FALL 2022 GRADUATE LEARNING SEMINAR
ARITHMETIC OF ALGEBRAIC DIFFERENTIAL EQUATIONS

The main reference is [Kat70]. This paper gives an arithmetic proof of the theorem that the Picard–Fuchs equations (or in other words, relative de Rham cohomology with the Gauss–Manin connection) have regular singularities and the local monodromy around the singularities are quasi-unipotent.
The organizational meeting is 8/31 (with a short teaser talk by Sug Woo).

(1) **9/7 (Yunqing)** Longer introduction.
(2) **9/14 and 9/21** Review of local systems, connections (with log singularities), and the de Rham complex loosely following [Kat70, §§1-4] and [Kat76].
The goals of the talks are:
   - Explain Gauss–Manin connection and Pichard–Fuchs equation;
   - Explain Deligne's Riemann–Hilbert correspondence ([Kat76, pp. 545-547]; here one introduces the notion of regular singularities using connections with log singularities in [Kat70, §4]; see also [Kat70, 11.2] and [Kat76, p. 546]; you may also consult Deligne's book [Del70, §§II.4-II.5]);
   - Explain "exponents" at the singularities (you may start from the analytic definition [Kat70, Remark 12.3]; see also [Kat76, pp. 538-539, 548]; in general, you may also state Manin's theorem [Kat70, Theorem 12.0] without proof and then define exponents as in [Kat70, Remark 12.2]);
   - Work out some examples. For instance, rank 1 differential equation on [Kat76, p.539] and its local monodromy; example of irregular singularities on [Kat76, p.541]; Gauss–Manin connection of the Legendre family mentioned in [Kat70, (0.5.2)] (for the computation, see for instance [Ked08, §3, Example 3.4]).

(3) **9/28 and 10/5** Connections in characteristic p following [Kat70, §§5-6].
The goals of the talks are:
   - Define p-curvature of a vector bundle with connection in char p; Cartier's theorem [Kat70, Theorem 5.1];
   - Basic properties of the p-curvature [Kat70, Proposition 5.2];
   - Define the notion "nilpotent of exponents $\leq n$" [Kat70, Cor 5.5, Def 5.6];
   - Nilpotence under pullback and higher direct image [Kat70, Thms 5.9, 5.10];
   - Generalization to connections with log singularities [Kat70, §6].

(4) **10/12** De Rham cohomology in char p following [Kat70, §7].
The goal of the talk is:
   - Explain Cartier isomorphism [Kat70, Thm 7.2] with examples;

- prove Cartier isomorphism. (For some motivation of the construction, more precisely its relation with Frobenius lifting, see for instance the discussion from 3.7 in Illusie's paper in the volume [BDIP02].

[Ogu04, Theorem 1.2.1] gives a generalization of the classical Cartier isomorphism for the de Rham cohomology sheaves of a vector bundle with flat connection (the classical Cartier isomorphism is for $(\mathcal{O}_X, d)$). One could also discuss this generalization if time permits.

(5) **10/19** Estimates on nilpotence following [Kat70, §§7,9,10].
The goal of the talk is:
- Explain how to use Cartier isomorphism to give the estimate of nilpotent exponent in char p (the conclusion is [Kat70, Cor 7.5];
- Define global nilpotence [Kat70, §9];
- Estimate the global nilpotence exponent from the local ones above [Kat70, §10].

(6) **10/26** Regular singularities following first half of [Kat70, §11].
The goal of this talk is:
- Recall the definition of regular singularities given in talk 9/14 and relate to [Kat70, (11.2)]; [Kat70] focuses on the 1-dimensional case; define the notion of regular singularities in high dimension, if not mentioned in talk 9/14 (or recall if alreday mentioned) and explain that for our purpose, it is enough to consider 1-dimensional base. See for instance [Kat71, §§II-III];
- Basic properties [Kat70, (11.3),(11.5)] (extension, sub, quotient, reduction to cyclic ones);
- Define property (J) (Jurkat's Estimate) [Kat70, (11.6)] and prove that it is independent of the choice of the bases [Kat70, (11.6.3)]; state basic properties [Kat70, (11.7),(11.8.1)].

(7) **11/2** Turritin's Theorem following second half of [Kat70, §11].
The goal of this talk is:
- State and prove Turrittin's theorem [Kat70, (11.10)]; the proof uses induction argument and reduces to cyclic connections. The cyclic case is discussed in [Kat70, (11.9)];
- If time permits, mention some direct consequences such as [Kat70, (11.9.20),(11.11),(11.12)].

(8) **11/9** Local monodromy following [Kat70, §12]; additional reference for examples and computations is [DGS94, III.7-8, V Lemma 2.4].[1]
The goal of this talk:
(a) Recall the geometric/complex analytic definition of local monodromy/exponents from talks 9/14 and 9/21 and give some example(s). One may start from [Kat70, (12.5)].
(b) State Manin's theorem [Kat70, Thm (12.0)] and define the local monodromy $C(\mathfrak{p})$. Give more examples such as the Gauss–Manin connection of the Legendre family, and more generally the differential equation of Gauss hypergeometric equations. The computation is buried in

[1] Although [DGS94] formulated some results in terms of scalar differential equations (of high order) instead of vector bundle with connections, one can freely translate the result from one language to another as one can always write a cyclic vector bundle with connection into a scalar differential equation (after possibly restricting to a Zariski open neighborhood of the singularity in question).

the proof of Manin's theorem. A reference is [DGS94, III.8, V Lemma 2.4]. Explain [Kat70, Remark (12.2)] (see also the computation in [Kat70, Remark (12.6)].

(c) Define quasi-unipotency and exponent of nilpotence of the local monodromy [Kat70, (12.4)] and discuss this property under inverse and direct image [Kat70, Prop (12.7)].

(d) If time permits, explain how to use local monodromy to construct the canonical extension of vector bundle with connection in Deligne's Riemann–Hilbert correspondence following [Kat76, pp. 548-549].

COMMENT: if the audience is interested in seeing more examples and giving some more detailed discussion on canonical extension, we may also break this talk into two talks.

(9) **11/16** Applications I: global nilpotence implies regular singularities and quasi-unipotent local monodromy following [Kat70, §13].
The goal of the talk is to state and prove [Kat70, Thm (13.0)]. Note that from the proof in *loc. cit.*, in order to obain regular singularities, we only need to assume that the mod $\mathfrak{p}$ connection to be nilpotent for infinitely many $\mathfrak{p}$; in order to obtain quasi-unipotent local monodromy, we only need to assume that the mod $\mathfrak{p}$ connection to be nilpotent for all $\mathfrak{p}$ above a density 1 set of rational primes.
An easy corollary that is worth mentioning is that if the p-curvature vanishes for all but finitely many $\mathfrak{p}$, then this vector bundle with connection has regular singularities and all local monodromies are finite. This may be viewed as the first step towards the Grothendieck–Katz p-curvature conjecture (see [Kat82, Conjecture 10.1] and the introduction of [Kat72]).
([Kat70, Thm (13.0)] is also discussed in [DGS94, III.2 and theorem III.6.1].)

(10) **11/30** Applications II: the Gauss–Manin connection following [Kat70, §14].
The goal of is talk is to state and prove [Kat70, Thms (14.1),(14.3)]; actually the proofs are just combining the main theorems from talk 10/19 and talk 11/16. If the speaker of talk 10/19 did not manage to cover the details of the proofs, then another goal of this talk is to fill in all the details for talk 10/19. If there is no leftover from previous talks, then another goal of this talk is to discuss the p-curvature conjecture for Gauss–Manin connections. The conjecture in this case is proved by Katz in [Kat72]. The goal is to explain the proof of [Kat72, Theorem 5.1] (we only consider the simplified case with G trivial and D empty) assuming [Kat72, Theorem 3.2] (the relation between p-curvature and Kodaira–Spencer map). (For a summary of the proof in this special case, see for instance the introduction of [Men19]; a good alternative reference for [Kat72, Theorem 3.2] is [Ogu78, Theorem 2.9].)

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