langle 6(g-1) \rangle} \xrightarrow{\mathbb{L}_G} \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std} \overset{\Delta}{\otimes} \text{Sym}^\bullet \text{Std})) \otimes \omega_{\text{Loc}_{\check{G}}} \in \text{QCoh}(\text{Loc}_{\check{G}}).$$

Proof. The Whittaker normalization (5.1.2) in this case reads

$$\text{Dmod}(\text{Bun}_G) \ni \mathscr{W}_{\langle 6(g-1) \rangle} \xrightarrow{\mathbb{L}_G} \omega_{\text{Loc}_{\check{G}}} \in \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}}). \quad (8.4.5)$$

By Example 5.2.2, specialized to $r = 2$, \mathbb{L}_{GL_2} intertwines the tensoring action of

$$\text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std})) \in \text{QCoh}(\text{Loc}_2) \text{ on } \text{IndCoh}_{\text{Nilp}}(\text{Loc}_2)$$

with the Hecke convolution action of

$$\prod_{n \geq 0} \underline{k}^{\langle n \rangle} \in \text{Shv}(\text{Hk}_{2, \text{Sym}^\bullet(C)}^{\text{Std}})$$

Hence by Lemma 8.4.2(2), we deduce that \mathbb{L}_G intertwines the tensoring action of

$$\text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std} \overset{\Delta}{\otimes} \text{Sym}^\bullet \text{Std})) \in \text{QCoh}(\text{Loc}_{\check{G}}) \text{ on } \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$$

with the Hecke convolution action of

$$\prod_{n \geq 0} \underline{k}^{\langle 2n \rangle} \in \text{Shv}(\text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \overset{\Delta}{\otimes} \text{Std}})$$

Acting by these operators on either side of (8.4.5) gives the result. \square

8.4.3. *Conclusion.* Twisting (8.3.31) by $\langle 8(g-1) \rangle$, we get an exact triangle in $\text{Shv}(\text{Bun}_G)$:

$$(\text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \overset{\Delta}{\otimes} \text{Std}}) \star \mathscr{W}_{\langle 6(g-1) + \text{deg} \rangle} \rightarrow \pi_{\text{aut}}!(\underline{k}_{\text{Bun}_G^X})^{\langle 8(g-1) \rangle} \rightarrow \text{Eis}_{B \times B}(g! \underline{k}_{\text{Bun}_1})^{\langle 8(g-1) + 2 \text{deg} \rangle}$$

where we used that $2r_E + 8(g-1) = 2(1-g) + \text{deg} + 8(g-1) = 6(g-1) + \text{deg}$.

Therefore, we have proved the following theorem.

Theorem 8.4.4. *The Geometric Langlands equivalence $\mathbb{L}_G: \text{Dmod}(\text{Bun}_G) \rightarrow \text{IndCoh}_{\text{Nilp}}(\text{Loc}_{\check{G}})$ intertwines the indicated objects in the diagram below, where each row is exact*

$$\begin{array}{ccccc} \text{Hk}_{G, \text{Sym}^\bullet(C)}^{\text{Std} \overset{\Delta}{\otimes} \text{Std}} \star \mathscr{W}_{\langle 6(g-1) \rangle} & \longrightarrow & \mathcal{P}_X^{\langle 8(g-1) \rangle} & \longrightarrow & \text{Eis}_{B \times B}(g! \underline{k}_{\text{Bun}_1})^{\langle 6(g-1) + \text{deg} \rangle} \\ \text{Lemma 8.4.3} \downarrow \mathbb{L}_G & & \downarrow \mathbb{L}_G? & & \downarrow \mathbb{L}_G \quad (8.4.2) \\ \text{Loc}^{\text{spec}}(\text{Fact}(\text{Sym}^\bullet \text{Std} \overset{\Delta}{\otimes} \text{Sym}^\bullet \text{Std})) \not\parallel & \longrightarrow & \mathcal{L}_{\check{X}} & \longrightarrow & \text{Eis}_{B \times B}^{\text{spec}}(g_*^{\text{spec}} \omega_{\text{Loc}_A \times \text{Loc}_1 \times \text{Loc}_1}) \end{array} \quad (8.4.6)$$

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