ARITHMETIC VOLUMES OF MODULI STACKS OF SHTUKAS

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Dedicated to the memory of Dick Gross

Abstract. We define and study "tautological classes" in the cohomology of moduli stacks of shtukas, pursuing two directions of applications. First, we prove a formula relating the "arithmetic volume" of tautological classes to higher derivatives of Artin L-functions, which can be viewed as an arithmetic analog of Hirzebruch's Proportionality principle. Second, we define and analyze the structure of the "phantom tautological ring", using a general relation between Hecke correspondences and Vinberg's degeneration, and give applications to a function field analog of Colmez's Conjecture.

Contents

1. Introduction

1.1. **Volume and Hirzebruch proportionality for locally symmetric spaces.** Hirzebruch's Proportionality Theorem [Hir58] (in the compact case), extended by Mumford to the non-compact case [Mum77], asserts that the integral (also known as the "volume") of any Chern class polynomial of automorphic vector bundles on a locally symmetric space is proportional to the integral of the corresponding Chern classes on a partial flag variety. Moreover, the proportionality constant is essentially the special value of a certain product of (shifts of) zeta or Hecke L-functions.

More precisely, let (G, D) be a Shimura datum, where G is a reductive group over $\mathbf{Q}$ and $D \simeq G(\mathbf{R})/K$ is a Hermitian symmetric domain with K a maximal compact subgroup of $G(\mathbf{R})$. Let $G_c \subset G(\mathbf{C})$ be a compact form containing K, and $D^\vee = G_c/K$ the compact dual of D. Note that $D^\vee$ can be realized as the $\mathbf{C}$-points of a partial flag variety $G(\mathbf{C})/P_\mu(\mathbf{C})$, where P_μ is a parabolic subgroup associated to the Shimura cocharacter μ. We have an analytic open embedding $D \hookrightarrow D^\vee$. For any discrete subgroup $\Gamma \subset G(\mathbf{R})$, we have a natural map, which we call the *Hodge morphism*[1],

$$\mathrm{ev} : \Gamma\backslash D \longrightarrow G(\mathbf{C})\backslash D^\vee \cong \mathbb{B}P_\mu(\mathbf{C}). \tag{1.1.1}$$

Automorphic vector bundles on the source are by definition the pull-back of vector bundles on the target. Then the map ev induces a homomorphism on the cohomology rings

$$\mathrm{ev}^* : \mathrm{H}^*(\mathbb{B}P_\mu(\mathbf{C}), \mathbf{Q}) = R_{\mathbf{Q}}^{W_\mu} \longrightarrow \mathrm{H}^*(\Gamma\backslash D, \mathbf{Q})\ . \tag{1.1.2}$$

[1]Terminology based on [WZ23, Introduction] where the authors call the target the *Hodge stack*.

Here $R_{\mathbf{Q}} = \mathrm{H}^*(\mathbb{B}T(\mathbf{C}), \mathbf{Q})$, where T is a maximal torus of G, and W_μ is the Weyl group of the Levi subgroup associated to μ, acting on T in the natural way. Chern–Weil theory implies that the cohomology classes coming from $\mathrm{H}^{>0}(\mathbb{B}G(\mathbf{C}), \mathbf{Q}) = R^W_{\mathbf{Q},+}$ (here W is the Weyl group of G) are mapped to zero under ev^*. Thus the ring homomorphism ev^* factors through a homomorphism

$$\rho : \mathrm{H}^*(D^\vee, \mathbf{Q}) = R_{\mathbf{Q}}^{W_\mu}/(R^W_{\mathbf{Q},+}) \longrightarrow \mathrm{H}^*(\Gamma\backslash D, \mathbf{Q}) \ . \tag{1.1.3}$$

Now suppose $\Gamma\backslash D$ is compact and let $N = \dim D = \dim D^\vee$. The map ρ then restricts to an isomorphism on the top cohomology

$$\rho^{2N} : \mathrm{H}^{2N}(D^\vee, \mathbf{Q}) \xrightarrow{\sim} \mathrm{H}^{2N}(\Gamma\backslash D, \mathbf{Q}).$$

Now both sides are one-dimensional vector spaces, and they are each equipped with canonical trivialization (up to a Tate twist) given by pairing with the fundamental cycles of $D^\vee$ and of $\Gamma\backslash D$ respectively. Therefore, there is a unique constant $c \in \mathbf{Q}^\times$ (cf. [WZ23, Corollary 7.21]) making the following diagram commutative

$$\begin{array}{ccc} \mathrm{H}^{2N}(D^\vee, \mathbf{Q}) & \xrightarrow[\sim]{\rho^{2N}} & \mathrm{H}^{2N}(\Gamma\backslash D, \mathbf{Q}) \\ \Big\downarrow{\scriptstyle \int_{D^\vee}} & & \Big\downarrow{\scriptstyle \int_{\Gamma\backslash D}} \\ \mathbf{Q}(-N) & \xrightarrow{\quad c \quad} & \mathbf{Q}(-N) \end{array} \tag{1.1.4}$$

This is the famous *Hirzebruch Proportionality principle* [Hir58]. We may suggestively write the proportionality principle as

$$\int_{\Gamma\backslash D} \mathrm{ev}^*\eta = c \int_{D^\vee} \widetilde{\eta} \tag{1.1.5}$$

for any class $\eta \in \mathrm{H}^{2N}(\mathbb{B}P_\mu(\mathbf{C}), \mathbf{Q})$, where $\widetilde{\eta}$ is the pull back of η along the map $D^\vee \to G(\mathbf{C})\backslash D^\vee \cong \mathbb{B}P_\mu(\mathbf{C})$. When $\Gamma\backslash D$ is non-compact, Mumford proved in [Mum77] that the same assertion goes through after replacing it by any smooth toroidal compactification.

Moreover, the proportionality constant c in (1.1.5) can be interpreted as the special value of an L-function attached to the "motive of G" in the sense of Gross [Gro97] (up to an index factor depending on Γ).

1.2. **Arithmetic volumes of Shimura varieties.** When there are natural integral models for Shimura varieties and their automorphic vector bundles, there is an "arithmetic" analogue of the volume of Chern classes, defined via the arithmetic intersection theory, which we call the *arithmetic volume*. In this case, the arithmetic volume has only been computed for specific families of examples, such as in the case where G is the isometry group of a quadratic space [HÏ4] over $\mathbf{Q}$ or a hermitian space [BH, Guo25] with respect to an imaginary quadratic field extension of a totally real field with signature $(1, n-1)$ at one place and definite in all other places. In these works, the arithmetic volumes are shown to be essentially a special value of the first derivative of the L-function of G.

1.3. **Arithmetic volumes of moduli of Shtukas.** In this paper we investigate the analogous question over function fields, namely the arithmetic volumes of "automorphic vector bundles" on moduli spaces of shtukas for a general reductive group. Our initial motivation was to extract the intersection number out of the constant term of the arithmetic theta series in the top degree case in our previous work [FYZ25]. The main difficulty is that these moduli spaces are almost never proper and their compactifications (when the number of legs is more than one) are more complicated than the toroidal compactification of Shimura varieties. As one of the key steps of this paper, we will define a "regularized" volume, via the trace of the endomorphism on the cohomology of Bun_G induced by certain natural correspondences.

Now let X be a smooth, projective, geometrically connected curve over $\mathbf{F}_q$. For simplicity, in the Introduction we consider only split reductive groups G over $\mathbf{F}_q$; the main text treats arbitrary quasisplit reductive group schemes over X (and we explain in Remark 7.1.1 that we can reduce arbitrary reductive group schemes over X to the quasisplit case, for the purpose of computing arithmetic volumes). Let Bun_G be the moduli stack of G-bundles on X.

1.3.1. *Moduli of shtukas.* Let $\mu = (\mu_1, \cdots, \mu_r)$ be an admissible (see (5.2.1)) r-tuple of modification types for G. There is an associated moduli stack of shtukas Sht_G^μ. It is neither proper nor of finite type, hence the Chern numbers of Sht_G^μ are not a priori defined. We now outline a regularized definition of Chern numbers, and relate them to higher derivatives of the L-function attached to the motive of G (in the sense of Gross [Gro97]) over the curve X.

The general setup is as follows. We have a Hecke stack

$$\begin{array}{ccc} & \mathrm{Hk}_G^\mu & \\ {}^{h_0}\swarrow & & \searrow^{h_r} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{array}$$

defined using the same modification data μ as Sht_G^μ. Then Sht_G^μ is defined as the fibered product

$$\begin{array}{ccc} \mathrm{Sht}_G^\mu & \longrightarrow & \mathrm{Hk}_G^\mu \\ \downarrow & & \downarrow{\scriptstyle (h_0,h_r)} \\ \mathrm{Bun}_G & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array} \tag{1.3.1}$$

In other words, Sht_G^μ is the "Frobenius-twisted fixed points" of a correspondence on Bun_G. Therefore we regularize Chern numbers of Sht_G^μ as the trace of a suitable endomorphism on the cohomology of Bun_G.

1.3.2. *Endomorphisms of cohomology.* We have $\dim \mathrm{Sht}_G^\mu = r + \sum_{i=1}^r \langle 2\rho, \mu_j\rangle$. We write $D_{\mu_j} := \langle 2\rho, \mu_j\rangle$ and $D_\mu := \sum_{j=1}^r D_{\mu_j}$.

Henceforth, $\mathrm{H}^*(-)$ always means *geometric* cohomology of $\overline{\mathbf{Q}}_\ell$ (i.e., cohomology over the algebraic closure of the ground field), where ℓ is invertible in $\mathbf{F}_q$. Let $\eta_j \in \mathrm{H}^{2(D_{\mu_j}+1)}(\mathrm{Hk}_G^{\mu_j})$. The numerology is such that $\eta_j \cap [\mathrm{Hk}_G^{\mu_j}]$ is a Borel-Moore homology class in $\mathrm{Hk}_G^{\mu_j}$ of degree $2 \dim \mathrm{Bun}_G$. For the natural correspondence

$$\mathrm{Bun}_G \xleftarrow{h_{j-1}} \mathrm{Hk}_G^{\mu_j} \xrightarrow{h_j} \mathrm{Bun}_G$$

we may view $\eta \cap [\mathrm{Hk}_G^{\mu_j}]$ as a cohomological correspondence for the constant sheaf on Bun_G supported on $\mathrm{Hk}_G^{\mu_j}$. Using that h_j is proper, it induces a degree-preserving map on cohomology (of the geometric fiber with $\overline{\mathbf{Q}}_\ell$-coefficients)

$$\begin{aligned} \Gamma_{\eta_j}^{\mu_j} \quad : \quad & \mathrm{H}^*(\mathrm{Bun}_G) \to \mathrm{H}^*(\mathrm{Bun}_G) \\ & \theta \mapsto h_{j-1,*}(h_j^*(\theta) \cup \eta_j) \end{aligned}$$

Similarly, since h_j is proper it induces a degree-preserving map on compactly supported cohomology

$$\begin{aligned} {}_c\Gamma_{\eta_j}^{\mu_j} \quad : \quad & \mathrm{H}_c^*(\mathrm{Bun}_G) \to \mathrm{H}_c^*(\mathrm{Bun}_G) \\ & \theta \mapsto h_{j*}(h_{j-1}^*(\theta) \cup \eta_j) \end{aligned}$$

(The reversal of the roles of h_{j-1} and h_j in these definitions is due to the adjoint relationship of Γ_η^μ and ${}_c\Gamma_\eta^\mu$. This will be explained further in §5.1.3.)

1.3.3. *Tautological classes.* In analogy with the Shimura variety case, there is a natural way to supply such $\eta_j \in \mathrm{H}^{2(D_{\mu_j}+1)}(\mathrm{Hk}_G^{\mu_j})$, by taking Chern classes of automorphic vector bundles. We show in §3.1.3 that for each $i = 1, \ldots, r$ there is a canonical morphism

$$\mathrm{ev}_j : \mathrm{Hk}_G^{\mu_j} \longrightarrow \mathbb{B}P_{\mu_j}$$

where $P_{\mu_j} \subset G$ is the parabolic subgroup defined as the attracting locus of $\mu_j : \mathbb{G}_m \to G$. We also have the leg map $p_j : \mathrm{Hk}_G^{\mu_j} \to X$. We will call classes pulled back along the *enhanced Hodge morphism*

$$(\mathrm{ev}_j, p_j)^* : \mathrm{H}^{2(D_{\mu_j}+1)}(\mathbb{B}P_{\mu_j} \times X) \to \mathrm{H}^{2(D_{\mu_j}+1)}(\mathrm{Hk}_G^{\mu_j}) \tag{1.3.2}$$

tautological classes on $\mathrm{Hk}_G^{\mu_j}$.

1.3.4. *Arithmetic Volume.* We may now define the arithmetic volume of Sht_G^μ with respect to tautological classes. For the sake of simplicity, we present a simplified definition in the Introduction for the case where G is semisimple.

Definition 1.3.5. Assume G is semisimple. Let $\eta_j \in \mathrm{H}^{2(D_{\mu_j}+1)}(\mathbb{B}P_{\mu_j} \times X)$ for $j = 1, \cdots, r$. We define the volume of Sht_G^μ with respect to $\eta := (\eta_1, \cdots, \eta_r)$ to be

$$\mathrm{vol}(\mathrm{Sht}_G^\mu, \eta) := \mathrm{Tr}({}_c\Gamma_{\eta_r}^{\mu_r} \circ \cdots \circ {}_c\Gamma_{\eta_1}^{\mu_1} \circ \mathrm{Frob}^* \mid \mathrm{H}_c^*(\mathrm{Bun}_G)). \tag{1.3.3}$$

As $\mathrm{H}_c^*(\mathrm{Bun}_G)$ is infinite-dimensional, the convergence of the trace in the definition of $\mathrm{vol}(\mathrm{Sht}_G^\mu, \eta)$ needs to be justified, which we do in greater generality in Proposition 5.5.3.

Remark 1.3.6. For an r-tuple $\mu = (\mu_1, \cdots, \mu_r)$, the ev_j assemble to a morphism

$$\mathrm{ev}_\mu : \mathrm{Hk}_G^\mu \longrightarrow \textstyle\prod_{j=1}^r \mathbb{B}P_{\mu_j}$$

The pre-composition of ev_μ with $\mathrm{Sht}_G^\mu \to \mathrm{Hk}_G^\mu$ defines a morphism $\mathrm{Sht}_G^\mu \to \prod_j \mathbb{B}P_{\mu_j}$, which may be viewed as an analog of the Hodge morphism (1.1.1). Vector bundles on Sht_G^μ built from pullbacks of vector bundles on $\mathbb{B}P_{\mu_j}$ are the analogue of "automorphic vector bundles" on Shimura varieties. Therefore, our arithmetic volume may be viewed as the analog of the arithmetic volumes of automorphic vector bundles on Shimura varieties.

1.3.7. *Main results.* Our main result is a formula of the arithmetic volume in terms of (mixed) higher derivative of L-functions attached to the Gross motive of a reductive group [Gro97]. We present the formula in the simplified setting where G is split semisimple, although in the main text we treat general quasisplit $G \to X$, which witnesses more interesting L-functions.

Choose a maximal torus and Borel subgroup $T \subset B \subset G$ and denote the Weyl group $W(G,T)$ by W. Denote $R = \mathrm{H}^*(\mathbb{B}T)$, which is canonically isomorphic to the graded ring $\mathrm{Sym}(\mathbb{X}^*(T)_{\overline{\mathbf{Q}}_\ell}(-1))$, with the grading such that $\mathbb{X}^*(T)_{\overline{\mathbf{Q}}_\ell}(-1)$ is concentrated in degree 2. The *Gross motive of* G (or rather, its ℓ-adic realization) from §4.1.8 is the graded $\mathbf{Q}_\ell$-vector space with Frob-action

$$\mathbb{V}_G \simeq R_+^W/(R_+^W)^2 \tag{1.3.4}$$

where $R_+^W \subset R^W$ is the augmentation ideal. Let $(d_1, \ldots, d_n)$ be the multiset of degrees of a homogeneous basis $f_1, \cdots, f_n$ of $\mathbb{V}_G$. When G is almost simple, the multiset $(d_1, \ldots, d_n)$ coincides with the multiset $\{e_i + 1\}$, where $\{e_i\}$ are the exponents of G. We define the associated multivariate L-function to be[2]

$$\mathscr{L}_{X,G}(s_1, \cdots, s_n) := \prod_{i=1}^n \zeta_X(s_i + d_i).$$

For $j \in \{1, 2, \ldots, r\}$, let W_{μ_j} be the Weyl group of the Levi L_{μ_j} of P_{μ_j}. Via the pull-backs from the maps $\mathbb{B}T \to \mathbb{B}P_{\mu_j} \to \mathbb{B}G$, we have

$$R^W \simeq \mathrm{H}^*(\mathbb{B}G) \subset \mathrm{H}^*(\mathbb{B}P_{\mu_j}) \simeq R^{W_{\mu_j}}.$$

For each $\eta_j \in R^{W_{\mu_j}}$ of degree $2(D_{\mu_j} + 1)$, we define in (5.3.2) an endomorphism $\nabla_{\mu_j}^{\eta_j} : R^W \to R^W$. It is a derivation that carries R_+^W to R_+^W, and hence induces a graded linear endomorphism of $\mathbb{V}_G$:

$$\overline{\nabla}_{\mu_j}^{\eta_j} \in \mathrm{End}^{gr}(\mathbb{V}_G).$$

Now let $\mu = (\mu_1, \ldots, \mu_r)$ and $\eta = (\eta_1, \ldots, \eta_r)$ be r-tuples as above. We will assume (Commutativity Assumption 5.6.1) that the operators $\overline{\nabla}_{\mu_j}^{\eta_j}$ pairwise commute for $j = 1, \ldots, r$. The assumption holds automatically if the d_i's are all distinct, which holds for all simple groups except those of type D_{2n}, and also holds for GL_n. We may choose the homogeneous basis $f_1, \cdots, f_n$ of $\mathbb{V}_G$ to consist of (generalized) eigenvectors of $\overline{\nabla}_{\mu_j}^{\eta_j}$ for each $j \in \{1, \cdots, r\}$. Let $\epsilon_i(\eta_j, \mu_j)$ be the generalized eigenvalue of $\overline{\nabla}_{\mu_j}^{\eta_j}$ on f_i. Consider the differential operator

$$\mathfrak{d}_j := -(\log q)^{-1} \sum_{i=1}^n \epsilon_i(\eta_j, \mu_j) \partial_{s_i}.$$

[2]In the non-semisimple case the analogous definition needs to be slightly renormalized – see (5.6.2).

Some special cases of our main result for split semisimple G are as follows. First we consider tautological classes which come wholly from $\mathrm{H}^{2(D_{\mu_j}+1)}(\mathbb{B}P_{\mu_j})$.

Theorem 1.3.8. *Let $\mu = (\mu_1, \cdots, \mu_r)$ be an admissible sequence of minuscule dominant coweights of G. Let $\eta = (\eta_1, \ldots, \eta_r)$, where $\eta_j \in \mathrm{H}^{2(D_{\mu_j}+1)}(\mathbb{B}P_{\mu_j})$ satisfying the Commutativity Assumption 5.6.1. Then, with the notations introduced above, we have*

$$\mathrm{vol}(\mathrm{Sht}_G^\mu, \eta) = \#\pi_1(G) \cdot q^{\dim \mathrm{Bun}_G} \left(\prod_{j=1}^r \mathfrak{d}_j \right) \mathscr{L}_{X,G}(s_1, \cdots, s_n)\Big|_{s_1=s_2=\cdots=s_n=0}. \tag{1.3.5}$$

This theorem computes an analog of arithmetic volumes for (integral models of) Shimura varieties mentioned earlier. It relates the arithmetic volume of Sht_G^μ to higher derivatives of L-functions. We also prove a formula that may be viewed as an analog of the Hirzebruch Proportionality principle (1.1.5) for locally symmetric spaces. It is obtained by looking at tautological classes with "maximal contribution from the curve direction".

Theorem 1.3.9. *Let $\mu = (\mu_1, \cdots, \mu_r)$ be an admissible sequence of minuscule dominant coweights of G. Let $\eta_j' \in \mathrm{H}^{2D_{\mu_j}}(\mathbb{B}P_{\mu_j})$, and $\xi \in \mathrm{H}^2(X)$ be the class of a point. Let $\eta = (\eta_1'\xi, \ldots, \eta_r'\xi)$. Then with the notations above, we have*

$$\mathrm{vol}(\mathrm{Sht}_G^\mu, \eta) = \#\pi_1(G) \cdot q^{\dim \mathrm{Bun}_G} \mathscr{L}_{X,G}(0, \cdots, 0) \left(\prod_{j=1}^r \int_{G/P_{\mu_j}} \eta_j' \right). \tag{1.3.6}$$

Theorems 1.3.8 and 1.3.9 represent two extremes of tautological classes: those with "minimal" and "maximal" contribution from the cohomology of X. In general, we calculate $\mathrm{vol}(\mathrm{Sht}_G^\mu, \eta)$ for *all* tautological classes in the relevant degree (Theorem 5.6.9 for split G, and Theorem 7.4.12 for quasisplit G/X). We give a concrete illustration of Theorem 7.4.12 without spelling out all the definitions, deferring the full formulation to §7.6.

Theorem 1.3.10. *Let $\mathrm{Sht}_{\mathrm{U}(n)}^r$ be the moduli stack of Hermitian shtukas with respect to a finite étale double cover X'/X, and let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be the tautological line bundles on $\mathrm{Sht}_{\mathrm{U}(n)}^r$. Let $\mathcal{E}$ be a vector bundle of rank n on X'. With respect to these choices,*

$$\mathrm{vol}(\mathrm{Sht}_{\mathrm{U}(n)}^r, \prod_{j=1}^r c_n(p_j^*\mathcal{E}^* \otimes \mathcal{P}_j)) = 2\frac{q^{n^2(g-1)}}{(\log q)^r} \left(\frac{d}{ds}\right)^r \Big|_{s=0} (q^{-s \deg \mathcal{E}} L_{X,\mathrm{U}(n)}(2s)).$$

Remark 1.3.11. One initial motivation for this work was to investigate the *singular* terms in a conjectural Higher Siegel–Weil formula, generalizing [FYZ24] which proved such a formula for the "non-singular terms". On the other end of the spectrum, the "most singular term", contributing to the constant Fourier coefficient, is the arithmetic volume of certain tautological Chern classes, and Theorem 1.3.10 should provide the comparison to the corresponding piece of the higher derivative of the Eisenstein series. In fact, our method was already used in the paper [FHM25] which extends the higher Siegel–Weil formula to the corank-1 case, the "least degenerate" of the singular terms. Concretely, [FHM25, §7] can be viewed as specializing our proof of Theorem 7.6.6 to the case $n = 1$.

1.3.12. *The case $r = 0$.* When $r = 0$, μ is empty and Sht_G^μ identifies with the groupoid $\mathrm{Bun}_G(\mathbf{F}_q)$ viewed as a discrete stack. In this case, our main result specializes to the Tamagawa number formula of Gaitsgory–Lurie [GL14][3], generalizing earlier work such as [BD09]. Our proof of this special case is the same as that of [GL14]: both are based on the Atiyah–Bott description for the cohomology of Bun_G. We prove this description for general reductive group schemes over X, using the simply connected case proved in [GL14] as input.

[3] They assume a simply connected hypothesis, but it is easy to deduce the general semisimple case from this.

1.3.13. *Calculation of eigenweights.* To apply Theorem 1.3.8 in practice, we need to calculate the eigenweights $\epsilon_i(\eta_j, \mu_j)$. In §6, we work out the eigenweights in many examples for G of Type A, B, D. More precisely, we obtain the eigenweights when $G = \mathrm{GL}_n$ or the special orthogonal groups, and μ_i are all special minuscule coweights corresponding to the standard representation of the Langlands dual group of G. For $G = \mathrm{GL}_n$ we are also able to calculate the eigenweights for $\mu = (1, 1, 0, \cdots, 0)$, for which the answer takes a much more complicated shape, see Proposition 6.2.1.[4] The determination of the remaining minuscule coweights in Type A, as well as the spin coweights in Types C and D will be explained in the separate paper [Fen26], using different methods (also devised by AI). Finally, a work-in-progress of Zeyu Wang will generalize the calculation of both the arithmetic volume and eigenweights to all (not necessarily minuscule) coweights in all types.

1.4. **Tautological rings.** It is natural to consider the cohomology classes on Sht^{μ}_G obtained by pulling back along the enhanced Hodge morphism (1.3.2)

$$(\mathrm{ev}_j, p_j) : \mathrm{Sht}^{\mu}_G \to \mathbb{B}P_{\mu_j} \times X$$

for $1 \leq j \leq r$. From this we obtain a ring homomorphism

$$\widetilde{\rho} : \widetilde{C}^{\mu}_G := \bigotimes_{j=1}^{r} (\mathrm{H}^*(X) \otimes R^{W_{\mu_j}}) \to \mathrm{H}^*(\mathrm{Sht}^{\mu}_G).$$

The *tautological ring* for Sht^{μ}_G is the image of $\widetilde{\rho}$. When G is a constant reductive group scheme over X, we define an ideal I^{μ}_G with explicit generators given by (8.1.3), and we prove in Theorem 8.1.7 that the pull-back map $\widetilde{\rho}$ factors through the quotient ring $C^{\mu}_G := \widetilde{C}^{\mu}_G / I^{\mu}_G$, which we call the *phantom* tautological ring.

1.4.1. *Structure of the phantom tautological ring.* The phantom tautological ring has nice properties. In Proposition 8.3.1 we show that the ring C^{μ}_G is a flat deformation of $\otimes_{j=1}^{r} \mathrm{H}^*(G/P_{\mu_j})$ over the Artinian ring $\mathrm{H}^*(X^r)$. As a corollary, the top degree of C^{μ}_G is canonically isomorphic to the top degree cohomology of $\prod_{j=1}^{r}(X \times G/P_{\mu_j})$,

$$(C^{\mu}_G)_{2N} \xrightarrow{\sim} \bigotimes_{j=1}^{r} (\mathrm{H}^2(X) \otimes \mathrm{H}^{2D_{\mu_j}}(G/P_{\mu_j})),$$

where $N = \sum_{j=1}^{r}(D_{\mu_j} + 1) = \dim \mathrm{Sht}^{\mu}_G$. With this isomorphism, the pairing with the fundamental class of $\prod_{j=1}^{r}(X \times G/P_{\mu_j})$ defines a volume functional on $(C^{\mu}_G)_{2N}$, denoted by $\int_{\prod_{j=1}^{r}(X \times G/P_{\mu_j})}$. On the other hand, we also have the volume functional $\mathrm{vol}(\mathrm{Sht}^{\mu}_G, -)$ defined earlier by (1.3.3) (applicable to all elements in $(\widetilde{C}^{\mu}_G)_{2N}$). Proposition 8.4.3 shows that $\mathrm{vol}(\mathrm{Sht}^{\mu}_G, -)$ factors through the quotient $(C^{\mu}_G)_{2N}$ of $(\widetilde{C}^{\mu}_G)_{2N}$. Then a special case of our main result (i.e., (1.3.6)) compares these two linear functionals on $(C^{\mu}_G)_{2N}$ (see Proposition 8.4.9)

$$\mathrm{vol}(\mathrm{Sht}^{\mu}_G, -) = \#\pi_1(G) q^{\dim \mathrm{Bun}_G} \prod_{i=1}^{n} \zeta_X(d_i) \cdot \int_{\prod_{j=1}^{r}(X \times G/P_{\mu_j})} (-).$$

This may be viewed as the analogue of Hirzebruch Proportionality for the moduli stack of shtukas.

Remark 1.4.2. Due to the non-properness of Sht^{μ}_G, the induced map $\rho : C^{\mu}_G \to \mathrm{H}^*(\mathrm{Sht}^{\mu}_G)$ is not injective (e.g., the top degree map vanishes). Hence the tautological ring of Sht^{μ}_G is a further quotient of C^{μ}_G. However, we have reasons to believe that C^{μ}_G should be viewed as the "correct" tautological ring for Sht^{μ}_G, that "amends" the non-properness of Sht^{μ}_G. In fact, we show that C^{μ}_G satisfies Poincaré duality under the volume form and hence carries the structure of a Frobenius algebra (Proposition 8.4.12). We expect that the image of C^{μ}_G is large enough to account for the Hecke (generalized) eigenspace for the most non-tempered representation, namely the trivial representation $\mathbf{1}$, in the cohomology:

$$C^{\mu}_G / \ker(\rho) \simeq \mathrm{H}^*(\mathrm{Sht}^{\mu}_G)[\mathbf{1}].$$

[4] This result was first found by a brutal calculation of the authors, and then proved by a more elegant argument found by Gemini Deep Think (*IMO Gold version*); we will present Gemini's argument instead of our original one.

1.4.3. *The tautological ring of Shimura varieties.* Our results should also give a conceptual framework for the tautological ring of Shimura varieties (i.e., the subring of the Chow ring generated by the Chern classes of all automorphic bundles), a topic which has seen much study, for example in [vdG99, EH17, WZ23]. In [EH17], Esnault–Harris proved that the positive degree ℓ-adic Chern classes of flat automorphic bundles vanish and conjectured that the same vanishing result should hold in Chow groups. We prove an analog of this result as a Corollary 8.1.10 to Theorem 8.1.7, where the ℓ-adic Chern classes of "flat automorphic bundle" in [EH17] correspond to the pull-back from ev_i of classes in $R^W = \mathrm{H}^*(\mathbb{B}G)$. It is amusing to note that in [EH17] the proof uses the Hecke eigen-property of the Chern classes of flat automorphic bundles, while our proof of the function field case avoids the expected Hecke eigen-property in view of Remark 1.4.2.

The right analog of the volume given by (1.3.5) in the number field case should be the arithmetic volume of integral models of Shimura varieties with respect to (the arithmetic Chern classes of) Hermitian metrized bundles. In the case of Siegel moduli space, Köhler [KÖ5] has obtained a proportionality principle, which resembles our Theorem 8.1.7. However, since the Siegel moduli space is non-proper, it does not have an immediate definition of arithmetic volumes. A conjecture of Maillot and Roessler [MR02] may be viewed as the analogous relation (8.1.3) for certain PEL type Shimura varieties. In a future work we will pursue the analogous questions for more general Shimura varieties, guided by Theorem 8.1.7.

1.5. **An analog of the Colmez conjecture.** In [Col93] Colmez conjectured a formula relating the stable Faltings height of an abelian variety with Complex Multiplication (by the ring of integers of a CM field) to the special value at $s = 1$ of the logarithmic Artin L-function attached to a class function arising from the corresponding CM type. Though the averaged (over all CM type for a fixed CM field) version of the Colmez conjecture has been proved [AGHMP18, YZ18], the general case remains wide open. In §9 we obtain a function field analog of the Colmez conjecture. Let $\pi : X \to Y$ be a finite Galois étale covering with Galois group Σ. For an r-tuple $\sigma \in \Sigma^r = \Sigma \times \cdots \times \Sigma$, consider the twisted diagonal map:

$$\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_r) : X \to X^r$$

sending $x \in X$ to $(\sigma_i(x))_{i=1}^r$. We restrict Sht_G^μ to the twisted diagonal above:

$$\mathrm{Sht}_{G,\sigma}^\mu := \mathrm{Sht}_G^\mu \times_{X^r,\sigma} X.$$

This may be viewed as an analog of a Shimura variety of Hilbert-Blumenthal type. The element $\Phi = \sum_{i=1}^r \sigma_i \in \overline{\mathbf{Q}}_\ell[\Sigma]$ in the group algebra is an analog of the CM type. In principle, Theorem 8.1.7 could allow us to compute the volume of $\mathrm{Sht}_{G,\sigma}^\mu$ with respect to any top degree form. In Theorem 9.2.6, we treat the case $G = \mathrm{PGL}_n$, where the volume involves special values of (logarithmic) Artin L-functions attached to the CM type. In the simplest case $G = \mathrm{PGL}_2$, the formula takes the following shape (we defer the unexplained notation to §9):

$$\begin{aligned}\mathrm{vol}(\mathrm{Sht}_{\mathrm{PGL}_2,\sigma}^\mu, \eta) \quad = \quad & -\frac{(r+1)!}{3!} q^{3g_X} \zeta_X(2) \cdot \#\pi_1(G) \cdot \\ & \left(2\frac{\zeta_X'(2)}{\log q\, \zeta_X(2)} + |\Sigma|^2 \frac{L'_{Y,(\Phi*\Phi^\vee)^\natural}(2)}{\log q\, L_{Y,(\Phi*\Phi^\vee)^\natural}(2)} + (g_Y - 1)|\Sigma|^2 \Big((\Phi * \Phi^\vee)(1) - \frac{r}{|\Sigma|} \Big) \right).\end{aligned}$$

The recipe of the class function in Colmez conjecture (cf. [Col98, §2] for a more explicit formula involving $\Phi * \Phi^\vee$) is completely analogous to the one above, except that here we have the special value at $s = 2$. In a future work we will pursue a similar question for Shimura varieties.

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2. Notation

2.1. **Cohomology.** For an algebraic stack $\mathcal{Y}$ over a field k and a prime number ℓ not equal to $\mathrm{char}(k)$, we follow the formalism of [LZ17] for $\overline{\mathbf{Q}}_\ell$-sheaves on $\mathcal{Y}$. We write $\mathcal{D}(\mathcal{Y})$ for the constructible derived category of étale $\overline{\mathbf{Q}}_\ell$-sheaves on $\mathcal{Y}$ in the sense of *loc. cit.*.

When talking about ℓ-adic cohomology of $\mathcal{Y}$, we always mean *geometric cohomology*, i.e., the cohomology of the base change $\mathcal{Y}_{\overline{k}}$. We will abbreviate $\mathrm{H}^*(\mathcal{Y}) := H^*(\mathcal{Y}_{\overline{k}}; \overline{\mathbf{Q}}_\ell)$.

2.2. **The curve.** Fix a prime p and finite field $\mathbf{F}_q$ of characteristic p. Let X be a smooth, projective, geometrically connected curve over $\mathbf{F}_q$. We set some notation for invariants of X.

2.2.1. *Homology and cohomology of X.* Let $\xi \in \mathrm{H}^2(X)(1)$ be the cycle class of a point. Let $[X] \in \mathrm{H}_2(X)(-1) \cong \mathrm{H}^0(X)$ be the fundamental class. We use the notation

$$\int_X : \mathrm{H}^2(X)(1) \to \overline{\mathbf{Q}}_\ell$$

to denote the pairing with $[X]$. The cup product on $\mathrm{H}^1(X)$ gives a symplectic pairing

$$\begin{aligned} \langle -,- \rangle : \mathrm{H}^1(X) \times \mathrm{H}^1(X) &\to \overline{\mathbf{Q}}_\ell(-1) \\ (\zeta_1, \zeta_2) &\mapsto \int_X \zeta_1 \zeta_2. \end{aligned}$$

For $z \in \mathrm{H}_i(X)$ we often write its homological degree i as $|z|$; similarly, for $\zeta \in \mathrm{H}^i(X)$ we write its cohomological degree i as $|\zeta|$.

We use

$$\mathrm{PD} : \mathrm{H}_*(X) \xrightarrow{\sim} \mathrm{H}^{2-*}(X)(1)$$

to denote the Poincaré duality isomorphism. In particular,

$$\mathrm{PD}(1) = \xi \quad \text{and} \quad \mathrm{PD}([X]) = 1. \tag{2.2.1}$$

Writing $\langle -,- \rangle$ also for the evaluation pairing induced by $\mathrm{H}_i(X) \cong \mathrm{H}^i(X)^*$, the isomorphism PD is characterized by the identity

$$\int_X \mathrm{PD}(z) \cdot \zeta = \langle z, \zeta \rangle$$

for any $z \in \mathrm{H}_{|z|}(X)$ and $\zeta \in \mathrm{H}^{|z|}(X)$. In other words, we have $\langle z, \zeta \rangle = \langle \mathrm{PD}(z), \zeta \rangle$, justifying the abuse of notation.

2.3. **Reductive groups.** For a torus T, we denote by $\mathbb{X}^*(T)$ and $\mathbb{X}_*(T)$ the character and cocharacter groups of T, respectively. For a cocharacter $\lambda \in \mathbb{X}_*(T)$, we denote by $L_\lambda = C_G(\lambda)$ the corresponding Levi subgroup of G, and P_λ the attracting parabolic to L_λ under the adjoint action of Γ_m on G via λ.

Given a character χ of L_λ, the associated line bundle $\mathcal{O}(\chi)$ on the flag variety G/P_λ is $G \times^{P_\lambda} \mathbf{A}^1$ where the action of P_λ on $G \times \mathbf{A}^1$ is so that $(gp, t) = (g, \chi(p)t)$. Under this normalization, anti-dominant weights correspond to semiample line bundles on G/P_λ.

Example 2.3.1. Let $G = \mathrm{GL}_n$, T be the standard diagonal torus, and $\lambda = (1, 0, \dots, 0)$. Then P_λ is the subgroup of matrices of the form

$$\begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

and G/P_λ identifies with $\mathbb{P}^{n-1}$. Under our conventions the character $\chi = (1, 0, \dots, 0) \in \mathbb{X}^*(T)$ corresponds to $\mathcal{O}(-1)$ on G/P_λ.

Let G be a split (connected) reductive group over a field k. Choose a maximal torus and Borel subgroup $T \subset B \subset G$. This choice induces a notion of dominant coweights in $\mathbb{X}_*(T)$. Denote the Weyl group $W(G,T)$ by W.

Fix a prime $\ell \neq \mathrm{char} k$. Let $V_T := \mathrm{H}_1(T, \overline{\mathbf{Q}}_\ell) := \mathbb{X}_*(T)_{\overline{\mathbf{Q}}_\ell}(1)$ be the $\overline{\mathbf{Q}}_\ell$ version of the Tate module of T. Let

$$R := \mathrm{Sym}(\mathbb{X}^*(T)_{\overline{\mathbf{Q}}_\ell}(-1)) = \overline{\mathbf{Q}}_\ell[V_T]$$

with the grading such that $\mathbb{X}^*(T)_{\overline{\mathbf{Q}}_\ell}(-1)$ is concentrated in degree 2. Then there is a canonical isomorphism of graded $\overline{\mathbf{Q}}_\ell$-algebras

$$R \cong \mathrm{H}^*(\mathbb{B}T).$$

Under this isomorphism, the subring of W-invariants is identified with $\mathrm{H}^*(\mathbb{B}G)$,

$$\mathrm{H}^*(\mathbb{B}G) \cong R^W = \overline{\mathbf{Q}}_\ell[V_T]^W. \tag{2.3.1}$$

2.4. **Graded algebras.** Let E be a field and A be an E-algebra with an augmentation $\epsilon : A \xrightarrow{\epsilon} E$. The *augmentation ideal* is $\ker(\epsilon)$ and its associated $\ker(\epsilon)$-adic filtration is denoted $F^n_{\mathrm{aug}} A = \ker(\epsilon)^n$. The *associated graded algebra* of A (for its augmentation filtration) is

$$\mathrm{Gr}^\bullet_{\mathrm{aug}}(A) = \oplus_{n \geq 0} \ker(\epsilon)^n / \ker(\epsilon)^{n+1}.$$

In case A is $\mathbb{Z}_{\geq 0}$-graded, and ϵ is also a graded map, $\mathrm{Gr}^\bullet_{\mathrm{aug}}(A)$ inherits the $\mathbb{Z}_{\geq 0}$-grading from A, in addition to the natural grading as the associated graded algebra.

The same discussion works more generally in the setting of an augmented algebra object (A, ϵ) in a symmetric monoidal abelian E-linear category.

3. Characteristic classes under modification

In this section, we prove a general result about the behavior of characteristic classes under "modifications" of G-bundles on a stack. We fix an algebraically closed base field k.

3.1. **Modifications of G-bundles.** Let S be a stack over k and $\mathcal{F}_0, \mathcal{F}_1$ be two G-torsors over S. Let $D \subset S$ be a Cartier divisor. Let

$$\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D} \tag{3.1.1}$$

be an isomorphism of G-bundles. We call φ a *modification (of the G-bundles $\mathcal{F}_0, \mathcal{F}_1$) along D*.

For any $V \in \mathrm{Rep}_k(G)$, let $\mathcal{F}_{i,V}$ be the associated vector bundle $\mathcal{F}_i \times^G V$ over S. The isomorphism φ induces an isomorphism of vector bundles over $S - D$,

$$\varphi_V : \mathcal{F}_{0,V}|_{S-D} \xrightarrow{\sim} \mathcal{F}_{1,V}|_{S-D}.$$

For a T-bundle $\mathcal{F}$ on S and $\lambda \in \mathbb{X}_*(T)$, there is a unique T-bundle $\mathcal{F}(\lambda D)$ such that for any $\chi \in \mathbb{X}^*(T)$, the associated line bundle $\mathcal{F}(\lambda D)_\chi$ satisfies

$$\mathcal{F}(\lambda D)_\chi \cong \mathcal{F}_\chi(\langle \chi, \lambda \rangle D).$$

This generalizes the construction of twisting a line bundle by a Cartier divisor.

Definition 3.1.1. Let $\lambda \in \mathbb{X}_*(T)$ be a dominant coweight. We say the modification φ from (3.1.1) is *of type* λ if the following condition holds fppf locally on S: there are T-reductions $\mathcal{F}_{0,T}$ and $\mathcal{F}_{1,T}$ of $\mathcal{F}_0$ and $\mathcal{F}_1$ respectively, such that the map φ respects the T-reductions, and extends to an isomorphism of T-bundles over S

$$\widetilde{\varphi}_T : \mathcal{F}_{0,T}(\lambda D) \xrightarrow{\sim} \mathcal{F}_{1,T}. \tag{3.1.2}$$

Example 3.1.2. For $G = \mathrm{GL}_n$ and a G-bundle $\mathcal{F}$, we consider the rank n vector bundle $\mathcal{E} := \mathcal{F}_V$ associated to the standard representation V of GL_n. Let $\lambda = (1, 0, \cdots, 0) \in \mathbb{Z}^n = \mathbb{X}_*(T)$. A modification $\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D}$ of G-bundles has type λ if the isomorphism $\varphi_V : \mathcal{E}_0|_{S-D} \xrightarrow{\sim} \mathcal{E}_1|_{S-D}$ (where $\mathcal{E}_i = \mathcal{F}_{i,V}$) extends to a map of coherent sheaves $\mathcal{E}_0 \to \mathcal{E}_1$ on S that fits into an exact sequence

$$0 \to \mathcal{E}_0 \xrightarrow{\varphi} \mathcal{E}_1 \to i_{D*}\mathcal{P} \to 0 \tag{3.1.3}$$

where $\mathcal{P}$ is a line bundle on D. We call such a modification φ an "upper modification of $\mathcal{E}_0$ of colength 1 along D".

3.1.3. *Canonical parabolic reduction.* Let

$$\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D} \tag{3.1.4}$$

be a modification along D, of type λ. Let P_λ and $P_{-\lambda}$ be the parabolic subgroups of G defined as the attracting and repelling loci to the Levi $L_\lambda = C_G(\lambda)$, under the adjoint action of $\mathbb{G}_m$ on G via λ.

Proposition 3.1.4. *The restriction $\mathcal{F}_0|_D$ carries a canonical P_λ-reduction.*

Proof. By definition, locally for the fppf topology on S we have T-reductions $\mathcal{F}_{0,T}$ and $\mathcal{F}_{1,T}$ such that φ respects the T-reductions and extends to an isomorphism

$$\widetilde{\varphi}_T : \mathcal{F}_{0,T}(\lambda D) \xrightarrow{\sim} \mathcal{F}_{1,T}.$$

We claim that the P_λ-reduction $\mathcal{P} := (\mathcal{F}_{0,T}|_D) \times^T P_\lambda$ is independent of choices of the T-reductions. To see this, we observe that a P_λ-reduction $\mathcal{P}'$ of $\mathcal{F}_0|_D$ is determined by its adjoint bundle $\mathrm{Ad}(\mathcal{P}') \subset \mathrm{Ad}(\mathcal{F}_0)|_D$. Now $\mathrm{Ad}(\mathcal{P}) \subset \mathrm{Ad}(\mathcal{F}_0)|_D$ is equal to the image of the vector bundle map over D

$$(\mathrm{Ad}(\mathcal{F}_0) \cap \mathrm{Ad}(\mathcal{F}_1))|_D \to \mathrm{Ad}(\mathcal{F}_0)|_D.$$

Here, $\mathrm{Ad}(\mathcal{F}_0) \cap \mathrm{Ad}(\mathcal{F}_1)$ denotes the preimage of $\mathrm{Ad}(\mathcal{F}_1)$ under the map $\mathrm{Ad}(\mathcal{F}_0) \to j_*\mathrm{Ad}(\mathcal{F}_1)$ induced by φ (where $j : S - D \hookrightarrow S$).

This characterization of $\mathcal{P}$ proves its independence of the choice of the T-reduction of $\mathcal{F}_0$, which allows us to descend it from a fppf cover of D, as desired.

□

We thus obtain a canonical P_λ-reduction of $\mathcal{F}_0|_D$, which we denote by $\mathcal{F}_0|_{D,P_\lambda}$. Similarly, the restriction $\mathcal{F}_1|_D$ carries a canonical $P_{-\lambda}$-reduction that we denote by $\mathcal{F}_1|_{D,P_{-\lambda}}$.

Denote the induced L_λ-torsor of $\mathcal{F}_i|_{D,P_{\pm\lambda}}$ by $\mathcal{F}_i|_{D,L_\lambda}$. Then there is a canonical isomorphism of L_λ-torsors

$$\psi_D : \mathcal{F}_0|_{D,L_\lambda}(\lambda D) \cong \mathcal{F}_1|_{D,L_\lambda} \tag{3.1.5}$$

where, observing that λ factors tautologically through $Z(L_\lambda)$, $\mathcal{E}(\lambda D)$ denotes the *central twist* of an L_λ-bundle $\mathcal{E}$ by D. Explicitly, $\mathcal{E}(\lambda D)$ can be described as the unique L_λ-bundle such that for any representation V of L_λ where the center $Z(L_\lambda)$ acts through a character $\omega : Z(L_\lambda) \to \mathbb{G}_m$, the associated bundle $\mathcal{E}(\lambda D)_V = \mathcal{E}_V(\langle \omega, \lambda \rangle D)$, where $\langle \omega, \lambda \rangle \in \mathbb{Z} = \mathrm{End}(\mathbb{G}_m)$ is the composition

$$\mathbb{G}_m \xrightarrow{\lambda} Z(L_\lambda) \xrightarrow{\omega} \mathbb{G}_m.$$

This definition is evidently consistent with Definition 3.1.1.

Example 3.1.5. We continue with Example 3.1.2 where $\mathcal{E}_0 \hookrightarrow \mathcal{E}_1$ is an upper modification of colength 1 along D. Restricting (3.1.3) to D, we get a map

$$\varphi_V|_D : \mathcal{E}_0|_D \to \mathcal{E}_1|_D$$

of constant rank $n-1$ between rank n bundles over D. Denote its image by $\mathcal{H} \subset \mathcal{F}_{1,V}|_D$, a rank $n-1$ subbundle. This gives the $P_{-\lambda}$-reduction of $\mathcal{F}_1|_D$. Similarly, the P_λ-reduction of $\mathcal{F}_0|_D$ is given by the line subbundle $\ker(\varphi|_D)$, which is isomorphic to $\mathcal{P} \otimes \mathcal{O}(-D)|_D$ with $\mathcal{P}$ as in (3.1.3).

3.2. **Formulation of the theorem.** Recall from (2.3.1) that characteristic classes of G-bundles are given by W-invariant polynomial functions on V_T. In particular, for $f \in R^W = \overline{\mathbf{Q}}_\ell[V_T]^W$ and a G-torsor $\mathcal{F}$ on S corresponding to $b_{\mathcal{F}} : S \to \mathbb{B}G$, we denote its f-characteristic class to be

$$f(\mathcal{F}) := b_{\mathcal{F}}^* f \in \mathrm{H}^*(S).$$

Let W_λ be the Weyl group of L_λ, viewed as a subgroup of W. A coweight $\lambda \in \mathbb{X}_*(T)$ can be viewed as a vector in $V_T(-1)$. It thus makes sense to define the partial derivative

$$\partial_\lambda f \in \overline{\mathbf{Q}}_\ell[V_T]^{W_\lambda}(-1) = R^{W_\lambda}(-1).$$

Similarly we define higher derivatives in the direction of λ:

$$\partial_\lambda^n f \in R^{W_\lambda}(-n).$$

Since $\mathrm{H}^*(\mathbb{B}L_\lambda) = \overline{\mathbf{Q}}_\ell[V_T]^{W_\lambda}$, we may view $\partial_\lambda^n f$ as an element in $\mathrm{H}^*(\mathbb{B}L_\lambda)(-n)$.

Theorem 3.2.1. *Consider a modification of G-bundles $\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D}$ of type λ. Let*

$$\nu_D := c_1(\mathcal{O}(D))|_D \in \mathrm{H}^2(D)(1)$$

be the Chern class of the normal bundle of D, and

$$i_{D!} : \mathrm{H}^*(D)(-1) \to \mathrm{H}^{*+2}(S)$$

be the Gysin map for the regular embedding $i_D : D \hookrightarrow S$.[5]

Then for any characteristic class $f \in \mathrm{H}^*(\mathbb{B}G) = R^W$, we have

$$f(\mathcal{F}_1) - f(\mathcal{F}_0) = i_{D!}\left(\sum_{n\geq 1} \frac{1}{n!}(\partial_\lambda^n f)(\mathcal{F}_0|_{D,L_\lambda}) \cdot \nu_D^{n-1}\right) \in \mathrm{H}^*(S). \tag{3.2.1}$$

Note that $\partial_\lambda^n f$ can be viewed as a characteristic class for L_λ-torsors (up to a Tate twist), hence $(\partial_\lambda^n f)(\mathcal{F}_0|_{D,L_\lambda})$ makes sense as an element in $\mathrm{H}^*(D)(-n)$.

Example 3.2.2. We continue with Example 3.1.2 and 3.1.5. Identify R with $\overline{\mathbf{Q}}_\ell[x_1, \cdots, x_n]$, with the action of $S_n = W$ permuting the variables. Consider $f := e_i(x_1, \cdots, x_n) \in R^W$, the degree i elementary symmetric polynomial in $x_1, \cdots, x_n$, so that $f(\mathcal{F}) = c_i(\mathcal{E})$ for rank n vector bundles $\mathcal{E}$. We have

$$\partial_\lambda e_i(x_1, \cdots, x_n) = \partial_{x_1} e_i(x_1, \cdots, x_n) = e_{i-1}(x_2, \cdots, x_n).$$

The higher derivatives $\partial_{x_1}^{>1} e_i(x_1, \cdots, x_n)$ are zero. According to Example 3.1.5, the induced $L_\lambda = \mathbb{G}_m \times \mathrm{GL}_{n-1}$ bundle from the P_λ-reduction of $\mathcal{F}_0|_D$ is the pair $(\mathcal{P} \otimes \mathcal{O}(-D)|_D, \mathcal{H})$. In this case, Theorem 3.2.1 reads

$$c_i(\mathcal{F}_1) - c_i(\mathcal{F}_0) = i_{D!} c_{i-1}(\mathcal{H}), \quad 1 \leq i \leq n. \tag{3.2.2}$$

Example 3.2.3 (Pfaffians)**.** Consider the case $G = \mathrm{SO}_{2m}$ and $f \in \mathrm{H}^*(\mathbb{B}G) = \overline{\mathbf{Q}}_\ell[x_1, \cdots, x_m]^W$ is the Pfaffian $f = x_1 \cdots x_m$. (See §6.1.1 for more on the group theoretical description.)

Consider $\lambda = (1, 0, \cdots, 0)$ and an isomorphism $\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D}$ of G-torsors of type λ. Let V be the standard representation of G. Using φ to view $\mathcal{F}_{0,V}, \mathcal{F}_{1,V}$ as being subsheaves of their identified rationalizations, define the coherent sheaf $\mathcal{F}_{1/2}^\flat := \mathcal{F}_{0,V} \cap \mathcal{F}_{1,V}$. The image of $\mathcal{F}_{1/2}^\flat|_D \to \mathcal{F}_{1,V}|_D$ gives a co-isotropic hyperplane $\mathcal{H} \subset \mathcal{F}_{1,V}|_D$, hence corresponds to an SO_{2m-2}-bundle $\mathcal{H}/\mathcal{H}^\perp$ over D.

Note that

$$\partial_\lambda f = \partial_{x_1}(x_1 \cdots x_m) = x_2 \cdots x_m$$

is the Pfaffian for SO_{2m-2}. The higher derivatives $\partial_{x_1}^{>1}(x_1 \cdots x_m) = 0$. In this case, Theorem 3.2.1 reads

$$\mathrm{Pf}(\mathcal{F}_1) - \mathrm{Pf}(\mathcal{F}_0) = i_{D!}\,\mathrm{Pf}(\mathcal{H}/\mathcal{H}^\perp).$$

3.3. **Proof of Theorem 3.2.1.** We prove Theorem 3.2.1 using a series of reductions.

3.3.1. *Product compatibility.* We show that if Theorem 3.2.1 holds for two elements $f, g \in R^W$, then it holds for their product fg.

Abbreviate $\mathcal{H} = \mathcal{F}_0|_{D,L_\lambda}$. By assumption, we have

$$\begin{aligned} fg(\mathcal{F}_1) - fg(\mathcal{F}_0) = &\left(f(\mathcal{F}_0) + \sum_{n\geq 1} i_{D!}(\frac{1}{n!}\partial_\lambda^n f(\mathcal{H}) \cdot \nu_D^{n-1})\right)\left(g(\mathcal{F}_0) + \sum_{n\geq 1} i_{D!}(\frac{1}{n!}\partial_\lambda^n g(\mathcal{H}) \cdot \nu_D^{n-1})\right) \\ &- f(\mathcal{F}_0)g(\mathcal{F}_0). \end{aligned} \tag{3.3.1}$$

Expanding this expression, we encounter two types of terms:

(1) $f(\mathcal{F}_0) i_{D!}(h)$ for some $h \in \mathrm{H}^*(D)$. We use projection formula to rewrite it as

$$f(\mathcal{F}_0) i_{D!}(h) = i_{D!}(i_D^* f(\mathcal{F}_0) \cdot h).$$

Note that $i_D^* f(\mathcal{F}_0) = f(\mathcal{F}_0|_D) = f(\mathcal{H})$, we then have

$$f(\mathcal{F}_0) i_{D!}(h) = i_{D!}(f(\mathcal{H}) \cdot h).$$

[5] It is induced from the canonical map $\overline{\mathbf{Q}}_\ell[-2](-1) \to i_D^! \overline{\mathbf{Q}}_\ell$ by taking global sections.

Similarly we have

$$i_{D!}(h)g(\mathcal{F}_0) = i_{D!}(h \cdot g(\mathcal{H})).$$

(2) $i_{D!}(h_1 \cdot \nu_D^{n_1-1}) \cdot i_{D!}(h_2 \cdot \nu_D^{n_2-1})$, where $h_1, h_2 \in \mathrm{H}^*(D)$. Again using the projection formula, and the identity $i_D^* i_{D!}(-) = (-) \cdot \nu_D$, we have

$$i_{D!}(h_1\nu_D^{n_1-1}) \cdot i_{D!}(h_2\nu_D^{n_2-1}) = i_{D!}(h_1 h_2 \nu_D^{n_1+n_2-1}).$$

Using these observations, the expansion of (3.3.1) can be written as

$$\sum_{n\geq 1} i_{D!}\left(\left(\sum_{n_1+n_2=n} \frac{1}{n_1!}\partial_\lambda^{n_1} f(\mathcal{H}) \frac{1}{n_2!}\partial_\lambda^{n_2} g(\mathcal{H})\right) \cdot \nu_D^{n-1}\right)$$

and we recognize the inner summation gives $\frac{1}{n!}\partial_\lambda^n(fg)(\mathcal{H})$ by the Leibniz rule. This shows that

$$fg(\mathcal{F}_1) - fg(\mathcal{F}_0) = i_{D!}\left(\sum_{n\geq 1} \frac{1}{n!}\partial_\lambda^n(fg)(\mathcal{H}) \cdot \nu_D^{n-1}\right),$$

as desired. □

Corollary 3.3.2. *If Theorem 3.2.1 holds for two reductive groups G_1 and G_2, then it holds for $G = G_1 \times G_2$.*

Proof. Since $\mathrm{H}^*(\mathbb{B}G) \cong \mathrm{H}^*(\mathbb{B}G_1) \otimes \mathrm{H}^*(\mathbb{B}G_2)$, it suffices to treat elements of the form $f_1 \otimes f_2 = (f_1 \otimes 1)(1 \otimes f_2)$, where $f_i \in \mathrm{H}^*(\mathbb{B}G_i)$. By the discussion in §3.3.1, it suffices to consider the case of $f_1 \otimes 1$ and $1 \otimes f_2$ separately. The equality (3.2.1) for $f_1 \otimes 1$ is obtained by pullback from the same equality for the modification of the induced G_1-bundles for the class f_1, and similarly for f_2. □

3.3.3. *Reduction: central isogeny.* Let $\theta : G \to G'$ be a central isogeny of connected reductive groups over k. We show that if Theorem 3.2.1 holds for G', then it holds for G.

Indeed, if $\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D}$ is a modification of type λ, letting $\mathcal{F}'_i = \mathcal{F}_i \times^G G'$, it induces a modification $\varphi' : \mathcal{F}'_0|_{S-D} \xrightarrow{\sim} \mathcal{F}'_1|_{S-D}$ of G'-bundles of type λ (now viewed as a dominant coweight of T', a maximal torus of G' containing $\theta(T)$). The pullback map $\theta^* : \mathrm{H}^*(\mathbb{B}G') \to \mathrm{H}^*(\mathbb{B}G)$ is an isomorphism, so we may view $f \in R^W$ also as a class $f \in \mathrm{H}^*(\mathbb{B}G')$, and we have $f(\mathcal{F}_i) = f(\mathcal{F}'_i) \in \mathrm{H}^*(S)$, and $\partial_\lambda^n f(\mathcal{F}_i|_{D,L_\lambda}) = \partial_\lambda^n f(\mathcal{F}'_i|_{D,L'_\lambda}) \in \mathrm{H}^*(D)$. Formula (3.2.1) now holds for f and the modification φ', which can then be viewed as the same formula for f and φ.

This reduction allows us to reduce to the case where G is a product of an adjoint group and a torus. In view of Corollary 3.3.2, we further reduce the statement to the case where G is either simple and adjoint, or $G = \mathbb{G}_m$.

3.3.4. *Case $G = \mathbb{G}_m$.* In this case, $\mathcal{F}_0$ and $\mathcal{F}_1$ are line bundles over S, and $\lambda \in \mathbb{X}_*(\mathbb{G}_m) = \mathbb{Z}$ is an integer. A modification $\varphi : \mathcal{F}_0|_{S-D} \xrightarrow{\sim} \mathcal{F}_1|_{S-D}$ is of type λ if and only if φ extends to an isomorphism $\mathcal{F}_0(\lambda D) \cong \mathcal{F}_1$. Now $\mathrm{H}^*(\mathbb{B}\mathbb{G}_m) = \overline{\mathbf{Q}}_\ell[c_1]$, where $c_1 \in \mathrm{H}^2(\mathbb{B}\mathbb{G}_m)$ is the first Chern class of the universal line bundle on $\mathbb{B}\mathbb{G}_m$. In view of §3.3.1, it suffices to prove (3.2.1) for $f = c_1$. Then

$$f(\mathcal{F}_1) - f(\mathcal{F}_0) = c_1(\mathcal{F}_1) - c_1(\mathcal{F}_0) = c_1(\mathcal{O}(\lambda D)) = i_{D!}(\lambda)$$

which gives (3.2.1) since $\partial_\lambda c_1 = \lambda \in \mathrm{H}^0(\mathbb{B}\mathbb{G}_m)$ and higher derivatives of c_1 vanish.

It remains to deal with the case where G is of adjoint type, for which we need some preparation.

3.3.5. *Wonderful compactification.* Let G be a simple adjoint group. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $\{\alpha_i\}_{i\in I}$ be the set of simple roots. Let $G \hookrightarrow \overline{G}$ be the wonderful compactification [DCP83]. The $G \times G$-orbits on $\overline{G}$ are given by intersections of simple normal crossing divisors $\{D_i\}_{i\in I}$ indexed by I. Each divisor classifies a map $\overline{G} \to [\mathbb{A}^1/\mathbb{G}_m]$, so all of them together give a smooth map

$$\pi : \overline{G} \to [\mathbb{A}^I/\mathbb{G}_m^I]. \tag{3.3.2}$$

For each subset $J \subset I$, let $e_J \in \mathbb{A}^I$ be the point with j-coordinate 1 if $j \in J$ and 0 otherwise. Let P_J be the standard parabolic subgroup of G whose Levi L_J containing T has simple roots $\{\alpha_j : j \in J\}$. Let

P_J^- be the parabolic subgroup opposite to P_J containing L_J. Then there is a canonical $G\times G$-equivariant isomorphism

$$\pi^{-1}(e_J) \cong (G \times G)/(P_J \times_{L_J} P_J^-). \tag{3.3.3}$$

Let $\lambda \in \mathbb{X}_*(T)$ be a dominant coweight. Then it extends to a map $a_\lambda : \mathbb{A}^1 \to \mathbb{A}^I$. Base changing (3.3.2) along $a_\lambda : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^I/\mathbb{G}_m^I]$, we get a map

$$\pi_\lambda : \overline{G}_\lambda := \overline{G} \times_{[\mathbb{A}^I/\mathbb{G}_m^I], a_\lambda} [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m] \tag{3.3.4}$$

whose fiber over 1 is G and fiber over 0 is $\pi^{-1}(e_J)$, where $J := \{i \in I | \langle \alpha_i, \lambda\rangle = 0\}$. The Levi subgroup L_J is the fixed point subgroup L_λ of the adjoint action of λ, and the parabolic subgroup $P_J = P_\lambda$ (resp. $P_J^- = P_{-\lambda}$) is the attracting (resp. repelling) locus of L_λ under this action. Hence we can rewrite (3.3.3) as a $G \times G$-equivariant isomorphism

$$\widetilde{D}_\lambda := \pi_\lambda^{-1}(0) \cong G \times G/(P_\lambda \times_{L_\lambda} P_{-\lambda}).$$

The preimage $D_\lambda := \pi_\lambda^{-1}(\{0\}/\mathbb{G}_m) = \overline{G}_\lambda - G \subset \overline{G}_\lambda$ is a smooth divisor of $\overline{G}_\lambda$. Let $L_\lambda^\flat := L_\lambda/\mathbb{G}_m$ where $\mathbb{G}_m$ acts by multiplication via $\lambda : \mathbb{G}_m \to Z(L_\lambda) \subset T$ (which is not necessarily injective, so $L_\lambda^\flat$ may be a group stack). We can describe the $G \times G$-variety D_λ as

$$D_\lambda \cong (G \times G)/(P_\lambda \times_{L_\lambda^\flat} P_{-\lambda}). \tag{3.3.5}$$

Lemma 3.3.6. *For a modification of G-bundles $\mathcal{F}_0|_{S-D} \overset{\sim}{\to} \mathcal{F}_1|_{S-D}$ as in §3.1, there is a canonical map*

$$h : S \to [\overline{G}_\lambda/(G \times G)]$$

with the following additional structures:

1. *The composition $S \to [\overline{G}_\lambda/(G \times G)] \to \mathbb{B}G \times \mathbb{B}G$ is classified by the G-bundles $(\mathcal{F}_0, \mathcal{F}_1)$.*
2. *The composition $S \to [\overline{G}_\lambda/(G \times G)] \xrightarrow{\pi_\lambda} [\mathbb{A}^1/\mathbb{G}_m]$, which is the datum of a line bundle $\mathcal{L}$ on S and a section of $\mathcal{L}^{-1}$, is given by $\mathcal{L} = \mathcal{O}(-D)$ and the tautological section $1 \in \mathcal{L}^{-1} = \mathcal{O}(D)$.*
3. *In particular, we have*
$$h|_D : D \to [D_\lambda/(G \times G)] \cong \mathbb{B}(P_\lambda \times_{L_\lambda^\flat} P_{-\lambda})$$
which is the same datum of a P_λ-bundle $\mathcal{P}_0$ on D, a $P_{-\lambda}$-bundle $\mathcal{P}_1$ on D and an isomorphism $\psi : \mathcal{P}_{0,L_\lambda} \cong \mathcal{P}_{1,L_\lambda}$ of their induced $L_\lambda^\flat$-bundles. Then referring to the canonical parabolic reductions of §3.1.3, we have a canonical isomorphism of P_λ-bundles $\mathcal{P}_0 \cong \mathcal{F}_0|_{D,P_\lambda}$ and a canonical isomorphism of $P_{-\lambda}$-bundles $\mathcal{P}_1 \cong \mathcal{F}_1|_{D,P_{-\lambda}}$, under which ψ is the isomorphism induced by ψ_D in (3.1.5) *(noting that the central twisting is trivial under our assumption that G is of adjoint type).*

Proof. The data of $(\mathcal{F}_0, \mathcal{F}_1, \varphi)$ gives a map $h_0 : S - D \to [G/(G \times G)]$. Since $[\overline{G}_\lambda/(G \times G)]$ is separated over $\mathbb{B}(G \times G)$, the extension of h_0 to $h : S \to [\overline{G}_\lambda/(G \times G)]$ is unique if it exists. Therefore it suffices to check the existence of h fppf locally on S.

We may thus assume there are T-reductions $\mathcal{F}_{i,T}$ of $\mathcal{F}_i$ such that φ respects the T-reductions and induces an isomorphism $\widetilde{\varphi}_T$ as in (3.1.2). This is equivalent to a map $h_{0,T} : S - D \to [T/(T \times T)]$. It is well-known that $\overline{G}$ contains an open subset $\overline{G}^\circ$ isomorphic to $(G \times G)/(B \times_T B^-)$, where B^- is the opposite Borel to B containing T, and the closure $\overline{T}^\circ$ of T in $\overline{G}^\circ$ can be identified with $\mathbb{A}^I$ via coordinates given by simple roots, so that the projection

$$\pi|_{\overline{T}^\circ} : \overline{T}^\circ \cong \mathbb{A}^I \to [\mathbb{A}^I/\mathbb{G}_m^I]$$

is the tautological projection. Let $\overline{T}_\lambda^\circ \subset \overline{G}_\lambda$ be the base change of $\overline{T}^\circ$ along $a_\lambda : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^I/\mathbb{G}_m^I]$. Then $\overline{T}_\lambda^\circ$ contains T as an open subset. Our goal is to extend $h_{0,T}$ to a map

$$h_T : S \to [\overline{T}_\lambda^\circ/(T \times T)]. \tag{3.3.6}$$

A map $S \to [\overline{T}^\circ/(T\times T)]$ is the same datum as a collection of effective Cartier divisors $\{D_i\}_{i\in I}$ on S, two T-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ and maps $\varphi_i : \mathcal{E}_{0,\alpha_i}(D_i) \to \mathcal{E}_{1,\alpha_i}$ of coherent sheaves on S for each $i \in I$. Such a map has a canonical factorization into $[\overline{T}_\lambda^\circ/(T\times T)]$ if we are given a single effective Cartier divisor $D \subset S$ such that $D_i = \langle \alpha_i, \lambda\rangle D$ for all $i \in I$. In other words, a map h_T as in (3.3.6) is the same datum as an effective Cartier divisor $D \subset S$, a pair of T-bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ and maps $\varphi_i : \mathcal{E}_{0,\alpha_i}(\langle \alpha_i, \lambda\rangle D) \to \mathcal{E}_{1,\alpha_i}$ of coherent sheaves on S for each $i \in I$. Now our D, the T-bundles $\mathcal{E}_0 = \mathcal{F}_{0,T}, \mathcal{E}_1 = \mathcal{F}_{1,T}$ and $\varphi_i : \mathcal{F}_{0,T}(\langle \alpha_i, \lambda\rangle D) \to \mathcal{F}_{1,T}$ induced from the $\widetilde{\varphi}_T$ give such a datum, hence inducing a map h_T as in (3.3.6) extending $h_{0,T}$. This proves the existence of h.

The claimed properties of h then follow easily from the construction. □

3.3.7. *Adjoint case.* Now we show Theorem 3.2.1 for G adjoint by reducing it to the torus case. By Lemma 3.3.6, it suffices to prove the Theorem for the universal case $S = [\overline{G}_\lambda/(G \times G)]$, $D = [D_\lambda/(G \times G)] \cong \mathbb{B}(P_\lambda \times_{L_\lambda} P_{-\lambda})$, and the canonical modification between the pullbacks of the tautological G-bundles via the first and second projection $h_i : S \to \mathbb{B}G$, $i = 0, 1$. Since the map (3.3.4) π_λ is smooth, D_λ is a regularly embedded divisor in S, hence by purity we have $i_D^! \overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_\ell[-2](1)$. We thus have an excision distinguished triangle

$$R\Gamma_{G\times G}(D_\lambda)[-2](-1) \to R\Gamma_{G\times G}(\overline{G}_\lambda) \to R\Gamma_{G\times G}(G) \to \tag{3.3.7}$$

From (3.3.5), we see that $R\Gamma_{G\times G}(D_\lambda) \cong R\Gamma(\mathbb{B}L_\lambda)$ and $R\Gamma_{G\times G}(G) \cong R\Gamma(\mathbb{B}G)$ are both concentrated in even degrees, hence the long exact sequence attached to the cohomology groups of (3.3.7) splits into short exact sequences in each even degree $2n$

$$0 \to \mathrm{H}^{2n-2}_{G\times G}(D_\lambda)(-1) \xrightarrow{i_{D_\lambda !}} \mathrm{H}^{2n}_{G\times G}(\overline{G}_\lambda) \to \mathrm{H}^{2n}_{G\times G}(G) \to 0 \tag{3.3.8}$$

Let $f \in \mathrm{H}^{2n}(\mathbb{B}G)$. We need to show that

$$h_1^* f - h_0^* f = i_{D_\lambda !}\Big(\sum_{i\geq 1} \frac{1}{i!}\partial_\lambda^i f \cdot \nu_{D_\lambda}^{i-1}\Big) \in \mathrm{H}^{2n}_{G\times G}(\overline{G}_\lambda). \tag{3.3.9}$$

On the other hand, consider the closure $\overline{T}_\lambda$ of $T \subset G$ in $\overline{G}_\lambda$, which is stable under the action of $T \times T$. It is well-known that the closure $\overline{T}$ of T in $\overline{G}$ is a smooth toric compactification of T, and the projection $\pi|_{\overline{T}} : \overline{T} \to [\mathbb{A}^I/\mathbb{G}_m^I]$ is also smooth. Therefore $\overline{T}_\lambda$ is also smooth over $[\mathbb{A}^1/\mathbb{G}_m]$. In particular, the divisor $D_{T,\lambda} = \overline{T}_\lambda - T$ is regularly embedded into $\overline{T}_\lambda$. We have a counterpart of the short exact sequence (3.3.8) for T

$$0 \to \mathrm{H}^{2n-2}_{T\times T}(D_{T,\lambda})(-1) \to \mathrm{H}^{2n}_{T\times T}(\overline{T}_\lambda) \to \mathrm{H}^{2n}_{T\times T}(T) \to 0. \tag{3.3.10}$$

Restriction gives a map from (3.3.8) to (3.3.10) that is easily seen to be injective on both ends, therefore the restriction map $\mathrm{H}^*_{G\times G}(\overline{G}_\lambda) \to \mathrm{H}^*_{T\times T}(\overline{T}_\lambda)$ is also injective. Therefore, to check (3.3.9), it suffices to check that its image in $\mathrm{H}^{2n}_{T\times T}(\overline{T}_\lambda)$ holds. This reduces to the case of the tautological modification between the two universal T-bundles on $[\overline{T}_\lambda/(T \times T)]$. Note that the $D_{T,\lambda}$ is not necessarily connected but has connected components $D_{T,\lambda}(\lambda')$ in bijection with coweights $\lambda' \in \mathbb{X}_*(T)$ in the W-orbit $W \cdot \lambda$, such that the modification type of the universal T-torsors along $D_{T,\lambda}(\lambda')$ is $\lambda' \in \mathbb{X}_*(T)$. The torus case of Theorem 3.2.1 has been proved already (by Corollary 3.3.2 and §3.3.4). Therefore (3.3.9) holds. This finishes the proof of Theorem 3.2.1 in the adjoint case, hence in general. □

4. The Atiyah-Bott formula for reductive group schemes

In this section we work over an algebraically closed base field $k = \overline{k}$. For a smooth projective curve X/k and reductive group scheme $G \to X$, we prove a formula for the ℓ-adic cohomology of the associated stack Bun_G that resembles the famous formula of Atiyah–Bott [AB83] for Betti cohomology when X is replaced by a Riemann surface, and G is split and semisimple. It has since been generalized to various other settings, but we require greater generality (where G may fail to be semisimple, or split) than previously treated in the literature.

4.1. **The classifying stack.** Let S be a scheme over k. Let $G \to S$ be a connected reductive group scheme, i.e., a flat relatively affine group scheme all of whose geometric fibers are connected reductive.

We have the classifying stack $\pi \colon \mathbb{B}G \to S$ and the direct image complex $\mathrm{R}\pi_*(\mathbf{Q}_\ell)$ on S. We first show that, roughly speaking, $\mathbb{B}G$ only depends on the outer class of G.

4.1.1. *Canonical quasisplit form.* Assume that S is connected. Then there is a unique connected reductive group $\mathbb{G}$ over k such that each k-fiber of G is isomorphic to $\mathbb{G}$. Also fix a pinning of $\mathbb{G}$, which identifies $\mathrm{Out}(\mathbb{G}) = \mathrm{Aut}(\mathbb{G})/\mathbb{G}_{\mathrm{ad}}$ with the subgroup of pinned automorphisms of $\mathbb{G}$. Let $\mathcal{P} = \underline{\mathrm{Isom}}_S(\mathbb{G}, G)$, which is a right $\mathrm{Aut}(\mathbb{G})$-torsor over S. Let $\mathcal{P}_{\mathrm{Out}} = \mathcal{P}/\mathbb{G}_{\mathrm{ad}}$ be the induced $\mathrm{Out}(\mathbb{G}) = \mathrm{Aut}(\mathbb{G})/\mathbb{G}_{\mathrm{ad}}$-torsor. We have a canonical isomorphism over S,

$$\mathbb{B}G \cong \mathcal{P} \times^{\mathrm{Aut}(\mathbb{G})} \mathbb{B}\mathbb{G}. \tag{4.1.1}$$

Definition 4.1.2. The *canonical quasisplit* form in the inner class of G is

$$G_{\mathrm{qs}} := \mathcal{P}_{\mathrm{Out}} \times^{\mathrm{Out}(\mathbb{G})} \mathbb{G}.$$

Lemma 4.1.3. *Assume that $\mathbb{G}_{\mathrm{der}}$ is of adjoint form, i.e., $\mathbb{G}_{\mathrm{der}} \to \mathbb{G}_{\mathrm{ad}}$ is an isomorphism. Then there is a canonical isomorphism over S*

$$\mathbb{B}G \cong \mathbb{B}G_{\mathrm{qs}}.$$

Here $\mathrm{Out}(\mathbb{G})$ *acts on* $\mathbb{G}$ *(and hence on* $\mathbb{B}\mathbb{G}$*) via pinned automorphisms.*

Proof. The assumption $\mathbb{G}_{\mathrm{der}} \xrightarrow{\sim} \mathbb{G}_{\mathrm{ad}}$ implies that $\mathbb{G} = \mathbb{G}_{\mathrm{ad}} \times \mathbb{A}$ for some torus $\mathbb{A}$. The action of $\mathbb{G}_{\mathrm{ad}}$ on $\mathbb{B}\mathbb{G} = \mathbb{B}\mathbb{G}_{\mathrm{ad}} \times \mathbb{B}\mathbb{A}$ (induced by the conjugation action on $\mathbb{G}_{\mathrm{ad}}$) is *canonically* trivial. Therefore we may rewrite the right side of (4.1.1) as

$$\mathcal{P}_{\mathrm{Out}} \times^{\mathrm{Out}(\mathbb{G})} \mathbb{B}\mathbb{G}$$

which is $\mathbb{B}G_{\mathrm{qs}}$. □

4.1.4. *Comparison under central isogenies.* Let $\theta\colon G \to G'$ be a central isogeny of reductive group schemes over S, which induces a map $\mathbb{B}\theta : \mathbb{B}G \to \mathbb{B}G'$.

$$\begin{array}{ccc} \mathbb{B}G & \xrightarrow{\theta} & \mathbb{B}G' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & S & \end{array}$$

Lemma 4.1.5. *The pullback map along* $\mathbb{B}\theta$

$$\mathrm{R}\pi'_* \mathbf{Q}_{\ell,\mathbb{B}G'} \to \mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G}$$

is an isomorphism.

Proof. It suffices to check on stalks, so we may reduce to the case $S = \mathrm{Spec}\ k$. Let $\Gamma = \ker(\theta)$. The presentation $\mathbb{B}G' = [\mathbb{B}G/\mathbb{B}\Gamma]$ exhibits $\mathbb{B}G$ as a gerbe over $\mathbb{B}G'$ banded by the group scheme Γ. Since Γ is finite, pullback induces an isomorphism on cohomology with $\mathbf{Q}_\ell$-linear coefficients for any Γ-gerbe. □

Let $\pi^{\mathrm{qs}} : \mathbb{B}G_{\mathrm{qs}} \to S$ be the structure map.

Proposition 4.1.6. *There is a canonical isomorphism in* $D(S)$

$$\mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G} \cong \mathrm{R}\pi^{\mathrm{qs}}_* \mathbf{Q}_{\ell,\mathbb{B}G_{\mathrm{qs}}}.$$

Proof. Let $G' = G_{\mathrm{ad}} \times A$, where $A = G/G_{\mathrm{der}}$ is the quotient torus of G. We have maps

$$\mathbb{B}G \to \mathbb{B}G' \cong \mathbb{B}G'_{\mathrm{qs}} \leftarrow \mathbb{B}G_{\mathrm{qs}}.$$

Here the middle isomorphism is given by Lemma 4.1.3. Both $G \to G'$ and $G_{\mathrm{qs}} \to G'_{\mathrm{qs}}$ are central isogenies, hence by Lemma 4.1.5 they induce isomorphisms on the direct images of classifying stacks. Composing these isomorphisms gives the desired canonical isomorphism. □

4.1.7. *Canonical splitting of* $\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G}$. Choose a connected component of $\mathcal{P}_{\mathrm{Out}}$ and denote it by S'. It is a torsor for a finite subgroup $\mathrm{Out}(\mathbb{G})^\natural \subset \mathrm{Out}(\mathbb{G})$. Let $\nu : S' \to S$ be the projection. Then G_{qs} becomes the constant group $\mathbb{G} \times S'$ after base change along ν. Therefore, with $R^W = \mathrm{R}\Gamma(\mathbb{B}\mathbb{G})$ as in §2.3, we have a canonical isomorphism

$$\nu^*\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G} \cong \nu^*\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G_{\mathrm{qs}}} \cong \mathrm{R}\Gamma(\mathbb{B}\mathbb{G}) \otimes \mathbf{Q}_{\ell,S'} = R^W \otimes \mathbf{Q}_{\ell,S'}. \tag{4.1.2}$$

The descent datum of the left side is given by the natural $\mathrm{Out}(\mathbb{G})^\natural$-action on R^W. In particular, we conclude that $\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G}$ is canonically isomorphic to the direct sum of its cohomology sheaves

$$\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G} \cong \bigoplus_{n\geq 0} \mathrm{R}^{2n}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G}[-2n]. \tag{4.1.3}$$

Each local system $\mathrm{R}^{2n}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G}$ is equipped with an isomorphism

$$\nu^*\mathrm{R}^{2n}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G} \cong \mathrm{H}^{2n}(\mathbb{B}\mathbb{G}) \otimes \mathbf{Q}_{\ell,S'}$$

whose descent datum is the natural $\mathrm{Out}(\mathbb{G})^\natural$-action on the right side.

4.1.8. *The Gross motive.* Using notation from §2.4, we have the augmentation filtration of $R^W = \mathrm{R}\Gamma(\mathbb{B}\mathbb{G})$, and in particular a graded vector space

$$\mathbb{V}_{\mathbb{G}} = \mathrm{Gr}^1_{\mathrm{aug}} R^W = R^W_+/(R^W_+)^2. \tag{4.1.4}$$

When $\mathbb{G}$ is almost simple, the multiset of degrees of $\mathbb{V}_{\mathbb{G}}$ (i.e., d repeated $\dim \mathbb{V}_{\mathbb{G},2d}$ times) coincides with the multiset $\{e_i + 1\}$, where $\{e_i\}$ are the exponents of $\mathbb{G}$.

Under the isomorphism (4.1.2), the augmentation filtration on R^W descends to a filtration $F_{\mathrm{aug}}\mathrm{R}\pi_*\mathbf{Q}_\ell$ on $\mathrm{R}\pi_*\mathbf{Q}_\ell$ (by direct summands), and $\mathbb{V}_{\mathbb{G}} \otimes \mathbf{Q}_{\ell,S'}$ descends to

$$\mathbb{V}_G := \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{R}\pi_*(\mathbf{Q}_\ell) \in \mathcal{D}(S).$$

The canonical splitting (4.1.3) induces a splitting of $\mathbb{V}_G$,

$$\mathbb{V}_G = \bigoplus_{d\in\mathbb{Z}_{>0}} \mathbb{V}_{G,2d}[-2d]$$

where each $\mathbb{V}_{G,2d}$ is a local system on S.

Remark 4.1.9. In [Gro97], B. Gross introduced the motive $\mathbb{M}_G$ of a reductive group G over any base field K. In our setup, if $S = \mathrm{Spec}\ K$, then $\mathbb{V}_G$ as a $\mathrm{Gal}(K^s/K)$-module is the ℓ-adic realization of $\mathbb{M}_{G_K}(-1)$. Comparing with Gross's definition [Gro97, (1.5)], our grading on $\mathbb{V}_G$ has been doubled, and each $\mathbb{V}_{2d}$ is already Tate twisted by $\overline{\mathbf{Q}}_\ell(-d)$. Our $\mathbb{V}_G$ coincides with what Gaitsgory–Lurie denote $M(G)$ in [GL14, Remark 6.4.10].

Example 4.1.10. Let $\nu : S' \to S$ be an étale double cover. Let G be a unitary group over S defined as the isometries of a S'/S-Hermitian rank n vector bundle $\mathcal{F}$ over S'. Let $\mathcal{L} = (\nu_*\mathcal{O}_{S'})^{\sigma=-1}$ where $\sigma : S' \to S'$ is the nontrivial involution over S. Then we have a canonical isomorphism

$$\mathbb{V}_G \cong \bigoplus_{i=1}^n \mathcal{L}^{\otimes i}[-2i].$$

The following is an immediate consequence of Proposition 4.1.6.

Lemma 4.1.11. (1) *Let $\theta : G \to G'$ be a central isogeny. Then θ^* induces a canonical isomorphism*

$$\mathbb{V}_{G'} \cong \mathbb{V}_G.$$

(2) *Let $A = G/G_{\mathrm{der}}$ and $\theta : G \to G_{\mathrm{ad}} \times A$ be the canonical central isogeny. Then θ induces a canonical isomorphism*

$$\mathbb{V}_G \cong \mathbb{V}_{G_{\mathrm{ad}}} \oplus \mathbb{V}_A.$$

Moreover, $\mathbb{V}_A$ is concentrated in degree 2, and $\mathbb{V}_{G_{\mathrm{ad}}}$ is concentrated in even degrees ≥ 4.

4.1.12. *More on central isogenies.* The rest of the discussion in this subsection is only used in the reduction step §4.3.2 in the proof of the Atiyah-Bott formula.

Let $\theta\colon G \to G'$ be a central isogeny of reductive group schemes over a stack S with kernel Γ. Since Γ is abelian, its classifying stack $\mathbb{B}\Gamma$ is an abelian group stack over S. Furthermore, since Γ is central in G, the multiplication map $a\colon \Gamma \times_S G \to G$ is a group homomorphism, and induces a map of classifying stacks

$$a\colon \mathbb{B}\Gamma \times_S \mathbb{B}G \to \mathbb{B}G \tag{4.1.5}$$

which extends to an action of the abelian group stack $\mathbb{B}\Gamma$ on $\mathbb{B}G$. This realizes $\mathbb{B}G'$ as the quotient stack $\mathbb{B}G' \cong [\mathbb{B}G/\mathbb{B}\Gamma]$.

Lemma 4.1.13. *With a the action map from* (4.1.5) *and* $\mathrm{pr}_1 : \mathbb{B}G \times_S \mathbb{B}\Gamma \to \mathbb{B}G$ *the projection, we have* $\mathrm{pr}_1^* = a^*$ *as maps*

$$\mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G} \to \mathrm{R}\Pi_* \mathbf{Q}_{\ell,\mathbb{B}G\times_S\mathbb{B}\Gamma},$$

where $\Pi : \mathbb{B}G \times_S \mathbb{B}\Gamma \to S$ *is the structure map.*

Proof. Since Γ is finite, pullback along the neutral section $e\colon S \to \mathbb{B}\Gamma$ induces an isomorphism on cohomology with $\mathbf{Q}_\ell$-linear coefficients. Hence it suffices to check the statement after pullback along the neutral section $\mathbb{B}G \xrightarrow{\mathrm{Id}\times e} \mathbb{B}G \times_S \mathbb{B}\Gamma$. But the maps a and pr_1 agree when composed with the neutral section, so a^* and pr_1^* obviously agree after such pullback. □

4.1.14. *G-characteristic classes.* For any map of stacks $S' \to S$, a G-torsor on S' is classified by a map $S' \to \mathbb{B}G$. Any cohomology class $c \in \mathrm{H}^*(\mathbb{B}G)$ thus gives rise to a system of compatible cohomology classes associated to G-torsors on $S' \to S$, which we call *G-characteristic classes.*

In the situation of §4.1.12, let $\mathcal{L}$ be a Γ-torsor on S. Then $\mathcal{L}$ can be interpreted as a section of $\mathbb{B}\Gamma$, and composing it with the action map (4.1.5) gives a map $\mathbb{B}G \to \mathbb{B}G$, which we refer to as "twisting a G-torsor by $\mathcal{L}$".

Corollary 4.1.15. *The action of twisting by $\mathcal{L}$ has a trivial effect on the G-characteristic classes.*

Proof. It suffices to check in the universal case, where the base is $\mathbb{B}\Gamma \times_S \mathbb{B}G$. In this case, the assertion amounts to Lemma 4.1.13. □

4.2. **Formulation of Atiyah-Bott formula.** Let X be a smooth connected and projective curve over k. Let $G \to X$ be a connected reductive group scheme.

Let Bun_G be the moduli stack of G-bundles over X. Since Bun_G classifies sections to the structure map $\pi : \mathbb{B}G \to X$, it only depends on $\mathbb{B}G$.

When G is the constant group $\mathbb{G} \times X$, we also write Bun_G as $\mathrm{Bun}_{\mathbb{G}}$.

4.2.1. *Components of* Bun_G. Let $\pi_0(\mathrm{Bun}_G)$ be the set of connected components of Bun_G. When $G = \mathbb{G} \times X$ is constant, $\pi_0(\mathrm{Bun}_{\mathbb{G}})$ can be identified with the algebraic fundamental group $\pi_1^{\mathrm{alg}}(\mathbb{G})$, which in particular has a group structure. For a maximal torus $\mathbb{T} \subset \mathbb{G}$, there is a canonical surjection of abelian groups

$$\mathbb{X}_*(\mathbb{T}) \twoheadrightarrow \pi_0(\mathrm{Bun}_{\mathbb{G}}) \tag{4.2.1}$$

whose kernel is the coroot lattice.

For general G, we have a local system of abelian groups over X

$$\pi_1^{\mathrm{alg}}(G/X) := \mathcal{P}_{\mathrm{Out}} \times^{\mathrm{Out}(\mathbb{G})} \pi_1^{\mathrm{alg}}(\mathbb{G}).$$

Then there is a canonical isomorphism

$$\pi_0(\mathrm{Bun}_G) \cong \mathrm{H}_0(X, \pi_1^{\mathrm{alg}}(G/X)). \tag{4.2.2}$$

This is a reformulation of [Hei10, Theorem 2]: the right side above is the same as the coinvariants of $\pi_1^{\mathrm{alg}}(G/X)$ under the monodromy action of $\pi_1(X)$.

For $\omega \in \pi_0(\mathrm{Bun}_G)$ we denote by Bun_G^{ω} the corresponding connected component of Bun_G.

4.2.2. *The evaluation map.* The universal G-bundle is classified by the evaluation map

$$\mathrm{ev}\colon X \times \mathrm{Bun}_G \to \mathbb{B}G. \tag{4.2.3}$$

Then (4.2.3) induces a map

$$\mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G} \to \mathrm{R}\Gamma(\mathrm{Bun}_G) \otimes \mathbf{Q}_{\ell,X}. \tag{4.2.4}$$

Tensoring (4.2.4) with the dualizing complex $\mathbf{D}_X$ and taking cohomology, we obtain a map

$$\mathrm{R}\Gamma(X; \mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G} \otimes \mathbf{D}_X) \to \mathrm{R}\Gamma(X; \mathbf{D}_X) \otimes \mathrm{R}\Gamma(\mathrm{Bun}_G).$$

Composing this with the trace map $\mathrm{R}\Gamma(X, \mathbf{D}_X) \to \mathbf{Q}_\ell$, we obtain in particular a canonical map

$$\mathrm{R}\Gamma(X; \mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G} \otimes \mathbf{D}_X) \to \mathrm{R}\Gamma(\mathrm{Bun}_G). \tag{4.2.5}$$

For $\mathcal{K} \in \mathcal{D}(X)$, the *homology* of X with coefficients in $\mathcal{K}$ is defined[6] to be

$$\mathrm{H}_*(X; \mathcal{K}) := \mathrm{H}^*(X; \mathcal{K} \otimes \mathbf{D}_X).$$

In these terms, we can interpret the map obtained by passing to cohomology in (4.2.5) as

$$\mathfrak{ev}_G : \mathrm{H}_*(X; \mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G}) \to \mathrm{H}^*(\mathrm{Bun}_G). \tag{4.2.6}$$

We denote its further restriction to Bun_G^ω by $\mathfrak{ev}_G^\omega : \mathrm{H}_*(X; \mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}G}) \to \mathrm{H}^*(\mathrm{Bun}_G^\omega)$.

4.2.3. *The constant group case.* We make the above discussions more explicit when $G = \mathbb{G} \times X$ is a constant group scheme. The map (4.2.6) now reads

$$\mathfrak{ev}_G : \mathrm{H}_*(X) \otimes \mathrm{H}^*(\mathbb{B}\mathbb{G}) = \mathrm{H}_*(X) \otimes R^W \to \mathrm{H}^*(\mathrm{Bun}_G).$$

For $z \in \mathrm{H}_*(X)$ and $f \in R^W$, we write

$$f^z = \mathfrak{ev}_G(z \otimes f) \in \mathrm{H}^*(\mathrm{Bun}_G). \tag{4.2.7}$$

4.2.4. *Graded evaluation map.* We define

$$\mathrm{H}_*(X; \mathbb{V}_G)_+ := \bigoplus_{i<2j} \mathrm{H}_i(X; \mathbb{V}_{G,2j})[i-2j] \subset \mathrm{H}_*(X; \mathbb{V}_G) \tag{4.2.8}$$

to be the subspace of $\mathrm{H}_*(X; \mathbb{V}_G)$ with cohomological degree > 0. For any $\omega \in \pi_0(\mathrm{Bun}_G)$, we will construct a canonical "evaluation" map

$$\mathfrak{ev}_{\mathrm{aug},G}^\omega : \mathrm{H}_*(X; \mathbb{V}_G)_+ \to \mathrm{Gr}^1_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{4.2.9}$$

We first make some reductions. If $\theta : G \to G' = G_{\mathrm{ad}} \times A$ (where $A = G/G_{\mathrm{der}}$) is the canonical map, it induces a map $\theta_{\mathrm{Bun}} : \mathrm{Bun}_G^\omega \to \mathrm{Bun}_{G'}^{\omega'}$. If $\mathfrak{ev}_{\mathrm{aug},G'}^{\omega'}$ is defined, then we let $\mathfrak{ev}_{\mathrm{aug},G}^\omega$ be the composition

$$\mathrm{H}_*(X; \mathbb{V}_G)_+ \cong \mathrm{H}_*(X; \mathbb{V}_{G'})_+ \xrightarrow{\mathfrak{ev}_{\mathrm{aug},G'}^{\omega'}} \mathrm{Gr}^1_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}_{G'}^{\omega'}) \xrightarrow{\mathrm{Gr}^1_{\mathrm{aug}} \theta^*_{\mathrm{Bun}}} \mathrm{Gr}^1_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

Here we use Lemma 4.1.11, and the last map is the map on $\mathrm{Gr}^1_{\mathrm{aug}}$ induced by the ring homomorphism θ^*_{Bun}. Using the additivity $\mathbb{V}_{G'} \cong \mathbb{V}_{G_{\mathrm{ad}}} \oplus \mathbb{V}_A$, we thus reduce to consider separately the case G is adjoint (or more generally semisimple) and G is a torus.

4.2.5. *The torus case.* Let $G = A$ be a torus. In this case, $\mathbb{V}_A = \mathrm{R}^2\pi_* \mathbf{Q}_{\ell,\mathbb{B}A}[-2]$, which is canonically a summand of $\mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}A}$ by (4.1.3). We define $\mathfrak{ev}_{\mathrm{aug},A}^\omega$ to be the composition

$$\mathrm{H}_*(X; \mathbb{V}_A)_+ \subset \mathrm{H}_*(X; \mathrm{R}\pi_* \mathbf{Q}_{\ell,\mathbb{B}A})_+ \xrightarrow{\mathfrak{ev}_A^\omega} \mathrm{H}^{>0}(\mathrm{Bun}_A^\omega) \twoheadrightarrow \mathrm{Gr}^1_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}_A^\omega).$$

[6] A priori, the relative homology of a map f should be defined to be the left adjoint $f_\sharp$ of f^*, if it exists. In this case, Verdier duality identifies $f_\sharp(\mathcal{K}) \cong \mathrm{R}f_*(\mathcal{K} \otimes \mathbf{D}_X)$.

4.2.6. *The semisimple case.* Let G be semisimple. Let $\mathrm{R}^{>0}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G} = F^1_{\mathrm{aug}}(\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G})$ be the direct sum of the positive degree summands under the canonical splitting (4.1.3). By construction we have a canonical map that is a surjection in each degree,

$$\mathrm{R}^{>0}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G} \to \mathbb{V}_G.$$

The desired map (4.2.9) is characterized by the following lemma.

Lemma 4.2.7. *There is a unique map $\mathfrak{ev}^{\omega}_{\mathrm{aug},G}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{H}_*(X;\mathrm{R}^{>0}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G})_+ & \xrightarrow{\mathfrak{ev}^{\omega}_G} & \mathrm{H}^{>0}(\mathrm{Bun}^{\omega}_G) \\ \downarrow & & \downarrow \\ \mathrm{H}_*(X;\mathbb{V}_G)_+ & \xrightarrow{\mathfrak{ev}^{\omega}_{\mathrm{aug},G}} & \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^{\omega}_G) \end{array}$$

Here $\mathrm{H}_*(X;\mathrm{R}^{>0}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G})_+ \subset \mathrm{H}_*(X;\mathrm{R}^{>0}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G})$ is the positive degree part.

Proof. Uniqueness is clear because the vertical maps are surjective. It remains to check that

$$\mathfrak{ev}^{\omega}_G \text{ sends } \mathrm{H}_*(X;F^2_{\mathrm{aug}}\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G})_+ \text{ to } F^2_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^{\omega}_G). \tag{4.2.10}$$

For this we compare G with the constant group situation.

Let $\nu : X' = \mathcal{P}_{\mathrm{Out}} \to X$ as in §4.1.7, so that $\nu^*\mathbb{B}G_{\mathrm{ad}}$ is canonically isomorphic to $\mathbb{B}\mathbb{G}_{\mathrm{ad}} \times X'$. This induces a canonical map $\varphi : \mathrm{Bun}_G \to \mathrm{Bun}_{\mathbb{G}_{\mathrm{ad}},X'}$ by pulling back G-bundles along ν and pushout along $\mathbb{B}G \to \mathbb{B}G_{\mathrm{ad}} \cong \mathbb{B}\mathbb{G}_{\mathrm{ad}}$. We have a commutative diagram

$$\begin{array}{ccc} X' \times \mathrm{Bun}_G & \xrightarrow{\widetilde{\mathrm{ev}}} & \nu^*\mathbb{B}G \\ \downarrow{\scriptstyle \mathrm{id}_{X'}\times\varphi} & & \downarrow \\ X' \times \mathrm{Bun}_{\mathbb{G}_{\mathrm{ad}},X'} & \xrightarrow{\mathrm{ev}'} & \mathbb{B}\mathbb{G}_{\mathrm{ad}} \end{array} \tag{4.2.11}$$

Here $\widetilde{\mathrm{ev}}$ is the base change of $\mathrm{ev} : X \times \mathrm{Bun}_G \to \mathbb{B}G$ along ν. Applying the discussion of the constant group case in §4.2.3, (4.2.11) induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_*(X') \otimes R^W & \xrightarrow{\mathfrak{ev}^{\omega'}_{\mathbb{G}_{\mathrm{ad}},X'}} & \mathrm{H}^{>0}(\mathrm{Bun}^{\omega'}_{\mathbb{G}_{\mathrm{ad}},X'}) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ \mathrm{H}_*(X;\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G}) & \xrightarrow{\mathfrak{ev}^{\omega}_G} & \mathrm{H}^{>0}(\mathrm{Bun}^{\omega}_G) \end{array}$$

Here the map α is the canonical quotient to $\mathrm{Out}(\mathbb{G})$-coinvariants. Since φ^* respects the augmentation filtrations, and α maps $\mathrm{H}_*(X') \otimes F^2_{\mathrm{aug}}R^W$ surjectively to $\mathrm{H}_*(X;F^2_{\mathrm{aug}}\mathrm{R}\pi_*\mathbf{Q}_{\ell,\mathbb{B}G})$, it suffices to check that (4.2.10) holds for the constant group $\mathbb{G} \times X'$ over X'.

We henceforth rename X' to X and let $G = \mathbb{G} \times X$, where $\mathbb{G}$ is semisimple. In this case, (4.2.10) is equivalent to: for any $f_1, f_2 \in R^W_+$ and any $z \in \mathrm{H}_*(X)$, we have $(f_1f_2)^z \in F^2_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^{\omega}_G)$. Indeed, let $\Delta_*(z) \in \mathrm{H}_*(X \times X) \cong \mathrm{H}_*(X) \otimes \mathrm{H}_*(X)$ be the image of z under the diagonal map for X. Write $\Delta_*(z) = \sum_i z'_i \otimes z''_i$ for $z'_i, z''_i \in \mathrm{H}_*(X)$. Then

$$(f_1f_2)^z = \sum_i f_1^{z'_i} f_2^{z''_i}. \tag{4.2.12}$$

Since G is semisimple, f_1, f_2 have cohomological degrees ≥ 4 while z'_i, z''_i have homological degrees ≤ 2, so $f_1^{z'_i}$ and $f_2^{z''_i}$ both have positive degrees, hence belong to $\mathrm{H}^{>0}(\mathrm{Bun}^{\omega}_G)$. Therefore $(f_1f_2)^z \in F^2_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^{\omega}_G)$. This proves (4.2.10) and finishes the proof. □

Now we can state the Atiyah-Bott formula for the cohomology of Bun_G.

Theorem 4.2.8. *For any $\omega \in \pi_0(\mathrm{Bun}_G)$, the $\overline{\mathbf{Q}}_\ell$-algebra map*

$$\mathrm{Sym}^{\blacktriangleright}_{\overline{\mathbf{Q}}_\ell}(\mathrm{H}_*(X;\mathbb{V}_G)_+) \to \mathrm{Gr}^{\blacktriangleright}_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^\omega_G) \tag{4.2.13}$$

induced by (4.2.9) *is an isomorphism.*

4.2.9. *The constant group case.* When $G = G_0 \times X$, choose a basis of $\mathrm{H}_*(X)$:

$$z_0 = 1 \in \mathrm{H}_0(X), \quad z_1, \cdots, z_{2g} \in \mathrm{H}_1(X) \text{ and } z_{2g+1} = [X] \in \mathrm{H}_2(X).$$

Choose also homogeneous free generators $f_1, \cdots, f_n$ of R^W as a polynomial ring. Then Theorem 4.2.8 implies that $\mathrm{H}^*(\mathrm{Bun}^\omega_G)$ is the polynomial ring over $\overline{\mathbf{Q}_\ell}$ with free generators

$$f_i^{z_j}|_{\mathrm{Bun}^\omega_G}, \quad 0 \le j \le 2g+1, 1 \le i \le n, \text{ such that } (\deg f_i, |z_j|) \neq (2,2). \tag{4.2.14}$$

(Here $|z_j|$ is the homological degree of z_j.) In particular, we get canonical isomorphisms between the cohomology rings of different components of Bun_G, under which the classes with the same name f^z (in the constant group case) correspond to each other.

Remark 4.2.10. When X and G are defined over $\mathbf{F}_q$, Bun_G is also defined over $\mathbf{F}_q$. The Frobenius Fr acts on $\pi_0(\mathrm{Bun}_{G,\overline{\mathbf{F}}_q})$ and induces a map $\mathrm{Bun}^\omega_{G,\overline{\mathbf{F}}_q} \to \mathrm{Bun}^{\mathrm{Fr}(\omega)}_{G,\overline{\mathbf{F}}_q}$. From the construction of (4.2.13), we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^{\blacktriangleright}_{\overline{\mathbf{Q}}_\ell}(\mathrm{H}_*(X_{\overline{\mathbf{F}}_q};\mathbb{V}_G)_+) & \longrightarrow & \mathrm{Gr}^{\blacktriangleright}_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^{\mathrm{Fr}(\omega)}_{G,\overline{\mathbf{F}}_q}) \\ \downarrow{\scriptstyle \mathrm{Fr}^*} & & \downarrow{\scriptstyle \mathrm{Fr}^*} \\ \mathrm{Sym}^{\blacktriangleright}_{\overline{\mathbf{Q}}_\ell}(\mathrm{H}_*(X_{\overline{\mathbf{F}}_q};\mathbb{V}_G)_+) & \longrightarrow & \mathrm{Gr}^{\blacktriangleright}_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Bun}^{\omega}_{G,\overline{\mathbf{F}}_q}) \end{array} \tag{4.2.15}$$

In particular, if $\mathrm{Fr}(\omega) = \omega$ then the isomorphism (4.2.13) is automatically Fr-equivariant.

4.2.11. *Prior results.* There have been many partial results towards Theorem 4.2.8 in the literature. The formula is inspired by a formula for closely related moduli spaces of G-bundles on a Riemann surface, for split semisimple G, due to Atiyah–Bott [AB83]. The analogous formula for Bun_G on a complex algebraic curve, again for split semisimple G, was established by Teleman in [Tel98].

Turning towards $X/\mathbf{F}_q$, in the case where $G \to X$ is a constant semisimple group scheme (i.e., pulled back from a semisimple group $G_0/\mathbf{F}_q$), Theorem 4.2.8 is proved in [HS10]. In the case where $G \to X$ has simply connected semisimple generic fiber (which implies that Bun_G is geometrically connected), it is proved in [Gai19] based on the work of Gaitsgory–Lurie [GL14]; the same result appears in [Ho21, Theorem 6.1.3]. We will bootstrap from this last result to prove Theorem 4.2.8 in the general reductive case.

4.3. **Propagation of the Atiyah–Bott formula.** We prove that Theorem 4.2.8 is stable under certain operations. We denote by Bun^0_G the neutral component (i.e., the connected component of Bun_G containing the trivial G-bundle).

4.3.1. *Products.* Let G_1, G_2 be reductive group schemes over X, and $G = G_1 \times_X G_2$. Then Theorem 4.2.8 holds for both G_1 and G_2, if and only if it holds for G. Indeed, we have $\mathbb{B}G \cong (\mathbb{B}G_1) \times_X (\mathbb{B}G_2)$ and $\mathrm{Bun}_G \cong \mathrm{Bun}_{G_1} \times \mathrm{Bun}_{G_2}$. Thus both sides of (4.2.13) for G can be written canonically as the tensor product of their counterparts for G_1 and G_2.

4.3.2. *Central isogenies.* Let $\theta\colon G \to G'$ be a central isogeny, with kernel Γ. It induces a map $v : \mathrm{Bun}_G \to \mathrm{Bun}_{G'}$.

Proposition 4.3.3. *The map $v^0\colon \mathrm{Bun}^0_G \to \mathrm{Bun}^0_{G'}$ is surjective.*

We record some preliminary Lemmas which will be used in the proof.

Lemma 4.3.4. *Let $f\colon G_1 \to G'$ be any homomorphism of groups over S whose kernel $F := \ker(f)$ is central in G_1. (In particular, F is abelian, so that the category $\mathbb{B}F(S)$ of F-torsors on S has a natural abelian group structure in groupoids). Then the fiber of the category $\mathbb{B}G_1(S)$ over a given G'-torsor $\mathcal{F}' \in \mathbb{B}G'(S)$, is itself a torsor for $\mathbb{B}F(S)$.*

Proof. Let $\mathcal{F}_0$ be a G_1-torsor equipped with an isomorphism $\tau_0 \colon \mathcal{F}_0 \times^{G_1} G' \cong \mathcal{F}'$. If $\mathcal{F}$ is another G_1-torsor equipped with an isomorphism $\tau \colon \mathcal{F} \times^{G_1} G' \cong \mathcal{F}'$, then we claim that the sheaf $\underline{\mathrm{Isom}}((\mathcal{F}_0, \tau_0), (\mathcal{F}, \tau))$ of isomorphisms from $\mathcal{F}_0$ to $\mathcal{F}$ intertwining the identifications τ_0 and τ, is an F-torsor. Clearly F acts on it; to check that it is an F-torsor, we can work locally, thus reducing to the case where $\mathcal{F}, \mathcal{F}_0$ are trivial, where the assertion is clear. □

Lemma 4.3.5. *Let $f \colon G \to G'$ be a central isogeny of reductive groups over a curve. Then there is a factorization $G \to G_1 \to G'$ such that $\ker(G_1 \to G')$ is an induced torus.*

Proof. Let $K := \ker(f)$, a finite flat group scheme over $X_{\overline{k}}$ of multiplicative type. Then the Cartier dual $K^D = \underline{\mathrm{Hom}}(K, \mathbb{G}_m)$ is a finite étale group scheme over $X_{\overline{k}}$. By choosing a surjection onto K^D from a permutation module for $\pi_1(X_{\overline{k}})$ and then forming Cartier duals, we can embed K in an induced torus T. Now we let $G_1 := (G \times_X T)/K$, where K acts diagonally as

$$k \cdot (g, t) = (gk, k^{-1}t).$$

We then have a map $G_1 \to G'$ given by $(g, t) \mapsto f(g)$. Clearly the kernel is $(K \times_X T)/K \cong T$. □

Proof of Proposition 4.3.3. In the proof, we base change the situation to $\overline{k}$ without changing notation.

Choose a factorization $G \to G_1 \to G'$ as in Lemma 4.3.5. Let $T := \ker(G_1 \to G')$, which by assumption is an induced torus. First we claim that the map $\mathrm{Bun}_{G_1} \to \mathrm{Bun}_{G'}$ is surjective. It suffices to show this on geometric points, so let κ be a separably closed field over k. Then the map $\mathrm{Bun}_{G_1}(\kappa) \to \mathrm{Bun}_{G'}(\kappa)$ is identified with

$$H^1(X_\kappa; G_1) \to H^1(X_\kappa; G').$$

The obstruction classes for this map lie in $H^2(X_\kappa; T)$, which vanishes because T is an induced torus (by Shapiro's Lemma and the vanishing of Brauer groups of curves over separably closed fields [Gro68, Corollary 5.8]). This shows that $\mathrm{Bun}_{G_1} \twoheadrightarrow \mathrm{Bun}_{G'}$. Lemma 4.3.4 shows that this map is a Bun_T-torsor, so we obtain an isomorphism $\mathrm{Bun}_{G_1}/\mathrm{Bun}_T \xrightarrow{\sim} \mathrm{Bun}_{G'}$. Also, we observe that the map $\mathrm{Bun}^0_{G_1} \to \mathrm{Bun}^0_{G'}$ is surjective. Indeed, the exact sequence $\pi_1^{\mathrm{alg}}(T/X) \to \pi_1^{\mathrm{alg}}(G_1/X) \to \pi_1^{\mathrm{alg}}(G'/X) \to 0$ plus (4.2.2) show that for a geometric point of $\mathrm{Bun}^0_{G'}$, any lift to Bun_{G_1} can be translated to $\mathrm{Bun}^0_{G_1}$ by the action of Bun_T.

Let $D := G_1/G$, so that $G \to G_1 \to D$ is an exact sequence. Then we have a Cartesian square

$$\begin{array}{ccc} \mathrm{Bun}_G & \longrightarrow & \mathrm{Bun}_{G_1} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{Bun}_D \end{array} \tag{4.3.1}$$

where the bottom horizontal arrow is the map induced by trivial D-bundle. Let $\mathrm{Bun}^\natural_G \subset \mathrm{Bun}_G$ be the preimage of $\mathrm{Bun}^0_{G_1}$ under $\mathrm{Bun}_G \to \mathrm{Bun}_{G_1}$. Then $\mathrm{Bun}^\natural_G$ is a union of connected components of Bun_G. We have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Bun}^\natural_G & \longrightarrow & \mathrm{Bun}^0_{G_1} \\ \downarrow & & \downarrow d \\ \mathrm{pt} & \longrightarrow & \mathrm{Bun}^0_D \end{array}$$

The map d is equivariant under the translation actions of Bun^0_T. Since $T \to D$ is an isogeny, the induced map $\mathrm{Bun}^0_T \to \mathrm{Bun}^0_D$ is surjective, hence the translation action of Bun^0_T on Bun^0_D is transitively. This shows that d is surjective, and that $\mathrm{Bun}^\natural_G \to \mathrm{Bun}^0_{G_1}/\mathrm{Bun}^0_T$ is surjective. We observed earlier that $\mathrm{Bun}^0_{G_1} \to \mathrm{Bun}^0_{G'}$ is surjective; since this map factors through $\mathrm{Bun}^0_{G_1}/\mathrm{Bun}^0_T$, we deduce that $\mathrm{Bun}^0_{G_1}/\mathrm{Bun}^0_T \twoheadrightarrow \mathrm{Bun}^0_{G'}$ is also surjective. Composing this with the surjection $\mathrm{Bun}^\natural_G \twoheadrightarrow \mathrm{Bun}^0_{G_1}/\mathrm{Bun}^0_T$, we conclude that

$$\text{The map } v^\natural : \mathrm{Bun}^\natural_G \to \mathrm{Bun}^0_{G'} \text{ is surjective.} \tag{4.3.2}$$

In particular, we learn that the map $v' \colon v^{-1}(\mathrm{Bun}^0_{G'}) \to \mathrm{Bun}^0_{G'}$ is surjective. By Lemma 4.3.4, v' is a torsor for the abelian group stack Bun_Γ. In particular, it is flat. At the same time, we see that Bun_Γ is finite by presenting Γ as the kernel of an isogeny of tori. Therefore, v' is also finite. Restricting v' to the connected component Bun_G^0, we see that the map $v^0\colon \mathrm{Bun}_G^0 \to \mathrm{Bun}_{G'}^0$ is also finite and flat. This implies that the image of v^0 is both open and closed, and clearly non-empty, hence must be everything.

□

Return to the central isogeny $\theta\colon G \to G'$ with kernel Γ. Note that Bun_Γ is an abelian group stack. By the constructions of §4.1.12, Bun_Γ acts on Bun_G such that the map $\mathrm{Bun}_G \to \mathrm{Bun}_{G'}$ induced by θ is Bun_Γ-invariant. Let $\mathrm{Bun}_\Gamma^\natural$ be the subgroup stack of Bun_Γ that preserves the neutral component Bun_G^0, then we have a canonical map

$$\mathrm{Bun}_G^0/\mathrm{Bun}_\Gamma^\natural \to \mathrm{Bun}_{G'}^0. \tag{4.3.3}$$

Corollary 4.3.6. *The map* (4.3.3) *is an isomorphism.*

Proof. Lemma 4.3.4 shows that Bun_G is a $\mathrm{Bun}_\Gamma^\natural$-torsor over its image. Then it suffices to see that the map $\mathrm{Bun}_G^0 \to \mathrm{Bun}_{G'}^0$ is surjective, which is Proposition 4.3.3. □

We have a corresponding classifying map

$$f\colon \mathrm{Bun}_{G'}^0 \to \mathbb{B}\mathrm{Bun}_\Gamma^\natural.$$

Each direct image $\mathrm{R}^i f_*(\mathbf{Q}_\ell)$ is a local system on $\mathbb{B}\mathrm{Bun}_\Gamma^\natural$ with stalk $\mathrm{H}^i(\mathrm{Bun}_G^0)$, together with the action of $\pi_0(\mathrm{Bun}_\Gamma^\natural)$ coming from the action of $\mathrm{Bun}_\Gamma^\natural$ on Bun_G^0.

Lemma 4.3.7. *Suppose that Theorem 4.2.8 holds for G. Then the action of $\pi_0(\mathrm{Bun}_\Gamma^\natural)$ on $\mathrm{H}^i(\mathrm{Bun}_G^0)$ is trivial.*

Proof. According to the Atiyah–Bott formula for G, $\mathrm{H}^*(\mathrm{Bun}_G^0)$ is generated by the Künneth components of G-Chern classes of the universal G-bundle on $X \times \mathrm{Bun}_G$. But we saw in Corollary 4.1.15 that twisting by Γ-torsors has trivial effect on G-Chern classes. □

Example 4.3.8. Consider the universal cover $\mathrm{SL}_2 = G \to G' = \mathrm{PGL}_2$. Then $\Gamma = \mu_2$ and $\mathrm{Bun}_\Gamma = \mathrm{Bun}_\Gamma^\natural$ can be identified with the moduli stack of 2-torsion line bundles. The action on $\mathrm{Bun}_{\mathrm{SL}_2}$ is via tensoring with 2-torsion line bundles. Those all have trivial Chern class, so the action on the Atiyah–Bott description is evidently trivial.

Corollary 4.3.9. *Suppose that Theorem 4.2.8 holds for Bun_G^0. Then f induces an isomorphism*

$$\mathrm{H}^*(\mathrm{Bun}_{G'}^0) \xrightarrow{\sim} \mathrm{H}^*(\mathrm{Bun}_G^0),$$

and Theorem 4.2.8 holds for $\mathrm{Bun}_{G'}^0$.

Proof. From the presentation $\mathrm{Bun}_{G'}^0 = [\mathrm{Bun}_G^0/\mathrm{Bun}_\Gamma^\natural]$ we have a spectral sequence

$$\mathrm{H}^i(\mathbb{B}\mathrm{Bun}_\Gamma^\natural; \mathrm{H}^j(\mathrm{Bun}_G^0)) \implies \mathrm{H}^{i+j}(\mathrm{Bun}_{G'}^0).$$

Since Γ is finite and of multiplicative type, $\mathrm{Bun}_\Gamma^\natural$ is a finite abelian group stack, so the higher cohomology of $\mathbb{B}\mathrm{Bun}_\Gamma^\natural$ vanishes with rational coefficients. The spectral sequence therefore degenerates. Furthermore, since $\mathrm{H}^j(\mathrm{Bun}_G^0)$ is the trivial local system on $\mathrm{Bun}_\Gamma^\natural$ by Lemma 4.3.7, we obtain that the pullback map

$$\mathrm{H}^*(\mathrm{Bun}_{G'}^0) \to \mathrm{H}^*(\mathrm{Bun}_G^0)$$

is an isomorphism. Moreover, it is clear from construction and Lemma 4.1.5 that this isomorphism is compatible with the formulation of Theorem 4.2.8. □

4.3.10. *Pure inner twisting.* Let $G \to S$ be a reductive group scheme and $\mathcal{P} \in \mathbb{B}G(S)$. We may view $\mathcal{P}$ as a G-torsor in the étale site of S. Let $G' = \underline{\mathrm{Aut}}_S(\mathcal{P})$ be the (internal) automorphism group of $\mathcal{P}$ (as a G-torsor) in the étale site of S. Then by general nonsense, G' is a twist of G in the étale site of S. Furthermore, the *pure inner twisting* construction $\mathcal{F} \mapsto \underline{\mathrm{Isom}}_S^G(\mathcal{F}, \mathcal{P})$ carries G-torsors to G'-torsors, and induces an equivalence of groupoids

$$\mathbb{B}G(S) \xrightarrow{\sim} \mathbb{B}G'(S).$$

Now let Bun_G^{ω} be a connected component of Bun_G. Fix a G-bundle $\mathcal{P} \in \mathrm{Bun}_G^{\omega}(k)$. Letting $G' = \underline{\mathrm{Aut}}_X(\mathcal{P})$ be the corresponding pure inner twist of G, the pure inner twisting construction $\mathcal{F} \mapsto \underline{\mathrm{Isom}}_S^G(\mathcal{F}, \mathcal{P})$ induces an isomorphism

$$\mathrm{Bun}_G \xrightarrow{\sim} \mathrm{Bun}_{G'}$$

carrying $\mathcal{P}$ to the trivial G'-torsor, hence in particular carrying

$$\mathrm{Bun}_G^{\omega} \xrightarrow{\sim} \mathrm{Bun}_{G'}^0.$$

Lemma 4.3.11. *Let $\mathcal{C}$ be a class of connected reductive group schemes over X preserved by pure inner twisting, e.g., $\mathcal{C} = \{$semisimple $G/X\}$ or $\mathcal{C} = \{$connected reductive $G/X\}$. Suppose Theorem 4.2.8 holds for the neutral components Bun_G^0 of all groups $G \in \mathcal{C}$. Then Theorem 4.2.8 holds for all $G \in \mathcal{C}$.*

Proof. Pure inner twisting by $\mathcal{P}$ induces an isomorphism $\mathbb{B}G \xrightarrow{\sim} \mathbb{B}G'$ of classifying stacks over X, such that the diagram

$$\begin{array}{ccc} X \times \mathrm{Bun}_G^{\omega} & \xrightarrow{\mathrm{ev}} & \mathbb{B}G \\ \downarrow \wr & & \downarrow \wr \\ X \times \mathrm{Bun}_{G'}^0 & \xrightarrow{\mathrm{ev}} & \mathbb{B}G' \end{array}$$

commutes. Then it is clear that Theorem 4.2.8 for the bottom row implies Theorem 4.2.8 for the top row. ☐

4.4. **The simply connected case.** In the case where G is semisimple with simply connected generic fiber, Bun_G is connected and Theorem 4.2.8 is already proved in [Gai19, (0.7)], and later is reproved by similar methods in [Ho21, Theorem 6.13].

4.5. **The semisimple case.** We prove that Theorem 4.2.8 holds for semisimple groups.

Let G be a semisimple group scheme over X. Let $G_{\mathrm{sc}} \to G$ be the simply connected cover. Then we saw in §4.4 that Theorem 4.2.8 holds for $\mathrm{Bun}_{G_{\mathrm{sc}}}$. From Corollary 4.3.9 we obtain that Theorem 4.2.8 holds for Bun_G^0, for all semisimple groups G. Then from Lemma 4.3.11, we obtain that Theorem 4.2.8 holds for all semisimple groups.

4.6. **The torus case.** We will prove Theorem 4.2.8 when $G = T$ is a torus over X. Applying Lemma 4.3.11 to $\mathcal{C} = \{T\}$, it suffices to prove Theorem 4.2.8 for the neutral component Bun_T^0.

4.6.1. *Split tori.* First suppose $T \cong \mathbb{T} \times X$ is split. Then the result is classical, but let us spell it out. By the additive nature of both sides of (4.2.13), it suffices to treat the case $\mathbb{T} = \mathbb{G}_m$, in which case $\mathrm{Bun}_T^0 = \mathrm{Pic}_X^0$. Upon choosing $x_0 \in X(k)$, we have

$$\mathrm{Pic}_X^0 \cong \mathrm{Jac}_X \times_k \mathbb{B}\mathbb{G}_m.$$

Here Jac_X is the moduli of degree zero line bundles on X with a trivialization at x_0. It follows that

$$\mathrm{Gr}^1_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Pic}_X^0) \cong \mathrm{H}^1(\mathrm{Jac}_X)[-1] \oplus \mathrm{H}^2(\mathbb{B}\mathbb{G}_m)[-2], \tag{4.6.1}$$

and the natural map

$$\mathrm{Sym}^{\bullet}\mathrm{Gr}^1_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Pic}_X^0) \to \mathrm{Gr}^{\bullet}_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Pic}_X^0)$$

is an isomorphism. Thus, to prove Theorem 4.2.8 in this case, it suffices to show that

$$\mathfrak{ev}^0_{\mathrm{aug},\mathbb{G}_m} : (\mathrm{H}_*(X) \otimes \mathbb{V}_{\mathbb{G}_m})_+ \to \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{H}^*(\mathrm{Pic}_X^0)$$

is an isomorphism.

We have $\mathbb{V}_{\mathbb{G}_m} = \mathbf{Q}_\ell[-2](-1)$, so that

$$(\mathrm{H}_*(X) \otimes \mathbb{V}_{\mathbb{G}_m})_+ = \mathrm{H}_1(X)[-1](-1) \oplus \mathrm{H}_0(X)[-2](-1). \tag{4.6.2}$$

Under the isomorphisms (4.6.1) and (4.6.2), the map $\mathfrak{ev}^0_{\mathrm{aug},\mathbb{G}_m}$ respects each direct summand. On the first summand, as a map $\mathrm{H}_1(X)(-1) \to \mathrm{H}^1(\mathrm{Jac}_X)$, or equivalently, an element in $\mathrm{H}^1(X) \otimes \mathrm{H}^1(\mathrm{Jac}_X)(1)$, it is the Künneth component of $c_1(\mathcal{L}^{\mathrm{univ}})$ for the universal line bundle $\mathcal{L}^{\mathrm{univ}}$ on $X \times \mathrm{Jac}_X$. This shows that $\mathfrak{ev}^0_{\mathrm{aug},\mathbb{G}_m}$ is an isomorphism on the first summand. On the second summand, if we compose with restriction to x_0, the evaluation map becomes the identity map $\mathrm{H}^2(\mathbb{B}\mathbb{G}_m)[-2](-1) \to \mathbf{Q}_\ell[-2](-1) = \mathbb{V}_{\mathbb{G}_m}$. This shows $\mathfrak{ev}^0_{\mathrm{aug},\mathbb{G}_m}$ is an isomorphism, and completes the proof for split tori.

4.6.2. *Induced tori.* Let $\nu : Y \to X$ be a finite étale map, T_0 a torus over Y, and $T = \mathrm{Res}_{Y/X} T_0$ be the Weil restriction, a torus over X. We claim that if Theorem 4.2.8 holds for T_0 and Y, then it holds for T and X. Indeed, $\mathrm{H}_*(Y, \mathbb{V}_{T_0})$ is canonically isomorphic to $\mathrm{H}_*(X, \mathbb{V}_T)$, and $\mathrm{Bun}_{T,X}$ is canonically isomorphic to $\mathrm{Bun}_{T_0,Y}$.

In particular, by the split tori case already proved, Theorem 4.2.8 holds for any induced torus $T = \mathrm{Res}_{Y/X}(\mathbb{T} \times Y)$, where Y is finite étale over X.

4.6.3. *General tori.* In general, T splits over some finite étale Galois $\nu \colon Y \to X$, so that $\nu^* T \cong \mathbb{T} \times Y$ for some torus $\mathbb{T}$ over Spec k. Then we have a canonical map $\alpha : T \to T' = \mathrm{Res}_{Y/X}(\mathbb{T} \times Y)$ and a norm map $\beta : T' \to T$ such that the composition $\beta \circ \alpha = [N]$, where $N = \deg(\nu)$. Therefore one can find a torus S over X together with an isogeny $T' \to T \times_X S$. Since Theorem 4.2.8 holds for $\mathrm{Bun}^0_{T'}$ by §4.6.2, it also holds for $\mathrm{Bun}^0_{T \times_X S}$ by Corollary 4.3.9, hence also for Bun^0_T by §4.3.1. This finishes the proof in the torus case.

4.7. **The reductive case.** If G is reductive, let Z° be the connected center of G and G_{der} the derived subgroup of G. Then the map $Z^\circ \times_X G_{\mathrm{der}} \to G$ is a central isogeny. Theorem 4.2.8 has been established for Z° and G_{der}, hence also for $Z^\circ \times_X G_{\mathrm{der}}$. By Corollary 4.3.9, Theorem 4.2.8 holds for Bun^0_G.

At this point we know that Theorem 4.2.8 holds for the geometric neutral component Bun^0_G for every reductive G. Then by its propagation under pure inner twisting construction (Lemma 4.3.11) it holds for all the geometric connected components of Bun_G for every reductive G.

5. Arithmetic volume of shtukas for split groups

Fix a prime p. Henceforth our base field will be $k := \mathbf{F}_q$ where q is a power of p. Let X be a smooth, projective, geometrically connected curve over k, of genus g.

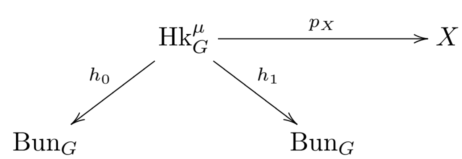
In this section, we formulate and prove our main formula for the "arithmetic volume" of the moduli stack of shtukas for a *split* reductive group scheme over X. (The nonsplit case will be treated later, in §7.) As we only work with constant group schemes over X in this section, we will denote by G a connected reductive group over $\mathbf{F}_q$ (which was denoted by $\mathbb{G}$ in §4); the discussions in §4 are applied to the constant group scheme $G \times X$ over X.

5.1. **Hecke correspondences.** Below we write $\mathbb{V}_G$ as $\mathbb{V}$. Let $\omega \in \pi_0(\mathrm{Bun}_G)$. Theorem 4.2.8 gives a bigraded isomorphism of algebras

$$\mathrm{AB}_\omega : \mathrm{Sym}^{\blacktriangleright}((\mathrm{H}_*(X) \otimes \mathbb{V})_+) \to \mathrm{Gr}^{\blacktriangleright}_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}^\omega_G). \tag{5.1.1}$$

We recall the notation $f^z \in \mathrm{H}^*(\mathrm{Bun}^\omega_G)$ from §4.2.3 for $f \in R^W$ and $z \in \mathrm{H}_*(X)$.

We will apply the general result in §3 to the Hecke correspondence for Bun_G. Let $\mu \in \mathbb{X}_*(T)^+$ be a dominant coweight. Consider the Hecke correspondence



that classifies modifications $(x, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1)$ at one moving point $x \in X$ of type *equal* to μ (rather than $\le \mu$).[7]

[7] Our main theorems will treat the case where μ is minuscule, in which case there is no distinction between these notions.

For a connected component Bun_G^ω, let ${}^\omega\mathrm{Hk}_G^\mu \subset \mathrm{Hk}_G^\mu$ be the preimage of Bun_G^ω under h_0. Then ${}^\omega\mathrm{Hk}_G^\mu$ has a natural correspondence structure

$$\begin{array}{ccccc} & & {}^\omega\mathrm{Hk}_G^\mu & \xrightarrow{p_X} & X \\ & \swarrow^{h_0} & & \searrow^{h_1} & \\ \mathrm{Bun}_G^\omega & & & & \mathrm{Bun}_G^{\omega'} \end{array} \tag{5.1.2}$$

where $\omega' = \omega + \overline{\mu}$, and $\overline{\mu}$ denotes the image of μ in $\pi_0(\mathrm{Bun}_{\mathbb{G}})$, the quotient of $\mathbb{X}_*(\mathbb{T})$ by the coroot lattice, cf (4.2.1).

5.1.1. *Tautological classes.* Consider $X \times \mathrm{Hk}_G^\mu$ with projection to the two factors denoted $q_X : X \times \mathrm{Hk}_G^\mu \to X$ and $p_{\mathrm{Hk}} : X \times \mathrm{Hk}_G^\mu \to \mathrm{Hk}_G^\mu$. Let $\Gamma(p_X) \subset X \times \mathrm{Hk}_G^\mu$ be the graph of $p_X : \mathrm{Hk}_G^\mu \to X$. Let $\mathcal{F}^{\mathrm{univ}}$ be the universal G-bundle over $X \times \mathrm{Bun}_G$. We have two G-bundles on $X \times \mathrm{Hk}_G^\mu$

$$\mathcal{F}_0 := (q_X \times h_0)^* \mathcal{F}^{\mathrm{univ}}, \quad \mathcal{F}_1 := (q_X \times h_1)^* \mathcal{F}^{\mathrm{univ}}$$

together with the universal Hecke modification of type μ along $\Gamma(p_X)$

$$\varphi^{\mathrm{univ}} : \mathcal{F}_0 \dashrightarrow \mathcal{F}_1. \tag{5.1.3}$$

By Proposition 3.1.4, the G-torsor $\mathcal{F}_0|_{\Gamma(p_X)}$ carries a canonical P_μ-reduction classified by a map

$$\mathrm{ev}_\mu : \mathrm{Hk}_G^\mu \to \mathbb{B}P_\mu. \tag{5.1.4}$$

We emphasize that the P_μ-reduction comes from $\mathcal{F}_0|_{\Gamma(p_X)}$, not $\mathcal{F}_1|_{\Gamma(p_X)}$.

Restricting to the ω-component, which we denote by ev_μ^ω, it induces a ring homomorphism

$$\mathrm{ev}_\mu^{\omega,*} : R^{W_\mu} \cong \mathrm{H}^*(\mathbb{B}P_\mu) \to \mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu) \tag{5.1.5}$$

which is easily seen to be injective. Combining this with pullback p_X^* along the leg map, we get a ring homomorphism

$$\mathrm{H}^*(X) \otimes R^{W_\mu} \to \mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu). \tag{5.1.6}$$

The image of this map can be regarded as "tautological classes on $\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$".

5.1.2. *Tautological endomorphisms of cohomology.* Now assume that $\mu \in \mathbb{X}_*(T)$ is minuscule and dominant. Let $D_\mu := \langle 2\rho, \mu\rangle = \dim G/P_\mu$. We consider classes in $\mathrm{H}^{2D_\mu+2}(\mathrm{Hk}_G^\mu)$ which are the image under (5.1.6) of

$$\eta + \xi\eta', \text{ where } \eta \in R^{W_\mu}_{2(D_\mu+1)}, \eta' \in R^{W_\mu}_{2D_\mu}.$$

Let $\omega' = \omega + \overline{\mu} \in \pi_0(\mathrm{Bun}_G)$, where we recall that $\overline{\mu}$ is the image of $\mu \in \mathbb{X}_*(T)$ in $\pi_0(\mathrm{Bun}_G)$. Consider the degree-preserving map

$$\Gamma_\mu^{\eta+\xi\eta'} : \mathrm{H}^*(\mathrm{Bun}_G^{\omega'}) \to \mathrm{H}^*(\mathrm{Bun}_G^\omega) \tag{5.1.7}$$

$$\theta \mapsto h_{0*}((\eta + \xi\eta') \cdot h_1^*\theta). \tag{5.1.8}$$

The pushforward map on cohomology h_{0*} is defined because h_0 is smooth and proper (recall that μ is assumed to be minuscule). The map $\Gamma_\mu^{\eta+\xi\eta'}$ is degree-preserving because the relative dimension of h_0 is $D_\mu + 1$.

5.1.3. *Variant on compactly supported cohomology.* With the same notation, we define

$${}_c\Gamma_\mu^{\eta+\xi\eta'} : \mathrm{H}_c^*(\mathrm{Bun}_G^\omega) \to \mathrm{H}_c^*(\mathrm{Bun}_G^{\omega'}) \tag{5.1.9}$$

$$\theta \mapsto h_{1*}((\eta + \xi\eta') \cdot h_0^*\theta). \tag{5.1.10}$$

The definition is arranged so that under the Poincaré duality pairing

$$\langle -, - \rangle : \mathrm{H}_c^i(\mathrm{Bun}_G^\omega) \times \mathrm{H}^{2\dim \mathrm{Bun}_G - i}(\mathrm{Bun}_G^\omega) \to \mathrm{H}_c^{2\dim \mathrm{Bun}_G}(\mathrm{Bun}_G^\omega) \cong \overline{\mathbf{Q}}_\ell(-\dim \mathrm{Bun}_G) \tag{5.1.11}$$

the map ${}_c\Gamma_\mu^{\eta+\xi\eta'}$ is adjoint to $\Gamma_\mu^{\eta+\xi\eta'}$, i.e.,

$$\langle {}_c\Gamma_\mu^{\eta+\xi\eta'}(\alpha), \beta\rangle = \langle \alpha, \Gamma_\mu^{\eta+\xi\eta'}(\beta)\rangle \tag{5.1.12}$$

for all $\alpha \in \mathrm{H}^i_c(\mathrm{Bun}^{\omega}_G)$ and $\beta \in \mathrm{H}^{2\dim \mathrm{Bun}_G - i}(\mathrm{Bun}^{\omega'}_G)$.

5.2. **Arithmetic volume of the moduli of shtukas.** We can now define the moduli spaces of shtukas and the notion of arithmetic volume of their tautological classes.

5.2.1. *Moduli of shtukas.* Let $r \in \mathbb{Z}_{\geq 0}$ and $\mu = (\mu_1, \cdots, \mu_r)$ be a sequence of minuscule dominant coweights of G that is *admissible* in the sense that

$$\mu_1 + \mu_2 + \cdots + \mu_r \text{ lies in the coroot lattice of } G. \tag{5.2.1}$$

In other words, we have $\sum_{i=1}^r \overline{\mu}_i = 0 \in \pi_0(\mathrm{Bun}_G)$, cf. (4.2.1).

Let Hk^{μ}_G be the fiber product

$$\mathrm{Hk}^{\mu_1}_G \times_{\mathrm{Bun}_G} \mathrm{Hk}^{\mu_2}_G \times_{\mathrm{Bun}_G} \cdots \times_{\mathrm{Bun}_G} \mathrm{Hk}^{\mu_r}_G$$

that classifies iterated modifications $(x_1, \cdots, x_r, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{F}_r)$, where $\mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ is a modification along x_i of type μ_i. For $0 \leq i \leq r$, let $h_i : \mathrm{Hk}^{\mu}_G \to \mathrm{Bun}_G$ record $\mathcal{F}_i$. Let ${}^{\omega}\mathrm{Hk}^{\mu}_G \subset \mathrm{Hk}^{\mu}_G$ be the preimage of Bun^{ω}_G under h_0.

The *moduli stack* Sht^{μ}_G *of G-shtukas with r legs and modification type* μ is the fiber product

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_G & \longrightarrow & \mathrm{Hk}^{\mu}_G \\ \downarrow & & \downarrow {\scriptstyle (h_0, h_r)} \\ \mathrm{Bun}_G & \xrightarrow{(\mathrm{id}, \mathrm{Fr})} & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array} \tag{5.2.2}$$

We decompose Sht^{μ}_G according to which connected component of Bun_G the first bundle $\mathcal{E}_0$ lies in. This gives a decomposition

$$\mathrm{Sht}^{\mu}_G = \coprod_{\omega \in \pi_0(\mathrm{Bun}_G)} {}^{\omega}\mathrm{Sht}^{\mu}_G$$

so that ${}^{\omega}\mathrm{Sht}^{\mu}_G$ is the preimage of ${}^{\omega}\mathrm{Hk}^{\mu}_G$. Let

$$\omega_j = \omega + \overline{\mu}_1 + \cdots + \overline{\mu}_j \in \pi_0(\mathrm{Bun}_G), \quad j = 1, 2, \cdots, r. \tag{5.2.3}$$

Then for $(x_1, \cdots, x_r, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{F}_r \cong {}^{\tau}\mathcal{F}_0) \in {}^{\omega}\mathrm{Sht}^{\mu}_G$, we have $\mathcal{F}_j \in \mathrm{Bun}^{\omega_j}_G$.

5.2.2. *Definition of Arithmetic volume.* For $j = 1, 2, \ldots, r$, let $D_{\mu_j} = \dim(G/P_{\mu_j}) = \langle 2\rho, \mu_j \rangle$ and choose a pair of classes

$$\eta_j \in R^{W_{\mu_j}}_{2(D_{\mu_j}+1)}, \quad \eta'_j \in R^{W_{\mu_j}}_{2D_{\mu_j}}.$$

We have a composite morphism

$$\mathrm{Sht}^{\mu}_G \to \mathrm{Hk}^{\mu}_G \to \mathrm{Hk}^{\mu_j}_G.$$

Composing this further with the map $\mathrm{Hk}^{\mu_j}_G \xrightarrow{\mathrm{ev}_j} \mathbb{B}P_{\mu_j}$ from (5.1.4) (corresponding to the canonical parabolic reduction of $\mathcal{F}_{j-1}|_{\Gamma(x_j)}$), we obtain the map

$$\mathrm{ev}_j : \mathrm{Sht}^{\mu}_G \to \mathrm{Hk}^{\mu_j}_G \xrightarrow{\mathrm{ev}_j} \mathbb{B}P_{\mu_j}.$$

For $1 \leq j \leq r$, let $p_j : \mathrm{Sht}^{\mu}_G \to X$ be the map recording the jth leg.

We let η be the r-tuple $(\eta_1 + \xi\eta'_1, \ldots, \eta_r + \xi\eta'_r)$. Recall the operators ${}_c\Gamma^{\eta_j + \xi\eta'_j}_{\mu_j}$ defined in (5.1.9). Write

$${}_c\Gamma^{\eta}_{\mu} := {}_c\Gamma^{\eta_r + \xi\eta'_r}_{\mu_r} \circ \ldots \circ {}_c\Gamma^{\eta_1 + \xi\eta'_1}_{\mu_1} : \mathrm{H}^*_c(\mathrm{Bun}^{\omega}_G) \to \mathrm{H}^*_c(\mathrm{Bun}^{\omega_r}_G) = \mathrm{H}^*_c(\mathrm{Bun}^{\omega}_G)$$

for the composition of the ${}_c\Gamma^{\eta_j + \xi\eta'_j}_{\mu_j}$; note that the target component ω agrees with the source component ω by the assumption (5.2.1).

Definition 5.2.3. Let $\mu = (\mu_1, \ldots, \mu_r)$ and $\eta = (\eta_1 + \xi\eta'_1, \ldots, \eta_r + \xi\eta'_r)$. The *arithmetic volume* of ${}^{\omega}\mathrm{Sht}^{\mu}_G$ with respect to η is the *graded* trace

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_G, \eta) := \mathrm{Tr}({}_c\Gamma^{\eta}_{\mu} \circ \mathrm{Frob} \mid \mathrm{H}^*_c(\mathrm{Bun}^{\omega}_G)). \tag{5.2.4}$$

Since $\mathrm{H}^*_c(\mathrm{Bun}^{\omega}_G)$ is infinite-dimensional, it is not clear that the RHS is well-defined; this will be justified in Proposition 5.5.3.

Remark 5.2.4. We justify why the definition (5.2.4) deserves to be viewed as an "arithmetic volume". The numerics are arranged so that $\prod_{j=1}^r(\mathrm{ev}_j^*\eta_j + p_j^*\xi \cdot \mathrm{ev}_j^*\eta_j')$ is a top-degree cohomology class of ${}^\omega\mathrm{Sht}_G^\mu$. If ${}^\omega\mathrm{Sht}_G^\mu$ were proper, then

$$\int_{{}^\omega\mathrm{Sht}_G^\mu} \prod_{j=1}^r(\mathrm{ev}_j^*\eta_j + p_j^*\xi \cdot \mathrm{ev}_j^*\eta_j') \tag{5.2.5}$$

would be well-defined, as the evaluation of the fundamental class of ${}^\omega\mathrm{Sht}_G^\mu$ on the top-degree cohomology class, and we would take this to be $\mathrm{vol}(\mathrm{Sht}_G^\mu, \eta)$. But ${}^\omega\mathrm{Sht}_G^\mu$ is almost never proper, so (5.2.5) has no a priori meaning.

If Bun_G^ω were proper, then a suitable version of the Grothendieck–Lefschetz trace formula (e.g., [FYZ24, Proposition 11.8]) would identify (5.2.5) with $\mathrm{Tr}(\mathrm{Frob} \circ {}_c\Gamma_\mu^\eta \mid \mathrm{H}_c^*(\mathrm{Bun}_G^\omega))$. The latter expression has the potential at least to make sense when Bun_G^ω is not proper (there is a convergence question, because $\mathrm{H}_c^*(\mathrm{Bun}_G^\omega)$ is infinite-dimensional), and we will show that it indeed is always well-defined.

Our main theorem (Theorem 5.6.9) computes the arithmetic volume in terms of differential operators applied to the L-function of the Gross motive of G (cf. §4.1.8). These differential operators are determined by local data, which we describe next.

5.3. **Local operators on characteristic classes.** We define some "local" operators on $R^W = \mathrm{H}^*(\mathbb{B}G)$.

5.3.1. *The operator* ∇_μ^η. Let $\mu \in \mathbb{X}_*(T)$. Then we have the partial derivative ∂_μ on $R = \mathrm{Sym}(\mathbb{X}^*(T)_{\mathbf{Q}_\ell}(-1))$ which lowers the grading by 2 and twist by 1. It carries the subring R^W to R^{W_μ}, so we may view it as a derivation

$$\partial_\mu : R^W \to R^{W_\mu}[-2](-1).$$

Let $P_\mu \subset G$ be the parabolic subgroup of G corresponding to μ. Pushforward along the proper smooth map $\mathbb{B}P_\mu \to \mathbb{B}G$ induces an R^W-linear map

$$\int_{G/P_\mu} : R^{W_\mu} = \mathrm{H}^*(\mathbb{B}P_\mu) \to \mathrm{H}^*(\mathbb{B}G)[-2D_\mu](-D_\mu) = R^W[-2D_\mu](-D_\mu) \tag{5.3.1}$$

where $D_\mu := \dim(G/P_\mu) = \langle 2\rho, \mu\rangle$.

Let $\eta \in R^{W_\mu}$ be homogeneous of degree $2(D_\mu + 1)$. Consider the map

$$\nabla_\mu^\eta \quad : \quad R^W \to R^W \tag{5.3.2}$$

$$f \mapsto \int_{G/P_\mu} \eta \cdot \partial_\mu f. \tag{5.3.3}$$

Then ∇_μ^η is a degree-preserving derivation.

5.3.2. *The operator* $\overline{\nabla}_\mu^\eta$. Since ∇_μ^η is a derivation that carries R_+^W to R_+^W, it also sends $R_+^W \cdot R_+^W$ to itself, hence induces a graded linear endomorphism of $\mathbb{V}$

$$\overline{\nabla}_\mu^\eta \in \mathrm{End}^{gr}(\mathbb{V}).$$

In other words, $\overline{\nabla}_\mu^\eta$ restricts to a linear endomorphism of each $\mathbb{V}_d$. The eigenvalues of $\overline{\nabla}_\mu^\eta$ are the crucial "structure constants" for the upcoming formulas for arithmetic volumes. We refer to them as *eigenweights.*

Example 5.3.3. In many examples of interest, $\mathbb{V}_{2d}$ has dimension at most one, and then the eigenweights can be calculated in the following way. By the assumption, we can choose homogeneous free generators $f_1, f_2, \cdots, f_n$ of R^W (which we know is a polynomial ring) of increasing degrees

$$2d_1 < 2d_2 < \cdots < 2d_n.$$

This means $\mathbb{V} = \bigoplus_{i=1}^n \mathbb{V}_{2d_i}$ and $\mathbb{V}_{2d_i}$ is one-dimensional spanned by the image of f_i. Then, for degree reasons, we necessarily have for each $i = 1, \cdots, n$,

$$\nabla_\mu^\eta(f_i) = \epsilon_i(\eta, \mu) f_i + (\text{polynomial in } f_1, \cdots, f_{i-1}).$$

The numbers $\{\epsilon_i(\eta, \mu)\}_{i=1}^n$ are the eigenweights of $\overline{\nabla}_\mu^\eta$.

5.4. **Example of calculating eigenweights.** We will calculate the operator ∇_μ^η in the example $G = \mathrm{GL}_n$ and $\mu = (1, 0, \cdots, 0)$, so that $D_\mu = n-1$. We identify $R = \overline{\mathbf{Q}}_\ell[x_1, \cdots, x_n]$ (cf. Example 3.2.2) with the obvious action of the Weyl group $W \cong S_n$ by permutations on the variables, so that

$$R^W = \overline{\mathbf{Q}}_\ell[e_1, \dots, e_n]$$

where the e_i are the elementary symmetric polynomials in the x_i.

Proposition 5.4.1. *Let $G = \mathrm{GL}_n$ and $\mu = (1, 0, \dots, 0)$. Choose $\eta = x_1^n \in R^{W_\mu}$. Then we have*

$$\nabla_\mu^\eta(e_i) = (-1)^{n-1} e_i \text{ for all } i = 1, 2, \dots, n.$$

In particular, we have $\epsilon_i(\eta, \mu) = (-1)^{n-1}$ for each $i = 1, \dots, n$.

Example 5.4.2. For $i = 1$, we want to calculate

$$\nabla_\mu^\eta(e_1) = \int_{G/P_\mu} x_1^n.$$

An R^W-module basis for R^{W_μ} is given by the powers $x_1^0, \dots, x_1^{n-1}$, and $\int_{G/P_\mu}$ extracts the coefficients of $(-x_1)^{n-1}$, by our convention that $-x_1$ corresponds to $\mathcal{O}(1)$ on G/P_μ (cf. Example 2.3.1). We have the characteristic relation

$$x_1^n - e_1 x_1^{n-1} + \dots + (-1)^n e_n = 0.$$

Thus

$$\int_{G/P_\mu} x_1^n = \int_{G/P_\mu} (e_1 x_1^{n-1} + \dots) = (-1)^{n-1} e_1.$$

For $i > 1$, the combinatorics become much more complicated to analyze directly in this way, so we will first develop some machinery.

5.4.3. *Combinatorial formula for the integration map.* We will first give a general description of the integration map (5.3.1) in more concrete terms, which will be useful for explicit calculations.

We maintain the notation of §5.3.1, so G is an arbitrary split reductive over $\mathbf{F}_q$ and $\mu \in \mathbb{X}_*(T)$. Consider the G-equivariant top Chern class of the tangent bundle of G/P_μ

$$c_{D_\mu}(T_{G/P_\mu}) = \mathfrak{R}_\mu := \prod_{\substack{\alpha \in \Phi(G) \\ \langle \alpha, \mu \rangle < 0}} \alpha \in R^{W_\mu} \tag{5.4.1}$$

where $\Phi(G)$ is the set of roots of G.

Lemma 5.4.4. *For any $f \in R^{W_\mu}$, we have*

$$\int_{G/P_\mu} f = \sum_{w \in W/W_\mu} w(f/\mathfrak{R}_\mu). \tag{5.4.2}$$

Here the sum is over a set of representatives of W/W_μ, which is well-defined since $f/\mathfrak{R}_\mu$ is W_μ-invariant. (The element $f/\mathfrak{R}_\mu$ is understood in the fraction field of R, yet the sum above will lie in R^W.)

Proof. Suppose the Lemma is proved whenever the parabolic subgroup is a Borel. Let B be the standard Borel subgroup contained in P_μ. Let $B_\mu := B \cap L_\mu$, a Borel subgroup of L_μ. Let $\mathfrak{R}^\mu$ be the analogous construction for $B_\mu < L_\mu$ (i.e., the L_μ-equivariant top Chern class of L_μ/B_μ), so that

$$\mathfrak{R} := \prod_{\substack{\alpha \in \Phi(G) \\ \alpha < 0}} \alpha = \mathfrak{R}_\mu \mathfrak{R}^\mu \in R.$$

By assumption, we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^*_G(G/B) & = & \mathrm{H}^*_{L_\mu}(L_\mu/B_\mu) & \xrightarrow{\sim} & R \\
\downarrow & & \twoheaddownarrow & & \downarrow\!\!\downarrow \sum_{w\in W_\mu} w((\cdot)/\mathfrak{R}^\mu) \\
\mathrm{H}^*_G(G/P_\mu) & = & \mathrm{H}^*_{L_\mu}(\mathrm{pt}) & \xrightarrow{\sim} & R^{W_\mu} \\
\downarrow & & & & \downarrow\!\!\downarrow \\
\mathrm{H}^*_G(\mathrm{pt}) & & \xrightarrow{\qquad\sim\qquad} & & R^W
\end{array}$$

where the composite right vertical arrow is $\sum_{w\in W} w((\cdot)/\mathfrak{R})$ (which is the content of the lemma for $P_\mu = B$). The description of the map $R \to R^{W_\mu}$ in the above diagram is the content of the lemma for $G = L_\mu$ with its Borel B_μ. This implies that the second right vertical arrow must be $\sum_{w\in W/W_\mu} w((\cdot)/\mathfrak{R}_\mu)$.

So it suffices to verify the Lemma in the case where $P_\mu = B$ is a Borel subgroup. By computing on simple reflections, which reduces to an SL_2 calculation, one checks that $\int_{G/B}$ is sign-equivariant under the W-action:

$$\int_{G/B} wf = (-1)^{\ell(w)} \int_{G/B} f, \quad \forall f \in R.$$

Let R^{sgn} be the sign-isotypic component of R under the W-action. Let

$$\begin{array}{rcl}
\mathrm{Av}_{\mathrm{sgn}} : R & \to & R^{\mathrm{sgn}} \\
f & \mapsto & \dfrac{1}{|W|} \displaystyle\sum_{w\in W} (-1)^{\ell(w)} wf
\end{array}$$

be the projector to R^{sgn}. Since $\int_{G/B}$ is sign-equivariant, it has a factorization as a composition

$$\int_{G/B} : R \xrightarrow{\mathrm{Av}_{\mathrm{sgn}}} R^{\mathrm{sgn}} \xrightarrow{\iota} R^W$$

for a unique R^W-linear map ι. It is well-known that R^{sgn} is a free R^W-module with basis $\mathfrak{R}$; hence, to determine ι, it suffices to compute $\iota(\mathfrak{R})$.

By the definition of $\int_{G/B}$ as a pushforward, we have

$$\int_{G/B} \mathfrak{R} = \int_{G/B} c_{\dim G/B}(T_{G/B}) = \chi(G/B) = |W|.$$

This implies $\iota(\mathfrak{R}) = |W|$, which in turn implies that

$$\int_{G/B} f = \iota(\mathrm{Av}_{\mathrm{sgn}}(f)) = |W| \mathrm{Av}_{\mathrm{sgn}}(f)/\mathfrak{R} = \sum_{w\in W} w(f/\mathfrak{R}),$$

proving (5.4.2). □

5.4.5. *Proof of Proposition 5.4.1.* Now we return to the setup of Proposition 5.4.1, so $G = \mathrm{GL}_n$ and $\mu = (1, 0, \ldots, 0) \in \mathbb{X}_*(T)$.

We have arranged that $\partial_\mu = \partial_{x_1}$. We choose $\eta = x_1^n$. In this case, $\mathbb{B}P_\mu \to \mathbb{B}G$ is a $\mathbb{P}^{n-1}$-bundle with x_1 being the first Chern class of the tautological bundle $\mathcal{O}(-1)$ in our conventions (cf. Example 2.3.1).

Each e_i spans the (1-dimensional) eigenspaces $\mathbb{V}_{2d_i}$, and we want to calculate

$$\nabla^\eta_\mu(e_i) = \int_{G/P_\mu} x_1^n \partial_{x_1} e_i.$$

Let $\widehat{e}_i \in R$ be the ith elementary symmetric polynomial in $x_2, \ldots, x_n$ (omitting x_1). Note that $\partial_{x_1}(e_i) = \widehat{e}_{i-1}$, so that we are interested in calculating $\int_{G/P_\mu} x_1^n \cdot \widehat{e}_{i-1}$.

Let t be a formal variable. Then we have

$$\prod_{j=2}^{n} (t - x_j) = \sum_{i=0}^{n-1} (-1)^i t^{n-1-i} \widehat{e}_i. \tag{5.4.3}$$

Identifying $W = S_n$ with the permutations of the $\{x_i\}$, the subgroup $W_\mu = S_{n-1}$ is the stabilizer of x_1. We claim that

$$\sum_{w \in S_n/S_{n-1}} w\left(x_1^n \frac{\prod_{j=2}^n (t-x_j)}{\prod_{j=2}^n (x_1 - x_j)}\right) = t^n - \prod_{i=1}^n (t - x_i). \tag{5.4.4}$$

Indeed, the left side is the Lagrange interpolation formula for the unique degree $n-1$ polynomial $P(t)$ such that $P(x_i) = x_i^n$ for each $i = 1, \ldots, n$, and the right side visibly has this same property, so they must agree.

Comparing coefficients of t^{n-i} in (5.4.4), and using (5.4.3), yields

$$\sum_{w \in S_n/S_{n-1}} w\left(\frac{x_1^n \cdot \widehat{e}_{i-1}}{\prod_{j=2}^n (x_1 - x_j)}\right) = e_i. \tag{5.4.5}$$

Note that the denominator $\prod_{j=2}^n (x_1 - x_j)$ is $(-1)^{n-1}\mathfrak{R}_\mu$ from (5.4.1). Invoking Lemma 5.4.4, we find that

$$\nabla_\mu^\eta(e_i) = \int_{G/P_\mu} x_1^n \widehat{e}_{i-1} = (-1)^{n-1} \sum_{w \in S_n/S_{n-1}} w\left(\frac{x_1^n \cdot \widehat{e}_{i-1}}{\prod_{j=2}^n (x_1 - x_j)}\right) \overset{(5.4.5)}{=} (-1)^{n-1} e_i, \tag{5.4.6}$$

as desired. □

5.5. **Convergence of trace.** Since the definition of arithmetic volume involves taking trace on an infinite-dimensional vector space, some analysis is required to see that the trace actually converges to a well-defined number; we undertake this analysis now in a more general context.

5.5.1. *More general arithmetic volumes.* Fix an admissible sequence $\mu = (\mu_1, \cdots, \mu_r)$ of dominant minuscule coweights of T. Recall $N = \sum_{j=1}^r (D_{\mu_j} + 1)$. For any $\theta \in \mathrm{H}^{2N}({}^\omega\mathrm{Hk}_G^\mu)$, we can define the operator Γ_μ^θ on $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ similarly as in §5.1.2 as the composition

$$\Gamma_\mu^\theta := h_{0*}(h_r^*(-) \cup \theta) : \mathrm{H}^*(\mathrm{Bun}_G^\omega) \to \mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

Similarly we have the version with compact support

$${}_c\Gamma_\mu^\theta := h_{r*}(h_0^*(-) \cup \theta) : \mathrm{H}_c^*(\mathrm{Bun}_G^\omega) \to \mathrm{H}_c^*(\mathrm{Bun}_G^\omega).$$

Lemma 5.5.2. *For each $i \in \mathbb{Z}$, we have*

$$\mathrm{Tr}({}_c\Gamma_\mu^\theta \circ \mathrm{Frob} \mid \mathrm{H}_c^i(\mathrm{Bun}_G^\omega)) = q^{\dim \mathrm{Bun}_G} \mathrm{Tr}(\mathrm{Frob}^{-1} \circ \Gamma_\mu^\theta \mid \mathrm{H}^{2\dim \mathrm{Bun}_G - i}(\mathrm{Bun}_G^\omega)).$$

Proof. By the analog of (5.1.12), under the Poincaré duality pairing (5.1.11), the endomorphism ${}_c\Gamma_\mu^\theta$ of $\mathrm{H}_c^i(\mathrm{Bun}_G^\omega)$ is adjoint to the endomorphism Γ_μ^θ of $\mathrm{H}^{2\dim \mathrm{Bun}_G - i}(\mathrm{Bun}_G^\omega)$. Also, the automorphism Frob on $\mathrm{H}_c^i(\mathrm{Bun}_G^\omega)$ is adjoint to the automorphism $q^{\dim \mathrm{Bun}_G}\mathrm{Frob}^{-1}$ on $\mathrm{H}^{2\dim \mathrm{Bun}_G - i}(\mathrm{Bun}_G^\omega)$. The conclusion follows. □

Proposition 5.5.3. *Fix an embedding of fields $\iota : \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$. For any $\theta \in \mathrm{H}^{2N}({}^\omega\mathrm{Hk}_G^\mu)$, the two series of complex numbers*

$$\sum_{i \in \mathbb{Z}} (-1)^i \iota\left(\mathrm{Tr}(\mathrm{Frob}^{-1} \circ \Gamma_\mu^\theta | \mathrm{H}^i(\mathrm{Bun}_G^\omega))\right) \text{ and } \sum_{i \in \mathbb{Z}} (-1)^i \iota\left(\mathrm{Tr}({}_c\Gamma_\mu^\theta \circ \mathrm{Frob} | \mathrm{H}_c^i(\mathrm{Bun}_G^\omega))\right) \tag{5.5.1}$$

are absolutely convergent. We denote their sums by

$$\mathrm{Tr}_\iota(\mathrm{Frob}^{-1} \circ \Gamma_\mu^\theta | \mathrm{H}^*(\mathrm{Bun}_G^\omega)) \text{ and } \mathrm{Tr}_\iota({}_c\Gamma_\mu^\theta \circ \mathrm{Frob} | \mathrm{H}_c^*(\mathrm{Bun}_G^\omega))$$

respectively.

By Lemma 5.5.2, it suffices to prove the absolute convergence of the first series in (5.5.1), whose proof occupies the rest of the subsection.

5.5.4. *The Ran grading.* For $m \geq 0$ define $\mathfrak{R}_m\mathrm{H}^*(\mathrm{Bun}_G^\omega) \subset \mathrm{H}^*(\mathrm{Bun}_G^\omega)$ to be the subspace spanned by elements of the form

$$f_1^{z_1} f_2^{z_2} \cdots f_s^{z_s} \quad \text{such that } \sum_i |z_i| = m \text{ and } \deg(f_i) > |z_i| \text{ for each } i = 1, \cdots, s.$$

Here the $f_i \in R_+^W$ are homogeneous. We note that $\deg(f_i) > |z_i|$ automatically holds unless $\deg(f_i) = 2 = |z_i|$, which is not allowed in the description of the Atiyah–Bott formula §4.2.9 (for in this case $f_i^{z_i} \in \overline{\mathbf{Q}}_\ell$).

From the definition, it is clear that $\mathfrak{R}_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ are multiplicative under cup product in that

$$\mathfrak{R}_m\mathfrak{R}_l \subset \mathfrak{R}_{m+l}.$$

Lemma 5.5.5. *We have a decomposition*

$$\mathrm{H}^*(\mathrm{Bun}_G^\omega) \cong \bigoplus_{m \geq 0} \mathfrak{R}_m\mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{5.5.2}$$

We call the resulting $\mathbb{Z}_{\geq 0}$-grading on $\mathrm{H}^*(\mathrm{Bun}_G)$ *the* Ran grading [8].

Proof. Abbreviate $\mathfrak{R}_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ simply by $\mathfrak{R}_m$. Let $f_1, \cdots, f_n$ be a set of free homogeneous generators of R^W. Let $z_0 = 1 \in \mathrm{H}_0(X)$, $z_1, \cdots, z_{2g} \in \mathrm{H}_1(X)$, and $z_{2g+1} = [X] \in \mathrm{H}_2(X)$ be a homogeneous basis of $\mathrm{H}_*(X)$. Then by Atiyah-Bott formula (see §4.2.9), $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ has a $\overline{\mathbf{Q}}_\ell$-basis consisting of

$$\prod_{i=1}^{n}\prod_{j=0}^{2g+1} (f_i^{z_j})^{e_{ij}} \quad \text{where} \quad e_{ij} \begin{cases} \in \mathbb{Z}_{\geq 0} & \text{if } j = 0 \text{ or } j = 2g+1, \\ \in \{0, 1\} & \text{if } j = 1, \cdots, 2g. \end{cases} \tag{5.5.3}$$

Let $\mathfrak{R}'_m$ be the subspace spanned by those monomials of the form (5.5.3) where

$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{2g} e_{ij} + 2e_{i,2g+1}\Big) = m.$$

Clearly $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ is the direct sum of $\mathfrak{R}'_m$. It remains to show that $\mathfrak{R}_m = \mathfrak{R}'_m$. The inclusion $\mathfrak{R}'_m \subset \mathfrak{R}_m$ is clear. Conversely, to show that $\mathfrak{R}_m \subset \mathfrak{R}'_m$, we observe that $\mathfrak{R}'_m$ are also multiplicative, so it suffices to show that

$$f^z \in \mathfrak{R}'_{|z|} \tag{5.5.4}$$

for any $f \in R^W$ and homogeneous $z \in \mathrm{H}_*(X)$.

Any $f \in R^W$ can be written as $f = P(f_1, \cdots, f_n)$ for a unique polynomial P in n variables. We prove (5.5.4) by induction on the degree of P. The case $\deg(P) \leq 1$ is clear. Suppose (5.5.4) holds for all f whose degree in $f_1, \cdots, f_n$ is $< d$. Let $f = P(f_1, \cdots, f_n)$ for $\deg(P) = d$. By linearity, it suffices to treat the case where P is a monomial of degree $d > 1$. In this case we write $P = P_1P_2$ where P_i are monomials of degree $< d$. Let $g_1 = P_1(f_1, \cdots, f_n)$ and $g_2 = P_2(f_1, \cdots, f_n)$. Then by (4.2.12) we have

$$f^z = (g_1g_2)^z = \sum_j (g_1)^{z'_j}(g_2)^{z''_j} \tag{5.5.5}$$

where $\Delta_*(z) = \sum_j z'_j \otimes z''_j \in \mathrm{H}_{|z|}(X \times X)$. By induction hypothesis $(g_1)^{z'_j} \in \mathfrak{R}'_{|z'_j|}$ and $(g_2)^{z''_j} \in \mathfrak{R}'_{|z''_j|}$. Therefore (5.5.5) implies $f^z \in \mathfrak{R}'_{|z|}$. This shows $\mathfrak{R}_m = \mathfrak{R}'_m$, completing the proof of the lemma. □

For each m, define the *Ran filtration* $F_\bullet\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ on $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ by

$$F_m\mathrm{H}^*(\mathrm{Bun}_G^\omega) \cong \bigoplus_{m' \leq m} \mathfrak{R}_{m'}\mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{5.5.6}$$

[8]The nomenclature comes from the fact that $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ is the factorization homology of $\mathrm{H}^*(\mathbb{B}G)$, a key observation of [GL14].

5.5.6. *Monomial basis for* $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. As in the proof of Lemma 5.5.5, we now fix a choice of homogeneous free generators $f_1, \cdots, f_n$ for R^W. Also choose a homogeneous basis $z_0 = 1, z_1, \cdots, z_{2g}, z_{2g+1} = [X]$ for $\mathrm{H}_*(X)$ that are also eigenvectors for the Frobenius action on $\mathrm{H}_*(X)$.[9] Let $\mathfrak{B}$ be the set of elements of the form (5.5.3); they form a basis for $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ that we call *the monomial basis.* We use the same notation $\mathfrak{B}$ for varying components $\omega \in \pi_0(\mathrm{Bun}_G)$.

For $\alpha \in \mathfrak{B}$ in the form (5.5.3), we call

$$d := \sum_{1\le i\le n,\ 0\le j\le 2g+1} e_{ij} \tag{5.5.7}$$

the *naive degree* of $\alpha \in \mathfrak{B}$.

5.5.7. *Hecke action on tautological classes.* From here until the end of Lemma 5.5.15, we will fix a dominant minuscule coweight μ of G, and consider the one-step Hecke stack ${}^\omega\mathrm{Hk}_G^\mu$.

First we need to prove a technical result about the behavior of tautological classes under Hecke correspondences. We follow the notation of §5.1.1.

Proposition 5.5.8. *For $f \in R^W$ and $z \in \mathrm{H}_{|z|}(X)$, we have*

$$h_1^*(f^z) - h_0^*(f^z) = \mathrm{PD}(z)\partial_\mu(f) + \begin{cases} (1-g)\langle z, \xi\rangle \xi \partial_\mu^2(f) & |z| = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We apply Theorem 3.2.1 to $S := X \times {}^\omega\mathrm{Hk}_G^\mu$, the two G-bundles $\mathcal{F}_0$ and $\mathcal{F}_1$ with the modification (5.1.3) along the divisor $D = \Gamma(p_X)$. Note that $D \hookrightarrow S$ is the pullback of the diagonal $\Delta_X \hookrightarrow X \times X$ under $(q_X, p_X \circ p_{\mathrm{Hk}}) : S \to X \times X$. Therefore the normal bundle of D is the pullback $p_X^* T_X$. In particular, its first Chern class is $\nu_D = (2-2g)\xi$, and $\nu_D^2 = 0$.

Let $\mathcal{H}$ be the L_μ-torsor on $\Gamma(p_X)$ obtained from the restriction $\mathcal{F}_0|_{\Gamma(p_X)}$ (cf. §3.1.3). Since $\nu_D^2 = 0$ as noted above, Theorem 3.2.1 says in this case that

$$f(\mathcal{F}_1) - f(\mathcal{F}_0) = i_{D!}\left((\partial_\mu f)(\mathcal{H}) + \frac{1}{2}(\partial_\mu^2 f)(\mathcal{H}) \cdot \nu_D\right). \tag{5.5.8}$$

By the Cartesian square

$$\begin{array}{ccc} D & \longrightarrow & \Delta_X \\ \downarrow{\scriptstyle i_D} & & \downarrow \\ X \times_k {}^\omega\mathrm{Hk}_G^\mu & \xrightarrow{\mathrm{Id} \times p_X} & X \times X \end{array}$$

we have

$$i_{D!}(\partial_\mu f)(\mathcal{H}) = i_{D!} i_D^*(p_{\mathrm{Hk}}^*(\partial_\mu f)(\mathcal{H})) = [D] \cdot p_{\mathrm{Hk}}^*((\partial_\mu f)(\mathcal{H})) = [\Delta_X] \cdot p_{\mathrm{Hk}}^*((\partial_\mu f)(\mathcal{H})).$$

By our convention (5.1.5), we are viewing $R^W \hookrightarrow \mathrm{H}^*(\mathrm{Hk}_\mu^\omega)$ embedded via the classifying map for $\mathcal{H}$, so we can simply write $\partial_\mu f(\mathcal{H})$ as $\partial_\mu f \in \mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$, which we will do below. Similarly, using $\nu_D = (2-2g)\xi$ we have

$$\begin{aligned} i_{D!}\Big(\frac{1}{2}(\partial_\mu^2 f)\nu_D\Big) &= i_{D!} i_D^*(p_{\mathrm{Hk}}^*((1-g)\xi\partial_\mu^2 f)) = [D] \cdot p_{\mathrm{Hk}}^*((1-g)\xi\partial_\mu^2 f) \\ &= (1-g)[\Delta_X] \cdot (1 \otimes \xi) p_{\mathrm{Hk}}^*(\partial_\mu^2 f) = (1-g)(\xi \otimes \xi) p_{\mathrm{Hk}}^*(\partial_\mu^2 f) \end{aligned}$$

where in the last step we used that $[\Delta_X] \cdot (1 \otimes \xi) = (\xi \otimes \xi) \in \mathrm{H}^4(X \times X)(2)$. Plugging these into (5.5.8) we get

$$f(\mathcal{F}_1) - f(\mathcal{F}_0) = [\Delta_X] \cdot p_{\mathrm{Hk}}^*(\partial_\mu f) + (1-g)(\xi \otimes \xi) p_{\mathrm{Hk}}^*(\partial_\mu^2 f).$$

Now we expand both sides using Künneth formula for $S = X \times {}^\omega\mathrm{Hk}_G^\mu$, and use $[\Delta_X] = \xi \otimes 1 - \beta + 1 \otimes \xi$; then contracting with $z \in \mathrm{H}_*(X)$ gives the desired formula. □

[9] We are using here the fact that Frob acts semisimply on $\mathrm{H}_1(X)$; however, without using this fact, the argument can still be made with slight modification towards the end.

5.5.9. *Ran filtration for Hecke stack.* We also define an increasing filtration on $\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ as follows. Let

$$\begin{aligned} F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) &= 0, \quad m < -2, \\ F_{-2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) &= p_X^*\mathrm{H}^2(X) \cdot R^{W_\mu}, \\ F_{-1}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) &= p_X^*\mathrm{H}^{\geq 1}(X) \cdot R^{W_\mu}, \\ F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) &= p_X^*\mathrm{H}^*(X) \cdot R^{W_\mu}, \quad \text{which is the image of (5.1.6)} \\ F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) &= F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) \cdot h_0^* F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega}), \quad m \geq 1. \end{aligned}$$

Both filtrations $F_\bullet\mathrm{H}^*(\mathrm{Bun}_G^{\omega})$ and $F_\bullet\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ are also multiplicative, i.e.,

$$F_i \cdot F_j \subset F_{i+j}.$$

In particular, $F_0\mathrm{H}^*(\mathrm{Bun}_G^{\omega})$ (resp. $F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$) is a subring of $\mathrm{H}^*(\mathrm{Bun}_G^{\omega})$ (resp. $\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$).

Lemma 5.5.10. *The map h_1^* sends $F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$ to $F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ for all $n \in \mathbb{Z}$.*

Proof. Consider $f^z \in F_{|z|}\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$. Then by Proposition 5.5.8 we have

$$h_1^*(f^z) - h_0^*(f^z) \in p_X^*\mathrm{H}^{\geq 2-|z|}(X) \cdot R^{W_\mu} = F_{|z|-2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}). \tag{5.5.9}$$

Since $h_0^*(f^z) \in F_{|z|}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ by definition, (5.5.9) implies that $h_1^*(f^z) \in F_{|z|}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$.

By construction, both filtrations on $\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$ and on $\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ are multiplicative. Since h_1^* preserves the filtrations on a set of ring generators, it preserves the filtrations on all elements of $\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$. □

Theorem 4.2.8 gives a canonical identification of the cohomology rings of different components of Bun_G. Using this identification, it makes sense to define a map

$$\begin{aligned} \Delta_\mu : \mathrm{H}^*(\mathrm{Bun}_G^{\omega}) &\to \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) \\ \theta &\mapsto h_1^*\theta - h_0^*\theta. \end{aligned}$$

Lemma 5.5.11. *The map Δ_μ sends $F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega})$ to $F_{m-2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ for all $n \in \mathbb{Z}$. The induced map on the associated graded*

$$\mathrm{Gr}_\bullet^F \Delta_\mu : \mathrm{Gr}_\bullet^F \mathrm{H}^*(\mathrm{Bun}_G^{\omega}) \to \mathrm{Gr}_{\bullet-2}^F \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$$

is a derivation with respect to the ring homomorphism

$$\mathrm{Gr}_\bullet^F(h_0^*) = \mathrm{Gr}_\bullet^F(h_1^*) : \mathrm{Gr}_\bullet^F \mathrm{H}^*(\mathrm{Bun}_G^{\omega}) \cong \mathrm{Gr}_\bullet^F \mathrm{H}^*(\mathrm{Bun}_G^{\omega'}) \to \mathrm{Gr}_\bullet^F \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}).$$

It is characterized by the property that

$$(\mathrm{Gr}_{|z|}^F \Delta_\mu)(f^z) = \mathrm{PD}(z)\partial_\mu(f) \in \mathrm{Gr}_{|z|-2}^F \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}).$$

Proof. By definition, a general element in $F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega})$ is a linear combination of $f = f_1^{z_1} f_2^{z_2} \cdots f_s^{z_s}$ where $|z_1| + \cdots + |z_s| \leq m$. We may write

$$h_1^* f - h_0^* f = \sum_{i=1}^{s} \left(\prod_{i' < i} h_1^* f_{i'}^{z_{i'}} \right) (h_1^* f_i^{z_i} - h_0^* f_i^{z_i}) \left(\prod_{i'' > i} h_0^* f_{i''}^{z_{i''}} \right). \tag{5.5.10}$$

By Lemma 5.5.10 and (5.5.9), each summand above lies in $F_{\sum_{i' \neq i} |z_{i'}| + |z_i| - 2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) = F_{m-2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$. Passing to associated graded, we learn that $h_0^* f$ has the same image as $h_1^* f$ in $\mathrm{Gr}_m^F \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$, hence (5.5.10) tells that Δ_μ is a derivation after passing to associated graded. The calculation of $(\mathrm{Gr}_{|z|}^F \Delta_\mu)(f^z)$ follows from Proposition 5.5.8. □

Lemma 5.5.12. *The map h_{0*} carries $F_{m-2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) \to F_m\mathrm{H}^{*-2D_\mu-2}(\mathrm{Bun}_G^{\omega})(-D_\mu - 1)$ for all $m \in \mathbf{Z}$.*

Proof. The canonical parabolic reduction fits into a commutative diagram

$$\begin{array}{ccc} {}^{\omega}\mathrm{Hk}_G^{\mu} & \longrightarrow & X\times\mathbb{B}P_\mu \\ {\scriptstyle p_X\times h_0}\Big\downarrow & & \Big\downarrow \\ X\times \mathrm{Bun}_G^\omega & \xrightarrow{\mathrm{ev}^\omega} & X\times \mathbb{B}G \\ \Big\downarrow & & \\ \mathrm{Bun}_G^\omega & & \end{array} \tag{5.5.11}$$

whose top square is moreover Cartesian, and whose vertical maps are fiber bundles for G/P_μ. Therefore, using the notation $\int_{G/P_\mu} : R^{W_\mu} \to R^W$ introduced in §5.3.1, we can rewrite h_{0*} on the image of $\mathrm{H}^*(X)\otimes \mathrm{H}^*(\mathbb{B}P_\mu) = \mathrm{H}^*(X)\otimes R^{W_\mu} \subset \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ as the composition

$$h_{0*} : \mathrm{H}^*(X)\otimes R^{W_\mu} \xrightarrow{\mathrm{id}\otimes\int_{G/P_\mu}} \mathrm{H}^*(X)\otimes R^W \xrightarrow{\mathrm{id}\otimes\mathrm{ev}^{\omega,*}} \mathrm{H}^*(X\times\mathrm{Bun}_G^\omega) \xrightarrow{\int_X} \mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{5.5.12}$$

For $n \leq 0$, we have

$$h_{0*}F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) = h_{0*}(\mathrm{H}^{\geq -m}(X)\otimes R^{W_\mu}) \subset \int_X (\mathrm{H}^{\geq -m}(X)\cdot R^W).$$

Here R^W is embedded into $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ via $\mathrm{ev}^{\omega,*}$. For $\zeta \in \mathrm{H}^{|\zeta|}(X)$ and $f \in R^W$, $\int_X(\zeta f)$ is a linear combination of f^z where $|z|+|\zeta| = 2$. This implies

$$\int_X (\mathrm{H}^{\geq -m}(X)\cdot R^W) \subset F_{m+2}.$$

Now consider the case $m > 0$. We abbreviate $F_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ simply as F_m. Using that h_{0*} is linear over $h_0^*\mathrm{H}^*(\mathrm{Bun}_G^\omega)$, and $h_{0*}F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) \subset F_2$ by the preceding paragraph, we conclude that

$$\begin{aligned} h_{0*}F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) &= h_{0*}(F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})\cdot h_0^*F_m) \\ &= h_{0*}F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})\cdot F_m \subset F_2\cdot F_m \subset F_{m+2}. \end{aligned}$$

□

Corollary 5.5.13. *For any $\beta \in \mathrm{H}^*(X\times\mathbb{B}P_\mu)$, Γ_μ^β sends $F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$ to $F_{\leq m+2}\mathrm{H}^*(\mathrm{Bun}_G^\omega)$, for all $m \in \mathbb{Z}$.*

Proof. Let $\alpha \in F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$. By Lemma 5.5.10, $h_1^*\alpha \in F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$. By definition, $\beta \in F_0\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$, hence $h_1^*(\alpha)\beta \in F_m\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$. Finally, by Lemma 5.5.12, $\Gamma_\mu^\beta(\alpha) = h_{0*}(h_1^*(\alpha)\beta) \in F_{\leq m+2}\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$. □

5.5.14. *Truncation of Γ_μ^β.* Let $\beta \in \mathrm{H}^*(X\times\mathbb{B}P_\mu)$ be homogeneous. For $i \in \mathbb{Z}$ let $\mathfrak{R}_i\Gamma_\mu^\beta : \mathrm{H}^*(\mathrm{Bun}_G^{\omega'}) \to \mathrm{H}^*(\mathrm{Bun}_G^\omega)$ be the Ran degree i part of Γ_μ^β: for $\alpha \in \mathfrak{R}_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$, $\mathfrak{R}_i\Gamma_\mu^\beta(\alpha)$ is the Ran degree $m+i$ piece of $\Gamma_\mu^\beta(\alpha)$. Define $\mathfrak{R}_{\geq i}\Gamma_\mu^\beta : \mathrm{H}^*(\mathrm{Bun}_G^{\omega'}) \to \mathrm{H}^*(\mathrm{Bun}_G^\omega)$ to be the sum

$$\mathfrak{R}_{\geq i}\Gamma_\mu^\beta = \sum_{i'\geq i}\mathfrak{R}_{i'}\Gamma_\mu^\beta.$$

The main estimate to prove the convergence of trace is the following.

Lemma 5.5.15. *Fix $i \geq 0$. For a monomial basis element $\alpha \in \mathfrak{B}$, write $\mathfrak{R}_{\geq -i}\Gamma_\mu^\beta$ as a linear combination of the monomial basis for $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$:*

$$\mathfrak{R}_{\geq -i}\Gamma_\mu^\beta(\alpha) = \sum_{\alpha'\in\mathfrak{B}} c_\alpha^{\alpha'}\alpha'$$

for $c_\alpha^{\alpha'} \in \mathbf{C}$. Then there is $C_{\beta,\mu,i} > 0$ depending only on β, μ, i (and not on α) such that

$$\sum_{\alpha'\in\mathfrak{B}} |c_\alpha^{\alpha'}|_\iota \leq C_{\beta,\mu,i}(d+1)^{i/2+1}, \text{ for all } \alpha\in\mathfrak{B} \text{ with naive degree } d.$$

Here $|\cdot|_\iota$ denotes the absolute value under the embedding $\iota : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$.

Proof. Write $\alpha = \prod_{s=1}^{d} f_{i_s}^{z_{j_s}}$, where d is the naive degree. Let $m = \sum_{s=1}^{d} |z_{j_s}|$ be the Ran degree of α. Let $\Delta_\mu(f^z) = h_1^*(f^z) - h_0^*(f^z)$. Expanding $h_1^*\alpha = \prod_s h_1^*(f_{i_s}^{z_{j_s}}) = \prod_s (h_0^*(f_{i_s}^{z_{j_s}}) + \Delta_\mu(f_{i_s}^{z_{j_s}}))$, we get

$$\begin{aligned} \Gamma_\mu^\beta(\alpha) &= h_{0*}(h_1^*\alpha\cdot\beta) = \sum_{J\subset\{1,2,\cdots,d\}} \pm h_{0*}\left(h_0^*(\prod_{s\notin J} f_{i_s}^{z_{j_s}})\cdot\prod_{s\in J}\Delta_\mu(f_{i_s}^{z_{j_s}})\beta\right) \\ &= \sum_{J\subset\{1,2,\cdots,d\}} \pm\left(\prod_{s\notin J} f_{i_s}^{z_{j_s}}\right) h_{0*}(\prod_{s\in J}\Delta_\mu(f_{i_s}^{z_{j_s}})\beta). \end{aligned} \tag{5.5.13}$$

By Lemma 5.5.11, $\prod_{s\in J}\Delta_\mu(f_{i_s}^{z_{j_s}}) \in F_{\sum_{s\in J}(|z_{j_s}|-2)}\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$. Since $\beta\in F_0\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$, by Lemma 5.5.12, we have

$$h_{0*}(\prod_{s\in J}\Delta_\mu(f_{i_s}^{z_{j_s}})\beta) \in F_{\le 2+\sum_{s\in J}(|z_{j_s}|-2)}\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu).$$

Therefore, the J-summand of (5.5.13) lies in $F_{\le m-2|J|+2}\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. Therefore, in computing $\mathfrak{R}_{\ge -i}\Gamma_\mu^\beta(\alpha)$, we only need to sum over those $J\subset\{1,2,\cdots,d\}$ with $m-2|J|+2\ge m-i$, i.e., $|J|\le i/2+1$. The number of such J is $\le (d+1)^{i/2+1}$. For fixed J, writing $h_{0*}(\prod_{s\in J}\Delta_\mu(f_{i_s}^{z_{j_s}})\beta)$ as a sum of elements in $\mathfrak{B}$, the sum of absolute values of coefficients have an upper bound $C_{\beta,\mu,i}$, since there are only finitely many possibilities for $\prod_{s\in J}\Delta_\mu(f_{i_s}^{z_{j_s}})$ when $|J|$ is bounded by i. Therefore for each $|J|\le i/2+1$, the corresponding summand in (5.5.13) will contribute at most $C_{\beta,\mu,i}$ to the total sum of absolute values of coefficients of $\mathfrak{R}_{\ge -i}\Gamma_\mu^\beta(\alpha)$, giving the desired upper bound. □

5.5.16. *Proof of Proposition 5.5.3–first reductions.* The map $(p_1,\cdots,p_r,h_0) : {}^\omega\mathrm{Hk}_G^\mu \to X^r\times\mathrm{Bun}_G^\omega$ is an iterated fibration with fibers G/P_{μ_j}. Therefore, as a module over $h_0^*\mathrm{H}^*(\mathrm{Bun}_G^\omega)$, $\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$ is generated by elements of the form $\beta_1\cdots\beta_r$, where $\beta_j\in\mathrm{H}^*(X\times\mathbb{B}P_{\mu_j})$ is pulled back to ${}^\omega\mathrm{Hk}_G^\mu$ via $(p_j,\mathrm{ev}_j):\mathrm{Hk}_G^\mu\to X\times\mathbb{B}P_{\mu_j}$. It is therefore sufficient to treat the case where $\theta=(h_0^*\alpha)\beta_1\cdots\beta_r$, where $\alpha\in\mathfrak{B}$ is a monomial basis element for $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ and $\beta_i\in\mathrm{H}^*(X\times\mathbb{B}P_{\mu_i})$. In this case Γ_μ^θ is the composition

$$\Gamma_\mu^\theta = (\cup\alpha)\circ\Gamma_{\mu_1}^{\beta_1}\circ\cdots\circ\Gamma_{\mu_r}^{\beta_r}.$$

Here $\cup\alpha$ is the endomorphism of $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ given by cup product with α.

Below we base change all $\overline{\mathbf{Q}}_\ell$-vector spaces to $\mathbf{C}$-vector spaces using ι. In particular, Γ_μ^θ can be represented as a matrix with entries in $\mathbf{C}$ under the monomial basis of $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. We can therefore talk about the diagonal matrix coefficients of Γ_μ^θ.

Lemma 5.5.17. *Let t be the Ran degree of α, and let $u=r(t/2+r)$. There exists $C_{\theta,\mu}>0$ (depending on θ and μ) such that, when writing Γ_μ^θ as a matrix using the monomial basis $\mathfrak{B}$, its diagonal entry at any $\alpha'\in\mathfrak{B}$ with naive degree $d\ge 0$ has absolute value bounded above by $C_{\theta,\mu}(d+1)^u$.*

Proof. Let $i=t+2(r-1)$. Write each $\Gamma_{\mu_j}^{\beta_j}$ as $\mathfrak{R}_{\ge -i}\Gamma_{\mu_j}^{\beta_j}+\mathfrak{R}_{<-i}\Gamma_{\mu_j}^{\beta_j}$ (where $\mathfrak{R}_{<-i}\Gamma_{\mu_j}^{\beta_j}$ is the part of $\Gamma_{\mu_j}^{\beta_j}$ that decreases the Ran degree by more than i). We expand

$$\Gamma_\mu^\theta = (\cup\alpha)\circ(\mathfrak{R}_{\ge -i}\Gamma_{\mu_1}^{\beta_1}+\mathfrak{R}_{<-i}\Gamma_{\mu_1}^{\beta_1})\circ\cdots\circ(\mathfrak{R}_{\ge -i}\Gamma_{\mu_r}^{\beta_r}+\mathfrak{R}_{<-i}\Gamma_{\mu_r}^{\beta_r})$$

into a sum of compositions of $\mathfrak{R}_{\ge -i}\Gamma_{\mu_j}^{\beta_j}$ and $\mathfrak{R}_{<-i}\Gamma_{\mu_j}^{\beta_j}$ with $\cup\alpha$. If $\mathfrak{R}_{<-i}\Gamma_{\mu_j}^{\beta_j}$ appears at the jth place, then the corresponding term will send $\mathfrak{R}_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ to $\mathfrak{R}_{<m-i+2(r-1)+t}\mathrm{H}^*(\mathrm{Bun}_G^\omega)=\mathfrak{R}_{<m}\mathrm{H}^*(\mathrm{Bun}_G^\omega)$, because each of the other terms $\mathfrak{R}_{\ge -i}\Gamma_{\mu_{j'}}^{\beta_{j'}}$ (there are $\le r-1$ of them) can at most increase the Ran grading by 2, and $\cup\alpha$ increases the Ran grading by m. Therefore, under the monomial basis, the diagonal entries of Γ_μ^θ are the same as the diagonal entries of

$$(\cup\alpha)\circ\mathfrak{R}_{\ge -i}\Gamma_{\mu_1}^{\beta_1}\circ\cdots\circ\mathfrak{R}_{\ge -i}\Gamma_{\mu_r}^{\beta_r}. \tag{5.5.14}$$

Now if $\alpha'\in\mathfrak{B}$ has naive degree d, by iterative use of Lemma 5.5.15, the sum of absolute values of coefficients of $\mathfrak{R}_{\ge -i}\Gamma_{\mu_1}^{\beta_1}\circ\cdots\circ\mathfrak{R}_{\ge -i}\Gamma_{\mu_r}^{\beta_r}(\alpha')$ in terms of $\mathfrak{B}$ is bounded above by

$$\prod_{s=1}^{r}\left(C_{\beta_s,\mu_s,i}(d+1+2(s-1))^{i/2+1}\right). \tag{5.5.15}$$

Since $\alpha \in \mathfrak{B}$, all matrix entries of $\cup\alpha$ in terms of the basis $\mathfrak{B}$ are either 0 or 1, we conclude that all diagonal entries of (5.5.14) are bounded by (5.5.15). Therefore the same is true for the diagonal entries of Γ^{θ}_{μ}. For a suitable constant $C_{\theta,\mu}$ depending only on θ and μ, one can bound (5.5.15) from above by $C_{\theta,\mu}(d+1)^{r(i/2+1)} = C(d+1)^u$. □

5.5.18. *Finish of the proof of Proposition 5.5.3.* Now decompose $\mathrm{H}^*(\mathrm{Bun}^{\omega}_G)$ according to the naive degrees of monomial basis elements:

$$\mathrm{H}^*(\mathrm{Bun}^{\omega}_G) = \bigoplus_{d\geq 0} \mathfrak{N}_d,$$

where $\mathfrak{N}_d$ is the span of $\alpha' \in \mathfrak{B}$ with naive degree d. Note that this grading splits the augmentation filtration on $\mathrm{H}^*(\mathrm{Bun}^{\omega}_G)$ introduced in §5.6.14. All the generators $f_i^{z_j} \in \mathfrak{B}$ have cohomological degrees between 2 and $2d_n$, where d_n is the largest degree of the generators $\{f_j\}$ of R^W. Hence a monomial of naive degree d has degree in $[2d, 2d_n d]$. Therefore

$$\mathrm{H}^i(\mathrm{Bun}^{\omega}_G) \subset \bigoplus_{i/(2d_n)\leq d\leq i/2} \mathfrak{N}_d.$$

In particular, by Lemma 5.5.17, the diagonal entries of Γ^{θ}_{μ} on $\mathrm{H}^i(\mathrm{Bun}^{\omega}_G)$ are bounded by $C(i/2+1)^u$. The Frobenius action is diagonal with respect to the monomial basis, and has eigenvalues with absolute value $q^{i/2}$ on $\mathrm{H}^i(\mathrm{Bun}^{\omega}_G)$. Therefore

$$|\mathrm{Tr}(\mathrm{Frob}^{-1}\circ \Gamma^{\theta}_{\mu}|\mathrm{H}^i(\mathrm{Bun}^{\omega}_G))| \leq C_{\theta,\mu} q^{-i/2}(i/2+1)^u \dim \mathrm{H}^i(\mathrm{Bun}^{\omega}_G). \tag{5.5.16}$$

Now $(i/2+1)^u \dim \mathrm{H}^i(\mathrm{Bun}^{\omega}_G)$ has polynomial growth in i while $q^{-i/2}$ decays exponentially, so the summation of (5.5.16) over i converges. This finishes the proof of Proposition 5.5.3. □

5.6. **Calculation of the arithmetic volume.** We will compute the arithmetic volume defined in Definition 5.2.3, at least under the following assumption.

Assumption 5.6.1 (Commutativity of local operators)**.** We assume that the operators $\overline{\nabla}^{\eta_j}_{\mu_j}$ pairwise commute for $j = 1, \ldots, r$. (When $r = 0$, we interpret this assumption as being vacuously true.)

Example 5.6.2. Assumption 5.6.1 is satisfied in many cases of interest. For example, under the degree splitting

$$\mathbb{V} \cong \bigoplus_d \mathbb{V}_{2d}$$

if the nonzero $\mathbb{V}_{2d}$ all have dimension 1, then Assumption 5.6.1 is automatically satisfied since the operators $\overline{\nabla}^{\eta_j}_{\mu_j}$ are degree-preserving. The one-dimensionality of graded pieces of $\mathbb{V}$ holds for all simple groups except those of type D_{2m}.

On the other hand, for $G = \mathrm{PSO}_{4m}$, there are three minuscule coweights. There is a natural choice of η coming from the Killing form of G (see §6). In this case, it was not known to the authors whether the $\overline{\nabla}^{\eta}_{\mu}$ commute; eventually a counterexample was found in [Fen26].

5.6.3. *The L-function.* Under Assumption 5.6.1, the endomorphisms $\{\overline{\nabla}^{\eta_j}_{\mu_j}\}$ of $\mathbb{V}$ have a common system of generalized eigenvalues. More precisely, one can find a basis $v_1, \cdots, v_n$ of $\mathbb{V}$ consisting of homogeneous elements, such that v_i belongs to a generalized eigenspace of $\overline{\nabla}^{\eta_j}_{\mu_j}$ for each $1 \leq j \leq r$. Let d_i be the degree of v_i, and let $\epsilon_i(\eta_j, \mu_j) \in \overline{\mathbf{Q}}_\ell$ be the generalized eigenvalue of $\overline{\nabla}^{\eta_j}_{\mu_j}$ on v_i. The n-tuple (of $(r+1)$-tuples)

$$(d_i, \epsilon_i(\eta_1,\mu_1), \cdots, \epsilon_i(\eta_r,\mu_r)), \quad i = 1, 2, \cdots, n$$

is independent of the choice of the basis v_i up to permutation of the index i.

To formulate the answer for our eventual calculation of $\mathrm{vol}(\mathrm{Sht}^{\mu}_G, \eta)$, we introduce the following L-function in n-variables:

$$\mathscr{L}_{X,G}(s_1, \cdots, s_n) := \prod_{i=1}^{n} \zeta_X(s_i + d_i). \tag{5.6.1}$$

We also consider the regularized version by removing all factors $(1-q^{-s_i})$ (i.e., where $d_i = 1$) in the denominator:

$$\mathscr{L}^*_{X,G}(s_1, \cdots, s_n) := \prod_{\substack{1\le i\le n \\ d_i\neq 1}} \zeta_X(s_i+d_i) \prod_{\substack{1\le i\le n \\ d_i=1}} \zeta^*_X(s_i+1), \tag{5.6.2}$$

5.6.4. *Invariants of the tautological classes.* Recall $\eta, \eta' \in R^{W_\mu}$ are homogeneous of degree $2(D_\mu+1)$ and $2D_\mu$ respectively. We will define numerical constants associated to η and η'.

Definition 5.6.5. By assumption, $\eta \in R^{W_\mu}$ has degree $2(D_\mu+1)$, hence $\int_{G/P_\mu} \eta \in R^W$ is of degree 2. Viewing R^W as a subring of $\mathrm{H}^*(X \times \mathrm{Bun}_G^\omega)$ via pullback along the tautological map, we can then apply $\int_X(-)$ to define (using notation from (4.2.7))

$$d^\omega_\mu(\eta) := \left(\int_{G/P_\mu} \eta\right)^{[X]} = \int_X \int_{G/P_\mu} \eta \in \mathrm{H}^0(\mathrm{Bun}_G^\omega) \cong \overline{\mathbf{Q}}_\ell.$$

This number $d^\omega_\mu(\eta)$ depends on the component ω of Bun_G.

Example 5.6.6. If G is semisimple, then the degree 2 part of R^W vanishes, so we automatically have $d^\omega_\mu(\eta) = 0$.

Definition 5.6.7. Recall that $\eta' \in R^{W_\mu}$ has degree $2D_\mu = 2\dim(G/P_\mu)$. Then we define

$$d_\mu(\eta') := \int_{G/P_\mu} \eta' \in \overline{\mathbf{Q}}_\ell.$$

5.6.8. *The differential operators.* Recall from (5.2.3) that $\omega_j = \omega + \overline{\mu}_1 + \cdots + \overline{\mu}_j \in \pi_0(\mathrm{Bun}_G)$. The definition is arranged so that ${}_c\Gamma_\mu^{\eta_j+\xi\eta'_j}$ goes from $\mathrm{H}^*_c(\mathrm{Bun}_G^{\omega_{j-1}})$ to $\mathrm{H}^*_c(\mathrm{Bun}_G^{\omega_j})$.

For $j = 1, \cdots, r$ consider the first order differential operator

$$\mathfrak{d}_j := d^{\omega_{j-1}}_{\mu_j}(\eta_j) + d_{\mu_j}(\eta'_j) - (\log q)^{-1} \sum_{i=1}^n \epsilon_i(\eta_j, \mu_j)\partial_{s_i}. \tag{5.6.3}$$

Here we refer to Definitions 5.6.5 and 5.6.7 for the numerical constants $d^{\omega_{j-1}}_{\mu_j}(\eta_j)$ and $d_{\mu_j}(\eta'_j)$.

Theorem 5.6.9. *Let $\mu = (\mu_1, \cdots, \mu_r)$ be a sequence of minuscule dominant coweights of G satisfying (5.2.1). Let $\eta_j \in \mathrm{H}^{2D_{\mu_j}+2}(\mathbb{B}L_{\mu_j})$ and $\eta'_j \in \mathrm{H}^{2D_{\mu_j}}(\mathbb{B}L_{\mu_j})$ for $j = 1, 2, \ldots, r$, satisfying Assumption 5.6.1. For $\eta = (\eta_1 + \xi\eta'_1, \ldots, \eta_r + \xi\eta'_r)$, we have*

$$\mathrm{vol}({}^\omega\mathrm{Sht}^\mu_G, \eta) = q^{\dim \mathrm{Bun}_G} \left(\prod_{j=1}^r \mathfrak{d}_j\right) \mathscr{L}^*_{X,G}(s_1, \cdots, s_n)\Big|_{s_1=s_2=\cdots=s_n=0}. \tag{5.6.4}$$

Remark 5.6.10. Even the case $r = 0$, where μ and η are empty, is interesting. In this case, the result collapses to the Tamagawa Number Formula proved by [GL14], at least in the case where G is semisimple and simply connected. Our proof of this special case is the same as theirs, being based on the Atiyah–Bott description.

Example 5.6.11. A common situation is: G is semisimple, all μ_i are equal to the same minuscule coweight μ, all η_j are equal to a common η, and all η'_j are equal to a common η'. Since G is semisimple, Example 5.6.6 implies that $d^{\omega_{j-1}}_{\mu_j}(\eta_j) = 0$ for all j. In this case, Assumption 5.6.1 is trivially satisfied (since all the endomorphisms are equal), and Theorem 5.6.9 can be written as

$$\mathrm{vol}({}^\omega\mathrm{Sht}^\mu_G, \eta) = q^{\dim \mathrm{Bun}_G} \left(d_\mu(\eta') - (\log q)^{-1}\frac{d}{ds}\right)^r\Big|_{s=0} \left(\prod_{i=1}^n \zeta_X(\epsilon_i(\eta,\mu)s + d_i)\right).$$

The remainder of this Section is devoted to the proof of Theorem 5.6.9. Our strategy is to compute the "eigenvalues" of the individual $\Gamma_{\mu_j}^{\eta_j+\xi\eta'_j}$, quotation marks because $\Gamma_{\mu_j}^{\eta_j+\xi\eta'_j}$ does not map $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ to itself, hence does not have an a priori notion of eigenvalues. However, as discussed in §4.2.9, $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$

and $\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$ can both be identified with the polynomial ring generated by the Künneth components of universal characteristic classes using the Atiyah-Bott description. We can use this identification to view $\Gamma_\mu^{\eta+\xi\eta'}$ as an endomorphism, and thus make sense of its eigenvalues. To do this, we will show that the Ran filtration on $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ introduced in (5.5.6) is stable under $\Gamma_\mu^{\eta+\xi\eta'}$ (in the above sense), and its eigenvalues on the associated graded are easy to calculate.

5.6.12. *Analysis of the associated graded.* Until §5.6.18, we denote by μ a minuscule coweight of G and consider the one-step Hecke stack ${}^\omega\mathrm{Hk}_G^\mu$. Let $\eta, \eta' \in R^{W_\mu}$ be homogeneous of degree $2(D_\mu+1)$ and $2D_\mu$ respectively. Recall the definitions of $d_\mu^\omega(\eta')$ and $d_\mu(\eta')$ from §5.6.4. The following is a refinement of Corollary 5.5.13.

Lemma 5.6.13. *Let* $\gamma_\mu^\eta : \mathrm{Gr}_\bullet^F \mathrm{H}^*(\mathrm{Bun}_G^{\omega'}) \to \mathrm{Gr}_\bullet^F \mathrm{H}^*(\mathrm{Bun}_G^\omega)$ *be the derivation characterized by the equation*

$$\gamma_\mu^\eta(f^z) = (\nabla_\mu^\eta(f))^z, \quad z \in \mathrm{H}_*(X), f \in R^W$$

where $\nabla_\mu^\eta(f)$ *was defined in (5.3.2). Then for all* $m \in \mathbf{Z}$*, the operator* $\Gamma_\mu^{\eta+\xi\eta'}$ *from (5.1.7) carries* $F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$ *to* $F_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$*. On the associated graded, it takes the form*

$$\mathrm{Gr}_\bullet^F \Gamma_\mu^{\eta+\xi\eta'}(\theta) = (d_\mu^\omega(\eta) + d_\mu(\eta'))\theta + \gamma_\mu^\eta(\theta).$$

Proof. Let $\theta \in F_m\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$. We have

$$\begin{aligned} \Gamma_\mu^{\eta+\xi\eta'}(\theta) &= h_{0*}((\eta+\xi\eta')(h_1^*\theta - h_0^*\theta) + (\eta+\xi\eta')h_0^*\theta) \\ &= h_{0*}((\eta+\xi\eta')\Delta_\mu(\theta)) + (h_{0*}(\eta+\xi\eta'))\cdot\theta. \end{aligned} \tag{5.6.5}$$

To make sense of the second summand, we identify θ with a class in $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ using the Atiyah–Bott presentation. By Lemma 5.5.11, we have $\Delta_\mu(\theta) \in F_{m-2}\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$. Since $\eta+\xi\eta' \in F_0\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$, the product $(\eta+\xi\eta')\Delta_\mu(\theta)$ also lies in $F_{m-2}\mathrm{H}^*({}^\omega\mathrm{Hk}_G^\mu)$. By Lemma 5.5.12, the first summand in (5.6.5) lies in $F_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. For the second summand, using (5.5.12) and the definitions of $d_\mu^\omega(\eta)$ and $d_\mu(\eta')$, we get

$$h_{0*}(\eta+\xi\eta') = d_\mu^\omega(\eta) + d_\mu(\eta').$$

Hence the second summand in (5.6.5) is $(d_\mu^\omega(\eta) + d_\mu(\eta'))\theta$, which also lies in $F_m\mathrm{H}^*(\mathrm{Bun}_G^\omega)$.

Now we calculate the effect of $\Gamma_\mu^{\eta+\xi\eta'}$ on the associated graded. Since $\xi\eta' \in F_{-2}$, Lemma 5.5.12 implies that $h_{0*}(\xi\eta'\Delta_\mu(\theta)) \in F_{m-2}$. Therefore, we have

$$\mathrm{Gr}_m^F \Gamma_\mu^{\eta+\xi\eta'}(\theta) \equiv h_{0*}(\eta\Delta_\mu(\theta)) + (d_\mu^\omega(\eta) + d_\mu(\eta'))\theta \mod F_{m-2}.$$

The map $\theta \mapsto h_{0*}(\eta\Delta_\mu(\theta))$ on $\mathrm{Gr}_n^F\mathrm{H}^*(\mathrm{Bun}_G^{\omega'})$ is given by

$$\delta_\mu^\eta(\theta) := (\mathrm{Gr}_{m-2}^F h_{0*})(\eta\cdot(\mathrm{Gr}_n^F\Delta_\mu)(\theta)).$$

By Lemma 5.5.11, $(\mathrm{Gr}_\bullet^F\Delta_\mu)(\theta)$ is a derivation while $\mathrm{Gr}_\bullet^F h_{0*}$ is $\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega)$-linear, hence we see that δ_μ^η is also a derivation. On f^z (where $z \in \mathrm{H}_*(X)$ and $f \in R^W$) we have by Lemma 5.5.11 and (5.5.12) that

$$\begin{aligned} \delta_\mu^\eta(f^z) &= (\mathrm{Gr}_\bullet^F h_{0*})(\mathrm{PD}(z)\eta\partial_\mu f) \\ &= \int_X \mathrm{PD}(z)\cdot\left(\int_{G/P_\mu}\eta\partial_\mu f\right) = (\nabla_\mu^\eta f)^z. \end{aligned}$$

Therefore $\delta_\mu^\eta = \gamma_\mu^\eta$. This finishes the proof. □

5.6.14. *Augmentation filtration.* The graded ring $\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ carries a natural augmentation by projecting to $\mathrm{Gr}_0^F\mathrm{H}^0(\mathrm{Bun}_G^\omega) = \overline{\mathbf{Q}}_\ell$. Note that this restricts to the natural augmentation on $\mathrm{Gr}_0^F\mathrm{H}^*(\mathrm{Bun}_G^\omega) \cong R^W$. We apply the construction in §2.4 to form the associated graded with respect to the adic filtration given by the augmentation ideal:

$$\mathrm{Gr}_{\mathrm{aug}}^\blacktriangleright \mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

This is a triply-graded ring, with the three gradings denoted $\blacktriangleright, \bullet$ and $*$.

Example 5.6.15. Abbreviating $\mathrm{Gr}_i^F = \mathrm{Gr}_i^F \mathrm{H}^*(\mathrm{Bun}_G^\omega)$, we have

$$\mathrm{Gr}^1_{\mathrm{aug}}\mathrm{Gr}_i^F = \begin{cases} \mathrm{Gr}^1_{\mathrm{aug}} R^W = \mathbb{V} & i=0; \\ \mathrm{Gr}_1^F/(R^W_+ \mathrm{Gr}_1^F) & i=1; \\ \mathrm{Gr}_2^F/(R^W_+\mathrm{Gr}_2^F + \mathrm{Gr}_1^F\cdot \mathrm{Gr}_1^F) & i=2. \end{cases}$$

For $z = 1 \in \mathrm{H}_0(X)$, the isomorphism $(-)^z : R^W \xrightarrow{\sim} \mathrm{Gr}_0^F \mathrm{H}^*(\mathrm{Bun}_G^\omega)$ induces an isomorphism

$$\mathrm{H}_0(X)\otimes \mathbb{V} \xrightarrow{\sim} \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{Gr}_0^F\mathrm{H}^*(\mathrm{Bun}_G^\omega) = \mathrm{Gr}^1_{\mathrm{aug}} R^W. \tag{5.6.6}$$

For $z\in \mathrm{H}_1(X)$, it is clear from the definition that the map $(-)^z : R^W \xrightarrow{\sim} F_1\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ is a derivation. In particular, $(f_1f_2)^z \in R^W_+ F_1\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. This induces a map

$$\mathrm{H}_1(X)\otimes\mathbb{V} \to \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{Gr}_1^F\mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{5.6.7}$$

For $z = [X] \in \mathrm{H}_2(X)$, it is clear from the definition that the map $(-)^z : R^W \to F_2\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ is a derivation modulo $F_{1,+}F_{1,+}$, where $F_{1,+}$ is the positive cohomological degree part of $F_1\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. Therefore it induces a map

$$\mathrm{H}_2(X)\otimes \mathbb{V}_{>2} \to \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{Gr}_2^F\mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{5.6.8}$$

The direct sum of (5.6.6), (5.6.7), and (5.6.8) gives a map

$$(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+ \to \mathrm{Gr}^1_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

This then induces a map of triply-graded commutative $\overline{\mathbf{Q}}_\ell$-algebras

$$\mathrm{Gr}_{\mathrm{aug}}\mathrm{AB}^\omega : \mathrm{Sym}^\blacktriangleright((\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+) \to \mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

Here the $\bullet$-grading and $*$-grading (cohomological) on the left is understood as follows: a monomial $(z_1\otimes f_1)(z_2\otimes f_2)\cdots(z_s\otimes f_s)$, where $z_i\in \mathrm{H}_{|z_i|}(X)$ and $f_i\in R^W$ homogeneous of degree $|f_i|$, has $\bullet$-degree $\sum |z_i|$ and $*$-degree $\sum|f_i|-|z_i|$.

Lemma 5.6.16. *The map* $\mathrm{Gr}_{\mathrm{aug}}\mathrm{AB}^\omega$ *is an isomorphism.*

Proof. As in §4.2.9, we have a set of free generators $f_i^{z_j}$ for $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$. Now $f_i^{z_j}\in F_{|z_j|}$ and we denote its image in $\mathrm{Gr}^F_{|z_j|}$ by $\overline{f}_i^{z_j}$. From the description in §4.2.9 of $\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ as the polynomial ring on generators (4.2.14), it is easy to see that $\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega)$ is a polynomial ring with free generators $\overline{f}_i^{z_j}$. Then the assertion follows from observing that the $z_j\otimes\overline{f}_i$ (with $|f_i|>|z_j|$) form a basis for $(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+$, mapping to $\overline{f}_i^{z_j}$ under $\mathrm{Gr}_{\mathrm{aug}}\mathrm{AB}^\omega$. □

Using Lemma 5.6.13, we can now describe the operator $\Gamma_\mu^{\eta+\xi\eta'}$ on $\mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}^F_\bullet$ explicitly.

Proposition 5.6.17. *The operator* $\mathrm{Gr}_\bullet^F\Gamma_\mu^{\eta+\xi\eta'}$ *preserves the adic filtration by the augmentation ideals and passes to the associated graded*

$$\mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\Gamma_\mu^{\eta+\xi\eta'} : \mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^{\omega'}) \to \mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

Let $(\mathrm{id}_{\mathrm{H}_\bullet(X)}\otimes\overline{\nabla}^\eta_\mu)_+$ *be the restriction of* $\mathrm{id}_{\mathrm{H}_\bullet(X)}\otimes\overline{\nabla}^\eta_\mu \in \mathrm{End}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})$ *to* $(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+$, *and* $(\mathrm{id}_{\mathrm{H}_\bullet(X)}\otimes\overline{\nabla}^\eta_\mu)^{\mathrm{der}}_+$ *be its unique extension to a derivation on* $\mathrm{Sym}^\blacktriangleright((\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+)$. *Then under the isomorphisms* $\mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{AB}^\omega$ *and* $\mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{AB}^{\omega'}$, $\mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\Gamma_\mu^{\eta+\xi\eta'}$ *is identified with the endomorphism of*

$$\mathrm{Sym}^\blacktriangleright((\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+)$$

given by

$$\mathrm{Gr}^\blacktriangleright_{\mathrm{aug}}\mathrm{Gr}_\bullet^F\Gamma_\mu^{\eta+\xi\eta'} = (d^\omega_\mu(\eta)+d_\mu(\eta'))\mathrm{id} + (\mathrm{id}_{\mathrm{H}_\bullet(X)}\otimes\overline{\nabla}^\eta_\mu)^{\mathrm{der}}_+.$$

5.6.18. *Completion of the proof.* For the sequence $\mu = (\mu_1, \dots, \mu_r)$ and a fixed $\omega \in \pi_0(\mathrm{Bun}_G)$, set $\omega_j := \omega + \overline{\mu}_1 + \dots + \overline{\mu}_j$. Abbreviate

$$H^i := \mathrm{H}^i(\mathrm{Bun}_G^\omega), \quad H^* := \oplus_{i\in\mathbb{Z}} H^i, \quad \text{and} \quad H_c^i := \mathrm{H}_c^i(\mathrm{Bun}_G^\omega).$$

By definition and Proposition 5.5.3,

$$\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu, \eta) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob} \circ {}_c\Gamma_\mu^{\eta+\xi\eta'} \mid H_c^i). \tag{5.6.9}$$

By Lemma 5.5.2, we have

$$\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu, \eta) = q^{\dim \mathrm{Bun}_G} \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}^{-1} \circ \Gamma_\mu^{\eta+\xi\eta'} \mid H^i), \tag{5.6.10}$$

where we write

$$\Gamma_\mu^{\eta+\xi\eta'} = \Gamma_{\mu_1}^{\eta_1+\xi\eta_1'} \circ \dots \circ \Gamma_{\mu_r}^{\eta_r+\xi\eta_r'}.$$

By Lemma 5.6.13, the Ran filtration on H^* is preserved by $\Gamma_\mu^{\eta+\xi\eta'}$. By Proposition 5.6.17, $\mathrm{Gr}_\bullet^F \Gamma_\mu^{\eta+\xi\eta'}$ preserves the augmentation filtration on $\mathrm{Gr}_\bullet^F H^*$. Hence the trace (5.6.10) can be calculated on its associated graded $\mathrm{Gr}_{\mathrm{aug}}^\blacktriangleright \mathrm{Gr}_\bullet^F H^*$. Again by Proposition 5.6.17, the action of $\Gamma_\mu^{\eta+\xi\eta'}$ on this double associated graded is (in terms of the Atiyah–Bott description) the composition of the endomorphisms

$$\left[(d_{\mu_j}^{\omega_{j-1}}(\eta_j) + d_{\mu_j}(\eta_j'))\mathrm{id} + (\mathrm{id}_{\mathrm{H}_\bullet(X)} \otimes \overline{\nabla}_{\mu_j}^{\eta_j})_+^{\mathrm{der}}\right] \tag{5.6.11}$$

for $j = 1, \dots, r$. By Assumption 5.6.1, $\mathbb{V}$ has a basis consisting of homogeneous elements that are simultaneous generalized eigenvectors for the operators $\{\overline{\nabla}_{\mu_j}^{\eta_j}\}_{1\le j\le r}$. We can thus choose a decomposition into lines

$$\mathbb{V} \cong \bigoplus_{l=1}^n \mathbb{L}_l$$

such that $\mathbb{L}_l \subset \mathbb{V}_{2d_l}$ (so as Frob-module $\mathbb{L}_l \cong \overline{\mathbf{Q}}_\ell(-d_l)$), and $\mathbb{L}_l$ is spanned by a generalized eigenvector v_l with eigenvalues $\epsilon_l(\eta_j, \mu_j)$ for $\overline{\nabla}_{\mu_j}^{\eta_j}$, for all $j = 1, \cdots, r$.

In the Atiyah–Bott description (5.1.1) of H^*, we have

$$\mathrm{H}^*(\mathrm{Bun}_G^\omega) \cong \mathrm{Sym}^\blacktriangleright(\mathrm{H}_\bullet(X) \otimes \mathbb{V})_+ \cong \mathrm{Sym}^\blacktriangleright\Big(\mathrm{H}_\bullet(X) \otimes (\oplus_{l=1}^n \mathbb{L}_l)\Big)_+ \tag{5.6.12}$$

$$\cong \Big(\bigotimes_{l=1}^n \mathrm{Sym}^\blacktriangleright(\mathrm{H}_\bullet(X) \otimes \mathbb{L}_l)\Big)_+. \tag{5.6.13}$$

In particular, abbreviating

$$\mathrm{Sym}^{N_1,\dots,N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+ := \Big(\mathrm{Sym}^{N_1}(\mathrm{H}_\bullet(X)\otimes\mathbb{L}_1) \otimes \dots \otimes \mathrm{Sym}^{N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{L}_n)\Big)_+,$$

each $\blacktriangleright$-graded piece of (5.6.12) splits as

$$\mathrm{Sym}^N(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+ \cong \bigoplus_{N_1+\dots+N_n=N} \mathrm{Sym}^{N_1,\dots,N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+.$$

The endomorphism $\mathrm{id}_{\mathrm{H}_\bullet(X)} \otimes \overline{\nabla}_{\mu_j}^{\eta_j}$ preserves each summand $\mathrm{Sym}^{N_1,\dots,N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+$, and acts on it with generalized eigenvalues $\sum_{l=1}^n N_l \epsilon_l(\eta_j, \mu_j)$. Thus the contribution of $\mathrm{Sym}^{N_1,\dots,N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+$ to the trace (5.6.10) is

$$q^{\dim \mathrm{Bun}_G} \prod_{j=1}^r \Big(d_{\mu_j}^{\omega_{j-1}}(\eta_j) + d_{\mu_j}(\eta_j') + \sum_{l=1}^n N_l \epsilon_l(\eta_j, \mu_j)\Big) \mathrm{Tr}(\mathrm{Frob}^{-1}, \mathrm{Sym}^{N_1,\dots,N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+). \tag{5.6.14}$$

Form the multivariate generating series

$$\Lambda(t_1, \dots, t_n) = \sum_{N_1,\dots,N_n\ge 0} \mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{Sym}^{N_1,\dots,N_n}(\mathrm{H}_\bullet(X)\otimes\mathbb{V})_+) t_1^{N_1} \dots t_n^{N_n}.$$

Note that for $t = q^{-s}$, we have $(-\log q)^{-1}\partial_s = t\partial_t$. Then the sum of (5.6.14) over all $(N_1, \cdots, N_n)$ becomes

$$q^{\dim \mathrm{Bun}_G} \left(\prod_{j=1}^{r} \mathfrak{d}_j \right) \Lambda(q^{-s_1}, \cdots, q^{-s_n})|_{s_1=\cdots=s_n=0}.$$

Therefore, it suffices to show that

$$\Lambda(q^{-s_1}, \dots, q^{-s_n}) = \mathscr{L}^*_{X,G}(s_1, \dots, s_n).$$

Both sides factor over each variable, reducing this to a one variable statement:

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{Sym}^{\blacktriangleright}(\mathrm{H}_\bullet(X) \otimes \mathbb{L}_l)_+) = \begin{cases} \zeta_X(s + d_l) & d_l > 1, \\ \zeta^*_X(s+1) & d_l = 1. \end{cases}$$

We now consider two cases, based on whether $d_l > 1$.

(1) If $d_l > 1$, then $(\mathrm{H}_\bullet(X) \otimes \mathbb{L}_l)_+ = \mathrm{H}_\bullet(X) \otimes \mathbb{L}_l \cong \mathrm{H}_\bullet(X)(-d_l)$. So we have

$$\Lambda(t) = \frac{\det(1 - t\mathrm{Frob}^{-1} \mid \mathrm{H}_1(X) \otimes \mathbb{L}_l)}{(1 - q^{-d_l}t)(1 - q^{-d_l+1}t)} = \frac{\det(1 - q^{-d_l}t\mathrm{Frob}^{-1} \mid \mathrm{H}_1(X))}{(1 - q^{-d_l}t)(1 - q^{-d_l+1}t)}. \tag{5.6.15}$$

The action of Frob^{-1} on $\mathrm{H}_1(X)$ is adjoint to the action of Frob on $\mathrm{H}^1(X)$. Hence (5.6.15) can be rewritten as

$$\frac{\det(1 - q^{-d_l}t\mathrm{Frob} \mid \mathrm{H}^1(X))}{(1 - q^{-d_l}t)(1 - q^{-d_l+1}t)} \tag{5.6.16}$$

and upon setting $t = q^{-s}$ we obtain $\zeta_X(s + d_l)$, as desired.

(2) If $d_l = 1$, then we exclude the summand $\mathrm{H}_2(X, \mathbb{L}_l)$ from $(\mathrm{H}_\bullet(X) \otimes \mathbb{L}_l)_+$. This implies that the denominator in (5.6.16) should be only $(1 - q^{-d_l}t)$, and since $d_l = 1$ this effectively means we multiply the entire expression by $(1 - t)$. Upon setting $t = q^{-s}$, we therefore obtain $\Lambda(q^{-s}) = \zeta^*_X(s + 1)$, as desired.

This finally completes the proof of Theorem 5.6.9. □

5.7. **An example.** Let $G = \mathrm{GL}_n$ and $\mu^\sharp := (1, 0, \dots, 0)$, $\mu^\flat := (0, 0, \dots, -1) \in \mathbb{X}_*(T)$. Then $\dim G/P_{\mu^\sharp} = \dim G/P_{\mu^\flat} = n - 1$. We identify $R = \overline{\mathbf{Q}}_\ell[x_1, \dots, x_n]$ and $W = S_n$ acting by permutation of the x_i. Then $W_{\mu^\sharp} \cong S_{n-1}$ is the subgroup fixing x_1, and $W_{\mu^\flat} \cong S_{n-1}$ is the subgroup fixing x_n.

5.7.1. *Hecke correspondences.* The Hecke stack $\mathrm{Hk}_G^{\mu^\sharp}$ is the moduli stack of upper modifications of rank n vector bundles of colength 1, in the terminology of [FYZ24, Definition 6.5]. This means that for a commutative k-algebra R, $\mathrm{Hk}_G^{\mu^\sharp}(R)$ parametrizes $x \in X(R)$ along with an injective map

$$\mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \tag{5.7.1}$$

where $\mathcal{E}_0, \mathcal{E}_1$ are rank n vector bundles on X_R and $\mathcal{E}_1/\mathcal{E}_0$ is a line bundle over the graph of x in X_R.

The Hecke stack $\mathrm{Hk}_G^{\mu^\flat}$ is defined similarly except that (5.7.1) is replaced by

$$\mathcal{E}_0 \hookleftarrow \mathcal{E}_1 \tag{5.7.2}$$

with cokernel a line bundle over the graph of x in X_R.

The connected components of Bun_G are indexed by $d \in \mathbf{Z}$. Concretely, Bun_G^d is the component of Bun_G parametrizing bundles of degree d. For $? \in \{\sharp, \flat\}$, ${}^d\mathrm{Hk}_G^{\mu^?}$ is the open-closed component of $\mathrm{Hk}_G^{\mu^?}$ where $\deg \mathcal{F}_0 = d$. Let $h_0, h_1 : \mathrm{Hk}_G^{\mu^?} \to \mathrm{Bun}_G$ be the maps recording $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. Then we have a correspondence

$$\begin{array}{ccccc} & & {}^d\mathrm{Hk}_G^{\mu} & & \\ & \overset{h_0}{\swarrow} & & \overset{h_1}{\searrow} & \\ \mathrm{Bun}_G^d & & & & \mathrm{Bun}_G^{d\pm 1} \end{array}$$

where the sign is $+1$ if $\mu = \mu^\sharp$ and -1 if $\mu = \mu^\flat$.

5.7.2. *Tautological bundles.* Under the map $\mathrm{Hk}_G^{\mu^\sharp} \to G/P_{\mu^\sharp} \cong \mathbb{P}^{n-1}$ from (5.1.4), the pullback of $\mathcal{O}(-1)$ is the "tautological line bundle" $\mathcal{P}^\sharp$ on $\mathrm{Hk}_G^{\mu^\sharp}$ whose fiber along an R-point (5.7.1) is $\ker(\mathcal{F}_0|_x \to \mathcal{F}_1|_x)$. Similarly, under the map $\mathrm{Hk}_G^{\mu^\flat} \to G/P_{\mu^\flat} \cong \mathbb{P}^{n-1}$, the pullback of $\mathcal{O}(-1)$ is the tautological line bundle $\mathcal{P}^\flat$ whose fiber along an R-point (5.7.2) is $(\mathcal{F}_0/\mathcal{F}_1)$.

By our conventions (cf. §2.3), these are arranged so that under the map $R^{W_\mu} \to \mathrm{H}^*(\mathrm{Hk}_G^\mu)$ induced by the canonical parabolic reduction, $c_1(\mathcal{P}^\sharp)$ agrees with the image of $x_1 \in R^{W_\mu}$ of $\mu = \mu^\sharp$, while $c_1(\mathcal{P}^\flat)$ agrees with the image of $-x_n$ if $\mu = \mu^\flat$.

5.7.3. *Operators.* For $\mu \in \{\mu^\sharp, \mu^\flat\}$, we write

$$[\mu] = \begin{cases} \sharp & \mu = \mu^\sharp \\ \flat & \mu = \mu^\flat \end{cases} \qquad \text{and} \qquad |\mu| = \begin{cases} 1 & \mu = \mu^\sharp \\ -1 & \mu = \mu^\flat \end{cases}$$

For a rank n vector bundle $\mathcal{E}$ on X, we define the map

$$\begin{aligned} \Gamma_\mu^{\mathcal{E}} \quad : \quad & \mathrm{H}^*(\mathrm{Bun}_G^{d\pm 1}) \to \mathrm{H}^*(\mathrm{Bun}_G^d) \\ & \theta \mapsto h_{0*}(h_1^*\theta \cup c_n(p_X^*\mathcal{E}^{[\mu]*} \otimes \mathcal{P}^{[\mu]})). \end{aligned}$$

where the source degree is $d+1$ if $\mu = \mu^\sharp$ and $d-1$ if $\mu = \mu^\flat$.

5.7.4. *Arithmetic volume.* Consider a sequence of modifications of type $\mu = (\mu_1, \cdots, \mu_r)$, where each $\mu_i \in \{\mu^\sharp, \mu^\flat\}$ with exactly half of each type (so in particular r must be even).

Fix a sequence of rank n vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_r$ on X. Consider the composition

$$\Gamma_\mu^{\mathcal{E}} := \Gamma_{[\mu_r]}^{\mathcal{E}_r} \circ \cdots \circ \Gamma_{[\mu_2]}^{\mathcal{E}_2} \circ \Gamma_{[\mu_1]}^{\mathcal{E}_1} \colon \mathrm{H}^*(\mathrm{Bun}_G^d) \to \mathrm{H}^*(\mathrm{Bun}_G^d)$$

and its compactly supported version

$${}_c\Gamma_\mu^{\mathcal{E}} := {}_c\Gamma_{[\mu_r]}^{\mathcal{E}_r} \circ \cdots \circ {}_c\Gamma_{[\mu_2]}^{\mathcal{E}_2} \circ {}_c\Gamma_{[\mu_1]}^{\mathcal{E}_1} \colon \mathrm{H}_c^*(\mathrm{Bun}_G^d) \to \mathrm{H}_c^*(\mathrm{Bun}_G^d).$$

We write

$$\mathrm{vol}({}^d\mathrm{Sht}_G^\mu, \prod_{j=1}^r c_n(p_j^*\mathcal{E}_j^* \otimes \mathcal{P}^{\mu_j})) = \mathrm{Tr}({}_c\Gamma_\mu^{\mathcal{E}} \circ \mathrm{Frob} \mid \mathrm{H}_c^*(\mathrm{Bun}_G^d))$$

for the corresponding arithmetic volume. Its value will be formulated in terms of the L-functions

$$L_{X,G}(s) := \mathscr{L}_{X,G}(s, s, \ldots, s) = \prod_{i=1}^n \zeta_X(s+i), \tag{5.7.3}$$

and

$$L_{X,G}^*(s) = \mathscr{L}_{X,G}^*(s, s, \ldots, s) = (1 - q^{-s}) L_{X,G}(s). \tag{5.7.4}$$

For $1 \le j \le r$, let $D_j := \deg \mathcal{E}_j$. We define coefficients $b_0^\mu, \ldots, b_r^\mu$ by writing the following product as a polynomial in N:

$$\prod_{j=1}^r (d - |\mu_r| - \cdots - |\mu_j| - |\mu_j| D_j + |\mu_j| N) = b_r^\mu N^r + b_{r-1}^\mu N^{r-1} + \cdots + b_0^\mu. \tag{5.7.5}$$

Each b_i^μ is itself a polynomial in d and $D_1, \ldots, D_r$.

Theorem 5.7.5. *For any $d \in \mathbf{Z}$, we have*

$$\mathrm{vol}({}^d\mathrm{Sht}_G^\mu, \prod_{j=1}^r c_n(p_j^*\mathcal{E}_j^* \otimes \mathcal{P}^{[\mu_j]})) = q^{n^2(g-1)} \sum_{i=0}^r b_i^\mu (-\log q)^{-i} \left(\frac{d}{ds}\right)^i \Big|_{s=0} L_{X,G}^*(s).$$

Proof. As discussed in Example 5.6.2, Assumption 5.6.1 is automatically satisfied because of the choice of group $G = \mathrm{GL}_n$. Hence we may apply Theorem 5.6.9 in order to calculate the left side. The first step is to rewrite the $\Gamma_\mu^{\mathcal{E}}$ in terms of the $\Gamma_\mu^{\eta+\xi\eta'}$ from §5.1.2.

We have

$$c_n(p_X^*\mathcal{E}_j^* \otimes \mathcal{P}^{[\mu_j]}) = c_1(\mathcal{P}^{[\mu_j]})^n + p_X^*(c_1(\mathcal{E}_j^*)) c_1(\mathcal{P}^{[\mu_j]})^{n-1} \in \mathrm{H}^*(\mathrm{Hk}_G^{\mu_j}). \tag{5.7.6}$$

Recall that canonical parabolic reduction induces a map $\mathrm{H}^*(\mathbb{B}P_{\mu_j}) = R^{W_{\mu_j}} \to \mathrm{H}^*(\mathrm{Hk}_G^{\mu_j})$. As explained in §5.7.2, the definition of $\mathcal{P}^{\sharp/\flat}$ is arranged so that $c_1(\mathcal{P}^{[\mu_j]})$ agrees with the image of $x_1 \in R^{W_{\mu_j}}$ if $[\mu_j] = \sharp$ and with the image of $-x_n$ if $[\mu_j] = \flat$.

Let us consider first the case $[\mu_j] = \sharp$. We may then rewrite (5.7.6) as

$$c_n(p_X^* \mathcal{E}_j^* \otimes \mathcal{P}^{[\mu_j]}) = x_1^n - D_j \xi x_1^{n-1}.$$

We thus have $\eta_j = x_1^n$ and $\eta_j' = -D_j x_1^{n-1}$. To apply Theorem 5.6.9, we need to calculate the constants $d_{\mu_j}^{\omega}(\eta)$, $d_{\mu_j}(\eta_j')$, and the eigenweights $\epsilon_i(\eta_j, \mu_j)$.

- Using (5.4.6) for $i = 1$, we find that $d_\mu^\omega(\eta) = (-1)^{n-1} e_1^{[X]} = (-1)^{n-1}\omega$, since Bun_G^ω parametrizes bundles of degree ω.
- Since x_1 corresponds to $\mathcal{O}(-1)$ on $G/P_\lambda \cong \mathbb{P}^{n-1}$ (cf. §2.3), we see that
$$d_\mu(\eta') = -D_j \int_{G/P_\mu} x_1^{n-1} = (-1)^n D_j.$$
- We have $\epsilon_i(\eta, \mu^\sharp) = \epsilon_i(x_1^n, \mu^\sharp) = (-1)^{n-1}$ by Proposition 5.4.1.

The operator $\Gamma_{\mu_j}^{\mathcal{E}_j}$ is applied to the component $\mathrm{Bun}_G^{\omega_j}$ for $\omega_j := d - |\mu_r| - \ldots - |\mu_j|$. Hence for $[\mu_j] = \sharp$, the differential operator (5.6.3) corresponding to $\Gamma_{\mu_j}^{\mathcal{E}}$ is

$$\mathfrak{d}_j = (-1)^{n-1}\Big(\omega_j - D_j - (\log q)^{-1} \sum_{i=1}^n \partial_{s_i}\Big) = (-1)^{n-1}\Big(\omega_j - |\mu_j| D_j + |\mu_j|(-\log q)^{-1} \sum_{i=1}^n \partial_{s_i}\Big).$$

Next we consider the case $[\mu_j] = \flat$. In this case, (5.7.6) becomes

$$c_n(p_X^* \mathcal{E}_j^* \otimes \mathcal{P}^{[\mu_j]}) = (-x_n)^n - D_j \xi (-x_n)^{n-1}.$$

We thus have $\eta_j = (-x_n)^n$ and $\eta_j' = -D_j(-x_n)^{n-1}$. We calculate the relevant constants.

- Repeating the proof of (5.4.6) with x_1 replaced by x_n, we find that $d_\mu^\omega(x_n^n) = e_1^{[X]} = \omega$. Hence $d_\mu^\omega((-x_n)^n) = (-1)^n \omega$.
- Since $-x_n$ corresponds to $\mathcal{O}(-1)$ on $G/P_\lambda \cong \mathbb{P}^{n-1}$ (cf. §2.3), we see that
$$d_\mu(\eta') = -D_j \int_{G/P_\mu} (-x_n)^{n-1} = (-1)^n D_j.$$
- Repeating the proof of Proposition 5.4.1 with x_1 replaced by $-x_n$, we have $\epsilon_i(\eta, \mu^\flat) = \epsilon_i((-x_n)^n, \mu^\flat) = (-1)^{n-1}$. (Another way to see this is from perspective of §6. Then $\epsilon_i(\Omega, \mu)$ is invariant under the $W = S_n$ action on μ. Also, we claim that replacing μ by $-\mu$ leaves the eigenweights unchanged. Indeed, we have $\mathfrak{R}_{-\mu} = (-1)^{D_\mu} \mathfrak{R}_\mu$, and the integrand changes from $t_\mu^{D_\mu+1} \partial_\mu$ to $(-t_\mu)^{D_\mu+1} \partial_{-\mu} = (-1)^{D_\mu} t_\mu^{D_\mu+1} \partial_\mu$).

Hence we have in this case that

$$\mathfrak{d}_j = (-1)^n \Big(\omega_j + D_j - (-\log q)^{-1} \sum_{i=1}^n \partial_{s_i}\Big) = (-1)^n \Big(\omega_j - |\mu_j| D_j + |\mu_j|(-\log q)^{-1} \sum_{i=1}^n \partial_{s_i}\Big).$$

Observing that $\dim \mathrm{Bun}_G = n^2(g-1)$, Theorem 5.6.9 says that

$$\begin{aligned}
&\mathrm{vol}({}^d\mathrm{Sht}_G^\mu, \prod_{j=1}^r c_n(p_j^* \mathcal{E}_j^* \otimes \mathcal{P}^{\mu_j})) \\
&= q^{n^2(g-1)} (-1)^{r/2} \prod_{j=1}^r \Big(\omega_j - |\mu_j| D_j + |\mu_j| (-\log q)^{-1} \sum_{i=1}^n \partial_{s_i}\Big) \mathscr{L}_{X,G}^*(s_1, \cdots, s_n)\Big|_{s_1 = s_2 = \cdots = s_n = 0}.
\end{aligned}$$

Examining (5.7.5), this expands as

$$q^{n^2(g-1)} \sum_{i=0}^r b_i^\mu (-\log q)^{-i} \Big(\sum_{j=1}^n \partial_{s_j}\Big)^i \mathscr{L}_{X,G}^*(s_1, \cdots, s_n)\Big|_{s_1 = s_2 = \cdots = s_n = 0}.$$

The result then follows from the elementary observation that for a smooth function of the form $f(s_1,\dots,s_n) = \prod_{j=1}^n f_j(s_j)$, we have

$$\Big(\sum_{j=1}^n \partial_{s_j}\Big)^i f(s_1,\dots,s_n)|_{s_1=\dots=s_n=s} = \partial_s^i f(s,\dots,s). \tag{5.7.7}$$

This completes the proof. □

6. Some calculations of eigenweights

In order to apply Theorem 5.6.9 in examples, we need to explicate the differential operators $\mathfrak{d}_j$ from (5.6.3). In practice, the constants appearing there are straightforward to calculate *except* for the eigenweights $\epsilon_i(\eta,\mu)$.

In this section, we will work out the eigenweights in more examples. The natural η of interest arise in the following way (up to sign). Let $\Omega \in R_4^W$ be a Casimir element corresponding to a fixed nondegenerate W-invariant quadratic form on $\mathbb{X}_*(T)_{\mathbf{Q}}$. The associated bilinear form gives an isomorphism

$$\iota_\Omega : \mathbb{X}_*(T)_{\mathbf{Q}} \xrightarrow{\sim} \mathbb{X}^*(T)_{\mathbf{Q}}.$$

Let $t_\mu := \partial_\mu \Omega \in R_2^{W_\mu}$, which is easily seen to be equal to $\iota_\Omega(\mu) \in \mathbb{X}^*(T)_{\mathbf{Q}}$. We consider $\eta := t_\mu^{D_\mu+1} \in R^{W_\mu}_{2(D_\mu+1)}$. This defines $\overline{\nabla}^\eta_\mu \in \mathrm{End}(\overline{\mathbb{V}})$ as in §5.3.2. Note that the example of §5.7 was of this form, up to sign normalizations.

Now let us explain for what families we are able to calculate the eigenweights.

- Let $G = \mathrm{GL}_n$. The minuscule coweights are $(1^m, 0^{n-m})$ for $1 \le m < n$. For $m = 1$, we calculated the eigenweights in Proposition 5.4.1. We treat $m = 2$ below in Proposition 6.2.1, where both the answer and proof are substantially more complicated than the case $m = 1$.
- For $G = \mathrm{SO}_{2m+1}$, there is a unique minuscule coweight, represented by $\mu = (1, 0^{m-1})$, and we compute the eigenweights.
- For $G = \mathrm{PSO}_{2n}$, there are three minuscule coweights. The *standard* coweight (corresponding to the standard representation of SO_{2n}) is represented by $\mu = (1, 0^{n-1})$, and we will calculate the corresponding eigenweights. There are also two *spin* coweights, corresponding to the two spin representations of Spin_{2n}.
- For exceptional groups, since there are only finitely many cases, the eigenweights may be found by a finite algorithm, although we have not carried it out.

The determination of the remaining minuscule coweights in Type A, as well as the spin coweights in Types C and D, proved more challenging. Eventually, a solution was found by an AI agent coded by the first author; it will be explained in the separate paper [Fen26].

6.1. **Special orthogonal groups.** We assume $p \neq 2$. Let G be the split special orthogonal group $\mathrm{SO}(V)$ where $\dim_k V = n$ and B is the non-degenerate symmetric bilinear pairing on V.

6.1.1. *Cohomology of* $\mathbb{B}G$. Suppose that $n = 2m+1$ is odd. Using the isotropic basis $v_1, v_2, \cdots, v_m, v_0, v_{-m}, \cdots, v_{-1}$ such that $B(v_i, v_j) = \delta_{i,-j}$, we get an identification $\mathbb{X}^*(T) \cong \mathbb{Z}^m$ on which $W = (\mathbb{Z}/2)^m \rtimes S_m$ acts by permuting and changing signs of coordinates. We have an isomorphism

$$\mathrm{H}^*(\mathbb{B}G) \cong \overline{\mathbf{Q}}_\ell[x_1,\cdots,x_m]^{(\mathbb{Z}/2)^m \rtimes S_m} = \overline{\mathbf{Q}}_\ell[e_1^{(2)}, e_2^{(2)}, \cdots, e_m^{(2)}]$$

Here x_i has cohomological degree 2, and $e_i^{(2)} \in \mathrm{H}^{4i}(\mathbb{B}G)$ is the i-th elementary symmetric polynomial in $x_1^2, \cdots, x_m^2$.

Next suppose that $n = 2m$ is even. Using an isotropic basis $v_1, \cdots, v_m, v_{-m}, \cdots, v_{-1}$ of V, we get an isomorphism $\mathbb{X}^*(T) \cong \mathbb{Z}^m$. The Weyl group $W \subset (\mathbb{Z}/2)^m \rtimes S_m$ is the kernel of the homomorphism $\chi : (\mathbb{Z}/2)^m \rtimes S_m \to \mathbb{Z}/2$ that is trivial on S_m and nontrivial on each copy of $\mathbb{Z}/2$; it acts by permutation and change of sign on the coordinates. We have an isomorphism

$$\mathrm{H}^*(\mathbb{B}G) \cong \overline{\mathbf{Q}}_\ell[x_1,\cdots,x_m]^W = \overline{\mathbf{Q}}_\ell[e_1^{(2)}, e_2^{(2)}, \cdots, e_{m-1}^{(2)}, \mathrm{Pf}] \tag{6.1.1}$$

where $\mathrm{Pf} = x_1 x_2 \cdots x_m$ is the Pfaffian. Note that $\mathrm{Pf}^2 = x_1^2 \cdots x_m^2$ is the m-th elementary symmetric polynomial $e_m^{(2)}$ in $x_1^2, \cdots, x_m^2$.

Remark 6.1.2. The class Pf for $\mathbb{B}G$ depends on the choice of the particular Lagrangian $\mathrm{Span}\{v_1, \cdots, v_m\}$ in V. The space of Lagrangians has two connected components; a choice of a Lagrangian lying in the other component changes Pf to $-\mathrm{Pf}$.

We take the Casimir element $\Omega := \frac{1}{2}\sum_{i=1}^m x_i^2 \in R^W$.

6.1.3. *Odd orthogonal case.* Let $\mu = (1, 0, \ldots, 0) \in \mathbf{Z}^m \cong \mathbb{X}_*(T)$. Then we have $\partial_\mu = \partial_{x_1}$ and $t_\mu = \partial_\mu(\Omega) = x_1$. As $D_\mu = \dim G/P_\mu = 2m-1$, we have $\eta = t_\mu^{2m} = x_1^{2m}$.

Lemma 6.1.4. *Suppose $n = 2m+1$ is odd. For $\eta = x_1^{2m}$ and $\mu = (1, 0, \ldots, 0)$, we have*

$$\nabla_\mu^\eta(e_i^{(2)}) = -4e_i^{(2)} \quad \text{for all } i = 1, 2, \ldots, m.$$

In particular, the eigenweights are all equal to -4.

Proof. Let $\widehat{e}_i^{(2)}$ be the ith elementary symmetric polynomial in $x_2^2, \ldots, x_m^2$. Then $\partial_{x_1} e_i^{(2)} = 2x_1 \widehat{e}_{i-1}^{(2)}$. We want to calculate

$$\nabla_\mu^\eta(e_i^{(2)}) = \int_{G/P_\mu} 2x_1^{2m+1}\widehat{e}_{i-1}^{(2)}.$$

We will use Lemma 5.4.4. In this case, the equivariant Chern class of (the tangent bundle of) G/P_μ is

$$\mathfrak{R}_\mu = -x_1 \prod_{j=2}^m (x_j - x_1)(-x_j - x_1) = (-1)^m x_1 \prod_{j=2}^m (x_j^2 - x_1^2).$$

Hence Lemma 5.4.4 says that

$$\int_{G/P_\mu} 2x_1^{2m+1}\widehat{e}_{i-1}^{(2)} = 2(-1)^m \sum_{w \in W/W_\mu} w\left(\frac{x_1^{2m+1}\widehat{e}_{i-1}^{(2)}}{x_1 \prod_{j=2}^m (x_1^2 - x_j^2)}\right) \tag{6.1.2}$$

$$= 2(-1)^m \sum_{w \in W/W_\mu} w\left(\frac{x_1^{2m}\widehat{e}_{i-1}^{(2)}}{\prod_{j=2}^m (x_1^2 - x_j^2)}\right). \tag{6.1.3}$$

The inner sum is the same expression calculated in §5.4.5, except replacing x_i by x_i^2 and also allowing substitutions $x_i \mapsto \pm x_i$, so that the answer is doubled compared to §5.4.5. Hence we conclude that

$$\nabla_\mu^\eta(e_i^{(2)}) = 4(-1)^m(-1)^{m-1}e_i^{(2)} = -4e_i^{(2)},$$

as desired. □

6.1.5. *Even orthogonal case.* Let $\mu = (1, 0, \cdots, 0) \in \mathbf{Z}^m \cong \mathbb{X}_*(T)$. Again we have $\partial_\mu = \partial_{x_1}$ and $t_\mu = x_1$. As $N = \dim G/P_\mu = 2m-2$, we have $\eta = t_\mu^{2m-1} = x_1^{2m-1}$.

Lemma 6.1.6. *Suppose $n = 2m$ is even. For $\eta = x_1^{2m-1}$ and $\mu = (1, 0, \ldots, 0)$, we have*

$$\nabla_\mu^\eta(e_i^{(2)}) = 4e_i^{(2)} \quad \text{for all } i = 1, 2, \ldots, m.$$

In particular, using homogeneous generators $e_1^{(2)}, e_2^{(2)}, \ldots, e_{m-1}^{(2)}, \mathrm{Pf}$ for R^W, the eigenweights are

$$4, 4, \ldots, 4, 2.$$

Proof. First we analyze the Pfaffian. A basis for R^{W_μ} over R^W is given by $1, x_1, \ldots, x_1^{2m-2}$ and $\widehat{\mathrm{Pf}} = x_2 \ldots x_m$. Since G/P_μ is embedded as a *quadric* hypersurface under the line bundle corresponding to $(-x_1)$, the map $\int_{G/P_\mu} : R^{W_\mu} \to R^W$ extracts *twice* the coefficient of $(-x_1)^{2m-2}$ in this basis. Hence

$$\nabla_\mu^\eta(\mathrm{Pf}) = \int_{G/P_\mu} x_1^{2m-1}\partial_{x_1}(x_1 x_2 \cdots x_m) = \int_{G/P_\mu} x_1^{2m-2}\,\mathrm{Pf} = 2\,\mathrm{Pf}.$$

Let $\widehat{e}_i^{(2)}$ be the ith elementary symmetric polynomial in $x_2^2, \ldots, x_m^2$. Then $\partial_{x_1} e_i^{(2)} = 2x_1\widehat{e}_{i-1}^{(2)}$. We want to calculate

$$\nabla_\mu^\eta(e_i^{(2)}) = \int_{G/P_\mu} 2x_1^{2m}\widehat{e}_{i-1}^{(2)}.$$

We will use Lemma 5.4.4. In this case, the equivariant Chern class of (the tangent bundle of) G/P_μ is

$$\mathfrak{R}_\mu = \prod_{j=2}^{m}(x_j - x_1)(-x_j - x_1) = \prod_{j=2}^{m}(x_1^2 - x_j^2).$$

Hence Lemma 5.4.4 says that

$$\int_{G/P_\mu} 2x_1^{2m}\widehat{e}_{i-1}^{(2)} = 2\sum_{w\in W/W_\mu} w\left(\frac{x_1^{2m}\widehat{e}_{i-1}^{(2)}}{\prod_{j=2}^{m}(x_1^2 - x_j^2)}\right).$$

The inner sum is the same expression calculated in §5.4.5, except replacing x_i by x_i^2 and also allowing substitutions $x_i \mapsto \pm x_i$, so that the answer is doubled compared to §5.4.5. Hence we conclude that

$$\nabla_\mu^\eta(e_i^{(2)}) = 4e_i^{(2)},$$

as desired.[10] □

6.2. **General linear groups: a case of non-minimal modification.** So far we have only computed eigenweights for "minimal" modification types μ, where the discrepancy between the bundles being modified is as small as possible. Beyond this case, the eigenweights quickly get very complicated.

We will illustrate this for $G = \mathrm{GL}_n$, where the minuscule coweights are of the form $\mu = (1^m, 0^{n-m})$ for $1 \le m < n$. We will calculate the corresponding eigenweights for $m = 2$, so we assume $n \ge 3$. We take the Casimir element $\Omega = \frac{1}{2}\sum_{i=1}^n x_i^2$, so $t_\mu = \partial_\mu(\Omega) = x_1 + x_2$. Then $\eta = t_\mu^{2(n-2)+1}$.

Proposition 6.2.1. *For $G = \mathrm{GL}_n$ ($n \ge 3$), $\mu = (1, 1, 0, \cdots, 0)$ and η as above, we have*

$$\epsilon_i(\eta,\mu) = \frac{1}{n}\binom{2n-2}{n-1} - \binom{2n-3}{n-i} + 2\binom{2n-3}{n-i-1} - \binom{2n-3}{n-i-2} \tag{6.2.1}$$

for $1 \le i \le n$.

After we first obtained the calculation (6.2.1), we contributed it to an internal benchmark at Google DeepMind, where it was eventually solved by Gemini Deep Think (*IMO Gold Version*). After comparing the two solutions, we actually preferred Gemini's argument and will present it (with our own edits) instead of our original one.

6.2.2. *Change of basis.* Recall that the *power sums* in $\{x_1, \dots, x_n\}$ are $p_k = \sum_{j=1}^n x_j^k$. Abbreviate I for the augmentation ideal of R^W.

Lemma 6.2.3. *Modulo I^2, the following identities hold for $k \ge 1$:*
1. $p_k \equiv (-1)^{k-1} k e_k \pmod{I^2}$
2. $p_k \equiv k h_k \pmod{I^2}$

Proof. (1) We start with Newton's identity:

$$ke_k = \sum_{i=1}^{k}(-1)^{i-1}e_{k-i}p_i = e_{k-1}p_1 - e_{k-2}p_2 + \cdots + (-1)^{k-1}e_0p_k.$$

Consider the terms modulo I^2. For any term $e_{k-i}p_i$ with $i < k$, both e_{k-i} and p_i are in the ideal I. Therefore, their product lies in I^2. This means all terms except the last one vanish modulo I^2. Hence we have

$$ke_k \equiv (-1)^{k-1}e_0p_k \pmod{I^2} = (-1)^{k-1}p_k \pmod{I^2}.$$

(2) The argument is analogous, using another of Newton's identities:

$$kh_k = \sum_{i=1}^{k} h_{k-i}p_i = h_{k-1}p_1 + h_{k-2}p_2 + \cdots + h_0p_k.$$

[10] As a sanity check, note that $\mathrm{Pf}^2 = e_m^{(2)}$. Since ∇_μ is a derivation, we have

$$\nabla_\mu^\eta(\mathrm{Pf}^2) = 2\,\mathrm{Pf}\,\nabla_\mu^\eta(\mathrm{Pf}) = 4\,\mathrm{Pf}^2.$$

which is consistent with the fact that $\nabla_\mu(e_m^{(2)}) = 4e_m^{(2)}$.

□

Thus, in the quotient space $\mathbb{V} = I/I^2$, the power sum and elementary polynomial bases are related by a simple scalar multiple. The advantage of working with the p_i is that their partial derivatives enjoy a simple recursive formula:

$$\partial_\mu p_i = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\sum_{j=1}^n x_j^i = i(x_1^{i-1} + x_2^{i-1}).$$

6.2.4. *Expression in terms of Divided Differences.* We prepare to apply Lemma 5.4.4. For $\mu = (1, 1, 0, \dots, 0)$, we have

$$\mathfrak{R}_\mu = \prod_{i>2}(x_i - x_1)(x_i - x_2).$$

Therefore, Lemma 5.4.4 says that for $f \in R^{W_\mu}$, we have

$$\int_{G/P_\mu} f = \sum_{1\le a<b\le n} \frac{w_{a,b}(f)}{\prod_{i\neq a,b}(x_a - x_i)(x_b - x_i)}, \tag{6.2.2}$$

where $w_{a,b}$ is the permutation exchanging $(x_1, x_2) \leftrightarrow (x_a, x_b)$ and fixing the other x_i. Let $A(z) := \prod_{k=1}^n (z - x_k)$, so that $A'(x_a) = \prod_{k\neq a}(x_a - x_k)$. Then we can rewrite

$$\frac{1}{\prod_{i\neq a,b}(x_a - x_i)(x_b - x_i)} = \frac{(x_a - x_b)(x_b - x_a)}{A'(x_a)A'(x_b)} = -\frac{(x_a - x_b)^2}{A'(x_a)A'(x_b)}.$$

Substituting this into (6.2.2), and then applying it to $f_i = \eta \cdot \partial_\mu p_i = i(x_1 + x_2)^{2n-3}(x_1^{i-1} + x_2^{i-1})$, we get:

$$\nabla^\eta_\mu(p_i) = -i\sum_{a<b} \frac{(x_a + x_b)^{2n-3}(x_a^{i-1} + x_b^{i-1})(x_a - x_b)^2}{A'(x_a)A'(x_b)}.$$

By symmetry, we can rewrite this as a sum over ordered pairs $a \neq b$:

$$\nabla^\eta_\mu(p_i) = -i\sum_{a=1}^n \frac{x_a^{i-1}}{A'(x_a)}\left(\sum_{b\neq a} \frac{(x_a + x_b)^{2n-3}(x_a - x_b)^2}{A'(x_b)}\right). \tag{6.2.3}$$

We will now invoke the theory of *divided differences*. For a polynomial $f(z)$, the divided difference $f[x_1, \dots, x_n]$ is determined recursively by $f[x_i] = f(x_i)$ and

$$f[x_r, \dots, x_s] = \frac{f[x_{r+1}, \dots, x_s] - f[x_r, \dots, x_{s-1}]}{x_s - x_r}.$$

For example, $f[x_1] = f(x_1)$ and $f[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. We may view the divided difference as a rational function in $x_1, \dots, x_n$, or as a number if specific values $x_1, \dots, x_n \in \overline{\mathbf{Q}}$ are chosen. Generally we adopt the former perspective, but the second will occasionally be useful for proofs; we will make clear which perspective is being taken.

Lemma 6.2.5. *Let $f \in \mathbf{Z}[x]$. Then we have an identity of rational functions in $\overline{\mathbf{Q}}(x_1, \dots, x_n)$,*

$$f[x_1, \dots, x_n] = \sum_{j=1}^n \frac{f(x_j)}{A'(x_j)}$$

Proof. This being an identity of rational functions over an infinite integral domain $\mathbf{Z}$, with denominators being products of factors $(x_i - x_j)$, it suffices to prove the statement for all specializations of $(x_1, \dots, x_n)$ to any n-tuple of pairwise distinct elements in $\mathbf{Z}$. So we fix any such specialization and treat the divided difference in $\mathbf{Q}$ for the proof.

Then another interpretation of the divided difference $f[x_1, \dots, x_n] \in \mathbf{Q}$ is as the leading coefficient of the unique degree $n-1$ polynomial $F(z)$ that passes through $(x_1, f(x_1)), \dots, (x_n, f(x_n))$. In fact, the divided differences can also be characterized in terms of the "Newton expansion" of $F(z)$ as

$$F(z) = \sum_{m=1}^{n} f[x_1, \dots, x_m] \prod_{i=1}^{m-1} (z - x_i),$$

which implies the preceding characterization. By the Lagrange interpolation formula, we also have

$$F(z) = \sum_{j=1}^{n} f(x_j) \frac{\prod_{i \neq j}(z - x_i)}{\prod_{i \neq j}(x_j - x_i)} = \sum_{j=1}^{n} f(x_j) \frac{A(z)}{(z - x_j)A'(x_j)}.$$

From this expression, it is clear that the leading coefficient of $F(z)$ is

$$\sum_{j=1}^{n} \frac{f(x_j)}{A'(x_j)},$$

so this equals $f[x_1, \dots, x_n]$, as desired. □

Lemma 6.2.6. *Let $P_j(z) = z^j$. Then we have*

$$P_j[x_1, \dots, x_n] = h_{j-n+1}(x_1, \dots, x_n) \in \mathbf{Z}[x_1, \dots, x_n][\prod_{r<s}(x_r - x_s)^{-1}],$$

where $h_{j-n+1}(x_1, \dots, x_n)$ is the complete homogeneous polynomial in $x_1, \dots, x_n$ of degree $j-n+1$ (which is 0 by definition if $j - n + 1 < 0$).

Proof. Consider the generating function

$$\sum_{j=0}^{\infty} h_j(x_1, \dots, x_n) t^j = \prod_{k=1}^{n} \frac{1}{(1 - x_k t)}. \tag{6.2.4}$$

Lemma 6.2.5 implies that

$$P_j[x_1, \dots, x_n] = \sum_{i=1}^{n} \frac{x_i^j}{\prod_{k \neq i}(x_i - x_k)}.$$

Consider another generating function.

$$\sum_{j \geq 0} P_j[x_1, \dots, x_n] t^j = \sum_{i=1}^{n} \frac{1}{\prod_{k \neq i}(x_i - x_k)} \frac{1}{1 - x_i t}.$$

We claim that the RHS is the partial fraction decomposition of $t^{n-1} \prod_{i=1}^{n} \frac{1}{1 - x_i t}$. Indeed, upon multiplying by $\prod_{i=1}^{n}(1 - x_i t)$ to clear denominators, we are comparing two polynomials of degree $\leq n-1$ in t, which when evaluated at each $t = 1/x_i$ gives $1/x_i^{n-1}$. The claim is proved, so we deduce that

$$\sum_{j \geq 0} P_j[x_1, \dots, x_n] t^j = t^{n-1} \prod_{i=1}^{n} \frac{1}{1 - x_i t} \tag{6.2.5}$$

and then the Lemma follows by comparing coefficients of the two generating functions (6.2.4) and (6.2.5). □

For $1 \leq a \leq n$, let $Q_a(z) := (x_a + z)^{2n-3}(x_a - z)^2$. By Lemma 6.2.5, we have

$$Q_a[x_1, \dots, x_n] = \sum_{b=1}^{n} \frac{Q_a(x_b)}{A'(x_b)}.$$

Since $Q_a(x_a) = 0$, we can omit the summand with $b = a$, and we are then left with the identity

$$Q_a[x_1, \dots, x_n] = \sum_{b \neq a} \frac{(x_a + x_b)^{2n-3}(x_a - x_b)^2}{A'(x_b)}. \tag{6.2.6}$$

Substituting this back into (6.2.3), we find that

$$\nabla^\eta_\mu(p_i) = -i\sum_{a=1}^n \frac{x_a^{i-1}}{A'(x_a)} Q_a[x_1,\dots,x_n]. \tag{6.2.7}$$

Define integers K_k by the identity $Q(t) = (1+t)^{2n-3}(1-t)^2 = \sum_k K_k t^k$. Comparing the definitions of $Q(t)$ and $Q_a(z)$, we see that

$$Q_a(z) = \sum_k K_k x_a^k z^{2n-1-k}. \tag{6.2.8}$$

By linearity of the formation of divided difference, we have

$$Q_a[x_1,\dots,x_n] = \sum_k K_k x_a^k (z^{2n-1-k}[x_1,\dots,x_n]) = \sum_k K_k x_a^k h_{n-k}$$

where we invoked Lemma 6.2.6 in the last equality. Substituting this back into (6.2.7) gives

$$\begin{aligned}\nabla^\eta_\mu(p_i) &= -i\sum_{a=1}^n \frac{x_a^{i-1}}{A'(x_a)}\left(\sum_k K_k x_a^k h_{n-k}\right) = -i\sum_k K_k h_{n-k}\left(\sum_a \frac{x_a^{i-1+k}}{A'(x_a)}\right)\\ &= -i\sum_k K_k h_{n-k} h_{i+k-n},\end{aligned} \tag{6.2.9}$$

where in the last step we applied Lemma 6.2.6 to the inner sum.

6.2.7. *Reduction modulo I^2.* We now consider (6.2.9) modulo I^2. Since $h_j \in I$ for $j \geq 1$, a product of the form $h_a h_b$ lies in I^2 unless $a = 0$ or $b = 0$. With $h_0 = 1$, the sum simplifies to:

$$\sum_k K_k h_{n-k} h_{i+k-n} \equiv K_n h_0 h_{i+n-n} + K_{n-i} h_{n-(n-i)} h_0 = (K_n + K_{n-i}) h_i \pmod{I^2}.$$

Putting this into (6.2.9), we have found that

$$\nabla^\eta_\mu(p_i) \equiv -i(K_n + K_{n-i}) h_i \pmod{I^2}.$$

Using the relation $h_i \equiv p_i/i \pmod{I^2}$ from Lemma 6.2.3, we can simplify this to

$$\nabla^\eta_\mu(p_i) \equiv -i(K_n + K_{n-i})\frac{p_i}{i} = -(K_n + K_{n-i}) p_i \pmod{I^2}.$$

This shows that $\epsilon_i(\eta,\mu) = -(K_n + K_{n-i})$.

6.2.8. *Explication of coefficients.* Now we match this up with the statement of Proposition 6.2.1. The coefficients K_i come from $Q(t) = (1+t)^{2n-3}(1-t)^2$. By the binomial formula, we have

$$K_i = \binom{2n-3}{i} - 2\binom{2n-3}{i-1} + \binom{2n-3}{i-2}.$$

For $i = n$, this specializes to

$$K_n = \binom{2n-3}{n} - 2\binom{2n-3}{n-1} + \binom{2n-3}{n-2} = -\frac{1}{n}\binom{2n-2}{n-1}.$$

Hence we have

$$\epsilon_i = -(K_n + K_{n-i}) = \frac{1}{n}\binom{2n-2}{n-1} - \left(\binom{2n-3}{n-i} - 2\binom{2n-3}{n-i-1} + \binom{2n-3}{n-i-2}\right).$$

This completes the proof of Proposition 6.2.1. □

7. Arithmetic volume of shtukas for non-split groups

In this section, we generalize the results of §5 on arithmetic volume to non-split reductive group schemes over X.

7.1. **Group schemes.** We define the class of reductive group schemes that we will consider. Let $\nu : X' \to X$ be a finite étale Galois cover with Galois group Γ. Let $\xi' \in \mathrm{H}^2(X')(1)$ be Poincaré dual to a point in each connected component of X'.[11]

Let G_0 be a split connected reductive group over k together with a split maximal torus T_0 and a Borel subgroup B_0 containing T_0. Suppose we are given an injective homomorphism

$$\tau : \Gamma \hookrightarrow \mathrm{Aut}(G_0, B_0, T_0).$$

Let G be the connected reductive group scheme over X defined as

$$G = (\mathrm{Res}_{X'/X}(G_0 \times X'))^{\Gamma}$$

where Γ acts diagonally on G_0 (via τ) and on X'. In other words, G is the descent of the constant group $G_0 \times X'$ to X using τ as the descent datum.

Remark 7.1.1. Our setup covers all quasisplit reductive groups over X. For the purpose of calculating arithmetic volumes, this essentially already captures the full generality of everywhere reductive group schemes over X; let us explain why. If G' is a pure inner twist of G, then as already observed in §4.3.10, Bun_G is canonically isomorphic to $\mathrm{Bun}_{G'}$. Therefore, there is no additional generality gained by considering pure inner twists. As explained in §4.1.1, every reductive $G \to X$ is an inner twist of a quasisplit $\mathbb{G} \to X$. In particular, if G is adjoint, then inner twists are automatically pure inner twists. In general, G is isogenous to an adjoint group times a torus, which is quasisplit, and one can reduce the volume computation from the quasisplit case.

7.2. **The moduli stack of G-bundles.** For $G \to X$ as in §7.1, let Bun_G be the moduli stack of G-bundles on X. Its set of *geometric* connected components is denoted $\pi_0(\mathrm{Bun}_G)$. We will define a Γ-invariant uniformization

$$\mathbb{X}_*(T_0) \to \pi_0(\mathrm{Bun}_G)$$

A result of Heinloth [Hei10, Theorem 2] identifies $\pi_0(\mathrm{Bun}_G)$ with the Γ-coinvariants $\pi_0(\mathrm{Bun}_{G_{0,X'}})_{\Gamma}$. For each $\lambda \in \mathbb{X}_*(T_0)$, choose any geometric connected component $X'_1 \subset X'_{\overline{k}}$, which induces a surjection $\mathbb{X}_*(T_0) \to \pi_0(\mathrm{Bun}_{G_{0,X'_1}})$. The target is a direct summand of $\pi_0(\mathrm{Bun}_{G_{0,X'}})$, and we further project it to the Γ-coinvariants. The resulting map

$$\mathbb{X}_*(T_0) \to \pi_0(\mathrm{Bun}_G)$$

is Γ-invariant, and is independent of the choice of the geometric connected component of X' (for different choices will be equalized after passing to Γ-coinvariants, because Γ permutes geometric components of X' transitively). We again denote this map by

$$\lambda \mapsto \overline{\lambda} \in \pi_0(\mathrm{Bun}_G).$$

7.2.1. *Atiyah–Bott formula.* We explicate the Atiyah–Bott formula in this case in a manner parallel to §4.2.3.

The universal G_0-bundle over $X' \times \mathrm{Bun}_G^{\omega}$ is classified by a map

$$\mathrm{ev}_{\omega} : X' \times \mathrm{Bun}_G^{\omega} \to \mathbb{B}G_0,$$

which then induces a map

$$\mathrm{H}_*(X') \otimes R_0^{W_0} \to \mathrm{H}^*(\mathrm{Bun}_G^{\omega}).$$

This map is invariant under the diagonal action of Γ on $\mathrm{H}_*(X')$ and $R_0^{W_0}$, hence factors through the coinvariants

$$\mathrm{ev}'_{\omega,\Gamma} : (\mathrm{H}_*(X') \otimes R_0^{W_0})_{\Gamma} \to \mathrm{H}^*(\mathrm{Bun}_G^{\omega}). \tag{7.2.1}$$

For $f \in R_0^{W_0}$ and $z \in \mathrm{H}_*(X')$, we denote

$$(f^z)_{\Gamma} := \mathrm{ev}'_{\omega,\Gamma}(z \otimes f) \in \mathrm{H}^*(\mathrm{Bun}_G^{\omega}).$$

[11] We want to allow X' to be geometrically disconnected to allow nonsplitness coming from k.

Remark 7.2.2. In general, for a local system $\mathbb{L}$ on X, we may form its homology

$$\mathrm{H}_*(X, \mathbb{L}).$$

If $\nu^*\mathbb{L}$ becomes a geometrically constant local system, i.e., it is pulled back from a local system $\mathbb{L}_0$ on Spec k with a Γ-action, then we have a canonical isomorphism

$$(\mathrm{H}_*(X') \otimes \mathbb{L}_0)_\Gamma \xrightarrow{\sim} \mathrm{H}_*(X, \mathbb{L}).$$

Remark 7.2.2 applies in particular to $\mathbb{L} = R^W$, so that (7.2.1) can be viewed as a map

$$\mathrm{ev}'_\omega : \mathrm{H}_*(X, R^W) \to \mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

Passing to associated graded induces the isomorphism

$$\mathrm{H}_{>0}(X, \mathbb{V}) \to \mathrm{Gr}^1_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}_G^\omega)$$

which then induces the isomorphism of bigraded algebras from Theorem 4.2.8:

$$\mathrm{AB}^\omega : \mathrm{Sym}^{\blacktriangleright}_{\mathbf{Q}_\ell}(\mathrm{H}_*(X; \mathbb{V})_+) \to \mathrm{Gr}^{\blacktriangleright}_{\mathrm{aug}} \mathrm{H}^*(\mathrm{Bun}_G^\omega). \tag{7.2.2}$$

7.3. **Hecke correspondences.** For a dominant[12] coweight $\mu \in \mathbb{X}_*(T_0)$, we define a Hecke correspondence

$$\begin{array}{ccccc} & & \mathrm{Hk}_G^\mu & \xrightarrow{p_{X'}} & X' \\ & \swarrow^{h_0} & & \searrow^{h_1} & \\ \mathrm{Bun}_G & & & & \mathrm{Bun}_G \end{array}$$

as follows. For a scheme $S/\mathbf{F}_q$, $\mathrm{Hk}_G^\mu(S)$ is the groupoid of $(x', \mathcal{F}_0, \mathcal{F}_1, \alpha)$ where

- $x' \in X'(S)$, with image under ν denoted $x \in X(S)$.
- $\mathcal{F}_i$ are G-bundles on $X \times S$, which pull back to G_0-bundles $\mathcal{F}'_i$ over $X' \times S$.
- $\alpha : \mathcal{F}_0|_{X \times S - \Gamma_x} \xrightarrow{\sim} \mathcal{F}_1|_{X \times S - \Gamma_x}$ is an isomorphism of G-torsors, such that its pullback to $X' \times S$:
$$\alpha' : \mathcal{F}'_0|_{X' \times S - \nu_S^{-1}(\Gamma_x)} \xrightarrow{\sim} \mathcal{F}'_1|_{X' \times S - \nu_S^{-1}(\Gamma_x)}$$
has relative position μ along $\Gamma_{x'} \subset \nu_S^{-1}\Gamma_x$. Here $\nu_S = \nu \times \mathrm{id}_S : X' \times S \to X \times S$. Note that
$$\nu_S^{-1}\Gamma_x = \coprod_{\gamma \in \Gamma} \Gamma_{\gamma(x')} \subset X' \times S$$
is the union of graphs of Γ-conjugates of x'. The relative position requirement along $\Gamma_{x'}$ then implies that the relative position of α' along $\gamma(x')$ is automatically $\gamma(\mu) := \tau(\gamma)(\mu)$, with notation as in §7.1.

For a connected component Bun_G^ω, let ${}^\omega\mathrm{Hk}_G^\mu \subset \mathrm{Hk}_G^\mu$ be the preimage of Bun_G^ω under h_0. Then ${}^\omega\mathrm{Hk}_G^\mu$ is a correspondence

$$\begin{array}{ccccc} & & {}^\omega\mathrm{Hk}_G^\mu & \xrightarrow{p_{X'}} & X' \\ & \swarrow^{h_0} & & \searrow^{h_1} & \\ \mathrm{Bun}_G^\omega & & & & \mathrm{Bun}_G^{\omega'} \end{array}$$

where $\omega' = \omega + \overline{\mu}$. We have a canonical isomorphism

$$\gamma_{\mathrm{Hk}} : {}^\omega\mathrm{Hk}_G^\mu \xrightarrow{\sim} {}^\omega\mathrm{Hk}_G^{\gamma(\mu)}$$

sending $(x', \mathcal{F}_0, \mathcal{F}_1, \alpha)$ to $(\gamma(x'), \mathcal{F}_0, \mathcal{F}_1, \alpha)$.

---

[12]Dominant with respect to B_0 as in §7.1

7.3.1. *Canonical parabolic reduction.* As in the split case, the pullback of the universal G_0-bundle $(p_{X'} \times h_0)^*\mathcal{F}^{\mathrm{univ}}$ on ${}^{\omega}\mathrm{Hk}_G^{\mu}$ has a canonical P_μ-reduction classified by the map

$$\mathrm{ev}_\mu^\omega : {}^{\omega}\mathrm{Hk}_G^{\mu} \to \mathbb{B}P_\mu.$$

These fit into a canonical commutative diagram

$$\begin{array}{ccc} {}^{\omega}\mathrm{Hk}_G^{\mu} & \xrightarrow{\mathrm{ev}_\mu^\omega} & \mathbb{B}P_\mu \\ \downarrow{\scriptstyle \gamma_{\mathrm{Hk}}} & & \downarrow{\scriptstyle \tau(\gamma)} \\ {}^{\omega}\mathrm{Hk}_G^{\gamma(\mu)} & \xrightarrow{\mathrm{ev}_{\gamma(\mu)}^\omega} & \mathbb{B}P_{\gamma(\mu)} \end{array} \tag{7.3.1}$$

We get embeddings

$$e_\gamma : \gamma_{\mathrm{Hk}}^* \circ \mathrm{ev}_{\gamma(\mu)}^{\omega,*} = \mathrm{ev}_\mu^{\omega,*} \circ \tau(\gamma)^* : R^{W_{\gamma(\mu)}} \to \mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu}) \tag{7.3.2}$$

7.4. **Arithmetic volume of moduli of shtukas.** Let $\mu = (\mu_1, \cdots, \mu_r)$ be a sequence of dominant coweights of T_0, such that

$$\overline{\mu}_1 + \cdots + \overline{\mu}_r = 0 \in \pi_0(\mathrm{Bun}_G). \tag{7.4.1}$$

As in §5.2.1, we define the iterated version of the Hecke stack Hk_G^{μ} defined in §7.3, and define Sht_G^{μ} by the Cartesian square (5.2.2). Note that the legs of points of Sht_G^{μ} are now in X'. We still denote by $p_i : {}^{\omega}\mathrm{Sht}_G^{\mu} \to X'$ map recording the ith leg for $1 \le i \le r$.

7.4.1. *Definition of arithmetic volume.* Let $D_{\mu_j} := \langle 2\rho, \mu_j \rangle = \dim G_0/P_{\mu_j}$. For each $1 \le j \le r$, fix $\eta_j \in R^{W_{\mu_j}}$ of degree $2(D_{\mu_j} + 1)$ and $\eta'_j \in R^{W_{\mu_j}}$ of degree $2D_{\mu_j}$.

Define ${}_c\Gamma_{\mu_j}^{\eta_j + \xi' \eta'_j}$ as in §5.1.2 (recall that $\xi' \in H^2(X')(1)$ is the fundamental class of the covering curve). For $\eta = (\eta_1 + \xi'\eta'_1, \dots, \eta_r + \xi'\eta'_r)$, define

$${}_c\Gamma_\mu^\eta := {}_c\Gamma_{\mu_r}^{\eta_r + \xi'\eta'_r} \circ \dots \circ {}_c\Gamma_{\mu_1}^{\eta_1 + \xi'\eta'_1} : \mathrm{H}_c^*(\mathrm{Bun}_G^\omega) \to \mathrm{H}_c^*(\mathrm{Bun}_G^\omega).$$

The target component ω agrees with the source component ω by the assumption (7.4.1).

Definition 7.4.2. Let ${}_c\Gamma_\mu^{\eta + \xi'\eta'}$ be as in §5.1.2. Let $\mu = (\mu_1, \dots, \mu_r)$ and $\eta = (\eta_1 + \xi'\eta'_1, \dots, \eta_r + \xi'\eta'_r)$. The *arithmetic volume* of ${}^{\omega}\mathrm{Sht}_G^{\mu}$ with respect to η is the *graded* trace

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}_G^{\mu}, \eta) := \mathrm{Tr}({}_c\Gamma_\mu^\eta \circ \mathrm{Frob} \mid \mathrm{H}_c^*(\mathrm{Bun}_G^\omega)) \tag{7.4.2}$$

Remark 7.4.3. In addition to the classes of the form $\mathrm{ev}_j^*\eta_j$, we could pull back classes using the evaluation maps $\mathrm{ev}_{j,\gamma} : \mathrm{Sht}_G^{\mu} \to \mathrm{Hk}_G^{\mu} \xrightarrow{\gamma_{\mathrm{Hk}}} \mathrm{Hk}_G^{\gamma(\mu)} \xrightarrow{\mathrm{ev}_{\gamma(\mu)}} \mathbb{B}P_{\gamma(\mu)}$ for any $\gamma \in \Gamma$. However, because of the diagram (7.3.1), one can rewrite such classes as pullbacks via ev_j^* again. Thus, as long as we are concerned with linear combinations of terms pulled back by $\mathrm{ev}_{j,\gamma}$, the form considered in (7.4.2) does not restrict the generality.

Remark 7.4.4. The convergence of the trace in the definition of $\mathrm{vol}({}^{\omega}\mathrm{Sht}_G^{\mu}, \eta)$ can be proved in a similar way as Proposition 5.5.3. We omit the proof here.

7.4.5. *Gross motive.* We follow the notation of §7.1. The constructions in the split case can be applied to G_0 together with its split maximal torus T_0 and Weyl group W_0 to give $R_0, R_0^{W_0}$ and $\mathbb{V}' := \mathrm{Gr}_{\mathrm{aug}}^1 R_0^{W_0}$, etc.

Since G is quasisplit as a group scheme over X, we have its abstract Cartan T as a torus over X. The abstract Weyl group W acting on T is now a finite group scheme over X. We can then form $R = \mathrm{Sym}(\mathbb{X}^*(T)_{\overline{\mathbf{Q}}_\ell}(-1))$, R^W and $\mathbb{V} = \mathrm{Gr}_{\mathrm{aug}}^1 R^W$, all as local systems over X, in the same way as in the split case §4.1.8.

For example, we view $\mathbb{V}'$ as a geometrically constant local system on X'. It carries a Γ-action induced from τ, which can be viewed as a Γ-equivariant structure on the constant local system $\mathbb{V}'$ on X'. Then $\mathbb{V}$ is the descent of $\mathbb{V}'$ to X. The generic stalk of $\mathbb{V}(1)$ is the $\overline{\mathbf{Q}}_\ell$-realization of the Gross motive $\mathbb{M}_G$, as a $\mathrm{Gal}(F^s/F)$-module.

7.4.6. *L-function.* Let $\mathbb{E}$ be a local system on X. The L-function of $\mathbb{E}$ is defined to be

$$L_X(s,\mathbb{E}) = \prod_{i\in\mathbb{Z}} \det(1-q^{-s}\mathrm{Frob}|\mathrm{H}^i(X,\mathbb{E}))^{(-1)^{i-1}}.$$

This is consistent with the usual definition of the L-function attached $\mathbb{E}$ as a $\mathrm{Gal}(F^s/F)$-module.

We will apply this to the graded local system $\mathbb{E} = \mathbb{V}^*$ dual to $\mathbb{V}$, but with a slight normalization. Recall that $\mathbb{V}^*$ has a negative grading $\oplus_d(\mathbb{V}_{2d})^*[2d]$. Let $\mathrm{H}^{<0}(X,\mathbb{V}^*)$ be the negative degree part of the cohomology. This removes exactly the summand $\mathrm{H}^2(X,(\mathbb{V}_2)^*)$, which is zero if $Z(G)$ does not contain a nontrivial split torus. Let

$$L_X^*(s,\mathbb{V}^*) = \prod_{i<0} \det(1-q^{-s}\mathrm{Frob}|\mathrm{H}^i(X,\mathbb{V}^*))^{(-1)^{i-1}}.$$

The normalization in the definition of $\mathbb{E}$ removes the pole of $L_X(s,\mathbb{V}^*)$ at $s=0$.

Example 7.4.7. For a local system $\mathbb{E}$ on X, we have $L_X(s,\mathbb{E}(1)) = L_X(s+1,\mathbb{E})$. Hence if $\mathbb{V} = \bigoplus_i \mathbf{Q}_\ell(-d_i)[-2d_i]$, then $\mathbb{V}^* = \bigoplus \mathbf{Q}_\ell(d_i)[2d_i]$ and $L_X(s,\mathbb{V}^*) = \prod_i \zeta_X(s+d_i)$. In particular, if $\mathbb{V}$ arises from a split group G/X as in §5, then $L_X(s,\mathbb{V}^*)$ agrees with the L-function $L_{X,G}(s)$ from (5.7.3), and $L_X^*(s,\mathbb{V}^*)$ agrees with the normalized $L^*_{X,G}(s)$ from (5.7.4).

7.4.8. *Local operators.* Recall the operator $\overline{\nabla}^{\eta_j}_{\mu_j}$ on $\mathbb{V}'$ from §5.3. For an Atiyah–Bott generator $f^z \in \mathrm{H}_*(X',\mathbb{V}')$ as in §7.2.1, consider the map

$$\begin{aligned} \widetilde{\mathbb{D}}^{\eta_j}_{\mu_j} : \mathrm{H}_*(X',\mathbb{V}') &\to \mathrm{H}_*(X',\mathbb{V}')^\Gamma \\ f^z &\mapsto \sum_{\gamma\in\Gamma} (\overline{\nabla}^{\eta_j}_{\mu_j}(\gamma^* f))^{\gamma_*^{-1}z} \in \mathrm{H}_*(X',\mathbb{V}')^\Gamma. \end{aligned}$$

It clearly factors through the coinvariants $\mathrm{H}_*(X',\mathbb{V}')_\Gamma = \mathrm{H}_*(X,\mathbb{V})$. Composing $\widetilde{\mathbb{D}}^{\eta_j}_{\mu_j}$ with the natural map from invariants to coinvariants $\mathrm{H}_*(X',\mathbb{V}')^\Gamma \to \mathrm{H}_*(X',\mathbb{V}')_\Gamma$, we obtain an endomorphism

$$\mathbb{D}^{\eta_j}_{\mu_j} \in \mathrm{End}^{gr}(\mathrm{H}_{>0}(X,\mathbb{V})).$$

It preserves the homological grading and the grading on $\mathbb{V}$. Taking the adjoint, we obtain graded endomorphisms of $\mathrm{H}^*(X,\mathbb{V}^*)$, and in particular endomorphisms of $\mathrm{H}^{<0}(X,\mathbb{V}^*)$. We still denote these endomorphisms by $\mathbb{D}^{\eta_j}_{\mu_j}$.

Assumption 7.4.9. We assume that the operators $\mathbb{D}^{\eta_j}_{\mu_j}$ pairwise commute for $j=1,\dots,r$.

Remark 7.4.10. Note that Assumption 7.4.9 is satisfied if the $\overline{\nabla}^{\eta_j}_{\mu_j}$ pairwise commute. This in turn is satisfied for many examples, as discussed in Example 5.6.2.

7.4.11. *The volume formula.* We will now introduce a multivariate version of $L^*(s,\mathbb{V}^*)$, taking into account the eigenvalues of the operators $\mathbb{D}^{\eta_j}_{\mu_j}$ on $\mathrm{H}^{<0}(X,\mathbb{V}^*)$. We decompose

$$\mathrm{H}^{<0}(X,\mathbb{V}^*) = \bigoplus_{\epsilon\in\overline{\mathbf{Q}}_\ell^r} \mathrm{H}^{<0}(X,\mathbb{V}^*)[\epsilon] \tag{7.4.3}$$

where for $\epsilon = (\epsilon_1,\cdots,\epsilon_r)$, $\mathrm{H}^{<0}(X,\mathbb{V}^*)[\epsilon]$ is the simultaneous generalized eigenspaces under $\mathbb{D}^{\eta_j}_{\mu_j}$ with generalized eigenvalue ϵ_j, for $1\le j\le r$. Let $\vec{s} = (s_\epsilon)$, with one variable for each $\epsilon\in\overline{\mathbf{Q}}_\ell^r$ with a nontrivial summand in (7.4.3). Let

$$L_X^*(\vec{s},\mathbb{V}^*) := \prod_\epsilon \prod_{i<0} \det(1-q^{-s_\epsilon}\mathrm{Frob}|\mathrm{H}^i(X,\mathbb{V}^*)[\epsilon])^{(-1)^{i-1}}.$$

For the sequence $\mu = (\mu_1,\dots,\mu_r)$ and a fixed $\omega\in\pi_0(\mathrm{Bun}_G)$, set $\omega_j := \omega + \overline{\mu}_1 + \ldots + \overline{\mu}_j$. For $1\le j\le r$, consider the first order differential operator

$$\mathfrak{d}_j = d^{\omega_{j-1}}_{\mu_j}(\eta_j) + d_{\mu_j}(\eta_j') - (\log q)^{-1} \sum_{\epsilon\in\overline{\mathbf{Q}}_\ell^r} \epsilon_j \partial_{s_{\epsilon_j}}.$$

Theorem 7.4.12. *Let $\mu = (\mu_1, \cdots, \mu_r)$ be a sequence of minuscule dominant coweights of G_0 satisfying (7.4.1). For $j = 1, \ldots, r$, let $\eta_j \in \mathrm{H}^{2D_{\mu_j}+2}(\mathbb{B}L_{\mu_j})$ and $\eta'_j \in \mathrm{H}^{2D_{\mu_j}}(\mathbb{B}L_{\mu_j})$ satisfying Assumption 7.4.9. For $\eta = (\eta_1 + \xi'\eta'_1, \ldots, \eta_r + \xi'\eta'_r)$, we have*

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_{G}, \eta) = q^{\dim \mathrm{Bun}_G} \Big(\prod_{j=1}^{r} \mathfrak{d}_j \Big) L^*_X(\vec{s}, \mathbb{V}^*) \Big|_{\vec{s}=0}.$$

7.5. **Proof of Theorem 7.4.12.** The proof of Theorem 7.4.12 follows the same strategy as in the split case. We first need the following generalization of Proposition 5.5.8 to the quasisplit case. Recall that in §7.3.1, specifically (7.3.2), we defined a map $R_0^{W_{\gamma(\mu)}} \to \mathrm{H}^*({}^{\omega}\mathrm{Hk}^{\mu}_{G})$.

Proposition 7.5.1. *Let g' be the genus of X'. For $f \in R_0^{W_0}$ and $z \in \mathrm{H}_{|z|}(X')$, we have*

$$h_1^*((f^z)_\Gamma) - h_0^*((f^z)_\Gamma) = \sum_{\gamma \in \Gamma} \gamma^* \mathrm{PD}(z) e_\gamma(\partial_{\gamma(\mu)} f) + \begin{cases} (1-g')\langle z, \xi' \rangle \xi' \sum_{\gamma \in \Gamma} e_\gamma(\partial^2_{\gamma(\mu)} f), & |z| = 2, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We need a slight extension of Theorem 3.2.1: the modification between G_0-bundles $\mathcal{F}_0$ and $\mathcal{F}_1$ on S is along a disjoint union of divisors $D = \coprod_{i \in I} D_i$, and it has type λ_i along D_i. Then the same proof of Theorem 3.2.1 shows that for $f \in R_0^{W_0}$, we have

$$f(\mathcal{F}_1) - f(\mathcal{F}_0) = \sum_{i \in I} i_{D_i !} \left(\sum_{n \geq 1} \frac{1}{n!} (\partial^n_{\lambda_i} f)(\mathcal{F}_1|_{D_i, L_{\lambda_i}}) \cup \nu^{n-1}_{D_i} \right). \tag{7.5.1}$$

Here again $\nu_{D_i} = c_1(\mathcal{O}(D_i))|_{D_i} \in \mathrm{H}^2(D_i)(1)$ is the Chern class of the normal bundle of D_i.

We apply this to the two pullbacks of the universal bundle $\mathcal{F}_i = (q_{X'} \times h_i)^* \mathcal{F}^{\mathrm{univ}}$, $i = 0, 1$, on $S := X' \times {}^{\omega}\mathrm{Hk}^{\mu}_{G}$. The modification is along the divisor

$$D = X' \times_X {}^{\omega}\mathrm{Hk}^{\mu}_{G} \to X' \times_X X' = \coprod_{\gamma \in \Gamma} \Delta_\gamma,$$

where $\Delta_\gamma = \{(\gamma x', x') | x' \in X'\} \subset X' \times X'$ is the graph of $\gamma \in \Gamma$. Hence $D = \coprod_{\gamma \in \Gamma} D_\gamma$ where D_γ is the preimage of Δ_γ in $X' \times_X {}^{\omega}\mathrm{Hk}^{\mu}_{G}$. We identify D_γ with ${}^{\omega}\mathrm{Hk}^{\mu}_{G}$ using the projection $p_{\mathrm{Hk}} : S \to {}^{\omega}\mathrm{Hk}^{\mu}_{G}$. Each D_γ is the graph of $\gamma \circ p_{X'} : {}^{\omega}\mathrm{Hk}^{\mu}_{G} \to X'$. Let

$$i_\gamma : D_\gamma \cong {}^{\omega}\mathrm{Hk}^{\mu}_{G} \xrightarrow{(\gamma \circ p_{X'}, \mathrm{id})} X' \times {}^{\omega}\mathrm{Hk}^{\mu}_{G} = S$$

be the inclusion of D_γ. The modification type of $\mathcal{F}_0 \dashrightarrow \mathcal{F}_1$ along D_γ is $\gamma(\mu)$, giving rise to a $P_{\gamma(\mu)}$-reduction of $\mathcal{F}_1|_{\Gamma(\gamma|X')}$ classified by the map

$$D_\gamma \cong {}^{\omega}\mathrm{Hk}^{\mu}_{G} \xrightarrow{\gamma_{\mathrm{Hk}}} {}^{\omega}\mathrm{Hk}^{\gamma(\mu)}_{G} \xrightarrow{\mathrm{ev}^{\omega}_{\gamma(\mu)}} \mathbb{B}P_{\gamma(\mu)}.$$

Then (7.5.1) implies

$$(q_{X'} \times h_1)^* f - (q_{X'} \times h_0)^* f = \sum_{\gamma \in \Gamma} i_{\gamma !} \Big(\sum_{n \geq 1} \frac{1}{n!} e_\gamma(\partial^n_{\gamma(\mu)} f) \cdot \nu^{n-1}_{D_\gamma} \Big).$$

The same calculation as in the proof of Proposition 5.5.8 gives

$$i_{\gamma !} \left(\sum_{n \geq 1} \frac{1}{n!} e_\gamma(\partial^n_{\gamma(\mu)} f) \cdot \nu^{n-1}_{D_\gamma} \right) = [\Delta_\gamma] \cdot e_\gamma(\partial_{\gamma(\mu)} f) + (1-g')(\xi' \otimes \xi') e_\gamma(\partial^2_{\gamma(\mu)} f).$$

Note that $[\Delta_\gamma] = (1 \times \gamma)^* [\Delta_{X'}]$. Extract Künneth components by pairing with $z \in \mathrm{H}_*(X')$, we get the desired formula. □

7.5.2. *Completion of the proof.* Passing from $\mathrm{H}^*_c(\mathrm{Bun}_G)$ to $\mathrm{H}^*(\mathrm{Bun}_G)$ via Poincaré duality, we may rewrite the arithmetic volume as

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_{G}, \eta) = q^{\dim \mathrm{Bun}_G} \mathrm{Tr}\Big(\mathrm{Frob}^{-1} \circ \prod_{j=1}^{r} \Gamma^{\eta_j + \xi' \eta'_j}_{\mu_j}, \mathrm{Sym}(\mathrm{H}^{>0}(X, \mathbb{V}^*))\Big)$$

We use the Atiyah–Bott formula to identify $\mathrm{H}^*(\mathrm{Bun}^{\omega_j}_G)$ for the different components ω_j appearing as the source/target of $\Gamma^{\eta_j+\xi'\eta'_j}_{\mu_j}$. Then we compute the eigenvalues of each $\Gamma^{\eta_j+\xi'\eta'_j}_{\mu_j}$ for $j = 1, \ldots, r$. To do this, we define the Ran filtration on $\mathrm{H}^*(\mathrm{Bun}^{\omega}_G)$ as in §5.5.9, and show that it is preserved by $\Gamma^{\eta_j+\xi'\eta'_j}_{\mu_j}$. Using Proposition 7.5.1, we find that the action of $\Gamma^{\eta+\xi'\eta'}_{\mu}$ on the doubly associated graded

$$\mathrm{Gr}^{\blacktriangleright}_{\mathrm{aug}} \mathrm{Gr}^{F}_{\bullet} \mathrm{H}^*(\mathrm{Bun}^{\omega}_G) \cong \mathrm{Sym}^{\blacktriangleright}(\mathrm{H}_{>0}(X, \mathbb{V}))$$

is the sum of the scalar operator $d^{\omega_{j-1}}_{\mu_j}(\eta_j) + d_{\mu_j}(\eta'_j)$ defined in the same way as in §5.6.4, plus the derivation whose action on $[f^z]_{\Gamma} \in \mathrm{H}_{>0}(X, \mathbb{V})$ is $\mathbb{D}^{\eta_j}_{\mu_j}$.

The rest of the proof is the same as in §5.6.18 for the split case. □

7.6. **Example: unitary groups.** We spell out the statement of Theorem 7.4.12 in the case of non-split unitary groups of the type occurring in [FYZ24, FYZ25]. Indeed, an original motivation of the present work was to address a singular version of the Arithmetic Siegel–Weil formula.

Let $\nu \colon X' \to X$ be a finite étale double cover. Let $\mathrm{Bun}_{\mathrm{U}(n)}$ be as in [FYZ24, §6.1], so that $\mathrm{Bun}_{\mathrm{U}(n)}(S)$ is the groupoid of rank n vector bundles $\mathcal{F}$ on $X' \times S$ plus a Hermitian structure $h \colon \mathcal{F} \xrightarrow{\sim} \sigma^* \mathcal{F}^*$.

Let $\mathrm{Hk}^1_{\mathrm{U}(n)}$ be the Hecke stack of modifications of colength 1 in the sense of [FYZ24, Definition 6.5]. For a commutative k-algebra R, $\mathrm{Hk}^1_{\mathrm{U}(n)}(R)$ is the groupoid of

$$\{x' \in X'(R), \mathcal{F}_0 \supset \mathcal{F}^{\flat}_{1/2} \subset \mathcal{F}_1 \mid \mathcal{F}_0, \mathcal{F}_1 \in \mathrm{Bun}_{\mathrm{U}(n)}(R)\} \tag{7.6.1}$$

such that $\mathcal{F}_0/\mathcal{F}^{\flat}_{1/2}$ (resp. $\mathcal{F}_1/\mathcal{F}^{\flat}_{1/2}$) is a line bundle along the graph of x' (resp. $\sigma(x')$). In the notation of §7.4, we have

$$\mathrm{Hk}^1_{\mathrm{U}(n)} = \mathrm{Hk}^{\mu^{\flat}}_{\mathrm{U}(n)} \text{for } \mu^{\flat} = (0, 0, \ldots, -1).$$

Let $h_0, h_1 : \mathrm{Hk}^1_{\mathrm{U}(n)} \to \mathrm{Bun}_{\mathrm{U}(n)}$ be the maps recording $\mathcal{F}_0$ and $\mathcal{F}_1$ respectively. Let $p_{X'} : \mathrm{Hk}^1_{\mathrm{U}(n)} \to X'$ be the map recording the support of $\mathcal{F}_0/\mathcal{F}^{\flat}_{1/2}$. Then we have a correspondence

$$\begin{array}{ccc} & \mathrm{Hk}^1_{\mathrm{U}(n)} & \\ {}^{h_0}\swarrow & & \searrow^{h_1} \\ \mathrm{Bun}_{\mathrm{U}(n)} & & \mathrm{Bun}_{\mathrm{U}(n)} \end{array}$$

7.6.1. *Tautological bundles.* Canonical parabolic reduction gives a map $\mathrm{Hk}^1_{\mathrm{U}(n)} \to G/P_{\mu^{\flat}} \cong \mathbb{P}^{n-1}$. The pullback of $\mathcal{O}(-1)$ is the "tautological line bundle" $\mathcal{P}$ whose fiber along an R-point (7.6.1) is $(\mathcal{F}_0/\mathcal{F}^{\flat}_{1/2})^*$.

Remark 7.6.2. This normalization is chosen to match with [FYZ24, FHM25] which takes the "tautological bundle" to be $\mathcal{F}_1/\mathcal{F}^{\flat}_{1/2}$, which is isomorphic to $\mathcal{F}^{\sharp}_{1/2}/\mathcal{F}_0$ where $\mathcal{F}^{\sharp}_{1/2} = \sigma^*(\mathcal{F}^{\flat}_{1/2})^*$. This is in turn dual to $\mathcal{F}_0/\mathcal{F}^{\flat}_{1/2}$.

7.6.3. *Moduli of shtukas.* As in [FYZ25, §2][13], let $\mathrm{Hk}^r_{\mathrm{U}(n)}$ be the iterated fibered product of $\mathrm{Hk}^1_{\mathrm{U}(n)}$ over $\mathrm{Bun}_{\mathrm{U}(n)}$, parametrizing

$$(x'_1, \ldots, x'_r, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1 \dashrightarrow \ldots \dashrightarrow \mathcal{F}_r \cong \mathrm{Frob}^*\mathcal{F}_0)$$

[13] but note that we are using different conventions on the similitude factor.

and let $\mathrm{Sht}^r_{\mathrm{U}(n)}$ be the fibered product

$$\begin{array}{ccc} \mathrm{Sht}^r_{\mathrm{U}(n)} & \longrightarrow & \mathrm{Hk}^r_{\mathrm{U}(n)} \\ \downarrow & & \downarrow{\scriptstyle (h_0,h_r)} \\ \mathrm{Bun}_{\mathrm{U}(n)} & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & \mathrm{Bun}_{\mathrm{U}(n)} \times \mathrm{Bun}_{\mathrm{U}(n)} \end{array}$$

7.6.4. *The L-function.* In this case, the L-function $L_X(s,\mathbb{E})$ specializes to

$$L_{X,\mathrm{U}(n)}(s) = \prod_{i=1}^{n} L(s+i, \chi^i_{X'/X})$$

where $\chi_{X'/X}$ is the quadratic character corresponding to X'. The multivariable version is

$$\mathscr{L}_{X,\mathrm{U}(n)}(s_1,\ldots,s_n) = \prod_{i=1}^{n} L(s_i+i, \chi^i_{X'/X}).$$

7.6.5. *Arithmetic Volume.* Let $\mathcal{E}$ be a rank n vector bundle on X'. We define

$$\begin{aligned} {}_c\Gamma^{\mathcal{E}}_1 \colon \mathrm{H}^*_c(\mathrm{Bun}_G) &\to \mathrm{H}^*_c(\mathrm{Bun}_G) \\ \theta &\mapsto h_{1*}(h_0^*\theta \cup c_n(p^*_{X'}\mathcal{E}^* \otimes \mathcal{P})) \end{aligned}$$

analogously to §5.7.3. We define ${}_c\Gamma^{\mathcal{E}}_r = ({}_c\Gamma^{\mathcal{E}}_1)^{\circ r}$. Finally, we define the arithmetic volume

$$\mathrm{vol}(\mathrm{Sht}^r_{\mathrm{U}(n)}, \prod_{i=1}^{r} c_n(p_i^*\mathcal{E}^* \otimes \mathcal{P})) := \mathrm{Tr}({}_c\Gamma^{\mathcal{E}}_r \circ \mathrm{Frob}, \mathrm{H}^*_c(\mathrm{Bun}_G))$$

where the trace is taken in the graded sense.

Theorem 7.6.6. *Let $\mathcal{E}$ be a rank n vector bundle on X' of degree D. If r is even, then we have*

$$\mathrm{vol}(\mathrm{Sht}^r_{\mathrm{U}(n)}, \prod_{i=1}^{r} c_n(p_i^*\mathcal{E}^* \otimes \mathcal{P})) = 2\frac{q^{n^2(g-1)}}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}(q^{-Ds}L_{X,\mathrm{U}(n)}(2s)).$$

Remark 7.6.7. In the case $n=1$, Theorem 7.6.6 recovers [FYZ25, Theorem 10.2] for $\mathfrak{L} = \omega_X^{-1}$.

Proof. There are two connected components $\mathrm{Bun}_{\mathrm{U}(n)} = \mathrm{Bun}^{\omega_0}_{\mathrm{U}(n)} \sqcup \mathrm{Bun}^{\omega_1}_{\mathrm{U}(n)}$, inducing a decomposition

$$\mathrm{Sht}^r_{\mathrm{U}(n)} = {}^{\omega_0}\mathrm{Sht}^r_{\mathrm{U}(n)} \sqcup {}^{\omega_1}\mathrm{Sht}^r_{\mathrm{U}(n)}.$$

We will show more precisely that

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}^r_{\mathrm{U}(n)}, \prod_{i=1}^{r} c_n(p_i^*\mathcal{E}^* \otimes \mathcal{P})) = \frac{q^{n^2(g-1)}}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}(q^{-Ds}L_{X,\mathrm{U}(n)}(2s)).$$

for each $\omega \in \{\omega_0,\omega_1\}$.

As discussed in Example 5.6.2, Assumption 7.4.9 is automatically satisfied. Hence we may apply Theorem 7.4.12 in order to calculate the left side.

We compute (e.g., by the splitting principle)

$$c_n(p^*_{X'}\mathcal{E}^* \otimes \mathcal{P}) = c_1(\mathcal{P})^n + p^*_{X'}c_1(\mathcal{E}^*)c_1(\mathcal{P})^{n-1}.$$

For the pullback map $\overline{\mathbf{Q}}_\ell[x_1,\ldots,x_n]^{S_{n-1}\times S_1} = R_0^{W_{\mu^\flat}} \to \mathrm{H}^*(\mathrm{Hk}^\mu_G)$ induced by canonical parabolic induction, our conventions are arranged so that $c_1(\mathcal{P})$ agrees with the image of $-x_n$ (cf. §5.7.2). Hence we have

$$c_n(p^*_{X'}\mathcal{E}^* \otimes \mathcal{P}) = (-x_n)^n - D\xi(-x_n)^{n-1} = (-1)^n(x_n^n + D\xi' x_n^{n-1}).$$

Hence we find that ${}_c\Gamma^{\mathcal{E}}_1 = {}_c\Gamma^{\eta}_{\mu^\flat}$ for $\eta = (-x_n)^n$ and $\eta' = -D(-x_n)^{n-1}$. To apply Theorem 5.6.9, we need to calculate the constants $d^{\omega}_{\mu^\flat}(\eta)$, $d_{\mu^\flat}(\eta')$, and $\epsilon_i(\eta,\mu^\flat)$.

- Using (5.4.6) for $i=1$, we find that $d^{\omega}_{\mu^\flat}(\eta) = -e_1^{[X]} = 0$.

- In this situation, $d_{\mu^\flat}(\eta')$ was already computed in the proof of Theorem 5.7.5. There we saw that $d_{\mu^\flat}(\eta') = -\int_{G/P_{\mu^\flat}} (-x_n)^{n-1} D = (-1)^n D$.
- We have $\nu^* \mathrm{U}(n) = \mathrm{GL}_n$, and $R_0^{W_0} = \overline{\mathbf{Q}}[e_1, \ldots, e_n]$. The eigenweights of $\overline{\nabla}^{\eta}_{\mu^\flat}$ were already computed in the proof of Theorem 5.7.5, where we saw that they were all equal to $(-1)^{n-1}$.

  We need to compute the eigenvalues of $\mathbb{D}^{\eta}_{\mu^\flat}$. The generator of $\Gamma \cong \mathbf{Z}/2\mathbf{Z}$ takes $e_i \mapsto (-1)^i e_i$. Write $\mathrm{H}^*(X') = \mathrm{H}^*(X')^{(+1)} \oplus \mathrm{H}^*(X')^{(-1)}$ for the eigenspace decomposition under Γ. We have $\mathbb{V}' = \bigoplus_{i=1}^n \mathbb{V}'_{2i}$ where Γ acts on $\mathbb{V}'_{2i}$ by the ith power of the sign character. Hence $\mathrm{H}_*(X', (\mathbb{V}')^*_{2i})^{(+)} = \mathrm{H}_*(X')^{(-1)^i}(i)$, on which the operator $\sum_{\gamma \in \Gamma} (\gamma^* f)^{\gamma_*^{-1} z}$ acts as multiplication by 2. Therefore the eigenvalues of $\mathbb{D}^{\eta}_{\mu^\flat}$ are all equal to $(-1)^{n-1}2$, and

$$\mathrm{H}^{<0}(X; \mathbb{V}^*)[\epsilon] = \bigoplus_{i \text{ even}} \mathrm{H}^{<0}(X')^{(+1)}(i)[2i] \oplus \bigoplus_{i \text{ odd}} \mathrm{H}^{<0}(X')^{(-1)}(i)[2i].$$

Hence we conclude that

$$\mathfrak{d}_j = (-1)^{n-1}(-D + 2(\log q)^{-1} \sum_{i=1}^n \partial_{s_i}).$$

Since r is even, we may ignore the sign when composing r such operators. Observing that $\dim \mathrm{Bun}_G = n^2(g-1)$, Theorem 5.6.9 says that

$$\begin{aligned} &\mathrm{vol}({}^{\omega_i}\mathrm{Sht}^r_{\mathrm{U}(n)}, \prod_{j=1}^r c_n(p_j^* \mathcal{E}^* \otimes \mathcal{P})) \\ &= q^{n^2(g-1)} \prod_{j=1}^r \Big(- D + 2(\log q)^{-1} \sum_{i=1}^n \partial_{s_i} \Big) \mathscr{L}^*_{X, \mathrm{U}(n)}(s_1, \cdots, s_n) \Big|_{s_1 = s_2 = \cdots = s_n = 0}. \end{aligned}$$

This agrees with the claimed formula by (5.7.7).

□

8. The phantom tautological ring

As in the study of other well-known moduli spaces, it is natural to consider the subring of the cohomology ring of Sht^μ_G generated by tautological classes. In this section, we construct a ring C^μ_G by generators and relations that maps to $\mathrm{H}^*(\mathrm{Sht}^\mu_G)$, with image consisting of tautological classes. We call C^μ_G the "phantom tautological ring" for Sht^μ_G, because it maps to $\mathrm{H}^*(\mathrm{Sht}^\mu_G)$ but the map is generally not injective. We also interpret the volume calculation in Section 5.2 as a linear functional on C^μ_G.

8.1. **The phantom tautological ring.** We assume until §8.5 that G is split and semisimple. We keep the setup as in §5.2.1.

Recall the following maps from Sht^μ_G: for $0 \le i \le r$ the map

$$h_i : \mathrm{Sht}^\mu_G \to \mathrm{Bun}_G$$

records $\mathcal{F}_i$; for $1 \le i \le r$ the map

$$p_i : \mathrm{Sht}^\mu_G \to X$$

records the ith leg. For the modification at the ith leg $\mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ of type μ_i, $\mathcal{F}_{i-1}$ carries a canonical reduction to P_{μ_i} at x_i. This gives map

$$\mathrm{ev}_i : \mathrm{Sht}^\mu_G \to \mathrm{Hk}^\mu_G \to \mathbb{B}P_{\mu_i}.$$

Pulling back along p_i and ev_i gives a ring homomorphism

$$\widetilde{\rho} : \widetilde{C}^\mu_G := \bigotimes_{i=1}^r (\mathrm{H}^*(X) \otimes R^{W_{\mu_i}}) \to \mathrm{H}^*(\mathrm{Hk}^\mu_G) \to \mathrm{H}^*(\mathrm{Sht}^\mu_G). \tag{8.1.1}$$

Abusing notation, we will use the notation $\widetilde{\rho}$ both for the pullback to $\mathrm{H}^*(\mathrm{Hk}^\mu_G)$, and the pullback to $\mathrm{H}^*(\mathrm{Sht}^\mu_G)$. If we equip $\widetilde{C}^\mu_G$ with the Frobenius action induced from each tensor factor, $\widetilde{\rho}$ is Frobenius equivariant.

Definition 8.1.1. For each $\omega \in \pi_0(\mathrm{Bun}_G)$, the *tautological ring* for ${}^\omega\mathrm{Sht}_G^\mu$ is the image of the composition

$$\widetilde{C}_G^\mu \xrightarrow{\widetilde{\rho}} \mathrm{H}^*(\mathrm{Sht}_G^\mu) \to \mathrm{H}^*({}^\omega\mathrm{Sht}_G^\mu).$$

Remark 8.1.2. We do not include pullbacks of classes from Bun_G in the source of the map $\widetilde{\rho}$. A posteriori, we show in Corollary 8.2.3 that the image of $\widetilde{\rho}$ contains all classes pulled back from Bun_G via h_i for all $0 \le i \le r$.

We will show that this map factors through a much smaller quotient ring of the left side.

Notation: below, we abbreviate ϕ for the Frobenius pullback endomorphism on the cohomology of stacks over $\mathbb{F}_q$.

8.1.3. *Classes on $X \times X$.* Let $\Delta_X \subset X \times X$ be the diagonal. For any integer $d \neq 0, 1$ we define the following classes in $\mathrm{H}^2(X \times X)$

$$\Xi_d = \left(\frac{1}{q^d\phi^{-1} - 1} \otimes \mathrm{id}\right)[\Delta_X] = \left(\mathrm{id} \otimes \frac{1}{q^{d-1}\phi - 1}\right)[\Delta_X].$$

The equality of the two expressions follows from the fact that ϕ and $q\phi^{-1}$ are adjoint under the Poincaré duality pairing on $\mathrm{H}^*(X)$.

Let $\{\zeta_j\}_{1 \le j \le 2g}$ be a basis for $\mathrm{H}^1(X)$ consisting of eigenvectors under ϕ with eigenvalues α_j, and let $\{\zeta^j\}$ be the basis of $\mathrm{H}^1(X)$ such that $\int_X \zeta_i \zeta^j = \delta_{ij}$. In terms of these bases we have $\Delta_X = 1 \otimes \xi - \sum_{j=1}^{2g} \zeta_j \zeta^j + \xi \otimes 1$, so we can rewrite Ξ_d as

$$\Xi_d = (q^d - 1)^{-1} 1 \otimes \xi + \sum_{j=1}^{2g} (1 - \alpha_j q^{d-1})^{-1} \zeta_j \otimes \zeta^j + (q^{d-1} - 1)^{-1} \xi \otimes 1.$$

Lemma 8.1.4. *For any integer $d \neq 0, 1$ we have*

$$\Delta_X^* \Xi_d = -\frac{\zeta_X'(d)}{\log(q)\zeta_X(d)} \xi.$$

Proof. We have

$$\begin{aligned} \Delta_X^* \Xi_d &= \xi \int_{X \times X} [\Delta_X] \cdot \Xi_d = \xi \int_{X \times X} [\Delta_X] \cdot \left(\frac{1}{q^d \phi^{-1} - 1} \otimes \mathrm{id}\right)[\Delta_X] \\ &= \mathrm{Tr}((q^d\phi^{-1} - 1)^{-1} | \mathrm{H}^*(X))\xi \end{aligned}$$

Thus it suffices to show the equality of rational functions in q^s

$$-\frac{d}{ds} \log \zeta_X(s) = \log(q) \mathrm{Tr}((q^s \phi^{-1} - 1)^{-1} | \mathrm{H}^*(X)).$$

Now note $\zeta_X(s) = \det(1 - q^{-s}\phi | \mathrm{H}^*(X))^{-1}$ (alternating product of determinant of graded pieces). Thus it suffices to show that for any finite-dimensional graded vector space $V = \oplus V_i$ over $\mathbf{C}$ and a graded automorphism ϕ of V, we have an equality in $\mathbf{C}(q^s)$

$$\frac{d}{ds} \log \det(1 - q^{-s}\phi | V) = \log(q) \mathrm{Tr}((q^s\phi^{-1} - 1)^{-1} | V). \tag{8.1.2}$$

Both sides are additive in (V, ϕ) in short exact sequences, therefore we reduce to the case $\dim V = 1$. Changing the parity of grading results in a negative sign on both sides, so we may assume $V = V_0$. In this case, ϕ acts on V through a scalar $\alpha \in \mathbf{C}^\times$. The equality (8.1.2) becomes

$$\frac{d}{ds} \log(1 - \alpha q^{-s}) = \frac{\log(q)}{\alpha^{-1} q^s - 1},$$

which is a direct calculation.

□

8.1.5. *An ideal in* $\widetilde{C}^\mu_G$. In the ring $\widetilde{C}^\mu_G$ from (8.1.1), for $\zeta \in \mathrm{H}^*(X)$, let $[\zeta]_i$ be the class ζ put at the ith factor of $\mathrm{H}^*(X)$. Similarly, for $1 \le i < i' \le r$ and $\Xi \in \mathrm{H}^*(X \times X)$ we have $[\Xi]_{i,i'}$ put at the ith and i'th factors. For $f \in R^{W_{\mu_i}}$, let $[f]_i$ be f put at the factor $R^{W_{\mu_i}}$.

Let $I^\mu_G \subset \widetilde{C}^\mu_G$ be the ideal generated by the following elements for varying homogeneous $f \in R^W$ of degree $2d > 2$ and $i = 1, 2, \cdots, r$

$$\begin{aligned} D_i(f) \quad &:= \quad [f]_i + \frac{\zeta'_X(d)}{\log(q)\zeta_X(d)}[\xi]_i \cdot [\partial_{\mu_i} f]_i \qquad (8.1.3)\\ &- \quad \sum_{i'=i+1}^{r} \left([\Xi_d]_{i,i'} \cdot [\partial_{\mu_{i'}} f]_{i'} + \frac{1-g}{q^{d-1}-1}[\xi]_i \otimes [\xi]_{i'} \cdot [\partial^2_{\mu_{i'}} f]_{i'} \right)\\ &+ \quad \sum_{i'=1}^{i-1} \left([\Xi_{1-d}]_{i',i} \cdot [\partial_{\mu_{i'}} f]_{i'} + \frac{1-g}{q^{1-d}-1}[\xi]_{i'} \otimes [\xi]_i \cdot [\partial^2_{\mu_{i'}} f]_{i'} \right). \end{aligned}$$

Note that the above element is an eigenvector under Frobenius with eigenvalue q^d. In particular, I^μ_G is stable under Frobenius.

Definition 8.1.6. The *phantom tautological ring* for Sht^μ_G is the quotient ring

$$C^\mu_G := \widetilde{C}^\mu_G / I^\mu_G$$

equipped with the action of Frob.

To write $D_i(f)$ in more manageable way, we introduce the following element in $\widetilde{C}^\mu_G$, for $f \in R^W$

$$\delta_i(f) = [\partial_{\mu_i} f]_i + \frac{1}{2}[c_1(TX)]_i[\partial^2_{\mu_i} f]_i = [\partial_{\mu_i} f]_i + (1-g)[\xi]_i[\partial^2_{\mu_i} f]_i.$$

We have introduced the notation $[-]_{i,i'}$ for $1 \le i < i' \le r$. For $i = i'$, define $[-]_{i,i}$ to be the composition

$$[-]_{i,i} : \mathrm{H}^*(X \times X) \xrightarrow{\Delta^*} \mathrm{H}^*(X) \xrightarrow{[-]_i} \widetilde{C}^\mu_G.$$

Using this notation and Lemma 8.1.4, we have

$$[\Xi_d]_{i,i} = [\Delta^*_X \Xi_d]_i = -\frac{\zeta'_X(d)}{\log(q)\zeta_X(d)}[\xi]_i.$$

Using these notations, we can rewrite $D_i(f)$ as

$$D_i(f) = [f]_i - \sum_{i'=i}^{r} [\Xi_d]_{i,i'} \delta_{i'}(f) + \sum_{i'=1}^{i-1} [\Xi_{1-d}]_{i',i} \delta_{i'}(f). \qquad (8.1.4)$$

Theorem 8.1.7. *The homomorphism* (8.1.1) *factors through the phantom tautological quotient* C^μ_G

$$\rho : C^\mu_G \to \mathrm{H}^*(\mathrm{Sht}^\mu_G). \qquad (8.1.5)$$

Restricting to each component of Sht^μ_G, we get a ring homomorphism from C^μ_G to the tautological ring of ${}^\omega\mathrm{Sht}^\mu_G$.

Remark 8.1.8. We will see from Proposition 8.3.1 that as a *module* over $\mathrm{H}^*(X^r)$ (but not as a ring), C^μ_G is isomorphic to $\bigotimes_{i=1}^r (\mathrm{H}^*(X \times G/P_{\mu_i}))$. The map ρ is an analog of the same-named map for Hermitian locally symmetric spaces in (1.1.3).

Remark 8.1.9. The map ρ is not injective: as we will see from Corollary 8.3.3, the top non-vanishing degree of C^μ_G is in degree $2r + 2\sum_{i=1}^r \dim G/P_{\mu_i} = 2\dim \mathrm{Sht}^\mu_G$. However, since Sht^μ_G is not proper, the top degree cohomology vanishes. However, we have reasons to believe that C^μ_G is the "correct" tautological ring for Sht^μ_G, supplying phantom cohomology classes that become zero in Sht^μ_G due to non-properness. This point of view can be used to give a meaning to the arithmetic volume calculation – see §8.4 and §8.4.11.

Corollary 8.1.10. *Let $\overline{\eta}_r$ be a geometric generic point of X^r. The restriction of the homomorphism ρ to the geometric generic fiber $\mathrm{Sht}^{\mu}_{G,\overline{\eta}_r}$ vanishes on $[f]_i$ for all $f \in R^W_+$ and $1 \le i \le r$.*

In particular, ρ induces a homomorphism $\rho_{\overline{\eta}_r}$:

$$\rho_{\overline{\eta}_r} : \bigotimes_{i=1}^{r} \mathrm{H}^*(G/P_{\mu_i}) \to \mathrm{H}^*(\mathrm{Sht}^{\mu}_{G,\overline{\eta}_r}). \tag{8.1.6}$$

Proof. The restriction to a geometric generic point of every term other than $[f]_i$ in the relation (8.1.3) obviously vanishes. □

8.2. **Proof of Theorem 8.1.7.** We need to check that elements of the form (8.1.3) are sent to zero under (8.1.1). Now we will abuse the notation to denote by $\delta_i(f)$ its image in $\mathrm{H}^*(\mathrm{Sht}^{\mu}_G)$, i.e.,

$$\delta_i(f) = \mathrm{ev}_i^*(\partial_{\mu_i} f) + \frac{1}{2} p_i^*(c_1(TX))\mathrm{ev}_i^*(\partial^2_{\mu_i} f).$$

For $1 \le i \le i' \le r$, we also use $[-]_{i,i'}$ to denote the pullback along $(p_i, p_{i'}) : \mathrm{Sht}^{\mu}_G \to X \times X$

$$[-]_{i,i'} : \mathrm{H}^*(X \times X) \to \mathrm{H}^*(\mathrm{Sht}^{\mu}_G).$$

Thus $[\Xi]_{i,i'} \in \mathrm{H}^*(\mathrm{Sht}^{\mu}_G)$ is the image of $[\Xi]_{i,i'} \in \widetilde{C}^{\mu}_G$ under $\widetilde{\rho}$. For $i = i'$, $[-]_{i,i} = p_i^* \Delta_X^*(-)$.

Thus we need to check for $f \in R^W$ of degree $2d > 2$ and $1 \le i \le r$ that

$$\mathrm{ev}_i^* f = \sum_{i'=i}^{r} [\Xi_d]_{i,i'} \delta_{i'}(f) - \sum_{i'=1}^{i-1} [\Xi_{1-d}]_{i',i} \delta_{i'}(f). \tag{8.2.1}$$

8.2.1. *The case $i = 1$.* We first check (8.2.1) for $i = 1$, in which case it reads

$$\mathrm{ev}_1^* f = \sum_{i'=1}^{r} [\Xi_d]_{1,i'} \delta_{i'}(f). \tag{8.2.2}$$

We use the same notation h_i, p_i and ev_i to denote the counterparts of h_i, p_i and ev_i as maps from Hk^{μ}_G. Recall that for $\mathcal{F}_\bullet \in \mathrm{Hk}^{\mu}_G$, $\mathrm{ev}_i(\mathcal{F}_\bullet)$ is the canonical P_{μ_i}-reduction of $\mathcal{F}_{i-1}$. This gives a commutative diagram for $1 \le i \le r$

$$\begin{array}{ccc} \mathrm{Hk}^{\mu}_G & \xrightarrow{\mathrm{ev}_i} & \mathbb{B}P_{\mu_i} \\ \downarrow{\scriptstyle (p_i, h_{i-1})} & & \downarrow \\ X \times \mathrm{Bun}_G & \xrightarrow{\mathrm{ev}} & \mathbb{B}G \end{array}$$

The right vertical map is induced by the inclusion $P_{\mu_i} \hookrightarrow G$. Therefore for $f \in R^W = \mathrm{H}^*(\mathbb{B}G)$, we have

$$(p_i, h_{i-1})^* \mathrm{ev}^* f = \mathrm{ev}_i^* f.$$

By Proposition 5.5.8, we have for any $f \in R^W$, $z \in \mathrm{H}_*(X)$ and $1 \le i \le r$

$$h_i^*(f^z) - h_{i-1}^*(f^z) = p_i^* \mathrm{PD}(z) \mathrm{ev}_i^*(\partial_{\mu_i} f) + (1-g)\langle z, \xi \rangle p_i^* \xi \cdot \mathrm{ev}_i^*(\partial^2_{\mu_i} f) = p_i^* \mathrm{PD}(z) \cdot \delta_i(f).$$

Adding these up for $i = 1, \cdots, r$ we get

$$h_r^*(f^z) - h_0^*(f^z) = \sum_{i=1}^{r} p_i^* \mathrm{PD}(z) \cdot \delta_i(f). \tag{8.2.3}$$

Now pulling back to Sht^{μ}_G and using that $h_r = \mathrm{Frob}_{\mathrm{Bun}_G} \circ h_0$, we get

$$(\phi - 1)(h_0^*(f^z)) = h_0^*(\phi(f^z) - f^z) = \sum_{i=1}^{r} p_i^* \mathrm{PD}(z) \cdot \delta_i(f). \tag{8.2.4}$$

Therefore

$$h_0^*(f^z) = (\phi - 1)^{-1} \sum_{i=1}^{r} p_i^* \mathrm{PD}(z) \cdot \delta_i(f). \tag{8.2.5}$$

Using that ϕ acts on $\delta_i(f)$ by q^{d-1} we get

$$h_0^*(f^z) = \sum_{i=1}^r p_i^* \left((q^{d-1}\phi - 1)^{-1}\mathrm{PD}(z)\right) \cdot \delta_i(f).$$

Choose a basis $\{z_j\}$ of $\mathrm{H}_*(X)$ with dual basis $\{\zeta^j\}$ of $\mathrm{H}^*(X)$, we get

$$\mathrm{ev}_1^* f = \sum_j \zeta^j h_0^*(f^{z_j}) = \sum_{i=1}^r \left[\sum_j \zeta^j \otimes (q^{d-1}\phi - 1)^{-1}\mathrm{PD}(z_j)\right]_{1,i} \delta_i(f). \tag{8.2.6}$$

Observe that

$$\sum_j \zeta^j \otimes (q^{d-1}\phi - 1)^{-1}\mathrm{PD}(z_j) = \left(\mathrm{id} \otimes \frac{1}{q^{d-1}\phi - 1}\right)[\Delta_X] = \Xi_d.$$

Thus (8.2.6) implies (8.2.2).

8.2.2. *Partial Frobenius and general i.* To deduce the general case from $i = 1$ case, we consider partial Frobenius on Sht_G^μ. Let $\mu' = (\mu_2, \cdots, \mu_r, \mu_1)$ be obtained from μ by a cyclic permutation. We have the partial Frobenius map

$$\mathfrak{P} : \mathrm{Sht}_G^\mu \to \mathrm{Sht}_G^{\mu'}$$

sending $(x_1, \cdots, x_r, \mathcal{F}_0, \cdots, \mathcal{F}_r = {}^\tau\mathcal{F}_0)$ to $(x_2, \cdots, x_r, \mathrm{Frob}(x_1), \mathcal{F}_1, \cdots, \mathcal{F}_r, \mathcal{F}_{r+1} = {}^\tau\mathcal{F}_1)$. We introduce the following notation: for $i' > r$ we define $\mu_{i'} = \mu_{i'-r}$, and

$$\begin{aligned} p_{i'} &= \mathrm{Frob}_X \circ p_{i'-r} : \mathrm{Sht}_G^\mu \to X, \\ h_{i'} &= \mathrm{Frob} \circ h_{i'-r} : \mathrm{Sht}_G^\mu \to \mathrm{Bun}_G, \\ \mathrm{ev}_{i'} &= \mathrm{Frob} \circ \mathrm{ev}_{i'-r} : \mathrm{Sht}_G^\mu \to \mathbb{B}P_{\mu_{i'}}. \end{aligned}$$

Using these notations we can define $\delta_{i'}(f)$ and $[-]_{i,i'}$ when $i' > r$. Therefore, for $r < i' < i + r$, we have

$$\delta_{i'}(f) = \phi(\delta_{i'-r}(f)), \quad [\alpha \otimes \beta]_{i,i'} = [\phi(\beta) \otimes \alpha]_{i'-r,i}, \quad \alpha, \beta \in \mathrm{H}^*(X). \tag{8.2.7}$$

From the definition it is easy to see equalities/canonical isomorphisms

$$p_i \circ \mathfrak{P} = p_{i+1}, \quad h_i \circ \mathfrak{P} \cong h_{i+1}, \quad \mathrm{ev}_i \circ \mathfrak{P} \cong \mathrm{ev}_{i+1}. \tag{8.2.8}$$

Thus for $i \le r < i' < i + r$

$$\delta_{i+1}(f) = \mathfrak{P}^* \delta_i(f), \quad [-]_{i+1,i'+1} = \mathfrak{P}^*[-]_{i,i'}. \tag{8.2.9}$$

Applying partial Frobenius $\mathfrak{P}^*$ to (8.2.2) $(i-1)$ times, using (8.2.9), we get for any $i \ge 1$

$$\mathrm{ev}_i^* f = \sum_{i'=i}^{r+i-1} [\Xi_d]_{i,i'} \delta_{i'}(f). \tag{8.2.10}$$

Using (8.2.7) we rewrite the terms involving $i' > r$ using $i' - r$, at the cost of an extra ϕ, we get for $r < i' < i + r$

$$\begin{aligned} [\Xi_d]_{i,i'}\delta_{i'}(f) &= \left[(\mathrm{id} \otimes \frac{1}{q^{d-1}\phi - 1})\Delta_X\right]_{i,i'} \delta_{i'}(f) = \left[(\frac{q^{d-1}\phi}{q^{d-1}\phi - 1} \otimes \mathrm{id})\Delta_X\right]_{i'-r,i} \delta_{i'-r}(f) \\ &= \left[(\frac{1}{1 - q^{1-d}\phi^{-1}} \otimes \mathrm{id})\Delta_X\right]_{i'-r,i} \delta_{i'-r}(f) = -[\Xi_{1-d}]_{i'-r,i}\delta_{i'-r}(f). \end{aligned}$$

Plugging into (8.2.10) we get (8.2.1). This finishes the proof of Theorem 8.1.7. □

Corollary 8.2.3. *The image of the ring homomorphism* (8.1.1) *contains the images of the pullback maps* $h_i^* : \mathrm{H}^*(\mathrm{Bun}_G) \to \mathrm{H}^*(\mathrm{Sht}_G^\mu)$ *for* $0 \le i \le r$.

Proof. For $i = 0$ the statement follows from (8.2.5). For general i, applying $\mathfrak{P}^*$ to both sides of (8.2.5) for i times, we get

$$h_i^*(f^z) = \sum_{i'=i+1}^{i+r} p_{i'}^* \left((q^{d-1}\phi - 1)^{-1}\mathrm{PD}(z)\right) \cdot \delta_{i'}(f).$$

Using (8.2.7) we rewrite the terms involving $i' > r$ using $i' - r$, and get

$$h_i^*(f^z) = \sum_{i'=i+1}^{r} p_{i'}^* \left(\frac{1}{q^{d-1}\phi - 1}\mathrm{PD}(z)\right) \cdot \delta_{i'}(f) + \sum_{i'=1}^{i} p_{i'}^* \left(\frac{q^{d-1}\phi}{q^{d-1}\phi - 1}\mathrm{PD}(z)\right) \cdot \delta_{i'}(f).$$

The right side is visibly in the image of (8.1.1). □

8.3. **Structure of the phantom tautological ring.** We show that C_G^μ is a flat deformation of $\otimes_{i=1}^r \mathrm{H}^*(G/P_{\mu_i})$ over the Artinian ring $\mathrm{H}^*(X^r)$. We will equip C_G^μ with a volume form that is compatible with our ad-hoc definition of $\int_{\mathrm{Sht}_G}$, and show that C_G^μ satisfies Poincaré duality under the volume form.

Proposition 8.3.1. *The phantom tautological ring C_G^μ is a free module over $\mathrm{H}^*(X^r)$, with a canonical* Frob^**-equivariant isomorphism*

$$C_G^\mu / \mathrm{H}^{>0}(X^r) \cdot C_G^\mu \cong \otimes_{i=1}^r \mathrm{H}^*(G/P_{\mu_i}). \tag{8.3.1}$$

Proof. We first check (8.3.1). The relations in I_G^μ modulo $\mathrm{H}^{>0}(X^r)$ become $[f]_i = 0$ for all $1 \le i \le r$ and $f \in R_+^W$. Therefore we get a canonical isomorphism

$$C_G^\mu / \mathrm{H}^{>0}(X^r) \cdot C_G^\mu \cong \otimes_{i=1}^r R^{W_{\mu_i}} / (R_+^W) \cong \otimes_{i=1}^r \mathrm{H}^*(G/P_{\mu_i}). \tag{8.3.2}$$

Now we show that C_G^μ is free over $\mathrm{H}^*(X^r)$. By Lemma 8.3.2 below, if we choose a set of homogeneous generators $f_1, \cdots, f_n$ for R^W, then I_G^μ is generated by

$$\{D_i(f_j)\}_{1 \le i \le r, 1 \le j \le n}. \tag{8.3.3}$$

We claim that the collection of elements (8.3.3) form a regular sequence in $\widetilde{C}_G^\mu$ in any order. Indeed, the Krull dimension of $\widetilde{C}_G^\mu$ is nr and there are nr elements in (8.3.3). The ideal they generate is I_G^μ and the quotient $C_G^\mu = \widetilde{C}_G^\mu / I_G^\mu$ is Artinian because it is finite over the Artinian ring $\mathrm{H}^*(X^r)$ by (8.3.2). Therefore (8.3.3) form a regular sequence in $\widetilde{C}_G^\mu$.

Let $P_t(V)$ denote the Hilbert polynomial of a graded vector space V, and $P_t(Y)$ denote the Poincaré polynomial of $\mathrm{H}^*(Y)$ for a stack Y. Let $2d_i$ be the degree of f_i. Since (8.3.3) is a regular sequence in $\widetilde{C}_G^\mu$, by using the Koszul complex we see that the Hilbert polynomial of C_G^μ is

$$P_t(C_G^\mu) = P_t(\widetilde{C}_G^\mu) \prod_{j=1}^n (1 - t^{2d_j})^r.$$

Using that $P_t(\mathbb{B}G) = \prod_{j=1}^n (1 - t^{2d_j})^{-1}$, we can write

$$P_t(C_G^\mu) = \prod_{i=1}^r \left(P_t(\mathbb{B}P_{\mu_i}) P_t(X) P_t(\mathbb{B}G)^{-1}\right).$$

Since $\mathrm{H}^*(\mathbb{B}P_{\mu_i})$ is free over $\mathrm{H}^*(\mathbb{B}G)$, we have $P_t(\mathbb{B}P_{\mu_i})/P_t(\mathbb{B}G) = P_t(G/P_{\mu_i})$, hence

$$P_t(C_G^\mu) = \prod_{i=1}^r P_t(X \times G/P_{\mu_i}).$$

In particular, taking $t = 1$ we get the total dimension

$$\dim C_G^\mu = \dim \mathrm{H}^*(X^r) \prod_{i=1}^r \dim \mathrm{H}^*(G/P_{\mu_i}). \tag{8.3.4}$$

By (8.3.2) and Nakayama's lemma, C_G^μ is a $\mathrm{H}^*(X^r)$-module generated by at most $\prod_{i=1}^r \dim \mathrm{H}^*(G/P_{\mu_i})$ elements. In view of the dimension equality (8.3.4), C_G^μ has to be free over $\mathrm{H}^*(X^r)$ with rank $\prod_{i=1}^r \dim \mathrm{H}^*(G/P_{\mu_i})$. □

Below are technical lemmas needed in the proof of Proposition 8.3.1. Recall the elements $D_i(f)$ (for $f \in R^W$ and $1 \le i \le r$) from (8.1.3) or (8.1.4).

Lemma 8.3.2. *For any two homogeneous elements $f, g \in R^W$, and $1 \le i \le r$, we have*

$$D_i(fg) \in (D_1(f), \cdots, D_r(f), D_1(g), \cdots, D_r(g)) \subset \widetilde{C}^{\mu}_G$$

where the right side means the ideal generated by the listed elements in $\widetilde{C}^{\mu}_G$.

Proof. Let $I_{f,g}$ be the ideal generated by $\{D_1(f), \cdots, D_r(f), D_1(g), \cdots, D_r(g)\}$ in $\widetilde{C}^{\mu}_G$. We first prove the statement for $i = 1$. To show $D_1(fg) \in I_{f,g}$, we need to show

$$[fg]_1 \equiv \sum_{i=1}^{r} [\Xi_{d+e}]_{1,i} \delta_i(fg) \mod I_{f,g}. \tag{8.3.5}$$

Note that

$$\delta_i(fg) = \delta_i(f)[g]_i + [f]_i \delta_i(g) + [c_1(TX)]_i \delta_i(f)\delta_i(g). \tag{8.3.6}$$

Let d and e be the degrees of f and g. Using (8.3.6), (8.3.5) is equivalent to

$$[f]_1[g]_1 \equiv \sum_{i=1}^{r} [\Xi_{d+e}]_{1,i} \left(\delta_i(f)[g]_i + [f]_i \delta_i(g) + [c_1(TX)]_i \delta_i(f)\delta_i(g)\right) \mod I_{f,g}.$$

Now we use the definition of $D_i(f)$ to replace $[f]_i$ above by expressions involving $D_i(f)$ and $\delta_{i'}(f)$, it suffices to show

$$\begin{aligned}
& \sum_{i=1}^{r} [\Xi_d]_{1,i} \delta_i(f) \sum_{i=1}^{r} [\Xi_e]_{1,i} \delta_i(g) \\
\equiv \quad & \sum_{i=1}^{r} [\Xi_{d+e}]_{1,i} \delta_i(f) \left(\sum_{i' \ge i} [\Xi_e]_{i,i'} \delta_{i'}(g) - \sum_{i' < i} [\Xi_{1-e}]_{i',i} \delta_{i'}(g) \right) \\
+ \quad & \sum_{i=1}^{r} [\Xi_{d+e}]_{1,i} \left(\sum_{i' \ge i} [\Xi_d]_{i,i'} \delta_{i'}(f) - \sum_{i' < i} [\Xi_{1-d}]_{i',i} \delta_{i'}(f) \right) \delta_i(g) \\
+ \quad & \sum_{i=1}^{r} [\Xi_{d+e}]_{1,i} [c_1(TX)]_i \delta_i(f)\delta_i(g) \mod I_{f,g}.
\end{aligned}$$

Comparing coefficients of $\delta_i(f)\delta_{i'}(g)$ on both sides, it reduces to showing

$$[\Xi_d]_{1,i}[\Xi_e]_{1,i'} = \begin{cases} [\Xi_{d+e}]_{1,i}[\Xi_e]_{i,i'} - [\Xi_{d+e}]_{1,i'}[\Xi_{1-d}]_{i,i'} & i < i' \\ -[\Xi_{d+e}]_{1,i}[\Xi_{1-e}]_{i',i} + [\Xi_{d+e}]_{1,i'}[\Xi_d]_{i',i} & i > i' \\ [\Xi_{d+e}]_{1,i}\left([\Xi_e]_{i,i} + [\Xi_d]_{i,i} + [c_1(TX)]_i\right) & i = i'. \end{cases} \tag{8.3.7}$$

Let us prove the above identity in the case $i < i'$. Write $x = q^{d-1}$ and $y = q^{e-1}$, the identity can be written as

$$\begin{aligned}
& \left(1 \otimes \frac{1}{\phi x - 1}\right) [\Delta]_{1,i} \left(1 \otimes \frac{1}{\phi y - 1}\right) [\Delta]_{1,i'} \\
= \quad & \left(1 \otimes \frac{1}{q\phi xy - 1}\right) [\Delta]_{1,i} \left(1 \otimes \frac{1}{\phi y - 1}\right) [\Delta]_{i,i'} \\
+ \quad & \left(1 \otimes \frac{1}{q\phi xy - 1}\right) [\Delta]_{1,i'} \left(\frac{\phi x}{\phi x - 1} \otimes 1\right) [\Delta]_{i,i'}.
\end{aligned} \tag{8.3.8}$$

Expand both sides in geometric series in x and y, the coefficient of $x^a y^b$ on the left side is

$$(1 \otimes \phi^a)[\Delta]_{1,i}(1 \otimes \phi^b)[\Delta]_{1,i'} = (1 \otimes \phi^a \otimes \phi^b)[\Delta]_{1,i,i'}. \tag{8.3.9}$$

Here we use that $[\Delta]_{1,i}[\Delta]_{1,i'}$ is the class of the small diagonal in $X \times X \times X$ (the $1, i, i'$-factors), which we denote by $[\Delta]_{1,i,i'}$. The coefficient of $x^a y^b$ on the right side of (8.3.8) is

$$\begin{cases} (1 \otimes q^b\phi^b)[\Delta]_{1,i'}(\phi^{a-b} \otimes 1)[\Delta]_{i,i'} & a > b \\ (1 \otimes q^a\phi^a)[\Delta]_{1,i}(1 \otimes \phi^{b-a})[\Delta]_{i,i'} & a \leq b. \end{cases} \tag{8.3.10}$$

When $a > b$, we have

$$(1\otimes q^b\phi^b)[\Delta]_{1,i'}(\phi^{a-b}\otimes 1)[\Delta]_{i,i'} = q^b(1\otimes\phi^b)[\Delta]_{1,i'}(\phi^{a-b}\otimes 1)[\Delta]_{i,i'} = (1\otimes\phi^b)[\Delta]_{1,i'}(\phi^{a-b}\otimes 1)(\phi\otimes\phi)^b[\Delta]_{i,i'}.$$

Here we use that $(\phi \otimes \phi)[\Delta]_{i,i'} = \phi[\Delta]_{i,i'} = q[\Delta]_{i,i'}$, because $[\Delta]$ is a divisor class. Therefore

$$(1 \otimes q^b\phi^b)[\Delta]_{1,i'}(\phi^{a-b} \otimes 1)[\Delta]_{i,i'} = (1 \otimes \phi^b)[\Delta]_{1,i'}(\phi^a \otimes \phi^b)[\Delta]_{i,i'} = (1 \otimes \phi^a \otimes \phi^b)[\Delta]_{1,i,i'},$$

which is equal to (8.3.9). Here we use that ϕ^a is a ring endomorphism of $\mathrm{H}^*(X)$. This shows that (8.3.10) and (8.3.9) are equal when $a > b$. The proof of their equality when $a \leq b$ is similar. Thus we have verified the $i < i'$ case of (8.3.7).

The case $i > i'$ of (8.3.7) is proved similarly.

Finally let us prove the $i = i'$ case of (8.3.7). When $i = 1 = i'$ both sides are zero. We therefore assume $i = i' > 1$. Then the equality (8.3.7) becomes

$$\left(1 \otimes \frac{1}{\phi x - 1}\right)[\Delta]_{1,i}\left(1 \otimes \frac{1}{\phi y - 1}\right)[\Delta]_{1,i} = \left(1 \otimes \frac{1}{q\phi xy - 1}\right)[\Delta]_{1,i}\cdot \mathrm{Tr}\left(\frac{1}{\phi x - 1} + \frac{1}{\phi y - 1} + 1 \Big| \mathrm{H}^*(X)\right)[\xi]_i, \tag{8.3.11}$$

Since ϕ and $q\phi^{-1}$ are adjoint under the Poincaré duality pairing on $\mathrm{H}^*(X)$, we have

$$\left(1 \otimes \frac{1}{\phi y - 1}\right)[\Delta]_{1,i} = \left(\frac{1}{q\phi^{-1}y - 1} \otimes 1\right)[\Delta]_{1,i}.$$

Therefore the left side of (8.3.11) is

$$\left(1 \otimes \frac{1}{\phi x - 1}\right)[\Delta]_{1,i}\left(\frac{1}{q\phi^{-1}y - 1} \otimes 1\right)[\Delta]_{1,i} = \mathrm{Tr}\left(\frac{1}{q\phi^{-1}y - 1} \cdot \frac{1}{\phi x - 1}\Big|\mathrm{H}^*(X)\right)[\xi \otimes \xi]_{1,i}.$$

We also have

$$\left(1 \otimes \frac{1}{q\phi xy - 1}\right)[\Delta]_{1,i} \cdot [\xi]_i = \frac{1}{qxy - 1}[\xi \otimes \xi]_{1,i}.$$

Therefore (8.3.11) is equivalent to the equality

$$\mathrm{Tr}\left(\frac{1}{q\phi^{-1}y - 1} \cdot \frac{1}{\phi x - 1}\right) = \frac{1}{qxy - 1}\mathrm{Tr}\left(\frac{1}{\phi x - 1} + \frac{1}{\phi y - 1} + 1\right). \tag{8.3.12}$$

where Tr means alternating trace on $\mathrm{H}^*(X)$. Again we may change $\frac{1}{\phi y - 1}$ on the right side above by its adjoint $\frac{1}{q\phi^{-1}y-1}$, and (8.3.12) follows from the equality of endomorphisms of $\mathrm{H}^*(X)$ before taking trace

$$\frac{1}{q\phi^{-1}y - 1} \cdot \frac{1}{\phi x - 1} = \frac{1}{qxy - 1}\left(\frac{1}{\phi x - 1} + \frac{1}{q\phi^{-1}y - 1} + 1\right).$$

This finishes the proof of the lemma for $i = 1$.

Now consider the case of general i. Let $\mu' = (\mu_2, \mu_3, \cdots, \mu_r, \mu_1)$ (so that $\mu'_i = \mu_{i+1}$, with the subscripts understood mod r). We define a partial Frobenius

$$\mathfrak{P}^* : \widetilde{C}_G^{\mu'} \to \widetilde{C}_G^{\mu} \tag{8.3.13}$$

that maps $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_r$ to $\phi(\alpha_r) \otimes \alpha_1 \otimes \cdots \otimes \alpha_{r-1}$, where $\alpha_i \in \mathrm{H}^*(X \times G/P_{\mu'_i})$. It is easy to check that $D_2(f) = \mathfrak{P}^*(D_1(f)) \in \widetilde{C}_G^{\mu}$, where $D_1(f)$ is viewed as an element of $\widetilde{C}_G^{\mu'}$. We have proved that $D_1(fg) \in I'_{f,g} := (D_1(f), \cdots, D_r(f), D_1(g), \cdots, D_r(g)) \subset \widetilde{C}_G^{\mu'}$. Applying $\mathfrak{P}^*$, we see that

$$D_2(fg) = \mathfrak{P}^* D_1(fg) \in \mathfrak{P}^* I'_{f,g} = I_{f,g} \subset \widetilde{C}_G^{\mu}.$$

Repeating this argument we see that $D_i(fg) \in I_{f,g}$ for all $i = 2, \cdots, r$. This finishes the proof of the lemma. □

Corollary 8.3.3. *Recall that* $D_{\mu_i} = \dim G/P_{\mu_i}$*, and write* $N := \dim \mathrm{Sht}^{\mu}_G = \sum_{i=1}^r (D_{\mu_i}+1)$*. Then the top non-vanishing degree of* C^{μ}_G *is* $2N$*, and* $\dim(C^{\mu}_G)_{2N} = 1$*. There is a canonical isomorphism*

$$\bigotimes_{i=1}^r (\mathrm{H}^2(X)\otimes \mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i})) \xrightarrow{\sim} (C^{\mu}_G)_{2N} \tag{8.3.14}$$

sending $\otimes(\xi\otimes\eta'_i)$ *(where* $\eta'_i \in \mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i})$*) to the image of* $\otimes(\xi\otimes\widetilde{\eta}'_i) \in \widetilde{C}^{\mu}_G$ *for any lifting* $\widetilde{\eta}'_i \in R^{W_{\mu_i}}$ *of* η'_i*.*

8.4. **The volume functional.** We consider two linear functionals on $(C^{\mu}_G)_{2N}$.

8.4.1. By the description of the top degree component of C^{μ}_G in (8.3.14), it carries a standard linear functional given by pairing with the fundamental class of $\prod_{i=1}^r (X\times G/P_{\mu_i})$

$$\int_{\prod_{i=1}^r (X\times G/P_{\mu_i})} : (C^{\mu}_G)_{2N} \cong \bigotimes_{i=1}^r (\mathrm{H}^2(X)\otimes \mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i})) \to \overline{\mathbf{Q}}_\ell(-N).$$

This is an isomorphism.

8.4.2. Pick an embedding $\iota : \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$. By the convergence result proved in Proposition 5.5.3, it makes sense to define a $\mathbf{C}$-valued linear map

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_G, -) \quad : \quad (\widetilde{C}^{\mu}_G)_{2N} \to \mathbf{C} \tag{8.4.1}$$

$$\theta \mapsto \mathrm{Tr}_\iota(\phi^{-1}\circ \Gamma_{\mu}^{\widetilde{\rho}(\theta)} | \mathrm{H}^*(\mathrm{Bun}^{\omega}_G)). \tag{8.4.2}$$

Later in Proposition 8.4.9, we will show that $\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_G, -)$ takes values in $\overline{\mathbf{Q}}_\ell$ and is independent of the choice of ι.

We will first prove:

Proposition 8.4.3. *The linear functional* $\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_G, -)$ *on* $(\widetilde{C}^{\mu}_G)_{2N}$ *factors through the quotient* $(C^{\mu}_G)_{2N}$*.*

The proof will be given after some preparations.

Lemma 8.4.4. *Let* $V = \oplus_{n\in\mathbb{Z}} V_n$ *and* $W = \oplus_{n\in\mathbb{Z}} W_n$ *be two graded* $\mathbf{C}$*-vector spaces such that* $\dim V_n$ *and* $\dim W_n$ *are finite for all* n *and zero for all* $n \ll 0$*. Let* $A : V \to W$ *be a linear map of degree* d*, and* $B : W \to V$ *be a linear map of degree* $-d$*, where* $d \in \mathbb{Z}$*.*

(1) *The series* $\sum_n \mathrm{Tr}(BA|V_n)$ *is absolutely convergent if and only if the series* $\sum_n \mathrm{Tr}(AB|W_n)$ *is absolutely convergent.*
(2) *When the condition in part (1) is satisfied, we have*

$$\sum_n (-1)^n \mathrm{Tr}(BA|V_n) = (-1)^d \sum_n (-1)^n \mathrm{Tr}(AB|W_n).$$

Proof. For any $n\in\mathbb{Z}$ we have maps $A|_{V_n} : V_n \to W_{n+d}$ and $B|_{W_{n+d}} : W_{n+d} \to V_n$. Therefore

$$\mathrm{Tr}(BA|V_n) = \mathrm{Tr}(AB|W_{n+d}).$$

The rest of the statements follow easily. □

Let $\nu := (\mu_2, \cdots, \mu_r)$. Note that

$$\mathrm{Hk}^{\mu}_G = \mathrm{Hk}^{\mu_1}_G \times_{\mathrm{Bun}_G} \mathrm{Hk}^{\nu}_G \tag{8.4.3}$$

Let $h^{\nu}_i : \mathrm{Hk}^{\nu}_G \to \mathrm{Bun}_G$ $(1\le i\le r)$ be the projections; let $h^{\mu_1}_0, h^{\mu_1}_1 : \mathrm{Hk}^{\mu_1}_G \to \mathrm{Bun}_G$ be the two projections. The fiber product in (8.4.3) uses $h^{\mu_1}_1$ and h^{ν}_1.

Suppose $\theta_1 \in \mathrm{H}^*(\mathrm{Hk}^{\mu_1}_G)$ and $\theta' \in \mathrm{H}^*(\mathrm{Hk}^{\nu}_G)$. We denote by $\theta_1\theta' \in \mathrm{H}^*(\mathrm{Hk}^{\mu}_G)$ the cup product of the pullbacks of θ_1 and θ' to Hk^{μ}_G via the two projections from Hk^{μ}_G using (8.4.3).

Let $\mu' = (\mu_2, \mu_3, \cdots, \mu_r, \mu_1)$ and $\omega' = \omega + \overline{\mu}_1$. We have

$$\mathrm{Hk}^{\mu'}_G = \mathrm{Hk}^{\nu}_G \times_{\mathrm{Bun}_G} \mathrm{Hk}^{\mu_1}_G \tag{8.4.4}$$

using the maps $h^{\nu}_r : \mathrm{Hk}^{\nu}_G \to \mathrm{Bun}_G$ and $h^{\mu_1}_0 : \mathrm{Hk}^{\mu_1}_G \to \mathrm{Bun}_G$. Similarly we view $\theta'\theta_1$ as a cohomology class on $\mathrm{Hk}^{\mu'}_G$ via the cup product of the pullbacks along the projections in (8.4.4).

Lemma 8.4.5. *Let $\theta_1 \in \mathrm{H}^*(\mathrm{Hk}_G^{\mu_1})$ and $\theta' \in \mathrm{H}^*(\mathrm{Hk}_G^{\nu})$ be homogeneous elements so that $\theta_1\theta'$ has degree $2N$. Then we have*

$$\mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu}^{\phi(\theta_1)\theta'} | \mathrm{H}^*(\mathrm{Bun}_G^{\omega})) = (-1)^{|\theta_1|} q^{D_{\mu_1}+1} \mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu'}^{\theta'\theta_1} | \mathrm{H}^*(\mathrm{Bun}_G^{\omega})).$$

Proof. Below all traces mean traces of endomorphisms of $\mathrm{H}^*(\mathrm{Bun}_G^{\omega})$. Observe that $\Gamma_{\mu}^{\theta_1\theta'} = \Gamma_{\mu_1}^{\theta_1} \circ \Gamma_{\nu}^{\theta'}$, therefore by the absolute convergence of trace proved in Proposition 5.5.3 together with Lemma 8.4.4,

$$\mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu}^{\phi(\theta_1)\theta'}) = \mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu_1}^{\phi(\theta_1)} \circ \Gamma_{\nu}^{\theta'}) = (-1)^{|\theta_1|} \mathrm{Tr}(\Gamma_{\nu}^{\theta'} \circ \phi^{-1} \circ \Gamma_{\mu_1}^{\phi(\theta_1)}).$$

On the other hand, the same reasoning gives

$$\mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu'}^{\theta'\theta_1}) = \mathrm{Tr}(\phi^{-1} \circ \Gamma_{\nu}^{\theta'} \circ \Gamma_{\mu_1}^{\theta_1}) = \mathrm{Tr}(\Gamma_{\nu}^{\theta'} \circ \Gamma_{\mu_1}^{\theta_1} \circ \phi^{-1}).$$

Thus it suffices to show

$$\phi^{-1} \circ \Gamma_{\mu_1}^{\phi(\theta_1)} = q^{D_{\mu_1}+1} \Gamma_{\mu_1}^{\theta_1} \circ \phi^{-1} \in \mathrm{End}(\mathrm{H}^*(\mathrm{Bun}_G)).$$

Equivalently, we need to show

$$\Gamma_{\mu_1}^{\phi(\theta_1)} \circ \phi = q^{D_{\mu_1}+1} \phi \circ \Gamma_{\mu_1}^{\theta_1}. \tag{8.4.5}$$

Writing the two projections $h_0^{\mu_1}, h_1^{\mu_1} : \mathrm{Hk}_G^{\mu_1} \to \mathrm{Bun}_G$ simply as h_0 and h_1, we have for any $\alpha \in \mathrm{H}^*(\mathrm{Bun}_G^{\omega})$

$$\Gamma_{\mu_1}^{\phi(\theta_1)} \circ \phi(\alpha) = h_{0*}(h_1^*(\phi(\alpha)) \cup \phi(\theta_1)) = h_{0*}\phi(h_1^*\alpha \cup \theta_1).$$

On the other hand,

$$\phi \circ \Gamma_{\mu_1}^{\theta_1}(\alpha) = \phi h_{0*}(h_1^*\alpha \cup \theta_1).$$

Therefore (8.4.5) follows from the identity

$$h_{0*}\phi = q^{\dim h_0} \phi h_{0*} : \mathrm{H}^*(\mathrm{Hk}_G^{\mu_1}) \to \mathrm{H}^*(\mathrm{Bun}_G)$$

where both sides are adjoint to $h_0^*\phi = \phi h_0^*$ with respect to Poincaré duality. Here we use the fact that h_0 is a smooth projective fiber bundle of relative dimension $D_{\mu_1} + 1$. □

Recall the map $\mathfrak{P}^* : \widetilde{C}_G^{\mu'} \to \widetilde{C}_G^{\mu}$ from (8.3.13). We have the following corollary of Lemma 8.4.5.

Corollary 8.4.6. *For any $\theta \in (\widetilde{C}_G^{\mu'})_{2N}$, we have*

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}_G^{\mu}, \mathfrak{P}^*\theta) = q^{D_{\mu_1}+1} \mathrm{vol}({}^{\omega'}\mathrm{Sht}_G^{\mu'}, \theta).$$

Remark 8.4.7. Formally, the above formula is consistent with the fact that the partial Frobenius $\mathfrak{P} : {}^{\omega}\mathrm{Sht}_G^{\mu} \to {}^{\omega'}\mathrm{Sht}_G^{\mu'}$ has degree $q^{D_{\mu_1}+1}$.

8.4.8. *Proof of Proposition 8.4.3.* We need to show that, for any $f \in R_{2d}^W$, $\theta \in (\widetilde{C}_G^{\mu})_{2N-2d}$ and $1 \le i \le r$, the (graded) trace vanishes,

$$\mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu}^{D_i(f)\theta} | \mathrm{H}^*(\mathrm{Bun}_G^{\omega})) = 0. \tag{8.4.6}$$

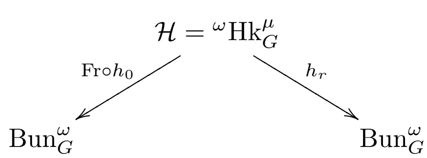
We first prove (8.4.6) in the case $i = 1$. For $z \in \mathrm{H}_{|z|}(X)$, write

$$\gamma_1(f^z) := h_0^*(\phi(f^z) - f^z) - \sum_{i=1}^{r} p_i^* \mathrm{PD}(z) \delta_i(f).$$

By the deduction from (8.2.4) to (8.2.6), $D_i(f)$ belongs to the ideal in $\mathrm{H}^*({}^{\omega}\mathrm{Hk}_G^{\mu})$ generated by $\gamma_1(f^z)$ for all z, therefore it suffices to show that for any $\theta' \in \mathrm{H}^{2N-2d+|z|}({}^{\omega}\mathrm{Hk}_G^{\mu})$, we have

$$\mathrm{Tr}(\phi^{-1} \circ \Gamma_{\mu}^{\gamma_1(f^z)\theta'} | \mathrm{H}^*(\mathrm{Bun}_G^{\omega})) = 0. \tag{8.4.7}$$

View ${}^{\omega}\mathrm{Hk}_G^{\mu}$ as a self-correspondence of Bun_G with the maps $\mathrm{Fr} \circ h_0$ and h_r:



Then $\phi^{-1}\circ\Gamma_\mu^\theta$ is the composition

$$\phi^{-1}\circ\Gamma_\mu^\theta : \mathrm{H}^*(\mathrm{Bun}_G^\omega)\xrightarrow{h_r^*}\mathrm{H}^*(\mathcal{H})\xrightarrow{\cup\theta}\mathrm{H}^*(\mathcal{H})\xrightarrow{(\mathrm{Fr}\circ h_0)_*}\mathrm{H}^*(\mathrm{Bun}_G^\omega).$$

From this we see that $\phi^{-1}\circ\Gamma_\mu^{\theta' h_r^*(f^z)}$ is the composition

$$\mathrm{H}^j(\mathrm{Bun}_G^\omega)\xrightarrow{\cup f^z}\mathrm{H}^{j+2d-|z|}(\mathrm{Bun}_G^\omega)\xrightarrow{\phi^{-1}\circ\Gamma_\mu^{\theta'}}\mathrm{H}^j(\mathrm{Bun}_G^\omega), \tag{8.4.8}$$

and $\phi^{-1}\circ\Gamma_\mu^{h_0^*(\phi(f^z))\theta'}$ is the composition

$$\mathrm{H}^j(\mathrm{Bun}_G^\omega)\xrightarrow{\phi^{-1}\circ\Gamma_\mu^{\theta'}}\mathrm{H}^{j-2d+|z|}(\mathrm{Bun}_G^\omega)\xrightarrow{\cup f^z}\mathrm{H}^j(\mathrm{Bun}_G^\omega). \tag{8.4.9}$$

The two maps in (8.4.8) and (8.4.9) differ only by the order of composition. By Lemma 8.4.4 and the absolute convergence of trace proved in Proposition 5.5.3, we have

$$\mathrm{Tr}(\phi^{-1}\circ\Gamma_\mu^{\theta' h_r^*(f^z)}) = (-1)^{|z|}\mathrm{Tr}(\phi^{-1}\circ\Gamma_\mu^{h_0^*(\phi(f^z))\theta'}),$$

or

$$\mathrm{Tr}(\phi^{-1}\circ\Gamma_\mu^{(h_0^*(\phi(f^z))-h_r^*(f^z))\theta'}) = 0. \tag{8.4.10}$$

By the identity (8.2.3) that holds in $\mathrm{H}^*(\mathrm{Hk}_G^\mu)$, we have

$$h_0^*(\phi(f^z)) - h_r^*(f^z) = \gamma_1(f^z).$$

Therefore (8.4.10) implies (8.4.7). This proves (8.4.6) in the case $\theta = D_1(f)\theta'$.

Now we prove (8.4.6) by induction on i. The case $i=1$ has been proved. Suppose $i>1$ and (8.4.6) holds for $D_{i-1}(f)\theta$. It is straightforward to check that $D_i(f)\theta = \mathfrak{P}^*(D_{i-1}(f)\theta')$, where $\theta'\in\widetilde{C}_G^{\mu'}$ is characterized by $\mathfrak{P}^*\theta'=\theta$. By Corollary 8.4.6, we have

$$\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu, D_i(f)\theta) = \mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu, \mathfrak{P}^*(D_{i-1}(f)\theta')) = q^{D_{\mu_1}+1}\mathrm{vol}({}^{\omega'}\mathrm{Sht}_G^{\mu'}, D_{i-1}(f)\theta') = 0.$$

This finishes the induction step and the proof is complete. □

The following result is the analogue of Hirzebruch proportionality (see diagram (1.1.4)) for the moduli stack of shtukas. Proposition 8.4.3 justifies also denoting by $\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu,-)$ the linear functional on the quotient $(C_G^\mu)_{2N}$ of $(\widetilde{C}_G^\mu)_{2N}$.

Proposition 8.4.9. *As linear functionals on $(C_G^\mu)_{2N}$, we have*

$$\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu,-) = q^{\dim\mathrm{Bun}_G}\prod_{j=1}^n\zeta_X(d_j)\cdot\int_{\prod_{i=1}^r(X\times G/P_{\mu_i})}(-).$$

Proof. Applying Theorem 5.6.9 to the integrand $\otimes_{i=1}^r(\xi\otimes\eta_i')\in\otimes_{i=1}^r(\mathrm{H}^2(X)\otimes\mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i}))\cong(C_G^\mu)_{2N}$, we get

$$\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu,\otimes_{i=1}^r(\xi\otimes\eta_i')) = q^{\dim\mathrm{Bun}_G}\left(\prod_{i=1}^r\int_{G/P_{\mu_i}}\eta_i'\right)\mathscr{L}_{X,G}(0,0,\cdots,0).$$

Note here Assumption 5.6.1 is trivially satisfied because all η_i are zero. Now $\mathscr{L}_{X,G}(0,0,\cdots,0) = \prod_{i=1}^n\zeta_X(d_i)$ by definition. The claim follows. □

Remark 8.4.10. From this Proposition, using the defining relations for C_G^μ, one should in principle be able to recover Theorem 5.6.9. However, the intricacy of the multiplicative structure of the ring C_G^μ seems to make this approach difficult to implement.

8.4.11. *Poincaré duality on C_G^μ.* Using the volume functional, we can define a bilinear pairing on elements of C_G^μ of complementary degrees:

$$\langle -,-\rangle : (C_G^\mu)_i \times (C_G^\mu)_{2N-i} \to \overline{\mathbf{Q}}_\ell(-N)$$

defined as

$$\langle \alpha, \beta\rangle = \mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu, \alpha\beta).$$

By Proposition 8.4.9, $\mathrm{vol}({}^\omega\mathrm{Sht}_G^\mu, -)$ is independent of the choice of ω.

The next Proposition says that the phantom tautological ring restores Poincaré duality that is missing on ${}^\omega\mathrm{Sht}_G^\mu$ due to non-properness.

Proposition 8.4.12. *The pairing $\langle -,-\rangle$ is perfect. In particular, C_G^μ has the structure of a Frobenius algebra.*

Proof. Let $\alpha, \alpha' \in \mathrm{H}^*(X^r)$ be of complementary degree; let $\beta, \beta' \in \otimes_{i=1}^r \mathrm{H}^*(G/P_{\mu_i})$ be of complementary degree. Let $\widetilde{\beta}, \widetilde{\beta}'$ be arbitrary liftings of β and β' to C_G^μ, viewing $\mathrm{H}^*(G/P_{\mu_i})$ as a quotient of C_G^μ by (8.3.1). Then from the construction of the pairing on C_G^μ we have

$$\langle \alpha\otimes\widetilde{\beta}, \alpha'\otimes\widetilde{\beta}'\rangle = \langle\alpha,\alpha'\rangle_{X^r}\langle\beta,\beta'\rangle_{\prod(G/P_{\mu_i})}. \tag{8.4.11}$$

Here $\langle -,-\rangle_Y$ means the Poincaré duality pairing on the smooth projective variety Y. Now let α run over a basis $\{\alpha_i\}$ of $\mathrm{H}^*(X^r)$ and let α' run over the dual basis $\{\alpha^i\}$ under the pairing $\langle -,-\rangle_{X^r}$; similarly let β run over a basis $\{\beta_j\}$ of $\mathrm{H}^*(\prod(G/P_{\mu_i}))$ and let β' run over the dual basis $\{\beta^j\}$. Take arbitrary liftings $\widetilde{\beta}_j$ and $\widetilde{\beta}^j$ in C_G^μ. The freeness assertion in Proposition 8.3.1 implies that both $\{\alpha_i\otimes\widetilde{\beta}_j\}$ and $\{\alpha^i\otimes\widetilde{\beta}^j\}$ are bases for C_G^μ; (8.4.11) shows that they are dual bases under $\langle -,-\rangle$. In particular, the pairing $\langle -,-\rangle$ is perfect. □

8.5. **The reductive case.** Here we extend the results on the phantom tautological ring to the case where G is split reductive over k. We otherwise keep the same setup as in §8.1. The ring $\widetilde{C}_G^\mu$ is defined in exactly the same way as in (8.1.1). For the ideal I_G^μ, we make the following modification. Let

$$\Xi_1^* := \left(\mathrm{id}\otimes\frac{1}{\phi-1}\right)([\Delta_X] - \xi\otimes 1) \in \mathrm{H}^2(X\times X),$$
$$\Xi_0^* := \left(\mathrm{id}\otimes\frac{1}{q^{-1}\phi-1}\right)([\Delta_X] - 1\otimes \xi) \in \mathrm{H}^2(X\times X).$$

For any $f\in R^W$ of degree 2, viewed as an element in $\mathbb{X}^*(T)^W_{\mathbf{Q}_\ell}$, the definition of $\delta_i(f)$ boils down to

$$\delta_i(f) = \langle\mu_i, f\rangle \in \overline{\mathbf{Q}}_\ell.$$

Define

$$D_i^*(f) = [f]_i - \sum_{i'=i}^r \langle\mu_{i'}, f\rangle[\Xi_1^*]_{i,i'} + \sum_{i'=1}^{i-1}\langle\mu_{i'}, f\rangle[\Xi_0^*]_{i',i} - \langle\omega+\mu_1+\cdots+\mu_{i-1}, f\rangle[\xi]_i, \quad \text{when } \deg(f)=2.$$

Here $\langle\omega, f\rangle$ makes sense because ω is well-defined up to the coroot lattice, on which $\langle -, f\rangle$ vanishes.

We then let I_G^μ be the ideal of $\widetilde{C}_G^\mu$ generated by $D_i^*(f)$ for all $f\in R_2^W$, $1\le i\le r$, and by $D_i(f)$ for all homogeneous $f\in R^W$ of degree >2 and $1\le i\le r$. Finally, define $C_G^\mu = \widetilde{C}_G^\mu/I_G^\mu$.

Example 8.5.1. Consider the case $G=\mathbb{G}_m$. In this case, $C_G^\mu \cong \mathrm{H}^*(X^r)$. The map $\rho: C_G^\mu \to \mathrm{H}^*({}^\omega\mathrm{Sht}_{\mathbb{G}_m}^\mu)$ is identified with the pullback by the leg map ${}^\omega\mathrm{Sht}_{\mathbb{G}_m}^\mu \to X^r$. Note that in this case ρ is injective, and its image is exactly the $\mathrm{Pic}_X^0(k)$-invariant part of $\mathrm{H}^*({}^\omega\mathrm{Sht}_{\mathbb{G}_m}^\mu)$.

All previous results in this section hold for split reductive groups G with the above definition of C_G^μ. The proofs are mostly the same, except that an extra calculation is needed alongside Lemma 8.3.2.

Lemma 8.5.2. *Recall that μ is admissible in the sense that $\mu_1+\cdots+\mu_r$ lies in the coroot lattice. Let $f\in R_2^W$ and $g\in R_+^W$ be homogeneous, and $1\le i\le r$.*

(1) *If $\deg(g)>2$, then $D_i(fg)\in(D_1^*(f),\cdots,D_r^*(f),D_1(g),\cdots,D_r(g))\subset\widetilde{C}_G^\mu$.*

(2) *If* $\deg(g)=2$, *then* $D_i(fg)\in(D_1^*(f),\cdots,D_r^*(f),D_1^*(g),\cdots,D_r^*(g))\subset\widetilde{C}^\mu_G$.

Proof. Using partial Frobenius it suffices to prove the statements for $i=1$. Denote the ideals in question by $I_{f,g}$. We prove only part (1), and the proof of part (2) is similar. Let $\deg(g)=2e$. Expanding $[f]_1[g]_1$ and $[fg]_1$ modulo $I_{f,g}$ as in the proof of Lemma 8.3.2, we write

$$[f]_1[g]_1-[fg]_1\equiv\sum_{i,i'=1}^r(a^*_{i,i'}-b^*_{i,i'})\langle\mu_i,f\rangle\delta_{i'}(g)+\sum_{i=1}^r c_i\langle\omega,f\rangle\delta_i(g)\mod I_{f,g}. \tag{8.5.1}$$

Here

$$\begin{aligned}
a^*_{i,i'} &= [\Xi_1^*]_{1,i}[\Xi_e]_{1,i'},\\
b^*_{i,i'} &= \begin{cases}[\Xi_{e+1}]_{1,i}[\Xi_e]_{i,i'}-[\Xi_{e+1}]_{1,i'}[\Xi_0^*]_{i,i'}+[\Xi_{e+1}]_{1,i'}[\xi]_{i'}, & i<i'\\ -[\Xi_{e+1}]_{1,i}[\Xi_{1-e}]_{i',i}+[\Xi_{e+1}]_{1,i'}[\Xi_1^*]_{i',i}, & i>i'\\ [\Xi_{e+1}]_{1,i}([\Xi_e]_{i,i}+[\Xi_1^*]_{i,i}+[c_1(TX)]_i), & i=i'\end{cases}\\
c_i &= [\Xi_e]_{1,i}[\xi]_1-[\Xi_{e+1}]_{1,i}[\xi]_i.
\end{aligned}$$

Direct calculation shows $c_i=0$. It remains to calculate $a^*_{i,i'}-b^*_{i,i'}$. As in the proof of Lemma 8.3.2 we let $y=q^{e-1}$. Let $a_{i,i'}$ denote the left side of (8.3.7), and $b_{i,i'}$ denote the right side of (8.3.7), both viewed as elements in $\overline{\mathbf{Q}}_\ell(x,y)\otimes\mathrm{H}^4(X^3)$. We also introduce the two-variable version of $a^*_{i,i'}$ and $b^*_{i,i'}$ by replacing Ξ_1^* with $\left(1\otimes\frac{1}{x\phi-1}\right)([\Delta]-\xi\otimes1)$, Ξ_0^* with $\left(1\otimes\frac{1}{q^{-1}x\phi-1}\right)([\Delta]-1\otimes\xi)$, and Ξ_{e+1} with $\left(1\otimes\frac{1}{qxy\phi-1}\right)[\Delta]$. We still denote these by $a^*_{i,i'}$ and $b^*_{i,i'}$. To calculate $a^*_{i,i'}-b^*_{i,i'}$, we calculate the differences $a_{i,i'}-a^*_{i,i'}$ and $b_{i,i'}-b^*_{i,i'}$ and find

$$\begin{aligned}
a_{i,i'}-a^*_{i,i'} &= \left(1\otimes\frac{1}{x\phi-1}\right)[\xi\otimes1]_{1,i}\left(1\otimes\frac{1}{y\phi-1}\right)[\Delta]_{1,i'}=\frac{1}{(x-1)(qy-1)}[\xi]_1[\xi]_{i'},\\
b_{i,i'}-b^*_{i,i'} &= \frac{1}{(x-1)(qxy-1)}[\xi]_1[\xi]_{i'}.
\end{aligned}$$

In the proof of Lemma 8.3.2 we showed that $a_{i,i'}=b_{i,i'}$, therefore

$$a^*_{i,i'}-b^*_{i,i'}=\left(\frac{1}{(x-1)(qxy-1)}-\frac{1}{(x-1)(qy-1)}\right)[\xi]_1[\xi]_{i'}=\frac{-qy}{(qy-1)(qxy-1)}[\xi]_1[\xi]_{i'}.$$

In particular, $a^*_{i,i'}-b^*_{i,i'}$ is independent of i (and equal to $\frac{-qy}{(qy-1)^2}[\xi]_1[\xi]_{i'}$ if we plug in $x=1$). Plugging into (8.5.1), using that $\langle\sum_i\mu_i,f\rangle=0$, we conclude that $[f]_1[g]_1\equiv[fg]_1$ modulo $I_{f,g}$. □

Corollary 8.5.3. *Let* $\varphi:G\to G'$ *be a surjection whose kernel is central. Let* $\mu=(\mu_1,\cdots,\mu_r)$ *be an admissible sequence of minuscule coweights of* G*, also viewed as an admissible sequence of minuscule coweights of* G'*. Then* φ *induces an isomorphism* $C^\mu_{G'}\xrightarrow{\sim}C^\mu_G$.

Proof. Both C^μ_G and $C^\mu_{G'}$ are free over $\mathrm{H}^*(X^r)$ by the reductive version of Proposition 8.3.1. It therefore suffices to check that φ induces an isomorphism $C^\mu_{G'}/\mathrm{H}^{>0}(X^r)C^\mu_{G'}\xrightarrow{\sim}C^\mu_G/\mathrm{H}^{>0}(X^r)C^\mu_G$. However both sides are canonically identified with $\otimes\mathrm{H}^*(G'/P'_{\mu_i})\cong\otimes\mathrm{H}^*(G/P_{\mu_i})$ again by Proposition 8.3.1. □

9. The Colmez conjecture over function fields

In this section, we consider a function field analog of the Colmez conjecture [Col93]. Let $\pi:X\to Y$ be a finite étale covering. For simplicity, we assume that it is Galois with Galois group Σ. Consider a map

$$\sigma=(\sigma_1,\sigma_2,\dots,\sigma_r):X\to X^r$$

sending $x\in X$ to $(\sigma_i(x))_{i=1}^r$, where $\sigma\in\Sigma^r=\Sigma\times\cdots\times\Sigma$ is an r-tuple. We restrict Sht^μ_G to the locus using the above map associated to $\sigma\in\Sigma^r$:

$$\mathrm{Sht}^\mu_{G,\sigma}:=\mathrm{Sht}^\mu_G\times_{X^r,\sigma}X. \tag{9.0.1}$$

In other words, all the legs are conjugate relative to $X \to Y$. Let $p : \mathrm{Sht}^{\mu}_{G,\sigma} \to X$ be the leg map (the second projection of (9.0.1)). Pulling back along p and ev_i gives a ring homomorphism

$$\widetilde{\rho}_\sigma : \widetilde{C}^{\mu}_{G,\sigma} := \mathrm{H}^*(X) \otimes \bigotimes_{i=1}^{r} R^{W_{\mu_i}} \to \mathrm{H}^*(\mathrm{Sht}^{\mu}_{G,\sigma}). \tag{9.0.2}$$

Let

$$C^{\mu}_{G,\sigma} = C^{\mu}_{G} \otimes_{\mathrm{H}^*(X^r)} \mathrm{H}^*(X)$$

where $\mathrm{H}^*(X)$ is viewed as a $\mathrm{H}^*(X^r)$-algebra via σ^*. Then ρ induces a canonical homomorphism

$$\rho_\sigma : C^{\mu}_{G,\sigma} \to \mathrm{H}^*(\mathrm{Sht}^{\mu}_{G,\sigma}).$$

9.1. **The one leg case.** Let G be split semisimple. Write ξ for the class $1 \otimes \xi \in C^{\mu}_{G,\sigma}$. Note that inside $C^{\mu}_{G,\sigma}$, for any $f \in R^W$ of degree $2d \geq 4$, Theorem 8.1.7 implies that

$$[f]_i = \xi \otimes \sum_{i'=1}^{r} c_{i,i'}(d)[\partial_{\mu_{i'}} f]_{i'}. \tag{9.1.1}$$

Here the constants $c_{i,i'}(d)$ are defined using

$$c_{i,i'}(d)\xi = \begin{cases} (\sigma_i, \sigma_{i'})^* \Xi_d, & i' \geq i, \\ -(\sigma_{i'}, \sigma_i)^* \Xi_{1-d}, & i' < i. \end{cases} \tag{9.1.2}$$

In particular, since $\xi^2 = 0$, we see that for any two $f, g \in R^W_+$ and $1 \leq i, i' \leq r$,

$$[f]_i [g]_{i'} = 0 \in C^{\mu}_{G,\sigma}. \tag{9.1.3}$$

We have a canonical isomorphism between the top degree of $C^{\mu}_{G,\sigma}$ and $\mathrm{H}^2(X) \otimes \left(\bigotimes_{i=1}^{r} \mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i}) \right)$, hence a canonical linear form $\int_{X \times \prod G/P_{\mu_i}}$ on $C^{\mu}_{G,\sigma}$ that is nonzero only on the top degree. We define

$$\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_{G,\sigma}, -) := q^{\dim \mathrm{Bun}_G} \prod_{j=1}^{n} \zeta_X(d_j) \int_{X \times \prod G/P_{\mu_i}} (-) : C^{\mu}_{G,\sigma} \to \overline{\mathbf{Q}}_\ell.$$

Remark 9.1.1. To see the structure of $C^{\mu}_{G,\sigma}$ more clearly, we focus on its even part $(C^{\mu}_{G,\sigma})^{\mathrm{even}}$, which is a free $\overline{\mathbf{Q}}_\ell[\xi]/(\xi^2)$-module with mod ξ reduction $A^{\mu}_G := \otimes_{i=1}^{r} \mathrm{H}^*(G/P_{\mu_i})$.

We can form the quotient B^{μ}_G of $\otimes_{i=1}^{r} R^{W_{\mu_i}}$ by imposing the relations (9.1.3). Then B^{μ}_G is a square zero extension of A^{μ}_G that fits into an exact sequence

$$0 \to \bigoplus_{i=1}^{r} \mathbb{V} \otimes A^{\mu}_G \to B^{\mu}_G \to A^{\mu}_G \to 0.$$

For $f \in \mathbb{V}$ we denote by $[f]_i \in B^{\mu}_G$ the element $f \otimes 1$ in the ith direct summand of the first term of the above sequence. Thus $(C^{\mu}_{G,\sigma})^{\mathrm{even}}$ can be identified with the pushout of B^{μ}_G along the A^{μ}_G-linear homomorphism $\bigoplus_{i=1}^{r} \mathbb{V} \otimes A^{\mu}_G \to \xi \otimes A^{\mu}_G$ sending $[f]_i$ to $\xi \otimes \sum_{i'=1}^{r} c_{i,i'}(d)[\partial_{\mu_{i'}} f]_{i'}$, if $f \in R^W$ is homogeneous of degree $2d$.

Remark 9.1.2. Let $d > D_{\mu_i}$. Since $R^{W_{\mu_i}}/(R^W_+) \cong \mathrm{H}^*(G/P_{\mu_i})$ has top degree $2D_{\mu_i}$, the homogeneous piece $R^{W_{\mu_i}}_{2d}$ lies in the ideal $(R^W_+) \subset R^{W_{\mu_i}}$. Using that $R^{W_{\mu_i}}$ is a free R^W-module, we have a canonical isomorphism

$$(R^W_+ R^{W_{\mu_i}})/(R^W_+ R^{W_{\mu_i}})^2 \cong \mathbb{V} \otimes \mathrm{H}^*(G/P_{\mu_i}).$$

In particular for $d > D_{\mu_i}$ there is a canonical map

$$\pi_d : R^{W_{\mu_i}}_{2d} \subset (R^W_+ R^{W_{\mu_i}}) \twoheadrightarrow (R^W_+ R^{W_{\mu_i}})/(R^W_+ R^{W_{\mu_i}})^2 \cong \mathbb{V} \otimes \mathrm{H}^*(G/P_{\mu_i}).$$

Lemma 9.1.3. *Let $\eta_i \in \otimes R^{W_{\mu_i}}$ be homogeneous of degree $2n_i$ for $1 \leq i \leq r$ and $\eta = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_r \in \widetilde{C}^{\mu}_G$. Assume $\sum_{i=1}^{r} n_i = 1 + \sum_{i=1}^{r} D_{\mu_i}$.*

(1) *If two or more i satisfy $n_i > D_{\mu_i}$, then η has zero image in $C^{\mu}_{G,\sigma}$.*

(2) *If two or more i satisfy $n_i < D_{\mu_i}$, then η has zero image in $C^{\mu}_{G,\sigma}$.*

(3) *If there is a unique $1 \le i \le r$ such that $n_i > D_{\mu_i}$, and there is a unique $1 \le i' \le r$ such that $n_{i'} < D_{\mu_{i'}}$, then we write $\pi_{n_i}(\eta_i) \in \mathbb{V}\otimes \mathrm{H}^*(G/P_{\mu_i})$ in the form $f_{\eta_i}\otimes\eta'_i+\cdots$ where $f_{\eta_i} \in \mathbb{V}_{2n_i-2D_{\mu_i}}$ and $\eta'_i \in \mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i})$, and the terms in $\cdots$ involve only non-top degrees of $\mathrm{H}^{2D_{\mu_i}}(G/P_{\mu_i})$. Then in $C^{\mu}_{G,\sigma}$ we have*

$$\eta \equiv c_{i,i'}(n_i - D_{\mu_i})\xi \otimes \eta'_i \otimes (\partial_{\mu_{i'}} f_{\eta_i} \cdot \eta_{i'}) \otimes (\otimes_{j\neq i,i'}\eta_j) \in C^{\mu}_{G,\sigma}.$$

(4) *If there is a unique $1 \le i \le r$ such that $n_i > D_{\mu_i}$, and all other i' satisfies $n_{i'} = D_{\mu_{i'}}$ (so that $n_i = D_{\mu_i}+1$), then we write $\pi_{n_i}(\eta_i) \in \mathbb{V}\otimes \mathrm{H}^*(G/P_{\mu_i})$ in the form $\sum_{j=1}^n f_j \otimes \eta_i^{(j)}$ using a homogeneous basis $\{f_j\}$ (of degree d_j) of $\mathbb{V}$, where $\eta_i^{(j)} \in \mathrm{H}^*(G/P_{\mu_i})$. Then*

$$\eta \equiv \xi \otimes \sum_{j=1}^{n} c_{i,i}(d_j)(\partial_{\mu_i} f_j \cdot \eta_i^{(j)}) \otimes (\otimes_{i'\neq i}\eta_{i'}) \in C^{\mu}_{G,\sigma}.$$

Proof. (1) Since $R^{W_{\mu_i}}/(R^W_+) \cong \mathrm{H}^*(G/P_{\mu_i})$ has top degree $2D_{\mu_i}$, and element $\eta_i \in R^{W_{\mu_i}}$ of degree $> 2D_{\mu_i}$ lies in the ideal (R^W_+), hence its image in $C^{\mu}_{G,\sigma}$ is divisible by ξ. Since $\xi^2 = 0$, if two η_i has degree $> 2D_{\mu_i}$, the image of η in $C^{\mu}_{G,\sigma}$ is zero.

(2)(3)(4) By (1) we may assume there is a unique i such that $n_i > D_{\mu_i}$. Write $\pi_{n_i}(\eta_i) \in \mathbb{V}\otimes\mathrm{H}^*(G/P_{\mu_i})$ as $\sum_j f_j \otimes \eta_i^{(j)}$ using a homogeneous basis $\{f_j\}$ (of degree d_j) of $\mathbb{V}$. Using the relation (9.1.1) for $[f_j]_i$ in $C^{\mu}_{G,\sigma}$, we have

$$\eta \equiv \xi \otimes \sum_{j=1}^{n}\sum_{i'=1}^{r} c_{i,i'}(d_j)\eta_i^{(j)} \otimes [\partial_{\mu_{i'}} f_j]_{i'} \otimes (\otimes_{i''\neq i}\eta_{i''}) \in C^{\mu}_{G,\sigma}. \tag{9.1.4}$$

We analyze the summand according as $i = i'$ or $i \neq i'$.

- Suppose the summand corresponding to $i = i'$ is nonzero in $C^{\mu}_{G,\sigma}$. Now $\sum_{j=1}^n c_{i,i}(d_j)\eta_i^{(j)} \otimes \partial_{\mu_i} f_j \in R^{W_{\mu_i}}$ has degree $n_i - 1$. If $n_i - 1 > D_{\mu_i}$, then this element lies in the ideal (R^W_+) hence its image in $C^{\mu}_{G,\sigma}$ is divisible by ξ, making the right side of (9.1.4) zero. Therefore we must have $n_i = D_{\mu_i}+1$. In this case $\sum_{i'\neq i} n_{i'} = \sum_{i'\neq i} D_{\mu_{i'}}$. By part (1), η is zero in $C^{\mu}_{G,\sigma}$ unless $n_{i'} = D_{\mu_{i'}}$ for all $i' \neq i$. This is exactly the situation described in part (4). To show part (4), we only need to check that the $i' \neq i$ summands on the right of (9.1.4) vanish in $C^{\mu}_{G,\sigma}$. Indeed, for $i' \neq i$, the i' factor of that summand has degree $d_j - 1 + D_{\mu_{i'}} > D_{\mu_{i'}}$ ($d_j > 1$ since G is assumed to be semisimple). This proves part (4).
- If for some $i' \neq i$ and some j, the summand $c_{i,i'}(d_j)\eta_i^{(j)} \otimes [\partial_{\mu_{i'}} f_j]_{i'} \otimes (\otimes_{i''\neq i}\eta_{i''})$ is nonzero in $C^{\mu}_{G,\sigma}$, then the same degree analysis above forces that $\eta_i^{(j)}$ has degree D_{μ_i}, f_j thus has degree $d_j = n_i - D_{\mu_i}$, $\eta_{i'}$ has degree $n_{i'} = D_{\mu_{i'}} - d_j + 1$, and all other $\eta_{i''}$ have degree $D_{\mu_{i''}}$. This is exactly the situation described in part (3).

When two or more i has $n_i < D_{\mu_i}$ we are in neither the situation of (3) nor (4), and the above analysis shows that η is zero in $C^{\mu}_{G,\sigma}$. □

9.2. **The case $G = \mathbf{PGL}_n$.** We now apply the above lemma to a particular example. Let $G = \mathrm{PGL}_n$. For R it is more convenient to identify it with the R for SL_n, i.e., $R = \overline{\mathbf{Q}}_\ell[x_1,\cdots,x_n]/(x_1+\cdots+x_n)$ with $W = S_n$ acting by permuting variables. Let $f_2,\cdots,f_n \in R^W$ be images of elementary symmetric polynomials in R^W.

We consider the case where each μ_i is either $\mu_+ = (1,0,\cdots,0)$ or $\mu_- = (0,0,\cdots,0,-1)$. Note that G/P_{μ_i} is isomorphic to $\mathbb{P}^{n-1}$. Let $t_i \in R_2^{W_{\mu_i}} = \mathrm{H}^2_G(G/P_{\mu_i})$ be $1/n$ of the G-equivariant Chern class of $\mathcal{O}_{\mathbb{P}^{n-1}}(n)$. When $\mu_i = \mu_+$, G/P_{μ_i} classifies lines, and $t_i = -x_1$ (as x_1 is the Chern class of $\mathcal{O}(-1)$); when $\mu_i = \mu_-$, G/P_{μ_i} classifies hyperplanes, and $t_i = x_n$. We have $\int_{G/P_{\mu_i}} t_i^{n-1} = 1$.

Note that $\dim \mathrm{Sht}^{\mu}_{G,\sigma} = (n-1)r+1$. Let

$$\eta = (t_1+t_2+\cdots+t_r)^{(n-1)r+1} \in C^{\mu}_{G,\sigma}.$$

Proposition 9.2.1. *The volume* $\mathrm{vol}({}^{\omega}\mathrm{Sht}^{\mu}_{G,\sigma},\eta)$ *is equal to* $q^{(n^2-1)(g-1)}\zeta_X(2)\cdots\zeta_X(n)$ *multiplied by the sum of the following quantities:*

$$-\sum_{i=1}^{r}\sum_{j=2}^{n}c_{i,i}(j)\binom{(n-1)r+1}{n,n-1,\cdots,n-1}$$
$$-\sum_{1\le i\ne i'\le r}\sum_{j=2}^{n}(-1)^{j\nu(i,i')}c_{i,i'}(j)\binom{(n-1)r+1}{n+j-1,n-j,n-1,\cdots,n-1}.$$

Here

$$\nu(i,i')=\begin{cases}0 & \text{if } \mu_i=\mu_{i'}\\ 1 & \text{if } \mu_i\ne\mu_{i'}.\end{cases}$$

Proof. Expand η in monomials $t_1^{n_1}\otimes\cdots\otimes t_r^{n_r}$ with total degree $(n-1)r+1$. By Lemma 9.1.3, this monomial has nonzero image in $C^{\mu}_{G,\sigma}$ only in the following cases:

- There is exactly one i such that $n_i=n$, and the rest i' satisfies $n_{i'}=n-1$. When $\mu_i=\mu_+$ we have
$$t_i^n=-(f_2t_i^{n-2}+f_3t_i^{n-3}+\cdots+f_n).$$
The right side may be considered as the image of t_i^n in $\mathbb{V}\otimes\mathrm{H}^*(G/P_{\mu_i})$. By Lemma 9.1.3(4), we have[14]
$$t_1^{n_1}\otimes\cdots\otimes t_r^{n_r}=t_i(t_1\cdots t_r)^{n-1}\equiv\xi\otimes\sum_{j=2}^{n}c_{i,i}(j)[-\partial_{\mu_+}f_j\cdot t_i^{n-j}]_i\otimes(\otimes_{i'\ne i}t_{i'}^{n-1})\in C^{\mu}_{G,\sigma}.$$
We have $\partial_{\mu_+}f_j\cdot t_i^{n-j}=\partial_{x_1}f_j\cdot t_i^{n-j}=c_{j-1}(Q_{n-1})t_i^{n-j}\in\mathrm{H}^{2n-2}(\mathbb{P}^{n-1})$, where $Q_{n-1}\cong\mathcal{O}^n/\mathcal{O}(-1)$ is the universal quotient bundle. The relation $c(Q_{n-1})(1-t_i)=1$ in $\mathrm{H}^*(\mathbb{P}^{n-1})$ implies $c_{j-1}(Q_{n-1})=t_i^{j-1}$ for $2\le j\le n$. Therefore $\partial_{x_1}f_j\cdot t_i^{n-j}=t_i^{n-1}\in\mathrm{H}^{2n-2}(\mathbb{P}^{n-1})$. We conclude that when $\mu_i=\mu_+$
$$t_1^{n_1}\otimes\cdots\otimes t_r^{n_r}\equiv-\sum_{j=2}^{n}c_{i,i}(j)\cdot\xi\otimes t_1^{n-1}\otimes\cdots\otimes t_r^{n-1}\in C^{\mu}_{G,\sigma}. \tag{9.2.1}$$
When $\mu_i=\mu_-$ we have
$$t_i^n=-f_2t_i^{n-2}+f_3t_i^{n-3}+\cdots+(-1)^{n-1}f_n.$$
Similar calculation gives the same formula as (9.2.1).
- There is exactly one i such that $n_i\ge n$, and exactly one i' such that $n_{i'}<n-1$, and the rest of $n_{i''}$ are equal to $n-1$. Note that $n_i-n+n_{i'}=n-1$. When $\mu_i=\mu_+$ we have $\pi_{n_i}(t_i^{n_i})=-f_{n_i-n+1}t_i^{n-1}+\text{lower powers of } t_i\in\mathbb{V}\otimes\mathrm{H}^*(\mathbb{P}^{n-1})$. Lemma 9.1.3(3) implies
$$t_1^{n_1}\otimes\cdots\otimes t_r^{n_r}\equiv-c_{i,i'}(n_i-n+1)\xi\otimes[\partial_{\mu_{i'}}f_{n_i-n+1}\cdot t_{i'}^{n_{i'}}]_{i'}\otimes(\otimes_{i''\ne i'}t_{i''}^{n-1})\in C^{\mu}_{G,\sigma}.$$
When $\mu_{i'}=\mu_+$ we have $\partial_{\mu_+}f_{n_i-n+1}\cdot t_{i'}^{n_{i'}}=\partial_{x_1}f_{n_i-n+1}\cdot t_{i'}^{n_{i'}}=c_{n_i-n}(Q_{n-1})t_{i'}^{n_{i'}}=t_{i'}^{n-1}$. Thus when $\mu_i=\mu_{i'}=\mu_+$ we have
$$t_1^{n_1}\otimes\cdots\otimes t_r^{n_r}\equiv-c_{i,i'}(n_i-n+1)\xi\otimes t_1^{n-1}\otimes\cdots\otimes t_r^{n-1}\in C^{\mu}_{G,\sigma}.$$
When $\mu_{i'}=\mu_-$ we have $\partial_{\mu_-}f_{n_i-n+1}\cdot t_{i'}^{n_{i'}}=-\partial_{x_n}f_{n_i-n+1}\cdot t_{i'}^{n_{i'}}=-c_{n_i-n}(S_{n-1})t_{i'}^{n_{i'}}$ where $S_{n-1}=\ker(\mathcal{O}^n\to\mathcal{O}(1))$ is the universal hyperplane bundle over $\mathbb{P}^{n-1}$. The relation $c(S_{n-1})(1+t_{i'})=1\in\mathrm{H}^*(\mathbb{P}^{n-1})$ implies $c_{n_i-n}(S_{n-1})=(-1)^{n_i-n}t_{i'}^{n_i-n}$. Thus in this case $\partial_{\mu_-}f_{n_i-n+1}\cdot t_{i'}^{n_{i'}}=(-1)^{n_i-n+1}t_{i'}^{n-1}$. Thus when $\mu_i=\mu_+$ and $\mu_{i'}=\mu_-$ we have
$$t_1^{n_1}\otimes\cdots\otimes t_r^{n_r}\equiv(-1)^{n_i-n}c_{i,i'}(n_i-n+1)\xi\otimes t_1^{n-1}\otimes\cdots\otimes t_r^{n-1}\in C^{\mu}_{G,\sigma}.$$

[14]The astute reader may notice a subtle issue here: since we identified R^W with $\mathrm{H}^*(\mathbb{B}\mathrm{SL}_n)$ rather than $\mathrm{H}^*(\mathbb{B}\mathrm{PGL}_n)$, $\mu_+=(1,0,\cdots,0)\in\mathbb{Z}^n/\Delta(\mathbb{Z})$ should be identified with the rational coweight $(\frac{n-1}{n},-\frac{1}{n},\cdots,-\frac{1}{n})$ of SL_n, hence $\partial_{\mu_+}f=\frac{n-1}{n}\partial_{x_1}f-\frac{1}{n}\partial_{x_2}f-\cdots-\frac{1}{n}\partial_{x_n}f$. However, because $\partial_{\mu_+}f\equiv\partial_{x_1}f \mod (R^W_+)$ and we only care about the image of $\partial_{\mu_+}f_j$ in $\mathrm{H}^*(G/P_{\mu_i})$ for the this calculation, we may replace $\partial_{\mu_+}f$ with $\partial_{x_1}f$ in the calculation.

Similar calculations show: when $\mu_i = \mu_-$ and $\mu_{i'} = \mu_+$

$$t_1^{n_1} \otimes \cdots \otimes t_r^{n_r} \equiv (-1)^{n_i - n} c_{i,i'}(n_i - n + 1)\xi \otimes t_1^{n-1} \otimes \cdots \otimes t_r^{n-1} \in C^\mu_{G,\sigma}.$$

When $\mu_i = \mu_-$ and $\mu_{i'} = \mu_-$

$$t_1^{n_1} \otimes \cdots \otimes t_r^{n_r} \equiv -c_{i,i'}(n_i - n + 1)\xi \otimes t_1^{n-1} \otimes \cdots \otimes t_r^{n-1} \in C^\mu_{G,\sigma}.$$

Summarizing the contribution of all monomials, we get that the image of η in $C^\mu_{G,\sigma}$ is $\xi \otimes t_1^{n-1} \otimes \cdots \otimes t_r^{n-1}$ multiplied by the constants in the statement of the proposition. This completes the proof. □

9.2.2. *Connection to Artin L-values.* We now rewrite the result in Proposition 9.2.1 in terms of special values of Artin L-functions. For an irreducible representation ρ of Σ, we have an Artin L-function defined by

$$L_{Y,\rho}(s) = \det(1 - q^{-s}\phi | \mathrm{H}^*(Y, \mathbb{L}_\rho)) = \prod_i \det(1 - q^{-s}\phi | \mathrm{H}^i(Y, \mathbb{L}_\rho))^{(-1)^{i+1}},$$

where $\mathbb{L}_\rho$ is the local system on Y corresponding to ρ.

We first relate the constants (9.1.2) to such L-values. We have the following generalization of Lemma 8.1.4.

Lemma 9.2.3. *Let $\sigma_X : X \to X \times X$ be the graph*[15] *For any integer $d \neq 0, 1$ we have*

$$\sigma_X^* \Xi_d = -\sum_\rho \chi_{\rho^\vee}(\sigma) \frac{L'_{Y,\rho}(d)}{\log(q) L_{Y,\rho}(d)} \xi \in \mathrm{H}^2(X)$$

where the sum runs over all irreducible representations ρ of Σ, and $\chi_{\rho^\vee}$ denotes the character of $\rho^\vee$, the dual representation to ρ.

Proof. We have

$$\begin{aligned} \sigma_X^* \Xi_d &= \xi \int_{X \times X} [\sigma_X(X)] \cdot \Xi_d = \xi \int_{X \times X} [\sigma_X(X)] \cdot \left(\frac{1}{q^d \phi^{-1} - 1} \otimes \mathrm{id} \right) [\Delta_X] \\ &= \mathrm{Tr}(\sigma^* (q^d \phi^{-1} - 1)^{-1} | \mathrm{H}^*(X)) \xi. \end{aligned}$$

Here we note that σ and ϕ commute as endomorphisms of $\mathrm{H}^*(X)$.

Decompose the local system $\pi_* \overline{\mathbf{Q}}_\ell$ on Y, as representations of Σ,

$$\pi_* \overline{\mathbf{Q}}_\ell = \bigoplus_\rho \rho^\vee \boxtimes \mathbb{L}_\rho,$$

where the sum runs over all irreducible representations ρ of Σ. Accordingly, we have

$$\mathrm{H}^*(X) = \bigoplus_\rho \rho^\vee \boxtimes \mathrm{H}^*(Y, \mathbb{L}_\rho)$$

as $\Sigma \times \langle \phi \rangle$-modules. We then have

$$\begin{aligned} \mathrm{Tr}(\sigma^* (q^d \phi^{-1} - 1)^{-1} | \mathrm{H}^*(X)) &= \sum_\rho \chi_{\rho^\vee}(\sigma) \mathrm{Tr}((q^d \phi^{-1} - 1)^{-1} | \mathrm{H}^*(Y, \mathbb{L}_\rho)) \\ &= -\sum_\rho \chi_{\rho^\vee}(\sigma) \frac{L'_{Y,\rho}(d)}{\log(q) L_{Y,\rho}(d)} \end{aligned}$$

as desired. □

[15]To be clear, we mean $\sigma_X(x) = (x, \sigma(x))$.

Definition 9.2.4. We define the logarithmic Artin L-function associated to a class function φ on Σ. Let $\varphi = \sum_\rho a_\rho \chi_\rho$ be any class function, where $a_\rho \in \overline{\mathbf{Q}}_\ell$ and ρ are irreducible representations of Σ. Then we define

$$\frac{L'_{Y,\varphi}(s)}{L_{Y,\varphi}(s)} := \sum_\rho a_\rho \frac{L'_{Y,\rho}(s)}{L_{Y,\rho}(s)}.$$

9.2.5. *Colmez's Conjecture.* We have an involution on $\overline{\mathbf{Q}}_\ell[\Sigma]$ induced by the inversion on Σ: for any element $\varphi \in \overline{\mathbf{Q}}_\ell[\Sigma]$, define $\varphi^\vee(g) = \varphi(g^{-1})$. Define the convolution of two functions

$$(\varphi * \varphi')(g) := \int_\Sigma \varphi(gh^{-1})\varphi'(h)\, dh.$$

Here the Haar measure on Σ is chosen such that $\mathrm{vol}(\Sigma) = 1$.

For $j \in \mathbf{Z}$, we define

$$\Phi_j = \sum_{i=1}^r \mathrm{sign}(\mu_i)^j \sigma_i \in \overline{\mathbf{Q}}_\ell[\Sigma], \quad \text{where } \mathrm{sign}(\mu_i) = \begin{cases} +1 & \mu_i = \mu_+, \\ -1 & \mu_i = \mu_-. \end{cases}$$

In particular, Φ_j depends only on the parity of j, and if j is even, we have

$$\Phi_j = \Phi = \sum_{i=1}^r \sigma_i.$$

For any $\varphi \in \overline{\mathbf{Q}}_\ell[\Sigma]$, let $\varphi^\natural$ be the projection to the space of class functions on Σ. In terms of the characters we have

$$\varphi^\natural = \sum_\rho \langle \varphi, \chi_{\rho^\vee}\rangle \chi_\rho.$$

Here the pairing $\langle \varphi, \varphi'\rangle$ is bilinear and we have $\langle \chi_\rho, \chi_{\rho^\vee}\rangle = 1$.

Theorem 9.2.6. *The volume* $\mathrm{vol}({}^\omega\mathrm{Sht}^\mu_{\mathrm{PGL}_n,\sigma}, \eta)$ *is equal to* $q^{\dim \mathrm{Bun}_{\mathrm{PGL}_n}} \prod_{d=2}^n \zeta_X(d)$ *multiplied by*

$$\begin{aligned}
&- \sum_{j=2}^n \left(\binom{(n-1)r+1}{n, n-1, \cdots, n-1} - \binom{(n-1)r+1}{n+j-1, n-j, n-1, \cdots, n-1} \right) \frac{\zeta'_X(j)}{\log q\, \zeta_X(j)} \\
&- |\Sigma|^2 \sum_{j=2}^n \binom{(n-1)r+1}{n+j-1, n-j, n-1, \cdots, n-1} \frac{L'_{Y,(\Phi_j * \Phi_j^\vee)^\natural}(j)}{\log q\, L_{Y,(\Phi_j * \Phi_j^\vee)^\natural}(j)} \\
&- (g_Y - 1)|\Sigma|^2 \sum_{j=2}^n \binom{(n-1)r+1}{n+j-1, n-j, n-1, \cdots, n-1} ((\Phi_j * \Phi_j^\vee)(1) - r/|\Sigma|).
\end{aligned}$$

Remark 9.2.7. In the number field case, Colmez conjecture [Col93] relates the stable Faltings height of an abelian variety with CM by the ring of integers of a CM field to the special value at $s = 1$ (or equivalently at $s = 0$ by functional equation) of the logarithmic Artin L-function attached to a class function arising from the corresponding CM type. The recipe of the class function in Colmez conjecture (cf. [Col98, §2] for a more explicit formula involving $\Phi * \Phi^\vee$) is completely analogous to the one above, except that here we have the special value at $s = j \geq 2$.

Proof. We apply Lemma 9.2.3 to get

$$c_{i,i}(j) \quad = \quad -\sum_\rho \chi_{\rho^\vee}(1) \frac{L'_{Y,\rho}(j)}{\log(q) L_{Y,\rho}(j)}.$$

We note that, by the factorization $\zeta_X(s) = \prod_\rho L_{Y,\rho}(s)^{\dim \rho}$,

$$\sum_\rho \chi_{\rho^\vee}(1) \frac{L'_{Y,\rho}(j)}{L_{Y,\rho}(j)} = \frac{\zeta'_X(j)}{\zeta_X(j)}.$$

Similarly, we have

$$c_{i,i'}(j) \quad = \quad -\sum_{\rho} \chi_{\rho^\vee}(\sigma_i^{-1}\sigma_{i'}) \frac{L'_{Y,\rho}(j)}{\log(q) L_{Y,\rho}(j)}$$

when $i' > i$, and

$$c_{i,i'}(j) \quad = \quad \sum_{\rho} \chi_{\rho^\vee}(\sigma_{i'}^{-1}\sigma_i) \frac{L'_{Y,\rho}(1-j)}{\log(q) L_{Y,\rho}(1-j)}$$

when $i' < i$.

By the functional equation

$$L_{Y,\rho}(s)\epsilon(\rho, 1/2) q^{s(2g_Y-2)\dim\rho} = L_{Y,\rho^\vee}(1-s)$$

we have the following relation between logarithmic derivatives of $L_{Y,\rho}(s)$ at d and $1-d$, by

$$\frac{L'_{Y,\rho}(d)}{\log(q) L_{Y,\rho}(d)} + \frac{L'_{Y,\rho^\vee}(1-d)}{\log(q) L_{Y,\rho^\vee}(1-d)} = -(2g_Y - 2)\dim\rho.$$

We may rewrite the case $i' < i$ as

$$c_{i,i'}(j) = -\sum_{\rho} \chi_{\rho^\vee}(\sigma_i^{-1}\sigma_{i'}) \frac{L'_{Y,\rho}(j)}{\log q\, L_{Y,\rho}(j)} - (2g_Y - 2)\sum_{\rho} \chi_{\rho^\vee}(\sigma_{i'}^{-1}\sigma_i)\dim\rho.$$

Then the displayed expression in Proposition 9.2.1 is the sum

$$\begin{aligned}
- &\sum_{j=2}^{n} \left(\binom{(n-1)r+1}{n, n-1, \cdots, n-1} - \binom{(n-1)r+1}{n+j-1, n-j, n-1, \cdots, n-1} \right) \frac{\zeta'_X(j)}{\log q\, \zeta_X(j)} \\
- &\sum_{j=2}^{n} \binom{(n-1)r+1}{n+j-1, n-j, n-1, \cdots, n-1} \sum_{\rho} \frac{L'_{Y,\rho}(j)}{\log q\, L_{Y,\rho}(j)} \sum_{(i,i')} \chi_{\rho^\vee}(\sigma_i^{-1}\sigma_{i'})(-1)^{j\nu(i,i')} \\
- &\sum_{j=2}^{n} \binom{(n-1)r+1}{n+j-1, n-j, n-1, \cdots, n-1} \sum_{\rho} \sum_{(i,i'),i>i'} \chi_{\rho^\vee}(\sigma_{i'}^{-1}\sigma_i)(-1)^{j\nu(i,i')}\dim\rho.
\end{aligned}$$

Now we note that

$$\sum_{(i,i')} \chi_{\rho^\vee}(\sigma_i^{-1}\sigma_{i'})(-1)^{j\nu(i,i')} = |\Sigma|^2 \langle \Phi_j * \Phi_j^\vee, \chi_{\rho^\vee}\rangle$$

and

$$\begin{aligned}
\sum_{\rho} \sum_{(i,i'),i>i'} \chi_{\rho^\vee}(\sigma_{i'}^{-1}\sigma_i)(-1)^{j\nu(i,i')}\dim\rho &= \sum_{\rho} \sum_{(i,i'),i>i'} \chi_{\rho^\vee}(\sigma_{i'}^{-1}\sigma_i)(-1)^{j\nu(i,i')}\chi_\rho(1) \\
&= |\Sigma| \cdot \sum_{i>i',\sigma_i=\sigma_{i'}} (-1)^{j\nu(i,i')} \\
&= |\Sigma|(|\Sigma|(\Phi_j * \Phi_j^\vee)(1) - r)/2.
\end{aligned}$$

These identities together complete the proof.

□

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