

ALGEBRA QUAL PREP: FIELDS AND GALOIS THEORY

TONY FENG

BACKGROUND

In the next two sections we briefly review some of important background knowledge about field extensions, which will be needed for the problem assigned.

Finite field extensions. Important facts to know:

- There is a *unique* extension of \mathbf{F}_q with degree n , denoted \mathbf{F}_{q^n} . (It has q^n elements by linear algebra.)
- All extensions of \mathbf{F}_q are *automatically* Galois, and their Galois groups are always *cyclic*, generated by the “Frobenius” automorphism $x \mapsto x^q$.

Cyclotomic extensions. Important facts to know:

- If ζ_n is an n th root of unity, then $\mathbf{Q}(\zeta_n)/\mathbf{Q}$ is a *cyclic Galois extension* with Galois group $\text{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q}) \xrightarrow{\sim} (\mathbf{Z}/n)^\times$. The isomorphism sends $\sigma \in \text{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q})$ to its action on $\mu_n \cong \mathbf{Z}/n$, noting that $\text{Aut}(\mathbf{Z}/n) = (\mathbf{Z}/n)^\times$.

Attempt the following problems.

1. SPLITTING FIELDS

- (1) Spring 2010 M4
- (2) Spring 2011 M2
- (3) Fall 2015 M3

2. ABSTRACT FIELD EXTENSIONS

- (1) Fall 2010 A3
- (2) Spring 2012 M3
- (3) Fall 2012 A7

3. GALOIS THEORY

- (1) Spring 2014 A3
- (2) Spring 2010 A1
- (3) Fall 2010 M4
- (4) Fall 2011 A5