MODULARITY OF HIGHER THETA SERIES I: COHOMOLOGY OF THE GENERIC FIBER

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Abstract. In a previous paper we constructed *higher* theta series for unitary groups over function fields, and conjectured their modularity properties. Here we prove the generic modularity of the ℓ-adic realization of higher theta series in cohomology. The proof debuts a new type of Fourier transform, occurring on the Borel-Moore homology of moduli spaces for shtuka-type objects, that we call the *arithmetic Fourier transform*. Another novelty in the argument is a *sheaf-cycle correspondence* extending the classical sheaf-function correspondence, which facilitates the deployment of sheaf-theoretic methods to analyze algebraic cycles. Although the modularity property is a statement within classical algebraic geometry, the proof relies on derived algebraic geometry, especially a nascent theory of *derived Fourier analysis* on derived vector bundles, which we develop.

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1. Introduction

The modularity of theta series has a long and storied history, beginning with Poisson, who famously applied Poisson summation to prove the modularity of Jacobi's theta series.[1] From the modern perspective of automorphic forms, theta series can be constructed much more generally, for all reductive groups fitting into *dual reductive pairs*, and essentially the same Poisson summation argument generalizes to prove their modularity. So, to make a long story short, the modularity of theta series can ultimately be seen as a relatively simple (by modern standards) consequence of Fourier duality.

Kudla introduced an analogue of theta series in arithmetic geometry, called *arithmetic theta series*. These objects are again constructed as Fourier series, but with coefficients being *algebraic cycles* rather than numbers. They are also conjectured to be modular, but that turns out to be much more difficult to prove

[1]The proof of the modularity appears in Jacobi's paper [Jac28], where he credits it to Poisson; cf. [Edw01, p.15, footnote ‡].

(and even to formulate, in the ideal generality). For example, the modularity of arithmetic theta series of divisors on (the integral models of) unitary Shimura varieties was proved only recently, in [BHK+20], and has interesting applications such as to the proof of the "Arithmetic Fundamental Lemma" [Zha21]. For another example, the modularity of arithmetic theta series on (the integral models of) orthogonal Shimura varieties was proved even more recently, in [HMP20] (for divisors) and [HM22], and has interesting applications such as to the study of exceptional jumps of Picard ranks of K3 surfaces over number fields [SSTT22].

As these examples illustrate, the modularity of *arithmetic* theta series has so far only been accessible through the codimension one (i.e., divisor) case; the reason for this is immediately clear from the proof strategy, which will be recalled below. In particular, it has so far been inaccessible in situations with no arithmetic theta series of codimension one[2], such as unitary groups with signature (p, q) where both $p, q > 1$. Moreover, this modularity has important and far-ranging consequences for other problems in arithmetic geometry.

In the papers [FYZ21a] and [FYZ21b], the authors investigated theta series and arithmetic theta series over function fields, and discovered in that context that the story extends further: for each $r \geq 0$, there are *higher theta series* $\widetilde{Z}^r$ that specialize to classical theta functions when $r = 0$, and arithmetic theta series when $r = 1$. The adjective "higher" refers to the fact, established in [FYZ21a], that these higher arithmetic theta series are related to *higher derivatives* of Siegel-Eisenstein series, a generalization of the Siegel-Weil and arithmetic Siegel-Weil formulas.

The main objective of [FYZ21b] was the construction of higher theta series, and the precise formulation of a Modularity Conjecture asserting their modularity property. The construction is itself a substantial task, because certain so-called "singular" Fourier coefficients comprising the Fourier series are especially subtle and complicated (the authors did not know how to define the singular Fourier coefficients at the time of writing [FYZ21a]). In particular, we emphasize that the singular Fourier coefficients appear to be *more* complicated in the function field setting than their counterparts over number fields (see the discussion in §1.2), at least from the perspective of classical algebraic geometry. A new insight of [FYZ21b], however, was that *all* Fourier coefficients – singular or not – have a uniform and concise description in terms of *derived algebraic geometry*; we refer to the Introduction of *loc. cit.* for more discussion of these issues. As was anticipated there, this phenomenon extends to the more traditional number field context of arithmetic theta series: Madapusi has recently found an interpretation of the virtual fundamental classes of special cycles on Shimura varieties in terms of derived algebraic geometry [Mad23].

In the present paper, we prove the modularity of the higher theta series for all r, after realization in ℓ-adic cohomology and restriction to the "generic fiber" (whose technical meaning will be explained below in §1.1). Notably, the argument is completely uniform in all parameters, including both r and the codimension of the cycles, unlike what one has in the number field setting. We also believe that it will apply with little change for symplectic-orthogonal dual pairs, as well as unitary groups with different "signatures", although for comprehensibility we have not written it in the maximum generality here. The proof employs an *arithmetic* incarnation of Fourier duality, which can be seen as a natural generalization to algebraic cycles of the Fourier-analytic argument for the modularity of the classical theta series (i.e., the case $r = 0$). In particular, our proof is completely different from existing proofs of modularity for arithmetic theta series.

1.1. **Formulation of the results.** We turn next to a precise formulation of our results. Let $X' \to X$ be an étale double cover of smooth projective curves over a finite field $\mathbf{F}_q$ of characteristic $p > 2$. Fix integers $n \geq m \geq 1$, and $r \geq 0$.

We recall the following definitions from [FYZ21b, §4.5]:

- Let $\mathrm{Bun}_{GU^-(2m)}$ be the moduli stack of triples $(\mathcal{G}, \mathfrak{M}, h)$ where $\mathcal{G}$ is a vector bundle of rank $2m$ over X', $\mathfrak{M}$ is a line bundle over X, and h is a skew-Hermitian isomorphism $h : \mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^\vee \otimes \nu^*\mathfrak{M} = \sigma^*\mathcal{G}^* \otimes \nu^*(\omega_X \otimes \mathfrak{M})$.
- Let $\mathrm{Bun}_{\widetilde{P}_m}$ be the moduli stack of quadruples $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E})$ where $(\mathcal{G}, \mathfrak{M}, h) \in \mathrm{Bun}_{GU(2m)}$, and $\mathcal{E} \subset \mathcal{G}$ is a Lagrangian sub-bundle (of rank m).
- Let $\mathrm{Sht}^r_{GU(n)}$ be the moduli stack of rank n similitude Hermitian shtukas.

[2]However, Kudla proved in [Kud21, Theorem 1.1] that the Beilinson-Bloch Conjecture can be used to deduce the modularity of the generating series for compact orthogonal Shimura varieties (on the generic fiber) even in signatures that do not admit special divisors. The argument relies on particular features of the Hodge diamond of orthogonal Shimura varieties, and does not apply for unitary Shimura varieties.

In [FYZ21b, §4], we constructed the *higher theta series*

$$\widetilde{Z}_m^r : \mathrm{Bun}_{\widetilde{P}_m}(k) \to \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}).$$

The value of $\widetilde{Z}_m^r$ on a tuple $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E})$ is defined as a Fourier series, with Fourier coefficients $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ where the Fourier parameter a is a Hermitian map $\mathcal{E} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*(\mathfrak{M})$.

The map $\mathrm{Bun}_{\widetilde{P}_m}(k) \to \mathrm{Bun}_{GU(2m)}(k)$, given by forgetting the Lagrangian sub-bundle $\mathcal{E} \subset \mathcal{G}$, is surjective, and [FYZ21b, Modularity Conjecture 4.15] predicts that $\widetilde{Z}_m^r$ descends through this map to induce a function $Z_m^r : \mathrm{Bun}_{GU(2m)}(k) \to \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)})$, as in the diagram below.

$$\begin{array}{ccc} \mathrm{Bun}_{\widetilde{P}_m}(k) & \xrightarrow{\widetilde{Z}_m^r} & \\ \downarrow & & \searrow \\ \mathrm{Bun}_{GU(2m)}(k) & \overset{Z_m^r}{\dashrightarrow} & \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}) \end{array}$$

In other words, the Modularity Conjecture says that the function $\widetilde{Z}_m^r$, which a priori depends on $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E})$, is actually independent of the Lagrangian sub-bundle $\mathcal{E} \subset \mathcal{G}$.

Next we proceed to describe the main theorem of this paper. For $\ell \neq p$, there is an ℓ-adic realization map

$$\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}) \to \mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^r_{GU(n)})$$

where $\mathrm{H}^{\mathrm{BM}}(Y)$ denotes the the ℓ-adic Borel-Moore homology of a space $Y \xrightarrow{\pi} \mathrm{Spec}\, k$, i.e., $\mathrm{H}^{\mathrm{BM}}_{2i}(Y) := \mathrm{H}^{-2i}(Y; \pi^! \mathbf{Q}_{\ell, \mathrm{Spec}\, k}(-i))$. We denote by $|\widetilde{Z}_m^r|_\ell$ the composition of the ℓ-adic realization map with the higher theta series, which is a function

$$|\widetilde{Z}_m^r|_\ell \colon \mathrm{Bun}_{\widetilde{P}_m}(k) \to \mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^r_{GU(n)}).$$

We consider a modification of $|\widetilde{Z}_m^r|_\ell$ according to the following structures:

- The stack $\mathrm{Sht}^r_{GU(n)}$ is locally of finite type, and admits a presentation as an inductive limit of finite type open substacks $\mathrm{Sht}^{r, \leq \mu}_{GU(n)}$ where μ is a Harder-Narasimhan polygon for $GU(n)$. Hence we have restriction maps
$$\mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^r_{GU(n)}) \to \mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^{r, \leq \mu}_{GU(n)})$$
for every μ.
- The stack $\mathrm{Sht}^r_{GU(n)}$ admits a "leg map" $\mathrm{Sht}^r_{GU(n)} \to (X')^r$. Let $\eta = \mathrm{Spec}\, F' \to X'$ be the generic point. Let $\eta^r = \mathrm{Spec}\,(F' \otimes_k \cdots \otimes_k F') \to (X')^r$. Note that η^r contains the generic point of $(X')^r$ but it also contains many more points such as the generic point of the diagonal X'. Hence we have a restriction map
$$\mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^r_{GU(n)}) \to \mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^r_{GU(n)} \times_{(X')^r} \eta^r).$$

Our main result is that the function $|\widetilde{Z}_m^r|_\ell$ is modular after composing with the restriction maps of the bullet points above:

Theorem 1.1.1. *The composition*

$$\mathrm{Bun}_{\widetilde{P}_m}(k) \xrightarrow{|\widetilde{Z}_m^r|_\ell} \mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^r_{GU(n)}) \to \varprojlim_{\mu} \mathrm{H}^{\mathrm{BM}}_{2r(n-m)}(\mathrm{Sht}^{r, \leq \mu}_{GU(n)} \times_{(X')^r} \eta^r)$$

descends through $\mathrm{Bun}_{\widetilde{P}_m}(k) \twoheadrightarrow \mathrm{Bun}_{GU(2m)}(k)$. *In other words, its value on* $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E}) \in \mathrm{Bun}_{\widetilde{P}_m}(k)$ *is independent of the Lagrangian sub-bundle* $\mathcal{E} \subset \mathcal{G}$.

Remark 1.1.2. Theorem 1.1.1 implies the modularity of $|\widetilde{Z}_m^r|_\ell$ restricted to the generic point of $(X')^r$ (hence also the geometric generic fiber of $(X')^r$), but it contains more information. For example, it also implies the modularity of $|\widetilde{Z}_m^r|_\ell$ restricted to the generic point of the diagonal $\Delta(X') \hookrightarrow (X')^r$ (i.e., all legs coincide), whose geometry is quite different from the generic fiber over $(X')^r$.

Remark 1.1.3. We emphasize that Theorem 1.1.1 can be formulated completely within classical algebraic geometry, while its proof will draw upon the theory of *derived algebraic geometry.*

1.2. **Comparison to number fields.** As hinted earlier, the case $r = 0$ of Theorem 1.1.1 is classical, while the case $r = 1$ of Theorem 1.1.1 is parallel to the modularity of arithmetic theta series on the generic fiber of Shimura varieties. Therefore it is natural to compare Theorem 1.1.1 to analogous results for arithmetic theta series on the generic fiber, which we refer to as "generic modularity".

One analogous result to (the $r = 1$ case of) Theorem 1.1.1 is the landmark work of Kudla-Millson [KM90], establishing modularity in Betti cohomology of Shimura varieties, which was a vast generalization of a theorem of Hirzebruch-Zagier [HZ76]. (Note however that this amounts to modularity in the *geometric* generic fiber, which is weaker than modularity in the generic fiber.) This will be discussed further below. For the analogous problem on orthogonal Shimura varieties, a similar result is the work of Borcherds [Bor99] which established the generic modularity in the Chow group for the codimension 1 case. Using Borcherds' work, the third author's thesis [Zha09] proved the generic modularity in arbitrary codimensions conditionally upon a convergence hypothesis. Finally, this convergence hypothesis was established by Bruinier-Westerholt-Raum [BWR15], completing the proof of the generic modularity in arbitrary codimensions (in the orthogonal case). Of course, these achievements built upon work of many other people, whom we have not mentioned.

Naturally, our initial attempts to prove the Modularity Conjecture started by looking to the proofs of the above results for inspiration. However, we did not find a way to adapt any of their ideas to the function field case, for reasons that we will briefly explain.

1.2.1. *The work of Kudla-Millson.* As mentioned above, the modularity of arithmetic theta series in the Betti cohomology of the geometric generic fiber was obtained by Kudla-Millson [KM90]. Roughly speaking, they imitate the proof of modularity for theta functions, but replacing functions by differential forms on the complex points of the relevant Shimura varieties, which are then uniformized by complex hermitian domains. Unfortunately for us, this argument relies fundamentally on features that do not exist in positive characteristic, such as:

- An "analytic description" of cohomology classes in terms of automorphic forms, coming from de Rham theory.
- The control of Betti cohomology of locally symmetric spaces provided by $(\mathfrak{g}, K)$-cohomology.

By contrast, we have no analogous "uniformization" of $\mathrm{Sht}^r_{GU(n)}$, we cannot represent their ℓ-adic cohomology classes by concrete objects close to automorphic functions, and their cohomology groups are comparatively very complicated (e.g., infinite dimensional).

We remark that although the statement of our Theorem 1.1.1 is formally analogous in the $r = 1$ case to the results of Kudla-Millson on geometric modularity in cohomology, the actual arguments seem to have nothing in common. In particular, the reason we restrict to the generic fiber has nothing to do with the previous paragraph; for us the point is that we need to add level structure along certain points on the curve (that we have no control over), and the level-structure cover is generically finite but may fail to be finite when these points coincide with the legs. If the cover were proper over the whole curve, then we would be able to execute our argument over the whole curve. Also, we prove modularity in absolute cohomology, i.e., without having to pass to the *geometric* generic fiber.

1.2.2. *The work of Borcherds, etc.* Except in low rank cases that can be analyzed explicitly, all other approaches to modularity of arithmetic theta series are based on the method of Borcherds [Bor99]. A summary of this method can be found in (for example) [BHK$^+$20, §1.2]. Roughly speaking, it proceeds by using *Borcherds products* to lift weakly holomorphic modular forms to meromorphic forms on the unitary Shimura variety. The divisor of each such form provides a relation in the Chow group of the Shimura variety. Applying this to the entire space of weakly holomorphic modular forms, of the correct weight and level, leads to a host of such relations, which comprise the content of modularity, by Borcherds' modularity criterion.

Unfortunately for us, no analogue of Borcherds lifting exists in positive characteristic.

Moreover, one might say that the strategy above relies implicitly on the fact that the zero-th Fourier coefficient of the generating series has a simple form: it is the negative of the first Chern class of the line bundle of modular forms ω. It is for this reason that constructing modular forms, i.e., sections of ω, produces the right relations. In general, one expects roughly that the *singular* Fourier coefficients of arithmetic theta series to be a power of this Chern class times a cycle that "looks like" a non-singular Fourier coefficient (see [Kud04] for more precise formulations).

By contrast, in [FYZ21b] we proposed a construction of the singular Fourier coefficients for higher theta series over function fields, which turned out to be much more complicated. For example, the constant term

$\mathcal{Z}_{\mathcal{E}}^r(0)$ of the higher arithmetic theta series has a decomposition into *infinitely* many (if $m > 1$) open-closed pieces, indexed by sub-bundles $\mathcal{K} \subset \mathcal{E}$. The piece labeled by the sub-bundle $\mathcal{K} = 0$ is what we call the "least degenerate stratum", while the piece labeled by $\mathcal{K} = \mathcal{E}$ dominates the whole moduli space of shtukas. Correspondingly, the virtual fundamental class $[\mathcal{Z}_{\mathcal{E}}^r(0)]$ is an infinite sum[3] of the form

$$[\mathcal{Z}_{\mathcal{E}}^r(0)] := \sum_{\mathcal{K} \subset \subset \mathcal{E}} \left(\left(\prod_{i=1}^{r} c_{\mathrm{top}}(p_i^* \sigma^* \mathcal{K}^* \otimes \ell_i) \right) \cap [\mathcal{Z}_{\mathcal{E}/\mathcal{K}}^{r,\circ}(\overline{0})] \right). \tag{1.2.1}$$

The notation is explained in [FYZ21b, §4]; we do not explain it here as we only want to refer to coarse aspects of its form.

- The summand indexed by $\mathcal{K} = \mathcal{E}$ contributes the top Chern class of a vector bundle, which is analogous to the Hodge bundle ω^{-1} in the number-field case.
- The summand indexed by $\mathcal{K} = 0$ (which we think of as the "least degenerate" piece) contributes a virtual class defined by certain, somewhat complicated, derived intersections of cycles.
- The intermediate terms, indexed by non-zero proper sub-bundles $\mathcal{K} \subset \mathcal{E}$, contribute some mixture of the above extremes: they are a Chern class times the virtual fundamental class of the "least degenerate" piece from a lower-dimensional situation.

From this perspective, what happens over number fields is that only one summand from (1.2.1) appears (namely, the one corresponding to the "most degenerate stratum"), because the other pieces are precluded by considerations at the archimedean place.

In summary, the vastly more complicated form of (1.2.1), as compared to the number field case, makes it difficult to imagine proving modularity by explicitly constructing all the necessary relations.

1.3. **New ingredients.** Having explained why the pre-existing approaches to modularity do not seem applicable in our setting, we now proceed to describe the novel ingredients featuring into our proof of Theorem 1.1.1.

1.3.1. *Derived fundamental classes.* The elementary but complicated definitions of the virtual fundamental cycles $[\mathcal{Z}_{\mathcal{E}}^r(a)]$, such as in (1.2.1), are too unwieldy for us to work with effectively. A key point is to find a more conceptual description, which is uniform in the Fourier parameter a.

An insight of [FYZ21b] is that the virtual fundamental classes admit an alternative description: they are the "naïve" fundamental classes from a more sophisticated perspective. Namely, recall that the higher theta series is defined as a Fourier series, with Fourier coefficients $[\mathcal{Z}_{\mathcal{E}}^r(a)]$ where a is the Fourier parameter. Here $\mathcal{Z}_{\mathcal{E}}^r(a)$ is a certain space which is finite over $\mathrm{Sht}_{GU(n)}^r$, but often of the "wrong" dimension, so the associated cycle class $[\mathcal{Z}_{\mathcal{E}}^r(a)]$ must be constructed as a *virtual* fundamental class. However, it was discovered in [FYZ21b] that "repeating" the definition of $\mathcal{Z}_{\mathcal{E}}^r(a)$ in the natural way within derived algebraic geometry produces a derived stack $\mathscr{Z}_{\mathcal{E}}^r$ which is *quasi-smooth* of the "correct" dimension. As explained in [Kha19], a quasi-smooth derived stack $\mathscr{S}$ has an intrinsic notion of *fundamental class* $[\mathscr{S}]$, which can be interpreted as a "virtual fundamental class" on its classical truncation. The classical truncation of $\mathscr{Z}_{\mathcal{E}}^r(a)$ is $\mathcal{Z}_{\mathcal{E}}^r(a)$, and we calculated that $[\mathscr{Z}_{\mathcal{E}}^r(a)]$ coincides with the elementary but complicated construction of the virtual class $[\mathcal{Z}_{\mathcal{E}}^r(a)]$; thus for example the "naïve" notion of fundamental class of the derived stack $[\mathscr{Z}_{\mathcal{E}}^r(0)]$ agrees with the unwieldy formula (1.2.1).

1.3.2. *Arithmetic Fourier transform.* As mentioned at the beginning, the modularity of classical theta series is based on the Fourier transform. In order to prove Theorem 1.1.1, we introduce an *arithmetic Fourier transform* over $\mathrm{H}^{\mathrm{BM}}(\mathrm{Sht}_{GU(n)}^r)$, which specializes to a relative version of the usual Fourier transform over finite fields when $r = 0$.

The construction of the arithmetic Fourier transform is formally analogous to that of the usual Fourier transform. We consider a level structure cover "Sht_V^r" $\to \mathrm{Sht}_{GU(n)}^r$, which is an $\mathbf{F}_q$-vector space in stacks over $\mathrm{Sht}_{GU(n)}^r$. (The main reason for working on the generic fiber is that the geometry of the level structure cover is relatively simple over the generic fiber.) Therefore, it has a dual cover "$\mathrm{Sht}_{\widehat{V}}^r$" $\to \mathrm{Sht}_{GU(n)}^r$, and an evaluation map

$$\mathrm{Sht}_V^r \times_{\mathrm{Sht}_{GU(n)}^r} \mathrm{Sht}_{\widehat{V}}^r \xrightarrow{\mathrm{ev}} \mathbf{F}_q.$$

[3]This is a well-defined cycle because $\mathrm{Sht}_{G(U)}^r$ is of infinite type. On any quasi-compact open substack $\mathrm{Sht}_{G(U)}^{r,\leq\mu}$, only finitely many of these summands are supported.

For a nontrivial additive character ψ of $\mathbf{F}_q$, the arithmetic Fourier transform

$$\mathrm{FT}^{\mathrm{arith}} \colon \mathrm{H}^{\mathrm{BM}}(\mathrm{Sht}^r_V) \to \mathrm{H}^{\mathrm{BM}}(\mathrm{Sht}^r_{\widehat{V}})$$

is defined in terms of the diagram

$$\begin{array}{ccccc} & & \mathrm{Sht}^r_V \times_{\mathrm{Sht}^r_{GU(n)}} \mathrm{Sht}^r_{\widehat{V}} & \xrightarrow{\mathrm{ev}} & \mathbf{F}_q \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathrm{Sht}^r_V & & & & \mathrm{Sht}^r_{\widehat{V}} \end{array}$$

by sending $\alpha \in \mathrm{H}^{\mathrm{BM}}(\mathrm{Sht}^r_V)$ to $\mathrm{pr}_{2!}(\mathrm{ev}^* \psi \cdot \mathrm{pr}_1^* \alpha) \in \mathrm{H}^{\mathrm{BM}}(\mathrm{Sht}^r_{\widehat{V}})$.

Recall that the Modularity Conjecture can be phrased as independence of the higher theta series $\widetilde{Z}^r_m$ on the Lagrangian sub-bundle $\mathcal{E} \subset \mathcal{G}$. In other words, its content is that for two different choices of Lagrangian sub-bundles $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{G}$, one has

$$\widetilde{Z}^r_m(\mathcal{G}, \mathfrak{M}, h, \mathcal{E}_1) = \widetilde{\mathcal{Z}}^r_m(\mathcal{G}, \mathfrak{M}, h, \mathcal{E}_2). \tag{1.3.1}$$

In the case of the classical theta series $r = 0$, one can prove this relation as follows. Assuming for simplicity that $\mathcal{E}_1$ and $\mathcal{E}_2$ are *transverse* Lagrangian sub-bundles, one can factor the special "cycles" through a level cover $\mathrm{Sht}^0_V \to \mathrm{Sht}^0_{GU(n)}$ (adding level structure along a subset depending on $\mathcal{E}_1, \mathcal{E}_2$), which is an $\mathbf{F}_q$-vector space over $\mathrm{Sht}^0_{GU(n)}$ equipped with a self-duality. One can show that the Fourier transform of the special "cycle" $\sum_{a_1} [\mathcal{Z}^0_{\mathcal{E}_1}(a_1)]$ for $\mathcal{E}_1$, which is really just a function on the discrete set $|\mathrm{Sht}^0_V|$, is essentially equal to the special cycle $\sum_{a_2} [\mathcal{Z}^0_{\mathcal{E}_2}(a_2)]$ for $\mathcal{E}_2$. The theta functions for $\mathcal{E}_1, \mathcal{E}_2$ are obtained by pairing the respective special cycles with a Gaussian, so then the equality (1.3.1) for $r = 0$ follows from the Plancherel formula (i.e., unitarity of the finite Fourier transform) and the Fourier self-duality of Gaussians.

We can formulate a generalization of this statement for higher r: the special cycles $\sum_{a_1} [\mathcal{Z}^r_{\mathcal{E}_1}(a_1)]$ and $\sum_{a_2} [\mathcal{Z}^r_{\mathcal{E}_2}(a_2)]$ factor through a certain self-dual level cover $\mathrm{Sht}^r_V \to \mathrm{Sht}^r_{GU(n)}$, and:

> The *arithmetic* Fourier transform $\mathrm{FT}^{\mathrm{arith}}$ of the special cycle $\sum_{a_1} [\mathcal{Z}^r_{\mathcal{E}_1}(a_1)]$ should be essentially equal to the special cycle $\sum_{a_2} [\mathcal{Z}^r_{\mathcal{E}_2}(a_2)]$. (1.3.2)

However, for $r > 0$ it is much less clear how one would prove such a statement, and this requires another innovation that we describe next.

Remark 1.3.1. The arithmetic transform does not depend on features specific to the function field context, such as the possibility of "multiple legs" or the existence of categorifications, so it makes sense even for Shimura varieties as in the traditional context of arithmetic theta series. It is therefore enticing to wonder how much of our strategy can be ported over to number fields. One new puzzle that arises when trying to do this is that the Archimedean place must be incorporated somehow (even when working on the generic fiber).

1.3.3. *The sheaf-cycle correspondence.* Grothendieck's sheaf-function correspondence associates to an ℓ-adic sheaf on a variety over a finite field $\mathbf{F}_q$, a function on its $\mathbf{F}_q$-points. This allows to bring the tools of sheaf theory to bear on of functions, and its utility is by now well-documented in myriad applications.

In fact, Grothendieck's formalism [SGA77] can also be applied to produce higher dimensional cohomology classes from sheaves, although we are not aware of any instance until now where this observation has been used. In order to prove a statement like (1.3.2), we extend Grothendieck's formalism to a framework that we call a *sheaf-cycle* correspondence, in order to bring the tools of sheaf theory to bear on the analysis of algebraic cycles.

To hint at what this entails, we recall that in the sheaf-function correspondence, one begins with an endomorphism of a sheaf, and then extracts a function by taking the trace of the endomorphism. In the sheaf-cycle correspondence, one begins with a *derived* endomorphism of a sheaf (i.e., a higher Ext class) and then extracts, by a generalization of the "trace" operation, a cycle class in cohomology. (By working with motivic sheaves, one can refine the trace to produce a class in the Chow group, but that is not considered in the present paper).

As usual, in practice it is useful to consider generalizations with complexes instead of sheaves, and correspondences instead of maps. Thus, in its general form, the sheaf-cycle correspondence applies a "trace" to extract a cycle class from a *cohomological correspondence*, which is a certain map of complexes.

To prove the precise statement underlying (1.3.2), we realize the virtual fundamental classes $\sum_{a_1}[\mathcal{Z}^r_{\mathcal{E}_1}(a_1)]$ and $\sum_{a_2}[\mathcal{Z}^r_{\mathcal{E}_2}(a_2)]$ as arising by the sheaf-cycle correspondence from cohomological correspondences $\mathfrak{c}_U$ and $\mathfrak{c}_{U^\perp}$, respectively. Then, we prove that a *sheaf-theoretic* Fourier transform essentially takes $\mathfrak{c}_U$ to $\mathfrak{c}_{U^\perp}$. Finally, from this sheaf-theoretic statement we extract (1.3.2) by taking the trace.

The above strategy uses essentially the additional flexibility afforded by sheaves (as opposed to cycles). Cohomological correspondences are maps of sheaves, and to show that the relevant two maps agree involves intricately dissecting them into pieces, Fourier transforming some of the pieces, etc. and then reassembling at the end.

Remark 1.3.2. The classical sheaf-function correspondence includes a compatibility with pushforward and pullback operations; in particular, the pullback compatibility is obvious there. For the sheaf-cycle correspondence, there is a form of pullback compatibility but it is much subtler, and seems to require derived geometry even to formulate (a reflection of the fact that pullback of algebraic cycles is a subtle operation, which is most robustly understood through derived geometry). Perhaps this is a reason why the sheaf-cycle correspondence has taken relatively long to materialize into applications.

1.3.4. *Derived Fourier analysis.* Via the sheaf-cycle correspondence, the duality (1.3.2) ultimately comes out of a new apparatus that we call *derived Fourier analysis.* This involves a generalization of Deligne-Laumon's theory of ℓ-adic Fourier transform, which takes place on vector bundles, to a context that we call "derived vector bundles", which are spaces built out of *perfect complexes*, generalizing how vector bundles are built from locally free coherent sheaves.

An example of a derived vector bundle is the *derived* fibered product of a morphism of classical vector bundles $E' \to E$ with the zero section of E. Derived vector bundles also include certain types of classical stacks as well. Derived vector bundles have duals, and this duality interchanges the "classical stacky" and "derived" directions of derived vector bundles. In particular, the dual of a classical stack can have non-trivial derived structure, and vice versa.

We can give a brief hint as to the role of derived vector bundles. For $r = 0$, the special cycle $\sum_{a_1} \mathcal{Z}^0_{\mathcal{E}_1}(a_1)$ is a counting function on the set of (Hermitian) vector bundles $\mathcal{F}$ over X, which sends $\mathcal{F}$ to the number of maps $\# \operatorname{Hom}_X(\mathcal{E}_1, \mathcal{F})$. For $r > 0$, we want to let $\mathcal{F}$ vary in moduli, but the vector spaces $\operatorname{Hom}_X(\mathcal{E}_1, \mathcal{F})$ do not assemble into a vector bundle as $\mathcal{F}$ varies, for example because their dimensions jump discontinuously with $\mathcal{F}$. However, the "derived vector spaces" $\operatorname{RHom}_X(\mathcal{E}_1, \mathcal{F})$ do (informally speaking) assemble into a derived vector bundle, which is locally of the form described in the first sentence of the preceding paragraph.

In particular, the previously discussed cohomological correspondences $\mathfrak{c}_U$ and $\mathfrak{c}_{U^\perp}$ live on derived vector bundles of the above sort, and are defined using the notion of relative fundamental class for a quasi-smooth map of derived schemes. We therefore develop the theory of ℓ-adic Fourier transform on derived vector bundles in order to compute with them. It turns out that there are several new technical challenges in the derived setting, which would be interesting for further study.

Remark 1.3.3. The primordial forms of derived Fourier analysis were discovered through computations in [FW], and the theory we develop here will be also be applied in *loc. cit.* (along with other ingredients) in order to categorify the Rankin-Selberg unfolding method for automorphic periods. Some of our results on the derived Fourier transform were inspired by work-in-progress of Adeel Khan investigating a derived Fourier transform for homogeneous sheaves.

1.4. **Organization of the paper.** We provide some commentary on the organization of the paper.

In the next section of the paper, §2, we explain a proof of modularity in the special case $r = 0$, as a template/toy model for the general case. Very roughly speaking, this proof will be geometrized from functions to sheaves, and then Theorem 1.1.1 will be extracted from the sheaf-theoretical level by an appropriate trace operation (which depends on r). The proof for $r = 0$ is in §2.3, and then in §2.4 we give an overview of the strategy for the general case, which relies on a setup that we call the "transverse Lagrangian ansatz". As the implementation of the strategy is quite long and involved, we recommend referring back to this overview repeatedly for guidance. In particular, we defer a discussion of the organization of some individual sections of this paper to §2.4.

Part I, consisting of §3 – §5, is devoted to the formalism of cohomological correspondences their interaction with algebraic cycles through the sheaf-cycle correspondence.

The notion of a cohomological correspondence, and the operation of extracting an algebraic cycle as the trace of a cohomological correspondence, are explained in §4.1. The majority of Part I is devoted to constructing the functoriality operations for cohomological correspondences, and establishing their compatibility with the formation of the trace. This story is much subtler than its analogue for the classical sheaf-function correspondence; in particular, derived geometry already arises naturally and crucially in the basic formulations.

Part II, consisting of §6 – §8, develops Fourier analysis in two new contexts.

In §6, we generalize the Deligne-Laumon Fourier transform for ℓ-adic sheaves to a derived setting. We recall the notion of *derived vector bundles*, which are built out of a perfect complex of coherent sheaves in a manner generalizing how vector bundles are built from locally free coherent sheaves. Then we define the *derived Fourier transform* and state its basic properties, with the proofs deferred to Appendix A. Actually, we are only able to establish one of these properties under a technical assumption of "global presentability", which appears to be an artefact of the proof. This is good enough for our purposes but it would be more satisfactory to remove it, which seems an interesting problem.

Then §7 studies the interaction of the derived Fourier transform with cohomological correspondences between derived vector bundles. Next §8 introduces the arithmetic Fourier transform, establishes its basic properties, and relates it to the derived Fourier transform through the sheaf-cycle correspondence.

Part III, consisting of §9 and §10, assembles the preceding ingredients to complete the proof Theorem 1.1.1. We postpone an overview of the contents of this Part to §2.4.

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1.6. **Notation.** Throughout the paper, let $k = \mathbf{F}_q$ be a finite field.

1.6.1. *Notation related to spaces.* In a previous article [FYZ21b], we took care to use calligraphic fonts like $\mathcal{Z}, \mathcal{M}$ for classical stacks and script fonts like $\mathscr{Z}, \mathscr{M}$ for derived stacks. Starting in this paper, we will always work with derived stacks over k by default (although many of them happen to have the *property* of being classical, i.e., the natural map from the classical truncation is an isomorphism), and we do not use script fonts for derived objects. Hence when we say "Cartesian square" we mean what might be called "derived Cartesian square" (sometimes we keep the adjective "derived" for emphasis), unless noted otherwise. In particular, the notation departs from that of [FYZ21b].

1.6.2. *Notation related to ℓ-adic sheaves.* Let Y be a derived Artin stack locally of finite type over k. Then tautologically the classical truncation Y_{cl} of Y is a *higher Artin stack* in the sense of [LZ17b, §5.4] (this notion goes back to Toën). We let $D(Y_{cl}) := D(Y_{cl}; \overline{\mathbf{Q}}_\ell)_a$ be the bounded derived category of constructible étale sheaves on Y_{cl} as constructed in [LZ17a, §1]. This is the bounded subcategory of a homotopy category of a certain stable ∞-category $\mathcal{D}(Y_{cl}; \overline{\mathbf{Q}}_\ell)_a$ constructed in *loc. cit.*, but we shall only need the six functors and their properties at the level of homotopy categories; for our purposes we prefer the framework of [LZ17b, LZ17a] because of their generality in handling higher Artin stacks.

Let f be a map of higher Artin stacks. In [LZ17a, §1.3] one finds the construction of f^* and f_* for general f, the construction of $f_!$ and $f^!$ for locally finite type f, and the construction of $- \otimes_Y -$ and $\mathcal{RH}om$. One also finds there ([LZ17a, Theorem 1.3.9 and Theorem 1.3.10]) the Künneth formula, the base change isomorphism, the projection formula, and other "usual" properties of the six functors when f is locally of finite type.

Since $Y_{cl} \to Y$ induces an isomorphism of étale sites by definition, we may set $D(Y) := D(Y_{cl})$. For a map $f \colon Y_1 \to Y_2$ of derived Artin stacks, we define f^*, f_* to be the corresponding functor on classical truncations; if f is locally finite type then we define $f^!$ and $f_!$ to be the corresponding functor on classical truncations. In this way we may bootstrap all of [LZ17b, LZ17a] to the setting of derived Artin stacks.

Remark 1.6.1. It may seem at first that consideration of derived structure is totally irrelevant to the categories of ℓ-adic sheaves. However this is not the case, as derived structures will be used to construct certain natural transformations of functors between such categories, namely the "Gysin natural transformations" associated to quasi-smooth morphisms f. This is analogous to how the Chow group of a derived stack is the same as that of its classical truncation, but the derived structure is still useful to construct a virtual fundamental class within the Chow group.

For a separated morphism $f\colon X \to Y$ of derived Artin stacks (meaning in particular that f is representable in derived schemes), we let $\mathrm{can}(f)$ be the natural transformation $f_! \to f_*$ of functors $D(X) \to D(Y)$. If f is proper, then $f_! = f_*$ and $\mathrm{can}(f)$ is the identity transformation.

For a quasi-smooth morphism $f\colon X \to Y$ of derived Artin stacks, we let $d(f)$ be the *virtual dimension* of f (i.e., Euler characteristic of its tangent complex), which we view as a locally constant function on the source. If $f\colon E \to S$ is a *derived vector bundle* in the sense of §6.1.1, then we call $d(f)$ the *virtual rank* of E, and denote it by $\mathrm{rank}(E)$.

For a locally finite type morphism $f\colon X \to Y$ of derived Artin stacks, we denote by $\mathbf{D}_{X/Y} := f^!\mathbf{Q}_{\ell,Y}$ the relative dualizing sheaf. If f is smooth of relative dimension d, then $f^! = f^*[2d](d)$. We denote by $\mathbf{D}_{X/Y}(-)\colon D(X) \to D(X)^{\mathrm{op}}$ the relative Verdier duality functor, which is represented by the object $\mathbf{D}_{X/Y}$. For $Y = \mathrm{Spec}\ k$, we abbreviate $\mathbf{D}_X := \mathbf{D}_{X/Y}$.

For any ℓ-adic complex $\mathcal{K} \in D(Y)$, we denote by $\mathcal{K}\langle i\rangle := \mathcal{K}[2i](i)$ the indicated shift and Tate twist.

1.6.3. *Notation related to coherent sheaves.* We let $\mathrm{Perf}(Y)$ be the triangulated category of perfect complexes on Y, i.e., the full subcategory of the derived category of quasicoherent sheaves on Y spanned by objects locally quasi-isomorphic to finite complexes of finite rank vector bundles.

For a torsion coherent sheaf Q on a curve X' we let D_Q be its scheme-theoretic support, viewed as a divisor on X', and $|Q| \subset X'$ its set-theoretic support.

In [FYZ21b] we distinguished between the notion of a $\mathrm{GL}(n)$-torsor $\mathcal{F}$ and the associated vector bundle $\mathrm{V}(\mathcal{F})$, because we wanted to consider maps of the associated vector bundles that are not isomorphisms (and so do not come from maps of torsors). However, this would be too much of a notational burden in the present paper, so we use the same notation for $\mathcal{F}$ and its associated vector bundle, trusting that context will make the usage clear.

2. Transverse Lagrangians ansatz

In this section, we will explain a proof of modularity of the higher theta series in the special case $r = 0$, as a toy model for the more general argument. In particular, the argument motivates the introduction of certain auxiliary spaces.

To give a more precise overview of this section:

(1) In §2.1 we review the formulation of the Modularity Conjecture, which says that a certain construction of higher theta series $\widetilde{Z}^r_m$, which a priori depends on a choice of a Lagrangian sub-bundle in a Hermitian bundle, is in fact independent of that choice.
(2) In §2.2, we reduce to the Modularity Conjecture for $\widetilde{Z}^r_m$ to a slightly weaker independence statement, namely that the values of $\widetilde{Z}^r_m$ on two *tranverse* Lagrangians coincide.
(3) In §2.3, we prove this independence statement in the case $r = 0$. This involves finite Fourier analysis on various auxiliary vector spaces.
(4) In §2.4, we outline the proof of the general case, indicating in particular the ansatz of spaces and maps that will be used to generalize the modularity argument from $r = 0$ to arbitrary r.

2.1. The modularity conjecture for higher theta series.

Recall from [FYZ21b, §4.5] that $\mathrm{Bun}_{GU(n)}$ parametrizes triples $(\mathcal{F}, \mathfrak{L}, h)$, where $\mathcal{F}$ is a vector bundle on X' of rank n, $\mathfrak{L}$ is a line bundle on X, and $h : \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^\vee \otimes \nu^*\mathfrak{L}$ is an $\mathfrak{L}$-twisted Hermitian structure (i.e., $\sigma^*h^\vee = h$).

Recall from [FYZ21b, §4.6] that $\mathrm{Bun}_{GU^-(2m)}$ parametrizes triples $(\mathcal{G}, \mathfrak{M}, h)$, where $\mathcal{G}$ is a vector bundle on X' of rank $2m$, $\mathfrak{M}$ is a line bundle on X, and $h : \mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^\vee \otimes \nu^*\mathfrak{M}$ is an $\mathfrak{M}$-twisted *skew-Hermitian* structure (i.e., $\sigma^*h^\vee = -h$). Alternatively, we can think of h as an $\mathcal{O}_{X'}$-bilinear perfect pairing

$$(\cdot,\cdot)_h : \mathcal{G} \times \sigma^*\mathcal{G} \to \nu^*(\mathfrak{M} \otimes \omega_X) \tag{2.1.1}$$

satisfying $(\sigma^*\beta, \sigma^*\alpha)_h = -\sigma^*(\alpha,\beta)_h$ for local sections α and β of $\mathcal{G}$ respectively.

Let $\mathrm{Bun}_{\widetilde{P}_m}$ be the moduli stack of quadruples $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E})$ where $(\mathcal{G}, \mathfrak{M}, h) \in \mathrm{Bun}_{GU^-(2m)}$, and $\mathcal{E} \subset \mathcal{G}$ is a Lagrangian sub-bundle (i.e., $\mathcal{E}$ has rank m and the composition $\mathcal{E} \subset \mathcal{G} \xrightarrow{h} \sigma^*\mathcal{G}^\vee \otimes \nu^*\mathfrak{M} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{M}$ is zero). In [FYZ21b, §4.6], we defined for each $r \geq 0$ and $m \leq n$ a *higher theta series*

$$\widetilde{Z}^r_m \colon \mathrm{Bun}_{\widetilde{P}_m}(k) \to \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}).$$

We briefly recall the definition of $\widetilde{Z}^r_m$. Let $\mathfrak{L} = \omega_X \otimes \mathfrak{M}$. Let $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ be the moduli stack of rank n Hermitian shtukas $\mathcal{F}_\bullet = ((x_i), (\mathcal{F}_i), (f_i), \varphi : \mathcal{F}_r \xrightarrow{\sim} {}^\tau\mathcal{F}_0)$ on X' with r legs and similitude line bundle $\mathfrak{L}$. For a vector bundle $\mathcal{E}$ on X' of rank m, we have the special cycle $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}$ parametrizing a point $\mathcal{F}_\bullet$ of $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$, and maps $t_i : \mathcal{E} \to \mathcal{F}_i$ $(0 \leq i \leq r)$ compatible with the shtuka structure on $\mathcal{F}_\bullet$. For details we refer to [FYZ21b, §2.3]. For a Hermitian map $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{L}$, let $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$ be the open-closed substack of $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}$ consisting of $(\mathcal{F}_\bullet, t_\bullet)$ such that the Hermitian form on $\mathcal{F}_\bullet$ induces the Hermitian map a on $\mathcal{E}$ via $t_\bullet$. Let $\zeta : \mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a) \to \mathrm{Sht}^r_{U(n),\mathfrak{L}} \subset \mathrm{Sht}^r_{GU(n)}$ be the map forgetting $t_\bullet$, which is known to be finite [FYZ21a, Proposition 7.5] and unramified.

In [FYZ21b, Definition 4.8] we have defined a *virtual fundamental class* $[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a))$. Pushing forward along ζ, we get Chow classes

$$\zeta_*[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n),\mathfrak{L}}).$$

The value of $\widetilde{Z}^r_m$ on $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E})$ (recall $\mathfrak{M} = \omega_X^{-1} \otimes \mathfrak{L}$), which we henceforth abbreviate as $(\mathcal{G}, \mathcal{E})$, is defined as

$$\widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}) = \chi(\det \mathcal{E}) q^{n(\deg \mathcal{E} - \deg \mathfrak{L} - \deg \omega_X)/2} \sum_{a \in \mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)} \psi(\langle e_{\mathcal{G},\mathcal{E}}, a \rangle) \zeta_*[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)]. \tag{2.1.2}$$

Here

- $\chi : \mathrm{Pic}_{X'}(k) \to \overline{\mathbf{Q}}_\ell^\times$ is a character whose restriction to $\mathrm{Pic}_X(k)$ is η^n, where $\eta : \mathrm{Pic}_X(k) \to \{\pm 1\}$ is the character corresponding to the double cover X'/X.
- $\psi : \mathbf{F}_q \to \overline{\mathbf{Q}}_\ell^\times$ is a nontrivial character.
- the summation of a runs over the set $\mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)$ of all Hermitian maps $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{L}$, including the singular ones.
- Let $\mathcal{E}' = \mathcal{G}/\mathcal{E}$. The pairing $(\cdot, \cdot)_h$ in (2.1.1) induces a perfect pairing $\mathcal{E} \times \sigma^*\mathcal{E}' \to \nu^*\mathfrak{L}$. This identifies $\mathcal{E}'$ with $\sigma^*\mathcal{E}^* \otimes \nu^*\mathfrak{L}$. We thus have a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \sigma^*\mathcal{E}^* \otimes \nu^*\mathfrak{L} \longrightarrow 0$$

  giving an extension class $e_{\mathcal{G},\mathcal{E}} \in \mathrm{Ext}^1(\sigma^*\mathcal{E}^* \otimes \nu^*\mathfrak{L}, \mathcal{E})$.
- The pairing $\langle -, - \rangle$ is the Serre duality pairing between $\mathrm{Ext}^1(\sigma^*\mathcal{E}^* \otimes \nu^*\mathfrak{L}, \mathcal{E})$ and $\mathrm{Hom}(\mathcal{E}, \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{L})$.

As explained after [FYZ21b, Conjecture 4.15], the modularity of $\widetilde{Z}^r_m$ can be formulated as the assertion that $\widetilde{Z}^r_m$ is actually independent of the choice of Lagrangian sub-bundle $\mathcal{E}$.

2.2. **Reduction to the case of transverse Lagrangians.** We first argue that it suffices to show that whenever $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{G}$ are two *transverse* Lagrangians in $\mathcal{G} \in \mathrm{Bun}_{GU^-(2m)}(k)$, meaning that their intersection in the vector bundle $\mathcal{G}$ is the 0-section, then we have

$$\widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}_1) = \widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}_2).$$

Note that the condition that $\mathcal{E}_1, \mathcal{E}_2$ are transverse is equivalent to their intersection being zero on the (geometric) generic fiber of X'.

Lemma 2.2.1. *Let F'/F be an extension of fields of characteristic not equal to 2 and V be a finite-dimensional F'/F-Hermitian space. Let L_1, L_2 be two Lagrangian subspaces of V. Then there exists a Lagrangian subspace $L \subset V$ such that*

$$L_1 \cap L = L_2 \cap L = 0.$$

Proof. Let $I := L_1 \cap L_2$, an isotropic subspace of V. Then (using that the characteristic of F is not 2) we may find an orthogonal decomposition $V \cong (I \oplus I^*) \oplus V'$ as Hermitian spaces, such that:

- I^* is Lagrangian in $I \oplus I^*$ and the Hermitian form on V induces a polarization $I \xrightarrow{\sim} I^*$.
- $L_1 = I \oplus L'_1$ and $L_2 = I \oplus L'_2$, with each L'_i being Lagrangian in V'.

We will take L to be of the form $I^* \oplus L'$, where L' is a Lagrangian in V' transverse to both L_1' and L_2'. To see that such L' exists, note that the Lagrangian polarization $V' \cong L_1' \oplus L_2'$ induces an identification of L_1' with $(L_2')^*$. Choosing any basis of L_1' induces a dual basis for L_2', and a corresponding decomposition of V' into a direct sum of 2-dimensional Hermitian spaces. This reduces to the case $\dim_{F'}(V') = 2$. In this case, we may arrange $l_1 \in L_1'$ and $l_2 \in L_2'$ whose non-zero pairing under the Hermitian form lies in $(F')^{\sigma=-1}$. Then $l_1 + l_2$ generates a Lagrangian subspace of V' which is transverse to both L_1' and L_2'.

With this choice of L, it is clear that L is tranverse to both L_1 and L_2. □

Corollary 2.2.2. *Suppose that for any $\mathcal{G} \in \mathrm{Bun}_{GU^-(2m)}(k)$ and any two transverse Lagrangian sub-bundles $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{G}$, we have*

$$\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_1) = \widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_2).$$

Then $\widetilde{Z}_m^r$ is modular.

Proof. The meaning of modularity is that $\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_1) = \widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_2)$ for any two (not necessarily transverse) Lagrangian sub-bundles $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{G}$. By Lemma 2.2.1, we can link any two Lagrangian sub-bundles by a Lagrangian sub-bundle which is transverse to both. □

Therefore, in order to establish Theorem 1.1.1, we are reduced to proving:

Theorem 2.2.3. *For any $\mathcal{G} \in \mathrm{Bun}_{GU^-(2m)}(k)$ and any two transverse Lagrangian sub-bundles $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{G}$, we have*

$$|\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_1)|_\ell = |\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_2)|_\ell \in \varprojlim_{\mu} \mathrm{H}^{\mathrm{BM}}_{2(n-m)r}(\mathrm{Sht}^{r,\leq\mu}_{GU(n)} \times_{(X')^r} \eta_r).$$

The formulation in [FYZ21b] involves a similitude line bundle $\mathfrak{L}$ on X'. For sanity of notation, *we will present the proof only in the case where $\mathfrak{L}$ is trivial*, so that it may be omitted entirely. The argument can be adapted to include $\mathfrak{L}$ in a completely straightforward manner. Accordingly, we assume henceforth that the similitude line bundle $\mathfrak{M}$ for $\mathcal{G}$ is ω_X^{-1}, i.e., the Hermitian form on $\mathcal{G}$ is an isomorphism $\mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^*$.

2.3. **The case $r = 0$.**

The proof of Theorem 2.2.3 will be long and complex. Some of the complications are caused by technical issues that are not present for $r = 0$. Therefore, we will illustrate the argument for $r = 0$, which can serve as a simplified model for the general case.

2.3.1. By definition [FYZ21b, Definition 4.13], the higher theta series for $r = 0$ is a function on $\mathrm{Bun}_{U(n)}(\mathbf{F}_q)$, whose value on $\mathcal{F} \in \mathrm{Bun}_{U(n)}(\mathbf{F}_q)$ is given by

$$\begin{aligned}\widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_1)_\mathcal{F} &:= \chi(\det \mathcal{E}_1) q^{n(\deg \mathcal{E}_1 - \deg \omega_X)/2} \sum_{a \in \mathcal{A}_\mathcal{E}(k)} \psi(\langle e_{\mathcal{G},\mathcal{E}_1}, a\rangle) \# \mathcal{Z}^0_{\mathcal{E}_1}(a)_\mathcal{F} \\ &= \chi(\det \mathcal{E}_1) q^{n(\deg \mathcal{E}_1 - \deg \omega_X)/2} \sum_{t \in \mathrm{Hom}(\mathcal{E}_1, \mathcal{F})} \psi(\langle e_{\mathcal{G},\mathcal{E}_1}, a(t)\rangle)\end{aligned}$$

where $a(t) \in \mathrm{Hom}(\mathcal{E}_1, \sigma^*\mathcal{E}_1^\vee)$ is the composition

$$\mathcal{E}_1 \xrightarrow{t} \mathcal{F} \xrightarrow{h_\mathcal{F}} \sigma^*\mathcal{F}^\vee \xrightarrow{\sigma^* t^\vee} \sigma^*\mathcal{E}_1^\vee.$$

We find it more psychologically convenient to rewrite the index of summation as $\mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*) = \mathrm{Hom}(\mathcal{E}_1, \mathcal{F})$.

Similarly, the value of $\widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_2)$ at $\mathcal{F}$ is

$$\widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_2)_\mathcal{F} := \chi(\det \mathcal{E}_2) q^{n(\deg \mathcal{E}_2 - \deg \omega_X)/2} \sum_{t \in \mathrm{Hom}(\mathcal{E}_2, \mathcal{F})} \psi(\langle e_{\mathcal{G},\mathcal{E}_2}, a(t)\rangle).$$

We will show that

$$\widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_1)_\mathcal{F} = \widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_2)_\mathcal{F}, \quad \forall \mathcal{F} \in \mathrm{Bun}_{U(n)}(\mathbf{F}_q), \tag{2.3.1}$$

whenever $\mathcal{E}_1$ and $\mathcal{E}_2$ are transversal Lagrangians in $\mathcal{G}$.

2.3.2. *Preliminaries.* We begin with some preliminaries that are not specific to $r = 0$. Recall that we assume the similitude line bundle $\mathfrak{L}$ is trivial, so that for $(\mathcal{G}, h) \in \mathrm{Bun}_{GU^-(2m)}$, h gives a perfect pairing

$$(\cdot,\cdot)_h : \mathcal{G} \times \sigma^*\mathcal{G} \to \mathcal{O}_{X'}. \tag{2.3.2}$$

Let $\mathcal{E}_1, \mathcal{E}_2$ be Lagrangian sub-bundles of $\mathcal{G}$. Each inclusion $\mathcal{E}_i \hookrightarrow \mathcal{G}$ induces a short exact sequence

$$0 \to \mathcal{E}_i \to (\mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^*) \to \sigma^*\mathcal{E}_i^* \to 0. \tag{2.3.3}$$

Here we are using the form $(\cdot,\cdot)_h$ to induce a perfect pairing $\mathcal{E}_i \times \sigma^*(\mathcal{G}/\mathcal{E}_i) \to \mathcal{O}_{X'}$, which in turn induces an isomorphism $\mathcal{G}/\mathcal{E}_i \cong \sigma^*\mathcal{E}_i^*$.

Another way to formulate the transversality of $\mathcal{E}_1, \mathcal{E}_2$ is as follows. If $\mathcal{E}_1 \cap \mathcal{E}_2$ vanishes, then the composition

$$b_{12} : \mathcal{E}_1 \to \mathcal{G} \to \sigma^*\mathcal{E}_2^*$$

has full rank generically, and therefore has torsion cokernel. Conversely, if the composite map $\mathcal{E}_1 \to \mathcal{E}_2^*$ has torsion cokernel, then $\mathcal{E}_1 \cap \mathcal{E}_2$ vanishes. We can think of b_{12} as given by the pairing

$$\mathcal{E}_1 \times \sigma^*\mathcal{E}_2 \to \mathcal{O}_{X'} \tag{2.3.4}$$

obtained by restricting $(\cdot,\cdot)_h$ from (2.3.2). Similarly we have $b_{21} : \mathcal{E}_2 \to \sigma^*\mathcal{E}_1^*$.

Since we are assuming that $\mathcal{E}_1, \mathcal{E}_2$ are transverse, we may define torsion sheaves Q_1 and Q_2 to fit into the short exact sequences

$$0 \to \mathcal{E}_1 \xrightarrow{b_{12}} \sigma^*\mathcal{E}_2^* \to Q_2 \to 0, \tag{2.3.5}$$

$$0 \to \mathcal{E}_2 \xrightarrow{b_{21}} \sigma^*\mathcal{E}_1^* \to Q_1 \to 0. \tag{2.3.6}$$

Let $\underline{F}'$ be the Zariski constant sheaf on X' with stalks F'. For a torsion sheaf $\mathcal{T}$, $\mathcal{T}^*$ is defined to be $\mathcal{H}om(\mathcal{T}, \underline{F}'/\mathcal{O}_{X'})$. Taking the linear dual (composed with σ^*) of (2.3.5), we get a short exact sequence

$$0 \to \mathcal{E}_2 \xrightarrow{\sigma^* b_{12}^\vee} \sigma^*\mathcal{E}_1^* \to \sigma^*Q_2^* \to 0. \tag{2.3.7}$$

Since $(\cdot,\cdot)_h$ is skew-Hermitian, and b_{12} and b_{21} can be interpreted as the restrictions of $(\cdot,\cdot)_h$ to $\mathcal{E}_1 \times \sigma^*\mathcal{E}_2$ and $\mathcal{E}_2 \times \sigma^*\mathcal{E}_1$ respectively, we have

$$\sigma^* b_{12}^\vee = -b_{21}. \tag{2.3.8}$$

Comparing (2.3.6) and (2.3.7), we get an isomorphism

$$\beta_{12} : Q_1 \xrightarrow{\sim} \sigma^*Q_2^* \tag{2.3.9}$$

compatible with the quotient maps $\sigma^*\mathcal{E}_1^* \to Q_1$ and $\sigma^*\mathcal{E}_1^* \to \sigma^*Q_2^*$ in (2.3.6) and (2.3.7).

Now switching the roles of $\mathcal{E}_1$ and $\mathcal{E}_2$, the same considerations give an isomorphism

$$\beta_{21} : Q_2 \xrightarrow{\sim} \sigma^*Q_1^* \tag{2.3.10}$$

compatible with the quotient maps $\sigma^*\mathcal{E}_2^* \to Q_2$ in (2.3.5) and $\sigma^*\mathcal{E}_2^* \to \sigma^*Q_1^*$ obtained by dualizing (2.3.6).

Lemma 2.3.1. *The maps β_{12} and β_{21} satisfy*

$$\sigma^*\beta_{12}^\vee = -\beta_{21}. \tag{2.3.11}$$

Proof. Let $\mathcal{G}^\sharp = \sigma^*\mathcal{E}_2^* \oplus \sigma^*\mathcal{E}_1^*$; then $\mathcal{G}$ is naturally a subsheaf of $\mathcal{G}^\sharp$ of the same rank. The form $(\cdot,\cdot)_h$ extends to a rational skew-Hermitian pairing

$$(\cdot,\cdot)_h : \mathcal{G}^\sharp \times \sigma^*\mathcal{G}^\sharp \to \underline{F}'. \tag{2.3.12}$$

This restricts to pairings:

$$\sigma^*\mathcal{E}_2^* \times \mathcal{E}_1^* \to \underline{F}' \tag{2.3.13}$$

$$\sigma^*\mathcal{E}_1^* \times \mathcal{E}_2^* \to \underline{F}' \tag{2.3.14}$$

which induce pairings

$$\gamma_{21} : Q_2 \times \sigma^*Q_1 \to \underline{F}'/\mathcal{O}_{X'}, \tag{2.3.15}$$

$$\gamma_{12} : Q_1 \times \sigma^*Q_2 \to \underline{F}'/\mathcal{O}_{X'}. \tag{2.3.16}$$

Unwinding the definitions, we see that β_{12} is induced from γ_{12} and β_{21} is induced from γ_{21}. Since $(\cdot,\cdot)_h$ is skew-Hermitian, we see that for local sections s_1 of Q_1 and s_2 of σ^*Q_2, we have

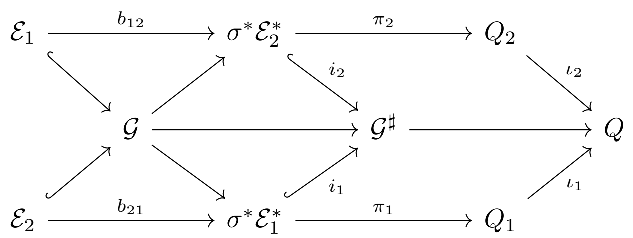
$$\gamma_{12}(s_1, s_2) = -\sigma^*\gamma_{21}(\sigma^*s_2, \sigma^*s_1) \tag{2.3.17}$$

This implies the desired equality for β_{12} and β_{21}. □

2.3.3. *Self-duality of Q.* Recall from the proof of Lemma 2.3.1 that $\mathcal{G}^\sharp = \sigma^*\mathcal{E}_2^* \oplus \sigma^*\mathcal{E}_1^*$ contains $\mathcal{G}$ as a subsheaf of the same rank. Introduce the torsion sheaf Q:

$$Q := \mathcal{G}^\sharp/\mathcal{G} = (\sigma^*\mathcal{E}_2^* \oplus \sigma^*\mathcal{E}_1^*)/\mathcal{G}. \tag{2.3.18}$$

Consider the commutative diagram of coherent sheaves on X',



(2.3.19)

To explain the maps in this diagram:

- The diagonal arrows in and out of $\mathcal{G}$ are as in (2.3.3).
- The horizontal sequences are short exact by definition.
- The maps i_1 and i_2 are the inclusions as a summand by the definition of $\mathcal{G}^\sharp$, and ι_j is induced from i_j.

Lemma 2.3.2. *Maintaining our assumption that the Lagrangians $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{G}$ are transverse, then both $\iota_1 : Q_1 \to Q$ and $\iota_2 : Q_2 \to Q$ are isomorphisms.*

Proof. The maps connecting the first and second rows of (2.3.19) give a map of short exact sequences. By definition $\mathcal{G}^\sharp$ is the pushout of $\mathcal{G}$ along b_{12} (and also along b_{21}), so the cokernels of the hook arrows $\mathcal{E}_1 \hookrightarrow \mathcal{G}$ and $\sigma^*\mathcal{E}_2^* \hookrightarrow \mathcal{G}^\sharp$ are both identified with $\sigma^*\mathcal{E}_1^*$, hence we conclude that ι_2 is an isomorphism. The same argument applied to the second and third rows of (2.3.19) shows that ι_1 is an isomorphism. □

Therefore, composing the isomorphisms in Lemma 2.3.2 with either β_{12} or β_{21}, we get isomorphisms

$$h_{12} : Q \xrightarrow{\iota_1^{-1}} Q_1 \xrightarrow{\beta_{12}} \sigma^*Q_2^* \xrightarrow{\sigma^*(\iota_2^{-1})^\vee} \sigma^*Q^* \tag{2.3.20}$$

$$h_{21} : Q \xrightarrow{\iota_2^{-1}} Q_2 \xrightarrow{\beta_{21}} \sigma^*Q_1^* \xrightarrow{\sigma^*(\iota_1^{-1})^\vee} \sigma^*Q^*. \tag{2.3.21}$$

Lemma 2.3.3. *Both h_{12} and h_{21} give Hermitian structures to Q, and*

$$h_{12} = -h_{21}. \tag{2.3.22}$$

Proof. Let

$$c_{12} : Q \times \sigma^*Q \to \underline{F}'/\mathcal{O}_{X'} \tag{2.3.23}$$

be the pairing induced by h_{12}. Similarly define c_{21}. Then for local sections s and s' of Q, we have

$$c_{12}(s, s') = \gamma_{12}(\iota_1^{-1}(s), \sigma^*(\iota_2^{-1}(s'))), \tag{2.3.24}$$

$$c_{21}(s, s') = \gamma_{21}(\iota_2^{-1}(s), \sigma^*(\iota_1^{-1}(s'))). \tag{2.3.25}$$

The equality $h_{12} = -h_{21}$ is equivalent to $c_{12} = -c_{21}$, which is equivalent to

$$\gamma_{12}(\iota_1^{-1}(s), \sigma^*(\iota_2^{-1}(s'))) + \gamma_{21}(\iota_2^{-1}(s), \sigma^*(\iota_1^{-1}(s'))) = 0. \tag{2.3.26}$$

Note $\mathcal{G}^\sharp \twoheadrightarrow Q_1 \oplus Q_2$. The rational pairing (2.3.12) induces a pairing

$$\gamma : (Q_1 \oplus Q_2) \times (\sigma^*Q_1 \oplus \sigma^*Q_2) \to \underline{F}'/\mathcal{O}_{X'} \tag{2.3.27}$$

whose restriction to $Q_i \times \sigma^*Q_i^*$ is zero, and whose restriction to $Q_1 \times \sigma^*Q_2$ (resp. $Q_2 \times \sigma^*Q_1$) is γ_{12} (resp. γ_{21}). Now the image $\overline{\mathcal{G}}$ of $\mathcal{G}$ in $Q_1 \oplus Q_2$ is isotropic under the above pairing, and both projections $\overline{\mathcal{G}} \to Q_i$ are isomorphisms. Therefore $\overline{\mathcal{G}}$ is the graph of a unique isomorphism $\varphi : Q_1 \xrightarrow{\sim} Q_2$. The fact that $\overline{\mathcal{G}}$ is isotropic implies that for local sections s_1, s_1' of Q_1, we have $\gamma(s_1 + \varphi(s_1), \sigma^*s_1' + \sigma^*(\varphi(s_1'))) = 0$, i.e.,

$$\gamma_{12}(s_1, \sigma^*\varphi(s_1')) + \gamma_{21}(\varphi(s_1), \sigma^*s_1') = 0. \tag{2.3.28}$$

Note that $Q = (Q_1 \oplus Q_2)/\overline{\mathcal{G}}$, hence

$$\iota_2^{-1} \circ \iota_1 = -\varphi : Q_1 \xrightarrow{\sim} Q_2. \tag{2.3.29}$$

We can rewrite (2.3.28) as

$$\gamma_{12}(s_1, \sigma^*(\iota_2^{-1}\iota_1(s_1'))) + \gamma_{21}(\iota_2^{-1}\iota_1(s_1), \sigma^* s_1') = 0, \tag{2.3.30}$$

which confirms (2.3.26) (letting $s = \iota_1(s_1), s' = \iota_1(s_1')$). This finishes the proof. $\square$

2.3.4. *Self-duality of* $\mathrm{Hom}(\mathcal{F}^*, \sigma^*Q)$. Let

$$V := \mathrm{Hom}(\mathcal{F}, \sigma^*Q). \tag{2.3.31}$$

The torsion sheaf $\mathcal{H}om(\mathcal{F}^*, \sigma^*Q) = \mathcal{F} \otimes \sigma^*Q$ carries a $\omega_{X'} \otimes \underline{F}'/\mathcal{O}_{X'}$-valued Hermitian form that is the tensor product of $h_{\mathcal{F}}$ on $\mathcal{F}$ and σ^*c_{12} on σ^*Q:

$$h_{\mathcal{F}} \otimes c_{12} : \mathcal{H}om(\mathcal{F}^*, \sigma^*Q) \times \sigma^*\mathcal{H}om(\mathcal{F}^*, \sigma^*Q) \to \omega_{X'} \otimes \underline{F}'/\mathcal{O}_{X'} = \omega_{F'}/\omega_{X'}. \tag{2.3.32}$$

Here, $\omega_{F'} = \omega_{X'} \otimes_{\mathcal{O}_{X'}} \underline{F}'$ is the sheaf of rational differentials on X'. Taking global sections and applying the residue map, this gives a perfect symmetric $\mathbf{F}_q$-bilinear pairing

$$\langle -,- \rangle_{12} \colon V \times V \to \mathbf{F}_q, \tag{2.3.33}$$

which induces a quadratic form

$$\mathfrak{q}_{12} \colon V \to \mathbf{F}_q.$$

Concretely, for $s : \mathcal{F}^* \to \sigma^*Q$, we have the composition

$$\mathcal{F} \otimes \omega_{X'}^{-1} \xrightarrow{h_{\mathcal{F}}} \sigma^*\mathcal{F}^* \xrightarrow{\sigma^* s} Q \xrightarrow{h_{12}} \sigma^*Q^* \xrightarrow{s^\vee} \mathcal{F}[1] \tag{2.3.34}$$

giving an element in $\mathrm{Ext}^1(\mathcal{F} \otimes \omega_{X'}^{-1}, \mathcal{F})$. Then $\mathfrak{q}_{12}(s)$ is the image of this element under the trace map $\mathrm{Ext}^1(\mathcal{F} \otimes \omega_{X'}^{-1}, \mathcal{F}) \to H^1(X', \omega_{X'}) = \mathbf{F}_q$.

Similarly, using h_{21} on σ^*Q instead of h_{12}, we obtain a perfect symmetric $\mathbf{F}_q$-bilinear pairing $\langle -,- \rangle_{21}$ on $\mathrm{Hom}(\mathcal{F}^*, \sigma^*Q)$ and a quadratic form $\mathfrak{q}_{21}$. By Lemma 2.3.3, we have

$$\mathfrak{q}_{12} = -\mathfrak{q}_{21}. \tag{2.3.35}$$

Lemma 2.3.4. *For* $t \in \mathrm{Hom}(\mathcal{E}_1, \mathcal{F})$*, let* $s \in V = \mathrm{Hom}(\mathcal{F}^*, \sigma^*Q)$ *be the composition*

$$\mathcal{F}^* \xrightarrow{t^\vee} \mathcal{E}_1^* \twoheadrightarrow \sigma^*Q_1 \xrightarrow{\sigma^*\iota_1} \sigma^*Q.$$

Then we have

$$\langle e_{\mathcal{G},\mathcal{E}_1}, a(t) \rangle = -\mathfrak{q}_{21}(s). \tag{2.3.36}$$

Proof. Let $\pi_i : \sigma^*\mathcal{E}_i^* \twoheadrightarrow Q_i$ be the projection. Let $e_1 \in \mathrm{Ext}^1(Q_2, \mathcal{E}_1)$ be the class of the top row of (2.3.19). Since the map $\mathcal{G} \to \sigma^*\mathcal{E}_1^*$ has a section over the subsheaf $\mathcal{E}_2 \hookrightarrow \sigma^*\mathcal{E}_1^*$, $e_{\mathcal{G},\mathcal{E}_1}$ is the image of e_1 under the map

$$\mathrm{Ext}^1(Q_2, \mathcal{E}_1) \cong \mathrm{Ext}^1(Q_1, \mathcal{E}_1) \to \mathrm{Ext}^1(\sigma^*\mathcal{E}_1^*, \mathcal{E}_1) \tag{2.3.37}$$

induced by the projection $\pi_1 \colon \sigma^*\mathcal{E}_1^* \twoheadrightarrow Q_1$ and the isomorphism $\iota_2^{-1}\iota_1 : Q_1 \xrightarrow{\sim} Q_2$. In other words, $e_{\mathcal{G},\mathcal{E}_1}$ is the composition

$$\sigma^*\mathcal{E}_1^* \xrightarrow{\pi_1} Q_1 \xrightarrow[\sim]{\iota_1} Q \xrightarrow[\sim]{\iota_2^{-1}} Q_2 \xrightarrow{e_1} \mathcal{E}_1[1] \tag{2.3.38}$$

Recall we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{b_{12}} & \sigma^*\mathcal{E}_2^* & \xrightarrow{\pi_2} & Q_2 & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle \beta_{21}} & & \\ 0 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{-\sigma^* b_{21}^\vee} & \sigma^*\mathcal{E}_2^* & \longrightarrow & \sigma^*Q_1^* & \longrightarrow & 0 \end{array} \tag{2.3.39}$$

This implies that $e_1 : Q_2 \to \mathcal{E}_1[1]$ can be written as a composition

$$Q_2 \xrightarrow{\beta_{21}} \sigma^*Q_1^* \xrightarrow{-\sigma^*\pi_1^\vee} \mathcal{E}_1[1]. \tag{2.3.40}$$

Using this and (2.3.38), we can rewrite $e_{\mathcal{G},\mathcal{E}_1}$ as the composition

$$\sigma^*\mathcal{E}_1^* \xrightarrow{\pi_1} Q_1 \xrightarrow[\sim]{\iota_1} Q \xrightarrow[\sim]{\iota_2^{-1}} Q_2 \xrightarrow{\beta_{21}} \sigma^*Q_1^* \xrightarrow{-\sigma^*\pi_1^\vee} \mathcal{E}_1[1] \tag{2.3.41}$$

Using the definition of h_{21} in (2.3.21), we see this is the composition

$$\sigma^*\mathcal{E}_1^* \xrightarrow{\iota_1\pi_1} Q \xrightarrow{h_{21}} \sigma^*Q^* \xrightarrow{-\sigma^*(\iota_1\pi_1)^\vee} \mathcal{E}_1[1] \tag{2.3.42}$$

Therefore $\langle e_{\mathcal{G},\mathcal{E}_1}, a(t)\rangle$ is the trace of

$$\mathcal{F}\otimes\omega_{X'}^{-1} \xrightarrow{h_{\mathcal{F}}} \sigma^*\mathcal{F}^* \xrightarrow{\sigma^*t^\vee} \sigma^*\mathcal{E}_1^* \xrightarrow{\iota_1\pi_1} Q \xrightarrow{h_{21}} \sigma^*Q^* \xrightarrow{-\sigma^*(\iota_1\pi_1)^\vee} \mathcal{E}_1[1] \xrightarrow{t} \mathcal{F}[1] \tag{2.3.43}$$

Using the definition of s, we can rewrite (2.3.43) as

$$\mathcal{F}\otimes\omega_{X'}^{-1} \xrightarrow{h_{\mathcal{F}}} \sigma^*\mathcal{F}^* \xrightarrow{\sigma^*s} Q \xrightarrow{h_{21}} \sigma^*Q^* \xrightarrow{-s^\vee} \mathcal{F}[1] \tag{2.3.44}$$

which is $-\mathfrak{q}_{21}(s)$ by comparing with (2.3.34). This proves the lemma. □

Remark 2.3.5. Combining Lemma 2.3.4 with (2.3.35), we see that

$$\langle e_{\mathcal{G},\mathcal{E}_1}, a(t)\rangle = \mathfrak{q}_{12}(s). \tag{2.3.45}$$

Switching the roles of $\mathcal{E}_1$ and $\mathcal{E}_2$ in Lemma 2.3.4, we have the following formula. Let $t \in \mathrm{Hom}(\mathcal{E}_2, \mathcal{F})$, let $s \in \mathrm{Hom}(\mathcal{F}^*, \sigma^*Q)$ be the composition

$$\mathcal{F}^* \xrightarrow{t^\vee} \mathcal{E}_2^* \twoheadrightarrow \sigma^*Q_2 \xrightarrow{\sigma^*\iota_2} \sigma^*Q.$$

Then we have

$$\langle e_{\mathcal{G},\mathcal{E}_2}, a(t)\rangle = \mathfrak{q}_{21}(s). \tag{2.3.46}$$

2.3.5. *Rewriting the theta series.* Denote by

$$f\colon \mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*) \to \mathrm{Hom}(\mathcal{F}^*, \sigma^*Q) = V$$

the map induced by the quotient $\sigma^*(\iota_1\circ\pi_1) : \mathcal{E}_1^* \twoheadrightarrow \sigma^*Q_1 \xrightarrow{\sim} \sigma^*Q$. Using (2.3.45), we may rewrite $\widetilde{Z}^0_m(\mathcal{G}, \mathcal{E}_1)_{\mathcal{F}}$ as

$$\begin{aligned} \widetilde{Z}^0_m(\mathcal{G}, \mathcal{E}_1)_{\mathcal{F}} &= \chi(\det\mathcal{E}_1)q^{n(\deg\mathcal{E}_1-\deg\omega_X)/2}\sum_{t\in\mathrm{Hom}(\mathcal{E}_1,\mathcal{F})}\psi(\langle e_{\mathcal{G},\mathcal{E}_1}, a(t)\rangle) \\ &= \chi(\det\mathcal{E}_1)q^{n(\deg\mathcal{E}_1-\deg\omega_X)/2}\sum_{s\in V}\psi(\mathfrak{q}_{12}(s))(f_!\mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)})(s) \end{aligned} \tag{2.3.47}$$

Here $\mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)}$ is the constant function with value 1 on the set $\mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)$, so that for $s \in V$, $(f_!\mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)})(s)$ is the number of maps $\mathcal{E}_1 \to \mathcal{F}$ lying in the fiber over s. For functions φ_1, φ_2 on V, we denote

$$\langle\varphi_1, \varphi_2\rangle_V := \sum_{v\in V}\varphi_1(v)\varphi_2(v)$$

so that (2.3.47) becomes

$$\widetilde{Z}^0_m(\mathcal{G}, \mathcal{E}_1)_{\mathcal{F}} = \chi(\det\mathcal{E}_1)q^{n(\deg\mathcal{E}_1-\deg\omega_X)/2}\langle\mathfrak{q}_{12}^*\psi, f_!\mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)}\rangle_V. \tag{2.3.48}$$

2.3.6. *More dualities.* Applying σ^* to the bottom row of (2.3.19), and using ι_1 to identify Q_1 with Q, we get a short exact sequence

$$0 \to \sigma^*\mathcal{E}_2 \to \mathcal{E}_1^* \to \sigma^*Q \to 0 \tag{2.3.49}$$

which induces a 5-term exact sequence

$$\mathrm{Hom}(\mathcal{F}^*, \sigma^*\mathcal{E}_2) \longrightarrow \mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*) \longrightarrow V \longrightarrow \mathrm{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_2) \longrightarrow \mathrm{Ext}^1(\mathcal{F}^*, \mathcal{E}_1^*). \tag{2.3.50}$$

From the top row of (2.3.19) we get another short exact sequence

$$0 \to \sigma^*\mathcal{E}_1 \to \mathcal{E}_2^* \to \sigma^*Q \to 0 \tag{2.3.51}$$

which induces a 5-term exact sequence

$$\mathrm{Hom}(\mathcal{F}^*, \sigma^*\mathcal{E}_1) \longrightarrow \mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_2^*) \longrightarrow V \longrightarrow \mathrm{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_1) \longrightarrow \mathrm{Ext}^1(\mathcal{F}^*, \mathcal{E}_2^*). \tag{2.3.52}$$

Serre duality exhibits certain dualities between the terms of (2.3.52) and (2.3.50), as indicated in the diagram below:

$$\begin{array}{ccccccccc} \operatorname{Hom}(\mathcal{F}^*, \sigma^*\mathcal{E}_2) & \longrightarrow & \operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_1^*) & \longrightarrow & V & \longrightarrow & \operatorname{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_2) & \longrightarrow & \operatorname{Ext}^1(\mathcal{F}^*, \mathcal{E}_1^*) \\ \operatorname{Hom}(\mathcal{F}^*, \sigma^*\mathcal{E}_1) & \longrightarrow & \operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_2^*) & \longrightarrow & V & \longrightarrow & \operatorname{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_1) & \longrightarrow & \operatorname{Ext}^1(\mathcal{F}^*, \mathcal{E}_2^*) \end{array} \tag{2.3.53}$$

where the dotted arrows connect spaces that are dual. For example, the bottom left term $\operatorname{Hom}(\mathcal{F}^*, \sigma^*\mathcal{E}_1) \cong \operatorname{Hom}(\sigma^*\mathcal{E}_1^*, \mathcal{F})$ is dual (via Serre duality) to

$$\operatorname{Ext}^1(\mathcal{F}, \sigma^*\mathcal{E}_1^* \otimes \omega_{X'}) \overset{h_{\mathcal{F}}}{\cong} \operatorname{Ext}^1(\sigma^*\mathcal{F}^\vee, \sigma^*\mathcal{E}_1^* \otimes \omega_{X'}) \cong \operatorname{Ext}^1(\mathcal{F}^*, \mathcal{E}_1^*),$$

which is the top right term. The self-duality of the middle term V has been explained in §2.3.4.

Lemma 2.3.6. *Under either the pairing $\langle -,-\rangle_{12}$ or $\langle -,-\rangle_{21}$ on V and the Serre duality pairings, the sequence of maps in the first row of* (2.3.53) *is dual to the sequence of maps in the second row, up to sign.*

Proof. This follows from applying $\operatorname{RHom}(\mathcal{F}^*, -)$ to the diagram (2.3.39). □

2.3.7. *Fourier transform over finite fields.* Next we consider the finite Fourier transform on $\operatorname{Hom}(\mathcal{F}^*, \sigma^*Q)$. First we will have to set up our normalizations.

Temporarily in this section, we let V denote a vector space of dimension r over the finite field $k = \mathbf{F}_q$, and $\widehat{V}$ the dual vector space over k. Recall ψ is a nontrivial additive character of k. We define the Fourier transform of a function φ on V by the formula

$$\mathrm{FT}_V(\varphi)(\widehat{v}) := (-1)^r \sum_{v \in V} \varphi(v)\psi\langle v, \widehat{v}\rangle, \quad \widehat{v} \in \widehat{V}.$$

This definition is compatible under the sheaf-function correspondence with the sheaf-theoretic Fourier transform to be defined in §6. With this normalization, we have the following properties of the Fourier transform:

- (Involutivity) $\mathrm{FT}_{\widehat{V}} \circ \mathrm{FT}_V(\varphi) = q^r[-1]^*\varphi$, where $[-1]_*\varphi(v) = \varphi(-v)$.
- (Plancherel formula)
$$q^r \sum_{v \in V} \varphi_1(v)\varphi_2(v) = \sum_{\widehat{v} \in \widehat{V}} \mathrm{FT}_V([-1]^*\varphi_1)(\widehat{v})\, \mathrm{FT}_V(\varphi_2)(\widehat{v}). \tag{2.3.54}$$
- (Gaussians) Suppose we have an isomorphism $h \colon V \xrightarrow{\sim} \widehat{V}$ satisfying $\widehat{h} = h$. This induces a quadratic form $\mathfrak{q} \colon V \to \mathbf{F}_q$ given by $\mathfrak{q}(v) = \langle v, h(v)\rangle$ and a quadratic form $\widehat{\mathfrak{q}} \colon \widehat{V} \to \mathbf{F}_q$ given by $\widehat{\mathfrak{q}}(\widehat{v}) = \langle h^{-1}(\widehat{v}), \widehat{v}\rangle$. Then we have
$$\mathrm{FT}_V(\mathfrak{q}^*\psi) = (-1)^r G(V, \mathfrak{q})(-\frac{1}{4}\widehat{\mathfrak{q}})^*\psi, \tag{2.3.55}$$
where $G(V, \mathfrak{q})$ is the Gauss sum
$$G(V, \mathfrak{q}) = \sum_{v \in V} \psi(\mathfrak{q}(v)). \tag{2.3.56}$$
- Let $f \colon V' \to V$ be a linear map between vector spaces of ranks r' and r respectively. This induces a morphism $\widehat{f} \colon \widehat{V} \to \widehat{V}'$ of dual spaces. Then we have
$$\mathrm{FT}_V(f_!\varphi') = (-1)^{r-r'} \widehat{f}^* \, \mathrm{FT}_{V'}(\varphi') \tag{2.3.57}$$
for all functions φ' on V', and
$$\mathrm{FT}_{V'}(f^*\varphi) = (-1)^{r'-r} q^{r'-r} \widehat{f}_! \, \mathrm{FT}_V(\varphi) \tag{2.3.58}$$
for all functions φ on V.

Example 2.3.7. Let $\mathbb{1}_V$ be the constant function on V with value 1 and δ_V be the delta function on V with value 1 at the origin. If $\dim_k(V) = r$ then we have

$$\mathrm{FT}(\mathbb{1}_V) = (-1)^r q^r \delta_{\widehat{V}}$$

and

$$\mathrm{FT}(\delta_V) = (-1)^r \mathbb{1}_{\widehat{V}}.$$

2.3.8. *The Gauss sum.* For a non-degenerate quadratic space $(V, \mathfrak{q}_V)$ over $\mathbf{F}_q$, the normalized Gauss sum (with respect to the fixed additive character ψ on $\mathbf{F}_q$) is defined as

$$\gamma(V, \mathfrak{q}_V) = q^{-\dim V/2} \sum_{v \in V} \psi(\mathfrak{q}_V(v)). \tag{2.3.59}$$

(Here $q^{-\dim V/2}$ means the positive square root of $q^{-\dim V}$.) We have the following well-known facts:

1. $\gamma(V, \mathfrak{q}_V)$ is a fourth root of unity in $\overline{\mathbf{Q}}_\ell$.
2. The function $(V, \mathfrak{q}_V) \mapsto \gamma(V, \mathfrak{q}_V)$ is additive, hence induces a homomorphism from the Witt group $\gamma : \mathrm{Witt}(\mathbf{F}_q) \to \mu_4(\overline{\mathbf{Q}}_\ell)$.
3. For the hyperbolic plane $H_+ := \mathbf{F}_q^2$ and $\mathfrak{q}_+(x, y) = xy$, we have $\gamma(H_+, \mathfrak{q}_+) = 1$. From this one deduces that if the quadratic form $\mathfrak{q}_V$ is split (i.e., there exists a Lagrangian subspace), then $\gamma(V, \mathfrak{q}_V) = 1$.
4. For $H_- := \mathbf{F}_{q^2}$ and $\mathfrak{q}_-(x) = \mathrm{Nm}_{\mathbf{F}_{q^2}/\mathbf{F}_q}(x)$, we have $\gamma(H_-, \mathfrak{q}_-) = -1$. From this one deduces that if V is a $\mathbf{F}_{q^2}$-vector space with a nondegenerate $\mathbf{F}_{q^2}/\mathbf{F}_q$-Hermitian form $(-,-)$, and $\mathfrak{q}_V(x) = \mathrm{Nm}_{\mathbf{F}_{q^2}/\mathbf{F}_q}(x, x)$, then $\gamma(V, \mathfrak{q}_V) = (-1)^{\dim_{\mathbf{F}_{q^2}} V}$.
5. Let k' be a finite extension of $\mathbf{F}_q$, and $(V', \mathfrak{q}_{V'})$ be a quadratic space over k'. On one hand, we can define the normalized Gauss sum $\gamma_{k'}(V', \mathfrak{q}_{V'})$ using the additive character $\psi_{k'} = \psi \circ \mathrm{Tr}_{k'/\mathbf{F}_q}$. On the other hand, we can view V' as a vector space over $\mathbf{F}_q$ equipped with the quadratic form $\underline{\mathfrak{q}}_{V'} := \mathrm{Tr}_{k'/\mathbf{F}_q} \circ \mathfrak{q}_{V'}$. Then we have $\gamma_{k'}(V', \mathfrak{q}_{V'}) = \gamma(V', \underline{\mathfrak{q}}_{V'})$.

Now we are back to the convention that $V = \mathrm{Hom}(\mathcal{F}^*, \sigma^* Q)$. We want to compute $\gamma(V, \mathfrak{q}_{12})$. Define the divisor D'_Q on X' to be the $\sum_{x'} d_{x'} x'$ where $d_{x'}$ is the length of Q at x'. Since Q carries a Hermitian form, we have $d_{x'} = d_{\sigma(x')}$, therefore $D'_Q = \nu^* D_Q$ for a unique divisor D_Q on X.

Lemma 2.3.8. *We have*

$$\gamma(\mathrm{Hom}(\mathcal{F}^*, \sigma^* Q), \mathfrak{q}_{12}) = \gamma(\mathrm{Hom}(\mathcal{F}^*, \sigma^* Q), \mathfrak{q}_{21}) = \eta_{F'/F}(D_Q)^n.$$

Proof. Decompose $Q = \oplus Q_x$ where Q_x is the summand supported over a place $x \in |X|$. Then $(V, \mathfrak{q}_{12}) := (\mathrm{Hom}(\mathcal{F}^*, \sigma^* Q), \mathfrak{q}_{12})$ is the orthogonal direct sum of quadratic spaces $(V_x, \mathfrak{q}_x)$ where $V_x = \mathrm{Hom}(\mathcal{F}^*, \sigma^* Q_x)$ and $\mathfrak{q}_{12,x} = \mathfrak{q}_{12}|_{V_x}$. By observation (2) above, it suffices to consider the case $Q = Q_x$ for some $x \in |X|$.

If x splits into x' and $x'' = \sigma(x')$ in X', then we can write $Q_x = Q_{x'} \oplus Q_{x''}$ according the support, and $V = V_{x'} \oplus V_{x''}$ (where $V_{x'} = \mathrm{Hom}(\mathcal{F}^*, Q_{x'})$ and $V_{x''} = \mathrm{Hom}(\mathcal{F}^*, Q_{x''})$) so that $V_{x'}$ and $V_{x''}$ are both isotropic. By observation (3) above, $\gamma(V, \mathfrak{q}_{12}) = 1$ in this case. On the other hand $\eta_{F'/F}(D_Q) = 1$, hence $\eta_{F'/F}(D_Q)^n = 1$.

Now we consider the case x is inert in X', and let x' be the unique place above x. By observation (5), we may rename the base fields, and thereby assume $k(x) = \mathbf{F}_q$ and $k(x') = \mathbf{F}_{q^2}$. Recall the quadratic form $\mathfrak{q}_{12}$ on V comes from a $\mathbf{F}_{q^2}/\mathbf{F}_q$ Hermitian form $(-,-)$ on V by taking $\mathfrak{q}_{12}(v) = (v, v)$. By observation (4), we have

$$\gamma(V, \mathfrak{q}_{12}) = (-1)^{\dim_{\mathbf{F}_{q^2}} V}.$$

Note that $\dim_{\mathbf{F}_{q^2}} V = nd$ where d is the multiplicity of D_Q. Therefore

$$\gamma(V, \mathfrak{q}_{12}) = (-1)^{nd} = \eta_{F'/F}(D_Q)^n.$$

The same argument works for $\mathfrak{q}_{21}$ in place of $\mathfrak{q}_{12}$. □

2.3.9. *Conclusion of the argument.* We return to the notation $V = \mathrm{Hom}(\mathcal{F}^*, \sigma^* Q)$. At the end of §2.3.5, we expressed

$$\widetilde{Z}^0_m(\mathcal{G}, \mathcal{E}_1)_{\mathcal{F}} = \chi(\det \mathcal{E}_1) q^{n(\deg \mathcal{E}_1 - \deg \omega_X)/2} \langle \mathfrak{q}_{12}^* \psi, f_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)} \rangle_V. \tag{2.3.60}$$

By the Plancherel formula (2.3.54), we have

$$\langle \mathfrak{q}_{12}^* \psi, f_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)} \rangle_V = \frac{1}{q^{\dim V}} \langle [-1]_* \mathrm{FT}(\mathfrak{q}_{12}^* \psi), \mathrm{FT}(f_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)}) \rangle_V.$$

So we turn to analyze $\mathrm{FT}(\mathfrak{q}_{12}^* \psi)$ and $\mathrm{FT}(f_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)})$. Below, we abbreviate $\hom(A, B) := \dim_{\mathbf{F}_q} \mathrm{Hom}(A, B)$ and $\mathrm{ext}^1(A, B) := \dim_{\mathbf{F}_q} \mathrm{Ext}^1(A, B)$.

- By (2.3.55), we have
$$[-1]_* \operatorname{FT}(\mathfrak{q}_{12}^*\psi) = (-1)^{\dim V} G(V, \mathfrak{q}_{12}) \cdot (-\frac{1}{4}\widehat{\mathfrak{q}})^*\psi.$$
As was explained in §2.3.8, especially Lemma 2.3.8, we have
$$G(V, \mathfrak{q}_{12}) = q^{\dim V/2}\gamma(V, \mathfrak{q}_{12}) = \eta_{F'/F}(D_Q)^n q^{\dim V/2}.$$
On the other hand, if we use $\mathfrak{q}_{12}$ to identify V with $\widehat{V}$, then $\widehat{\mathfrak{q}}_{12} = \mathfrak{q}_{12}$. By (2.3.35), we have $-\mathfrak{q}_{12} = \mathfrak{q}_{21}$. Therefore, under this identification, we have
$$[-1]_* \operatorname{FT}(\mathfrak{q}_{12}^*\psi) = (-1)^{\dim V}\eta_{F'/F}(D_Q)^n q^{\dim V/2} \cdot (\frac{1}{4}\mathfrak{q}_{21})^*\psi. \tag{2.3.61}$$
- To analyze the Fourier transform of $f_! \mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)}$, we have a long exact sequence
$$0 \to \operatorname{Hom}(\mathcal{F}^*, \sigma^*\mathcal{E}_2) \to \operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_1^*) \xrightarrow{f} \operatorname{Hom}(\mathcal{F}^*, \sigma^*Q) \xrightarrow{g} \operatorname{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_2) \tag{2.3.62}$$
coming from the applying $\operatorname{RHom}(\mathcal{F}^*, -)$ to the short exact sequence $\sigma^*\mathcal{E}_2 \to \mathcal{E}_1^* \to \sigma^*Q$. From this we get
$$f_! \mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_1)} = q^{\hom(\mathcal{F}^*,\sigma^*\mathcal{E}_2)} g^*\delta_{\operatorname{Ext}^1(\mathcal{F}^*,\sigma^*\mathcal{E}_2)}, \tag{2.3.63}$$
By (2.3.58), the Fourier transform on $V = \operatorname{Hom}(\mathcal{F}^*, \sigma^*Q)$ sends $g^*\delta_{\operatorname{Ext}^1(\mathcal{F}^*,\sigma^*\mathcal{E}_2)}$ to
$$\operatorname{FT}(g^*\delta_{\operatorname{Ext}^1(\mathcal{F}^*,\sigma^*\mathcal{E}_2)}) = (-1)^{\dim V} q^{\dim V - \operatorname{ext}^1(\mathcal{F}^*,\sigma^*\mathcal{E}_2)} \widehat{g}_! \mathbb{1}_{\operatorname{Ext}^1(\mathcal{F}^*,\sigma^*\mathcal{E}_2)^*} \tag{2.3.64}$$
$$= (-1)^{\dim V} q^{\dim V - \operatorname{ext}^1(\mathcal{F}^*,\sigma^*\mathcal{E}_2)} \widehat{g}_! \mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)} \tag{2.3.65}$$

where the last equality uses the duality between the upper left and lower right corners of the diagram (2.3.53), and Lemma 2.3.6.

Putting these equations together and collecting factors yields:
$$\begin{aligned}\widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_1)_{\mathcal{F}} &= \chi(\det \mathcal{E}_1) q^{n(\deg \mathcal{E}_1 - \deg \omega_X)/2} \langle \mathfrak{q}_{12}^*\psi, f_!\mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)}\rangle_V \\ &= \chi(\det \mathcal{E}_1)\eta_{F'/F}(D_Q)^n q^{?} \langle (\frac{1}{4}\widehat{\mathfrak{q}}_{21})^*\psi, \widehat{g}_!\mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)}\rangle_V\end{aligned} \tag{2.3.66}$$
where the exponent of q is
$$? = \frac{n}{2}(\deg \mathcal{E}_1 - \deg \omega_X) + \chi(\mathcal{F} \otimes \sigma^*\mathcal{E}_2) + \frac{1}{2}\dim V. \tag{2.3.67}$$
We want to show that this agrees with
$$\widetilde{Z}_m^0(\mathcal{G}, \mathcal{E}_2)_{\mathcal{F}} = \chi(\det \mathcal{E}_2) q^{n(\deg \mathcal{E}_2 - \deg \omega_X)/2} \langle \mathfrak{q}_{21}^*\psi, \widehat{g}_!\mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)}\rangle_V$$
so we separately compare the sign and the exponent of q.

First, we claim that
$$\langle (\frac{1}{4}\widehat{\mathfrak{q}}_{21})^*\psi, \widehat{g}_!\mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)}\rangle_V = \langle \mathfrak{q}_{21}^*\psi, \widehat{g}_!\mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)}\rangle_V. \tag{2.3.68}$$
Indeed, the factor $1/4 = (1/2)^2$ can be eliminated on the left side since $\widehat{g}_!\mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)}$ is invariant under the scaling action of $\mathbf{F}_q^\times$ on $\operatorname{Hom}(\mathcal{F}^*, Q^*)$.

Next we compare the exponents of q. We expand (2.3.67) as
$$? = \frac{1}{2}(n \deg \mathcal{E}_1 - n \deg \omega_X) + n\deg(\mathcal{E}_2) + m\deg(\mathcal{F}) - \frac{1}{2}mn \deg \omega_{X'} + \frac{1}{2}\dim V. \tag{2.3.69}$$
From (2.3.49) we have $-\deg \mathcal{E}_1 = \deg \mathcal{E}_1^* = \deg \mathcal{E}_2 + \deg Q$. Also $\dim V = n \deg Q$. Substituting these into (2.3.69) and simplifying yields
$$? = \frac{1}{2}(n \deg \mathcal{E}_2 - n \deg \omega_X) + m \deg \mathcal{F} - mn\frac{\deg \omega_{X'}}{2}. \tag{2.3.70}$$
Now, $\mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^\vee$ implies that $\deg \mathcal{F} = \frac{n \deg \omega_{X'}}{2}$, so (2.3.70) equals $\frac{1}{2}(n \deg \mathcal{E}_2 - n \deg \omega_X)$, as desired.

Next we compare the signs. From the top row of (2.3.19) we have
$$\chi(\det \mathcal{E}_1)\chi(\nu^* D_Q) = \chi(\det \sigma^*\mathcal{E}_2^*). \tag{2.3.71}$$

Since $\chi \circ \nu^* = \eta^n$, we have $\chi(\nu^* D_Q) = \eta_{F'/F}(D_Q)^n$, and we may rewrite (2.3.71) as

$$\chi(\det \mathcal{E}_1)\eta_{F'/F}(D_Q)^n = \chi(\det \sigma^* \mathcal{E}_2^*) = \chi(\det \sigma^* \mathcal{E}_2)^{-1}. \tag{2.3.72}$$

Finally, we note that

$$\chi(\det \mathcal{E}_2)\chi(\det \sigma^* \mathcal{E}_2) = \chi(\nu^* \operatorname{Nm} \det \mathcal{E}_2) = \eta(\operatorname{Nm} \det \mathcal{E}_2)^n = 1,$$

so that (2.3.72) agrees with $\chi(\det \mathcal{E}_2)$, as desired.

□

2.4. **Outline of the proof of Theorem 2.2.3.** We now give a brief summary of the proof of Theorem 2.2.3. It is a generalization of the $r = 0$ case in §2.3, but the steps are of course much more complicated. We also take the opportunity to indicate, in more detail than in the Introduction, the role of the individual sections of this paper in this strategy.

Choose transverse Lagrangians $\mathcal{E}_1, \mathcal{E}_2 \hookrightarrow \mathcal{G}$. Let Q be as above. We want to show that

$$\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_1) = \widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_2).$$

(1) We will realize the cycle $[\mathcal{Z}_{\mathcal{E}_1}^r]$, whose summands comprise the Fourier coefficients of $\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_1)$, as the *trace* of a *cohomological correspondence*[4] "$\mathfrak{c}_U$", where U is a derived vector bundle over $\operatorname{Bun}_{U(n)}$ geometrizing the $\mathbf{F}_q$-vector space $\operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)$ which appeared in §2.3.6.

Similarly, we will realize the cycle $[\mathcal{Z}_{\mathcal{E}_2}^r]$, whose summands comprise the Fourier coefficients of $\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_2)$, as the trace of a cohomological correspondence "$\mathfrak{c}_{U^\perp}$", where $U^\perp$ is a derived vector bundle over $\operatorname{Bun}_{U(n)}$ geometrizing the vector space $\operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_2^*)$ which appeared in §2.3.6.

The spaces U and $U^\perp$ are defined in §9.1, the cohomological correspondences $\mathfrak{c}_U$ and $\mathfrak{c}_{U^\perp}$ are defined in §9.4, and the computation of the traces is performed in §10.2, based on general results established in §4.7.

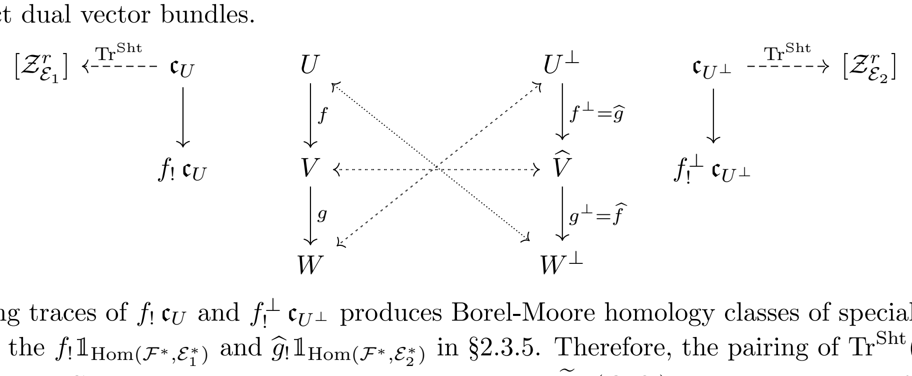
(2) We will construct maps $f\colon U \to V$ and $f^\perp\colon U^\perp \to \widehat{V}$, where V is a vector bundle over $\operatorname{Bun}_{U(n)}$ geometrizing the $\mathbf{F}_q$-vector space $\operatorname{Hom}(\mathcal{F}^*, Q)$ which appeared in §2.3.6. This occurs in §9.1.

(3) We will construct certain vector bundles W and $W^\perp$, geometrizing the $\mathbf{F}_q$-vector spaces $\operatorname{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_2)$ and $\operatorname{Ext}^1(\mathcal{F}^*, \sigma^*\mathcal{E}_1)$, respectively, which appeared in §2.3.6. In addition we will construct maps $g\colon V \to W$ and $g^\perp\colon \widehat{V} \to W^\perp$ such that g geometrizes the "g" from (2.3.62), and $g^\perp$ plays the analogous role with $\mathcal{E}_1$ and $\mathcal{E}_2$ interchanged. This all occurs in §9.1.

(4) We will prove that a certain pushforward cohomological correspondence "$f_! \, \mathfrak{c}_U$" agrees with a pullback cohomological correspondence "$g^* \, \mathfrak{c}_W$", geometrizing the identity (2.3.63). We note that while the identity (2.3.63) is trivial, its geometrization is highly non-obvious, and occupies the entirety of §5.

(5) We will prove that the sheaf-theoretic Fourier transform of cohomological correspondences (introduced in §7.1) takes the cohomological correspondence "$g^* \, \mathfrak{c}_W$" to the cohomological correspondence "$f_!^\perp \, \mathfrak{c}_{U^\perp}$" (up to shift and twist), geometrizing (2.3.64). This occurs in §9.4, based on general results established in §7.

At this point the situation is summarized by the following diagram, in which the dotted arrows connect dual vector bundles.



(6) Forming traces of $f_! \, \mathfrak{c}_U$ and $f_!^\perp \, \mathfrak{c}_{U^\perp}$ produces Borel-Moore homology classes of special cycles, generalizing the $f_! \mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_1^*)}$ and $\widehat{g}_! \mathbb{1}_{\operatorname{Hom}(\mathcal{F}^*, \mathcal{E}_2^*)}$ in §2.3.5. Therefore, the pairing of $\mathrm{Tr}^{\mathrm{Sht}}(f_! \, \mathfrak{c}_U)$ with an appropriate Gaussian produces the higher theta series $\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_1)$, while the pairing of $\mathrm{Tr}^{\mathrm{Sht}}(f_!^\perp \, \mathfrak{c}_{U^\perp})$ with an appropriate Gaussian produces the higher theta series $\widetilde{Z}_m^r(\mathcal{G}, \mathcal{E}_2)$. This is detailed in §10.3.

[4] These notions will be reviewed later, in §4.

$$\begin{array}{ccccccc}
 & & f_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)} & \longleftrightarrow & \widehat{g}_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)} & & \\
 & & \uparrow{\scriptstyle r=0} & & \uparrow{\scriptstyle r=0} & & \\
f_! \mathfrak{c}_U & \xrightarrow{\mathrm{Tr}^{\mathrm{Sht}}} & [\mathcal{Z}^r_{\mathcal{E}_1}] & \xleftrightarrow{\text{arithmetic Fourier transform}} & [\mathcal{Z}^r_{\mathcal{E}_2}] & \xleftarrow{\mathrm{Tr}^{\mathrm{Sht}}} & f_!^{\perp} \mathfrak{c}_{U^\perp}
\end{array}$$

$$\widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}_1) = \langle [\mathcal{Z}^r_{\mathcal{E}_1}], \mathrm{Gaussian}\rangle \qquad \langle [\mathcal{Z}^r_{\mathcal{E}_2}], \mathrm{Gaussian}\rangle = \widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}_2)$$

(7) We introduce in §8 an *arithmetic Fourier transform* on Borel-Moore homology classes, generalizing the finite Fourier transform from §2.3.7. The previous steps imply that the arithmetic Fourier transform sends the special cycle $[\mathcal{Z}^r_{\mathcal{E}_1}]$ for $\mathcal{E}_1$ to the special cycle $[\mathcal{Z}^r_{\mathcal{E}_2}]$ for $\mathcal{E}_2$ (up to signs and powers of q), generalizing the observation that the finite Fourier transform sends $f_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_1^*)}$ to $\widehat{g}_! \mathbb{1}_{\mathrm{Hom}(\mathcal{F}^*,\mathcal{E}_2^*)}$ (up to signs and powers of q) from §2.3.5. Noting again that the Gaussian $\mathfrak{q}^*\psi$ is essentially self-dual, we conclude that

$$\langle [\mathcal{Z}^r_{\mathcal{E}_1}], \mathrm{Gaussian}\rangle = \langle [\mathcal{Z}^r_{\mathcal{E}_2}], \mathrm{Gaussian}\rangle$$

using a version of the Plancherel formula (2.3.54) for the arithmetic Fourier transform. This is carried out in §10.3.

Part 1. Generalities on cohomological correspondences

3. Base change transformations

The purpose of this section is to establish situations in which we can push forward or pull back cohomological correspondences (to be recalled in the next section). This is important for functoriality in the sheaf-cycle correspondence.

We will start with a commutative square

$$\begin{array}{ccc}
A & \xrightarrow{g'} & B \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
C & \xrightarrow{g} & D
\end{array} \tag{3.0.1}$$

We will investigate various situations in which we have base change maps between various combinations of pull/push functors in this diagram, and the compatibilities between them.

Remark 3.0.1. Recall that restriction induces an equivalence of étale sites of a derived stack and its classical truncation. Therefore, one might ask why we discuss derived stacks at all in this section, since all categories and functors are determined by the underlying classical stack. The answer is that when discussing pullbacks (in §3.5) we will need to invoke the relative fundamental class of a quasi-smooth morphism, which is a derived notion.

3.1. **Pushable and pullable squares.** Let $\widetilde{B} = C \times_D B$ (derived fiber product) such that the square (3.0.1) decomposes into a derived Cartesian square and two triangles

$$\begin{array}{ccccc}
A & & & & \\
 & \searrow^{a} & & \overset{g'}{\searrow} & \\
 & & \widetilde{B} & \xrightarrow{\widetilde{g}} & B \\
 & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\
 & & C & \xrightarrow{g} & D
\end{array} \tag{3.1.1}$$

so that $g' = \widetilde{g} \circ a$ and $f' = \widetilde{f} \circ a$.

Definition 3.1.1. The outer square in (3.1.1) is called

- *pushable*, if a is proper.

- *pullable*, if a is quasi-smooth. In this case, we call the relative dimension $d(a)$ the *defect* of the pullable square (A, B, C, D).

Note that the notion of a pushable or pullable is invariant under flipping the square about the diagonal connecting A and D.

Example 3.1.2. Here are some examples of pushable squares:

(1) A square whose reduced classical truncation is Cartesian.
(2) A square where f and f' are proper.
(3) A square where g' is proper and and g is separated and representable in derived schemes.

The pushability of (2) is implied by the pushability of (3) using the flipping symmetry, but we highlight it to make contact with previous constructions in the literature (see Example 3.2.1 below). Some special cases of the above examples are observed in [Var07, §1.1.6].

Example 3.1.3. Here are some examples of pullable squares:

(1) A square whose reduced classical truncation is Cartesian.
(2) A square where f and f' are smooth.
(3) A square where g' is quasi-smooth and g is smooth.

Indeed, in the last case, $\widetilde{g} : \widetilde{B} \to B$ is also smooth since g is, hence a is quasi-smooth since $g' = \widetilde{g} \circ a$ is quasi-smooth.

The pullability of (2) is implied by that of (3) using the flipping symmetry, but we highlight it because it will be of special importance. Some special cases of the above examples are observed in [GV20, §A.2].

Remark 3.1.4. After releasing the first draft of this paper, we learned that the notion of pushability and its significance for pushing forward cohomological correspondences had already been identified in work of Lu-Zheng, [LZ22, Construction 2.10]. We are not aware that the notion of pullability, which is most useful in the context of derived geometry, has previously been identified.

3.2. **Push-pull.** Referring to the diagram (3.1.1), there are always base change natural transformations

$$g^* f_* \to f'_*(g')^* \tag{3.2.1}$$

and

$$f'_!(g')^! \to g^! f_!. \tag{3.2.2}$$

by adjunction. If the outer square in (3.1.1) is pushable, i.e., the map a is proper, then we have a natural transformation

$$g^* f_! \xrightarrow{\diamond} \widetilde{f}_! \widetilde{g}^* \to \widetilde{f}_! a_* a^* \widetilde{g}^* = \widetilde{f}_! a_! a^* \widetilde{g}^* = f'_!(g')^* \tag{3.2.3}$$

Here and below we always label the proper base change isomorphism by $\diamond$. The second map above is the unit map $\mathrm{Id} \to a_* a^*$, and the next step uses $a_! = a_*$ because a is proper.

We often denote such natural transformations coming from a pushable square by $\triangledown$:

$$g^* f_! \xrightarrow{\triangledown} f'_!(g')^*. \tag{3.2.4}$$

Example 3.2.1. In special cases, the map (3.2.3) has been observed before with a slightly different description. In particular, if f and f' are proper then $f_* = f_!$ and $f'_* = f'_!$, and Varshavsky [Var07, §1.1.6] observes that the base change map

$$g^* f_* \to f'_*(g')^* \tag{3.2.5}$$

therefore gives another natural transformation $g^* f_! \to f'_!(g')^*$. Since we will use some of Varshavsky's results concerning his natural transformation, we spell out for completeness why this natural transformation coincides with (3.2.3). Under these assumptions (3.2.3) is adjoint to the composite map

$$f_* \xrightarrow{\mathrm{unit}(g)} f_* \widetilde{g}_* \widetilde{g}^* = g_* \widetilde{f}_* \widetilde{g}^* \xrightarrow{\mathrm{unit}(a)} g_* \widetilde{f}_* a_* a^* \widetilde{g}^* = g_* f'_*(g')^*$$

which since $g' = g \circ a$ coincides with

$$f_* \xrightarrow{\mathrm{unit}(g')} f_*(g')_*(g')^* = g_* f'_*(g')^*$$

which is adjoint to (3.2.5).

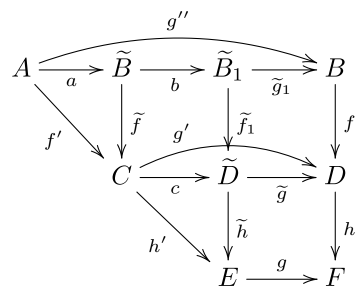
3.2.1. *Compositions.* Suppose we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g''} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g'} & D \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ E & \xrightarrow{g} & F \end{array} \tag{3.2.6}$$

Lemma 3.2.2 (2 out of 3 for pushable squares)**.** *Consider the commutative diagram* (3.2.6)*.*

(1) *If both the upper square and the lower square are pushable, then the outer square formed by* (A, B, E, F) *is also pushable.*
(2) *Suppose g and g' are separated. If the outer square is pushable, then the upper square is also pushable (recall that all maps are assumed to be separated).*

Proof. Introduce the base changes $\widetilde{D} = E \times_F D, \widetilde{B}_1 = \widetilde{D} \times_D B \cong E \times_F B$ and $\widetilde{B} = C \times_D B \cong C \times_{\widetilde{D}} \widetilde{B}_1$. We have a commutative diagram



(3.2.7)

where all squares are Cartesian.

In situation (1), both a and c are proper, hence b is proper (being a base change of c), therefore $b \circ a$ is proper, i.e., the outer square is pushable.

In situation (2), we know that $b \circ a$ is proper. Since g is separated, $\widetilde{g}$ is separated. Since $g' = \widetilde{g} \circ c$ is separated, c is separated. Hence b is separated, and $b \circ a$ proper implies a is proper. □

Assume both the upper square and the lower square are pushable. Then the outer square formed by (A, B, E, F) is also pushable by the above lemma. In this case, we have two maps $g^*(h \circ f)_! \to (h' \circ f')_!(g'')^*$: one given by (3.2.3) for the outer pushable square, the other as the composition

$$g^* h_! f_! \to h'_!(g')^* f_! \to h'_! f'_!(g'')^*$$

where both arrows are (3.2.3) applied to the upper and the lower squares. The next result says that these two maps are the same.

Proposition 3.2.3. *Assume that both the upper square and the lower square in* (3.2.6) *are pushable. Then the following diagram is commutative*

$$\begin{array}{ccccc} g^* h_! f_! & \xrightarrow{\triangledown f_!} & h'_!(g')^* f_! & \xrightarrow{h'_! \triangledown} & h'_! f'_!(g'')^* \\ \| & & & & \| \\ g^*(h \circ f)_! & & \xrightarrow{\quad\quad\triangledown\quad\quad} & & (h' \circ f')_!(g'')^* \end{array} \tag{3.2.8}$$

Proof. Consider the diagram (3.2.7), in which a, b and c are proper.

Consider the diagram of natural transformations

$$\begin{array}{ccccccc} g^*h_!f_! & \xrightarrow{\diamond} & \widetilde{h}_!\widetilde{g}^*f_! & \xrightarrow{u} & \widetilde{h}_!c_*c^*\widetilde{g}^*f_! & = & h'_!c^*\widetilde{g}^*f_! \\ & \searrow^{\diamond} & \downarrow{\scriptstyle\diamond} & & \downarrow{\scriptstyle\diamond} & & \downarrow{\scriptstyle\diamond} \\ & & \widetilde{h}_!\widetilde{f}_{1!}\widetilde{g}_1^* & \xrightarrow{u} & \widetilde{h}_!c_*c^*\widetilde{f}_{1!}\widetilde{g}_1^* & = & h'_!c^*\widetilde{f}_{1!}\widetilde{g}_1^* \\ & & \downarrow{\scriptstyle u} & \star & \downarrow{\scriptstyle\diamond} & & \downarrow{\scriptstyle\diamond} \\ & & \widetilde{h}_!\widetilde{f}_{1!}b_*b^*\widetilde{g}_1^* & = & \widetilde{h}_!c_!\widetilde{f}_!b^*\widetilde{g}_1^* & = & h'_!\widetilde{f}_!b^*\widetilde{g}_1^* \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ & & \widetilde{h}_!\widetilde{f}_{1!}b_*a_*a^*b^*\widetilde{g}_1^* & = & \widetilde{h}_!c_!\widetilde{f}_!a_*a^*b^*\widetilde{g}_1^* & = & h'_!\widetilde{f}_!a_*a^*b^*\widetilde{g}_1^* \end{array} \tag{3.2.9}$$

Here the arrows labelled by $\diamond$ are the proper base change isomorphisms, and the arrows labelled by u are the unit maps $\mathrm{Id} \to a_*a^*, \mathrm{Id} \to b_*b^*$ and $\mathrm{Id} \to c_*c^*$. Starting from the upper left corner, going along the top and then down to reach the lower right corner is the top row of (3.2.8). On the other hand, going along the diagonal first, then down and then right gives the bottom row of (3.2.8). Therefore it suffices to show that each square and triangle in (3.2.9) is commutative. These are almost all clear except possibly the square labelled $\star$. To show $\star$ commutes, we refer to the Cartesian square

$$\begin{array}{ccc} \widetilde{B} & \xrightarrow{b} & \widetilde{B}_1 \\ \downarrow{\scriptstyle\widetilde{f}} & & \downarrow{\scriptstyle\widetilde{f}_1} \\ C & \xrightarrow{c} & D \end{array} \tag{3.2.10}$$

and would like to show that

$$\begin{array}{ccc} \widetilde{f}_{1!} & \xrightarrow{u} & c_*c^*\widetilde{f}_{1!} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle\diamond} \\ \widetilde{f}_{1!}b_*b^* & = & c_!\widetilde{f}_!b^* \end{array} \tag{3.2.11}$$

is commutative. Note that b and c are proper. To check this, it suffices to check on the geometric stalks. Therefore we may reduce to the case where D is a geometric point, in which case both compositions are identified with the map $R\Gamma_c(\widetilde{B}_1, -) \to R\Gamma_c(\widetilde{B}_1, -) \otimes R\Gamma(C)$ given by $x \mapsto x \otimes 1$. □

The following variant of Proposition 3.2.3 will also be needed later.

Proposition 3.2.4. *Suppose we have two pushable squares*

$$\begin{array}{ccccc} A & \xrightarrow{g'} & B & \xrightarrow{h'} & C \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{g} & E & \xrightarrow{h} & F \end{array} \tag{3.2.12}$$

Then:

1. *(1) The outer square is also pushable.*
2. *(2) The following diagram is commutative*

$$\begin{array}{ccccc} g^*h^*f_! & \xrightarrow{g^*\triangledown} & g^*f'_!(h')^* & \xrightarrow{\triangledown(h')^*} & f''_!(g')^*(h')^* \\ \| & & & & \| \\ (h\circ g)^*f_! & & \xrightarrow{\triangledown} & & f''_!(h'\circ g')^* \end{array} \tag{3.2.13}$$

Proof. Part (1) is the same as Lemma 3.2.2(1). Part (2) follows from a similar argument as in the proof of Proposition 3.2.3.

□

3.3. **Push-push.** Suppose the square (3.0.1) is Cartesian. Then we have a base change map

$$f_! g'_* \xrightarrow{\diamond} g_* f'_! . \tag{3.3.1}$$

To construct this, we construct the adjoint map $g^* f_! g'_* \to f'_!$. Then we have the proper base change isomorphism $g^* f_! \xrightarrow{\diamond} f'_!(g')^*$, so composing with the counit of the $((g')^*, g'_*)$ adjunction gives a sequence of natural transformations

$$g^* f_! g'_* \xrightarrow{\diamond} f'_!(g')^* g'_* \to f'_! .$$

Lemma 3.3.1. *Consider the commutative square* (3.0.1)*. Suppose it is Cartesian and the maps g, g' are separated and locally of finite type.*

(1) The following diagram commutes:

$$\begin{array}{ccc} f_! g'_! & = & g_! f'_! \\ {\scriptstyle \mathrm{can}(g')}\downarrow & & \downarrow{\scriptstyle \mathrm{can}(g)} \\ f_! g'_* & \xrightarrow{(3.3.1)} & g_* f'_! \end{array}$$

(2) The following diagram commutes:

$$\begin{array}{ccc} f_! g'_* & \xrightarrow{(3.3.1)} & g_* f'_! \\ {\scriptstyle \mathrm{can}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{can}(f')} \\ f_* g'_* & = & g_* f'_* \end{array}$$

Proof. Since g is separated and locally finite type, we may compactify g into the composition of an open embedding followed by a proper map. The statement is obvious when g is proper, so we reduce to the case where g (hence also g') is an open embedding. It suffices to check that the adjoint diagram

$$\begin{array}{ccc} g^* f_! g'_! & \longrightarrow & f'_! \\ {\scriptstyle \mathrm{can}(g')}\downarrow & & \Vert{\scriptstyle \mathrm{Id}} \\ g^* f_! g'_* & = & f'_! \end{array} \tag{3.3.2}$$

commutes. By definition, the adjoint of the base change map $\diamond$ is the composition

$$g^* f_! g'_* \xrightarrow{\diamond} f'_!(g')^* g'_* \to f'_! .$$

But since g' is an open embedding, $(g')^* g'_!$ is the identity functor. Hence, after applying the isomorphism $g^* f_! \xrightarrow{\diamond} f'_!(g')^*$ to both of the left terms (3.3.2), the upper right path through (3.3.2) is the identity natural transformation $f'_! \xrightarrow{\mathrm{Id}} f'_!$. Similarly, since $(g')^* g'_*$ is also the identity functor, the lower left path through (3.3.2) also evidently the identity natural transformation $f'_! \xrightarrow{\mathrm{Id}} f'_!$. The map $\mathrm{can}(g') \colon (g')^* g'_! \to (g')^* g'_*$ is the identity natural transformation, so the diagram commutes.

(2) Similar to (1). □

Now, suppose that (3.0.1) is pushable. Then natural transformation (3.2.3) induces by adjunction a map

$$f_! g'_* \to g_* f'_! . \tag{3.3.3}$$

Unravelling the construction, it is the composition

$$f_! g'_* = f_! \widetilde{g}_* a_* \xrightarrow{\diamond} g_* \widetilde{f}_! a_* = g_* \widetilde{f}_! a_! \to g_* f'_! . \tag{3.3.4}$$

Here, the map $\diamond$ is (3.3.1).

The natural transformation (3.3.3) has similar compositional properties to (3.2.3).

Proposition 3.3.2. *Assuming both the upper square and the lower square in* (3.2.6) *are pushable. Then the following diagram commutes:*

$$\begin{array}{ccccc} h_! f_! g''_* & \xrightarrow{h_! \triangledown} & h_! g'_* f'_! & \xrightarrow{\triangledown f'_!} & g_* h'_! f'_! \\ \| & & & & \| \\ (h \circ f)_! g''_* & & \xrightarrow{\triangledown} & & g_* (h' \circ f')_! \end{array} \tag{3.3.5}$$

Proof. Similar to the proof of Proposition 3.2.3. □

3.4. **Recollections on relative fundamental classes.** We review some properties of relative fundamental classes.

Let $f \colon Y \to Z$ be a quasi-smooth map of derived stacks, of relative dimension $d(f)$. Then one has a relative fundamental class (also called "Gysin map")

$$f^* \mathbf{Q}_{\ell,Z} \xrightarrow{[f]} f^! \mathbf{Q}_{\ell,Z} \langle -d(f) \rangle. \tag{3.4.1}$$

In fact, (3.4.1) induces a natural transformation $f^* \to f^! \langle -d(f) \rangle$. To see this, we note that there is a base change natural transformation for $\mathcal{K}_1, \mathcal{K}_2 \in D(Z)$

$$f^*(\mathcal{K}_1) \otimes f^!(\mathcal{K}_2) \xrightarrow{\diamond} f^!(\mathcal{K}_1 \otimes \mathcal{K}_2) \tag{3.4.2}$$

arising from the Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{f \times \mathrm{Id}} & Y \times Z \\ \downarrow f & & \downarrow \mathrm{Id} \times f \\ Z & \xrightarrow{\Delta} & Z \times Z \end{array}$$

Taking $\mathcal{K}_2 = \mathbf{Q}_{\ell,Z}$ in (3.4.2), the relative fundamental class induces

$$f^*(\mathcal{K}_1) = f^*(\mathcal{K}_1) \otimes f^*(\mathcal{K}_2) \xrightarrow{(3.4.1)} f^*(\mathcal{K}_1) \otimes f^!(\mathcal{K}_2) \langle -d(f) \rangle \xrightarrow{(3.4.2)} f^!(\mathcal{K}_1 \otimes \mathcal{K}_2) \langle -d(f) \rangle = f^!(\mathcal{K}_1) \langle -d(f) \rangle.$$

The base change property of relative fundamental classes [Kha19, Theorem 3.13] says that for a derived Cartesian square of derived Artin stacks

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Z' \\ \downarrow s' & & \downarrow s \\ Y & \xrightarrow{f} & Z \end{array}$$

with f (hence also f') quasi-smooth of relative dimension $d(f) = d(f')$, the base change homomorphism $(s')^* f^! \to (f')^! s^*$ fits into a commutative diagram

$$\begin{array}{ccc} (s')^* f^* & = & (f')^* s^* \\ \downarrow [f] & & \downarrow [f'] \\ (s')^* f^! \langle -d(f) \rangle & \xrightarrow{\diamond} & (f')^! s^* \langle -d(f) \rangle \end{array} \tag{3.4.3}$$

and dually, the diagram below commutes

$$\begin{array}{ccc} (s')^! f^* & \xleftarrow{\diamond} & (f')^* s^! \\ \downarrow [f] & & \downarrow [f'] \\ (s')^! f^! \langle -d(f) \rangle & = & (f')^! s^! \langle -d(f) \rangle \end{array} \tag{3.4.4}$$

3.5. **Pull-pull.** Suppose the square (3.0.1) is Cartesian. Then we have a base change transformation

$$(f')^* g^! \xrightarrow{\diamond} (g')^! f^*. \tag{3.5.1}$$

To construct this, we construct the adjoint map $g^! \to f'_*(g')^! f^*$. We start with the proper base change natural isomorphism, $g^! f_* \xrightarrow{\sim} f'_*(g')^!$. Then we compose this with the unit of the (f^*, f_*)-adjunction:

$$g^! \xrightarrow{\text{unit}(f)} g^! f_* f^* \xrightarrow{\sim} f'_*(g')^! f^*.$$

Now suppose only that (3.0.1) is pullable, i.e., the map a in (3.1.1) is quasi-smooth. Then we have a natural transformation

$$(f')^* g^! = a^* \widetilde{f}^* g^! \xrightarrow{a^* \diamond} a^* \widetilde{g}^! f^* \xrightarrow{[a]} a^! \widetilde{g}^! f^* \langle -d(a) \rangle = (g')^! f^* \langle -d(a) \rangle. \tag{3.5.2}$$

Here the map $\diamond$ is induced from the proper base change isomorphism as in (3.5.1). We have also used the natural transformation

$$[a] : a^* \to a^! \langle -d(a) \rangle$$

induced by the derived fundamental class of the quasi-smooth map a.

3.5.1. *Gysin natural transformation.* We often denote such a natural transformation induced by a pullable square by $\triangle$,

$$(f')^* g^! \xrightarrow{\triangle} (g')^! f^* \langle -\delta \rangle \tag{3.5.3}$$

where $\delta = d(a)$ is the defect of the pullable square.

Remark 3.5.1. If (3.0.1) is pullable, the map (3.5.2) then induces the following maps by adjunction

$$g'_!(f')^* \to f^* g_! \langle -\delta \rangle. \tag{3.5.4}$$

This resembles the pull-push map (3.2.3) (with the square flipped), but the arrow is in the opposite direction and there is a shift and twist.

Example 3.5.2. If f is smooth and f' is quasi-smooth, then the square (3.0.1) is pullable with defect $d(f) - d(f')$. In this case, the base change map (3.5.2) can be alternatively described as the composition

$$(f')^* g^! \xrightarrow{[f']} (f')^! g^! \langle -d(f') \rangle = (g')^! f^! \langle -d(f') \rangle \cong (g')^! f^* \langle d(f) - d(f') \rangle \tag{3.5.5}$$

where the last isomorphism is induced from the inverse of the isomorphism $[f] : f^* \xrightarrow{\sim} f^! \langle -d(f) \rangle$ since f is smooth. The agreement of the composition above and (3.5.2) can be proved by a similar argument to that of Example 3.2.1.

3.5.2. *Compositions.* The setup is the same as in §3.2.1.

Lemma 3.5.3. *Consider the commutative diagram* (3.2.6).

(1) If both the upper square and the lower square are pullable, then the outer square formed by (A, B, E, F) *is also pullable.*

(2) If $\delta_{\text{upp}}, \delta_{\text{low}}$ *and* δ_{out} *denote the defects of the upper, lower and outer squares respectively, then*

$$\delta_{\text{out}} = \delta_{\text{upp}} + \delta_{\text{low}}. \tag{3.5.6}$$

Proof. (1) Consider the commutative diagram (3.2.7), in which all squares are Cartesian. Both a and c are quasi-smooth, hence b is also quasi-smooth. It follows that $b \circ a$ is quasi-smooth, i.e., the outer square is also pullable.

(2) The defect equality follows from the fact that $\delta_{\text{upp}} = d(a)$, $\delta_{\text{low}} = d(c) = d(b)$ and $\delta_{\text{out}} = d(b \circ a)$. ☐

Proposition 3.5.4. *Suppose in the commutative diagram* (3.2.6) *both the upper and the lower squares are pullable. Let* $\delta_{\text{upp}}, \delta_{\text{low}}$ *and* δ_{out} *denote the defects of the upper, lower and outer squares respectively. Then the following diagram is commutative*

$$\begin{array}{ccc} (f')^*(h')^* g^! \xrightarrow{f'^* \triangle} (f')^*(g')^! h^* \langle -\delta_{\text{low}} \rangle & \xrightarrow{\triangle h^*} & (g'')^! f^* h^* \langle -\delta_{\text{low}} - \delta_{\text{upp}} \rangle \\ \| & & \| \\ (h' \circ f')^* g^! & \xrightarrow{\triangle} & (g'')^! (h \circ f)^* \langle -\delta_{\text{out}} \rangle \end{array} \tag{3.5.7}$$

Proof. Consider the diagram (3.2.9). Let $\widetilde{g}' \colon \widetilde{B} \to B$ be the pullback of g'. Consider the diagram of natural transformations

$$\begin{array}{ccccccc}
(f')^*(h')^*g^! & = & (f')^*c^*\widetilde{h}^*g^! & & & & \\
\| & & \downarrow \diamond & & & & \\
 & & (f')^*c^*\widetilde{g}^!h^* & \xrightarrow{[c]} & (f')^*c^!\widetilde{g}^!h^*\langle -\delta_{\mathrm{low}}\rangle = a^*\widetilde{f}^*c^!\widetilde{g}^!h^*\langle -\delta_{\mathrm{low}}\rangle & & \\
\| & & \| & & \downarrow \diamond & & \\
a^*b^*\widetilde{f}_1^*\widetilde{h}^*g^! & & a^*b^*\widetilde{f}_1^*\widetilde{g}^!h^* & \xrightarrow{[b]} & a^*b^!\widetilde{f}_1^*\widetilde{g}^!h^*\langle -\delta_{\mathrm{low}}\rangle & \xrightarrow{[a]} & a^!b^!\widetilde{f}_1^*\widetilde{g}^!h^*\langle -\delta_{\mathrm{out}}\rangle \\
\downarrow \diamond & & \downarrow \diamond & & \downarrow \diamond & & \downarrow \diamond \\
a^*b^*\widetilde{g}_1^!f^*h^* & = & a^*b^*\widetilde{g}_1^!f^*h^* & \xrightarrow{[b]} & a^*b^!\widetilde{g}_1^!f^*h^*\langle -\delta_{\mathrm{low}}\rangle & \xrightarrow{[a]} & a^!b^!\widetilde{g}_1^!f^*h^*\langle -\delta_{\mathrm{out}}\rangle
\end{array} \tag{3.5.8}$$

Tracing along the top and right edges gives the top and right path of (3.5.7), while tracing along the left and bottom edges of (3.5.8) gives the bottom and left path of (3.5.7), using that $[b] \circ [a] = [b \circ a]$. Therefore, it suffices to verify that all the rectangles of (3.5.8) commute. For the rectangle on the left, this is because the proper base change natural isomorphisms are compatible for compositions of Cartesian squares. For the middle rectangle, the commutativity is an instance of (3.4.3). For the two rectangles in the bottom row, the commutativity is obvious.

□

Proposition 3.5.5. *Suppose we have two pullable squares*

$$\begin{array}{ccccc}
A & \xrightarrow{g'} & B & \xrightarrow{h'} & C \\
\downarrow f'' & & \downarrow f' & & \downarrow f \\
D & \xrightarrow{g} & E & \xrightarrow{h} & F
\end{array} \tag{3.5.9}$$

(1) The outer square is also pullable. Moreover, if we let δ_ℓ and δ_r be the defects of the left and right square, then the defect of the outer square $\delta_{\mathrm{out}} = \delta_\ell + \delta_r$.
(2) The following diagram is commutative

$$\begin{array}{ccccc}
(f'')^*g^!h^! & \xrightarrow{\vartriangle h^!} & (g')^!(f')^*h^! & \xrightarrow{(g')^!\vartriangle} & (g')^!(h')^!f^* \\
\| & & & & \| \\
(f'')^*(h \circ g)^! & & \xrightarrow{\quad\vartriangle\quad} & & (h' \circ g')^!f^*
\end{array} \tag{3.5.10}$$

Proof. (1) is the same as Lemma 3.5.3. The proof of (2) is similar to the proof of Proposition 3.5.4. □

4. The sheaf-cycle correspondence

As explained in the Introduction, an important part of our strategy is the realization of our cycle classes of interest as "traces" (in a suitable sense) of *cohomological correspondences*. Furthermore, we will need notions of pushforward/pullback of cohomological correspondences, and we will need to know in some situations that they interact well with the notion of pushforward/pullback of cycles. We will explain this yoga in this section; it constitutes a framework that we call the "sheaf-cycle correspondence", extending the classical sheaf-function correspondence.

In §4.1 we recall the notion of a cohomological correspondence, and definition of the trace of a cohomological correspondence. In §4.2 we explain a variant of this construction, incorporating a Frobenius twist, that applies over finite fields. The early parts of this section are similar to material already appearing in [SGA77] and [Var07, §1], but we need much more generality than is handled there.

In §4.3 and §4.4 we will review some situations in which cohomological correspondences can be pushed forward or pulled back; this uses the material of §3. In §4.5 we explain some situations in which this pushforward or pullback functoriality is compatible with the formation of traces, with proofs given in §4.6. In §4.7 we apply the theory to compute the trace for certain types of cohomological correspondences that will come up later. Finally, in §4.8 we introduce a dual notion of *cohomological correspondence*, which plays a role in later analysis of the interaction with Fourier transform.

4.1. **The trace of a cohomological correspondence.**

4.1.1. *Basic definitions.* Let Y_0 and Y_1 be derived Artin stacks. A *correspondence between Y_0 and Y_1* is a diagram of derived Artin stacks

$$\begin{array}{ccc} & C & \\ {}^{c_0}\swarrow & & \searrow^{c_1} \\ Y_0 & & Y_1 \end{array}$$

Let $\mathcal{K}_0 \in D(Y_0)$, $\mathcal{K}_1 \in D(Y_1)$. A *cohomological correspondence from $\mathcal{K}_0$ to $\mathcal{K}_1$ supported on C* is a map

$$c_0^* \mathcal{K}_0 \to c_1^! \mathcal{K}_1$$

in $D(C)$. Let

$$\mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_1) := \mathrm{Hom}_C(c_0^* \mathcal{K}_0, c_1^! \mathcal{K}_1) \tag{4.1.1}$$

denote the vector space of cohomological correspondence from $\mathcal{K}_0$ to $\mathcal{K}_1$ supported on C.

4.1.2. *Fixed points of a self-correspondence.* Now suppose that we have a fixed isomorphism $Y_0 \xrightarrow{\sim} Y_1$, which we will sometimes use to identify Y_0 with Y_1; however, it will also be convenient to distinguish them at times. Let $\Delta \colon Y_0 \to Y_0 \times Y_1$ be the diagonal embedding. Define $\mathrm{Fix}(C)$ as the fibered product

$$\begin{array}{ccc} \mathrm{Fix}(C) & \xrightarrow{\Delta'} & C \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} \\ Y_0 & \xrightarrow{\Delta} & Y_0 \times Y_1 \end{array} \tag{4.1.2}$$

where $c = (c_0, c_1)$.

4.1.3. *Trace of a cohomological self-correspondence.* Let $\mathcal{K} \in D(Y_0)$. Then, following [Var07, §1.2][5] we will define a trace map

$$\mathrm{Tr} \colon \mathrm{Corr}_C(\mathcal{K}, \mathcal{K}\langle -i \rangle) \to \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C)) := \mathrm{H}^{-2i}(\mathrm{Fix}(C), \mathbf{D}_{\mathrm{Fix}(C)}(-i)).$$

Observation 4.1.1. Recall that one has the following isomorphisms:

(1) For any $\mathcal{K}_0 \in D(Y_0), \mathcal{K}_1 \in D(Y_1)$, an isomorphism [Var07, §0.3]
$$\mathcal{RH}om_C(c_0^* \mathcal{K}_0, c_1^! \mathcal{K}_1) \cong c^! \mathcal{RH}om_{Y_0 \times Y_1}(\mathrm{pr}_0^* \mathcal{K}_0, \mathrm{pr}_1^! \mathcal{K}_1) \in D(C).$$

(2) For any $\mathcal{K}_0 \in D(Y_0), \mathcal{K}_1 \in D(Y_1)$, an isomorphism [Var07, §0.4]
$$\mathcal{RH}om_{Y_0 \times Y_1}(\mathrm{pr}_0^* \mathcal{K}_0, \mathrm{pr}_1^! \mathcal{K}_1) \cong \mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1 \in D(Y_0 \times Y_1).$$

Definition 4.1.2. Let $\mathfrak{c} \in \mathrm{Corr}_C(\mathcal{K}, \mathcal{K}\langle -i \rangle)$. We will define its *trace* $\mathrm{Tr}(\mathfrak{c}) \in \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C))$. By Observation 4.1.1(1), we have

$$\mathcal{RH}om_C(c_0^* \mathcal{K}, c_1^! \mathcal{K}\langle -i \rangle) \cong c^! \mathcal{RH}om_{Y_0 \times Y_1}(\mathrm{pr}_0^* \mathcal{K}, \mathrm{pr}_1^! \mathcal{K}\langle -i \rangle). \tag{4.1.3}$$

Then Observation 4.1.1(2) gives an isomorphism

$$c^! \mathcal{RH}om_{Y_0 \times Y_1}(\mathrm{pr}_0^* \mathcal{K}, \mathrm{pr}_1^! \mathcal{K}\langle -i \rangle) \cong c^!(\mathbf{D}(\mathcal{K}) \boxtimes \mathcal{K}\langle -i \rangle). \tag{4.1.4}$$

The evaluation map $\mathbf{D}(\mathcal{K}) \otimes \mathcal{K}\langle -i \rangle \to \mathbf{D}_{Y_0}\langle -i \rangle$ induces by adjunction a map

$$\mathbf{D}(\mathcal{K}) \boxtimes \mathcal{K}\langle -i \rangle \to \Delta_* \mathbf{D}_{Y_0}\langle -i \rangle.$$

Composing this with (4.1.3) and (4.1.4) gives a map

$$\mathcal{RH}om_C(c_0^* \mathcal{K}, c_1^! \mathcal{K}\langle -i \rangle) \to c^! \Delta_* \mathbf{D}_{Y_0}\langle -i \rangle. \tag{4.1.5}$$

Finally, using proper base change, we have isomorphisms

$$c^! \Delta_* \mathbf{D}_{Y_0}\langle -i \rangle \cong \Delta'_* (c')^! \mathbf{D}_{Y_0}\langle -i \rangle = \Delta'_* \mathbf{D}_{\mathrm{Fix}(C)}\langle -i \rangle. \tag{4.1.6}$$

We may regard $\mathfrak{c}$ as a global section of $\mathcal{RH}om_C(c_0^* \mathcal{K}, c_1^! \mathcal{K}\langle -i \rangle)$. Then $\mathrm{Tr}(\mathfrak{c}) \in \mathrm{H}^0(C, \Delta'_* \mathbf{D}_{\mathrm{Fix}(C)}\langle -i \rangle) \cong \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C))$ is defined as its image under the composition of (4.1.5) and (4.1.6).

[5]Technically, Varshavsky does not consider the variant with the shifted Tate twist $\langle -i \rangle$.

Remark 4.1.3. It may not be clear why Definition 4.1.2 is similar to the usual notion of "trace". The linear algebraic notion of trace has a natural generalization to symmetric monoidal categories, and it is possible to view the construction above as a special case of this general "categorical trace", at least when $i = 0$. For a development along these lines, see [LZ22].

4.1.4. *Shift and twist.* Let $\mathfrak{c} \in \mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_1)$. For $m, n \in \mathbf{Z}$, the same map $\mathfrak{c} : c_0^* \mathcal{K}_0 \to c_1^! \mathcal{K}_1$ induces a map

$$c_0^* \mathcal{K}_0[m](n) \to c_1^! \mathcal{K}_1[m](n)$$

which we denote by $\mathbb{T}_{[m](n)}\, \mathfrak{c}$. Then $\mathfrak{c} \mapsto \mathbb{T}_{[m](n)}\, \mathfrak{c}$ defines an isomorphism

$$\mathbb{T}_{[m](n)} : \mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_1) \xrightarrow{\sim} \mathrm{Corr}_C(\mathcal{K}_0[m](n), \mathcal{K}_1[m](n)).$$

When $Y_0 = Y_1$ and $\mathcal{K}_1 = \mathcal{K}_0\langle -i\rangle$, we have

$$\mathrm{Tr}(\mathbb{T}_{[m](n)}\, \mathfrak{c}) = (-1)^m \, \mathrm{Tr}(\mathfrak{c}) \in \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C)).$$

4.2. **Fix vs Sht.** We will be working with correspondences of objects over the *finite* field k. Therefore, our objects will have a Frobenius endomorphism Frob, which in terms of the functor points is the absolute Frobenius Frob_q on the test scheme. We will use this to twist the preceding construction by Frobenius.

For a correspondence

$$Y \xleftarrow{c_0} C \xrightarrow{c_1} Y$$

over k, we will let $\mathrm{Sht}(C)$ (or sometimes Sht_Y) be the derived fibered product

$$\begin{array}{ccc} \mathrm{Sht}(C) & \longrightarrow & C \\ \big\downarrow & & \big\downarrow{\scriptstyle (c_0, c_1)} \\ Y & \xrightarrow{(\mathrm{Id}, \mathrm{Frob})} & Y \times Y \end{array} \tag{4.2.1}$$

This derived fibered product can be also be presented with the derived Cartesian square

$$\begin{array}{ccc} \mathrm{Sht}(C) & \longrightarrow & C \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{Frob} \circ c_0, c_1)} \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array} \tag{4.2.2}$$

which is the "fixed point Cartesian square" for the correspondence datum

$$Y \xleftarrow{\mathrm{Frob} \circ c_0} C^{(1)} \xrightarrow{c_1} Y \tag{4.2.3}$$

where $C^{(1)} := C$ but with the left map twisted by Frob. In other words, we have a canonical identification

$$\mathrm{Sht}(C) = \mathrm{Fix}(C^{(1)}). \tag{4.2.4}$$

Given a cohomological correspondence $\mathfrak{c} \colon c_0^* \mathcal{K}_0 \to c_1^! \mathcal{K}_1$ on C, with $\mathcal{K}_i \in D(Y)$, plus the canonical Weil structure $\mathrm{Frob}^* \mathcal{K}_0 \cong \mathcal{K}_0$ (because Y is defined over $k = \mathbf{F}_q$), we have a cohomological correspondence $\mathfrak{c}^{(1)} \colon (\mathrm{Frob} \circ c_0)^* \mathcal{K}_0 \to c_1^! \mathcal{K}_1$. In this way we obtain a linear isomorphism

$$\mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_1) \xrightarrow{\sim} \mathrm{Corr}_{C^{(1)}}(\mathcal{K}_0, \mathcal{K}_1)$$

sending $\mathfrak{c}$ to $\mathfrak{c}^{(1)}$. If $\mathcal{K}_1 = \mathcal{K}_0\langle -i\rangle$, then we define

$$\mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}) := \mathrm{Tr}(\mathfrak{c}^{(1)}) \in \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C^{(1)})) = \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Sht}(C))$$

using the notion of trace in §4.1 for the cohomological correspondence $\mathfrak{c}^{(1)}$ supported on $C^{(1)}$.

Lemma 4.2.1. *The tangent complex* $\mathbf{T}_{\mathrm{Sht}(C)/\mathbf{F}_q}$ *is the restriction of* $\mathbf{T}_{c_1}$. *In particular, if* c_1 *is quasi-smooth, then* $\mathrm{Sht}(C)$ *is quasi-smooth (over* $\mathrm{Spec}\, \mathbf{F}_q$*), of virtual dimension equal to* $d(c_1)$.

Proof. The tangent complex of $\mathrm{Sht}(C)$ over $\mathbf{F}_q$ is the fibered product

$$\begin{array}{ccc} \mathbf{T}_{\mathrm{Sht}(C)/\mathbf{F}_q} & \longrightarrow & \mathbf{T}_{C/\mathbf{F}_q}|_{\mathrm{Sht}(C)} \\ \big\downarrow & & \big\downarrow{\scriptstyle d(\mathrm{Frob} \circ c_0, c_1)} \\ \mathbf{T}_{Y/\mathbf{F}_q}|_{\mathrm{Sht}(C)} & \xrightarrow{d\Delta} & \mathbf{T}_{Y/\mathbf{F}_q}|_{\mathrm{Sht}(C)} \oplus \mathbf{T}_{Y/\mathbf{F}_q}|_{\mathrm{Sht}(C)} \end{array}$$

Since Frob induces the zero map of tangent complexes, this fibered product simplifies to $\mathbf{T}_{c_1}|_{\mathrm{Sht}(C)}$. Since by assumption c_1 is quasi-smooth, this shows that $\mathrm{Sht}(C) \to \mathrm{Spec}\ \mathbf{F}_q$ is quasi-smooth. $\square$

Assume that c_1 is quasi-smooth, so that $[c_1] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(C/Y)$ exists. Then $\mathrm{Sht}(C)$ is quasi-smooth (over $\mathrm{Spec}\ \mathbf{F}_q$), so the derived fundamental class $[\mathrm{Sht}(C)] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Sht}(C))$ exists.

On the other hand, regarding $[c_1]$ as a map $c_1^*\mathbf{Q}_{\ell,Y} \to c_1^!\mathbf{Q}_{\ell,Y}\langle -d(c_1)\rangle$, we have a cohomological correspondence

$$c_0^*\mathbf{Q}_{\ell,Y} = \mathbf{Q}_{\ell,Y} = c_1^*\mathbf{Q}_{\ell,Y} \xrightarrow{[c_1]} c_1^!\mathbf{Q}_{\ell,Y}\langle -d(c_1)\rangle,$$

whose composition we call $\mathfrak{c}_Y$, and regard as an element of $\mathrm{Corr}_C(\mathbf{Q}_{\ell,Y}, \mathbf{Q}_{\ell,Y}\langle -d(c_1)\rangle)$. We equip $\mathbf{Q}_{\ell,Y}$ with the natural Weil structure $\mathrm{Frob}^*\,\mathbf{Q}_{\ell,Y} = \mathbf{Q}_{\ell,Y}$. Then we have $\mathrm{Tr}^{\mathrm{Sht}}_C([\mathfrak{c}_Y]) \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Sht}(C))$. It is natural to ask when it will be true that

$$\mathrm{Tr}^{\mathrm{Sht}}_C(\mathfrak{c}_Y) = [\mathrm{Sht}(C)] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Sht}(C)). \tag{4.2.5}$$

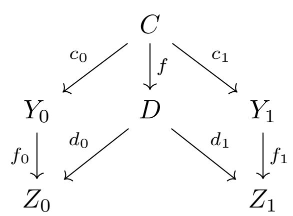
We will see below in §4.7 that (4.2.5) follows from Proposition 4.5.4 whenever Y is smooth. We expect that proving (4.2.5) in more generality will be important for integral modularity statements.

4.2.1. *Shift and twist.* For $\mathfrak{c} \in \mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_0\langle -i\rangle)$ and $m, n \in \mathbf{Z}$, the shifted and twisted cohomological correspondence $\mathbb{T}_{[m](n)}\,\mathfrak{c} \in \mathrm{Corr}_C(\mathcal{K}_0[m](n), \mathcal{K}_0[m](n)\langle -i\rangle)$ is defined in §4.1.4. When $Y_0 = Y_1$ and $\mathcal{K}_1 = \mathcal{K}_0\langle -i\rangle$, we have

$$\mathrm{Tr}^{\mathrm{Sht}}(\mathbb{T}_{[m](n)}\,\mathfrak{c}) = (-1)^m q^{-n}\,\mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}) \in \mathrm{H}^{\mathrm{BM}}_{2i}(\mathrm{Sht}(C)). \tag{4.2.6}$$

This is because the canonical Weil structure $\mathrm{Frob}^*(\mathcal{K}_0[m](n)) \xrightarrow{\sim} \mathcal{K}_0[m](n)$ of $\mathcal{K}_0[m](n)$ is q^{-n} times the Weil structure induced from that of $\mathcal{K}_0[m]$.

4.3. **Pushforward functoriality for cohomological correspondences.** Suppose we have a commutative diagram of correspondences



(4.3.1)

We assume that all morphisms are separated and representable in derived schemes.

Definition 4.3.1. The diagram of correspondences (4.3.1) is called *left pushable* if the square with vertices (C, Y_0, D, Z_0) is pushable in the sense of Definition 3.1.1. In other words, letting $\widetilde{D}_0 = D \times_{Z_0} Y_0$, the natural map $\widetilde{c}_0 = (f, c_0) : C \to \widetilde{D}_0$ is proper.

When (4.3.1) is left pushable, for any cohomological correspondence $c_0^*\mathcal{K}_0 \xrightarrow{\mathfrak{c}} c_1^!\mathcal{K}_1$, we shall construct a "pushforward correspondence" $f_!(\mathfrak{c}) : d_0^* f_{0!}\mathcal{K}_0 \to d_1^! f_{1!}\mathcal{K}_1$, yielding a linear map

$$f_! : \mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{Corr}_D(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1). \tag{4.3.2}$$

Given $c_0^*\mathcal{K}_0 \xrightarrow{\mathfrak{c}} c_1^!\mathcal{K}_1$, we get a map

$$f_! c_0^*\mathcal{K}_0 \xrightarrow{f_!(\mathfrak{c})} f_! c_1^!\mathcal{K}_1. \tag{4.3.3}$$

Recall from (3.2.2) that there is always a base change morphism

$$f_! c_1^!\mathcal{K}_1 \to d_1^! f_{1!}\mathcal{K}_1. \tag{4.3.4}$$

Since the left square of (4.3.1) is pushable, (3.2.3) gives a base change morphism

$$d_0^* f_{0!}\mathcal{K}_0 \to f_! c_0^*\mathcal{K}_0. \tag{4.3.5}$$

Precomposing (4.3.3) with (4.3.5) and post-composing it with (4.3.4), we get natural maps

$$d_0^* f_{0!}\mathcal{K}_0 \xrightarrow{(4.3.5)} f_! c_0^*\mathcal{K}_0 \xrightarrow{(4.3.3)} f_! c_1^!\mathcal{K}_1 \xrightarrow{(4.3.4)} d_1^! f_{1!}\mathcal{K}_1$$

whose composition is a cohomological correspondence from $f_{0!}\mathcal{K}_0$ to $f_{1!}\mathcal{K}_1$ that we denote by $f_!(\mathfrak{c})$.

4.4. **Pullback functoriality for cohomological correspondences.** Consider the diagram of correspondences in (4.3.1).

Definition 4.4.1. The diagram of correspondences (4.3.1) is called *right pullable* if the square with vertices (C, Y_1, D, Z_1) is pullable in the sense of Definition 3.1.1. In other words, letting $\widetilde{D}_1 = D \times_{Z_1} Y_1$, the natural map $\widetilde{c}_1 = (f, c_1) : C \to \widetilde{D}_1$ is quasi-smooth.

In this case, we define the *defect* δ_f of the map of correspondences $f : C \to D$ to be the defect of the square (C, Y_1, D, Z_1), i.e., the relative dimension of $\widetilde{c}_1 \colon C \to D \times_{Y_1} Z_1$.

When (4.3.1) is left pushable, for any cohomological correspondence $d_0^* \mathcal{K}_0 \xrightarrow{\mathfrak{c}} d_1^! \mathcal{K}_1$, we shall construct a "pullback correspondence" $f^*(\mathfrak{c}) : c_0^* f_0^* \mathcal{K}_0 \to c_1^! f_1^* \mathcal{K}_1\langle -\delta_f \rangle$, yielding a linear map

$$f^* : \mathrm{Corr}_D(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{Corr}_C(f_0^* \mathcal{K}_0, f_1^* \mathcal{K}_1\langle -\delta_f \rangle). \tag{4.4.1}$$

Given $d_0^* \mathcal{K}_0 \xrightarrow{\mathfrak{c}} d_1^! \mathcal{K}_1$, we get a map

$$f^* d_0^* \mathcal{K}_0 \xrightarrow{f^*(\mathfrak{c})} f^* d_1^! \mathcal{K}_1. \tag{4.4.2}$$

We have an obvious identification

$$f^* d_0^* \mathcal{K}_0 = c_0^* f_0^* \mathcal{K}_0. \tag{4.4.3}$$

Since the right square of (4.3.1) is pullable, we get a base change morphism (cf. §3.5)

$$f^* d_1^! \mathcal{K}_1 \to c_1^! f_1^* \mathcal{K}_1\langle -\delta_f \rangle. \tag{4.4.4}$$

Precomposing (4.4.2) with (4.4.3) and post-composing it with (4.4.4), we get natural maps

$$c_0^* f_0^* \mathcal{K}_0 \xrightarrow{(4.4.3)} f^* d_0^* \mathcal{K}_0 \xrightarrow{(4.4.2)} f^* d_1^! \mathcal{K}_1 \xrightarrow{(4.4.4)} c_1^! f_1^* \mathcal{K}_1\langle -\delta_f \rangle$$

whose composition is a cohomological correspondence between $f_0^* \mathcal{K}_0$ and $f_1^* \mathcal{K}_1\langle -\delta_f \rangle$ that we denote by $f^*(\mathfrak{c})$.

4.5. **Functoriality for the trace.** Now we examine some situations in which pushforwards or pullbacks of cohomological correspondences are compatible with the formation of trace.

Assume that in (4.3.1) we have fixed identifications

$$Y_0 \cong Y_1, Z_0 \cong Z_1 \text{ and } f_0 \cong f_1 \tag{4.5.1}$$

so that $\mathrm{Fix}(C)$ and $\mathrm{Fix}(D)$ are defined.

4.5.1. *Proper pushforward.* Assume the maps f_0 and f are proper. Then (4.3.1) is left pushable (cf. Example 3.1.2), and the pushforward $f_!$ on cohomological correspondences is defined. Moreover, the induced map $\mathrm{Fix}(f) \colon \mathrm{Fix}(C) \to \mathrm{Fix}(D)$ is proper, so that we have a map $\mathrm{Fix}(f)_! \colon \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C)) \to \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(D))$.

In this situation, we have the following compatibility between $f_!$ and the formation of trace, which generalizes [Var07, Proposition 1.2.5]and [LZ22, Corollary 2.22].

Proposition 4.5.1. *With notation as in* (4.3.1) *and* (4.5.1), *assume that* f_0, f, f_1 *are all proper and let* $\mathfrak{c} \in \mathrm{Corr}_C(\mathcal{K}, \mathcal{K}\langle -i \rangle)$. *Then we have*

$$\mathrm{Tr}(f_! \, \mathfrak{c}) = \mathrm{Fix}(f)_! \, \mathrm{Tr}(\mathfrak{c}) \in \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(D)).$$

Proof. The proof of Corollary [LZ22, Corollary 2.22] goes through verbatim in this slightly more general setting. □

Remark 4.5.2. Varshavsky's proof of [Var07, Proposition 1.2.5] does not generalize immediately to stacks (since he requires the existence of compactifications for all total spaces), but the proof of Lu-Zheng applies as is (cf. [LZ22, Remark 2.24]).

4.5.2. *Smooth pullback.* Assume that the map f_1 (hence also f_0) is smooth, and f is quasi-smooth. Then (4.3.1) is right pullable (cf. Example 3.1.3) of defect $\delta = d(f) - d(f_1)$, hence the map f^* on cohomological correspondences is defined.

On the other hand, we consider the map between fixed points.

Lemma 4.5.3. *Under the above assumptions, the map* $\mathrm{Fix}(f) : \mathrm{Fix}(C) \to \mathrm{Fix}(D)$ *is quasi-smooth of dimension* $\delta = d(f) - d(f_1)$.

Proof. Define $E = \mathrm{Fix}(D) \times_D C$. Then $\mathrm{Fix}(f)$ factors as

$$\mathrm{Fix}(f) : \mathrm{Fix}(C) \xrightarrow{\alpha} E \xrightarrow{\beta} \mathrm{Fix}(D).$$

We get an exact triangle of quasi-coherent sheaves on $\mathrm{Fix}(C)$ by taking cotangent complexes:

$$\alpha^* \mathbf{L}_\beta \to \mathbf{L}_{\mathrm{Fix}(f)} \to \mathbf{L}_\alpha. \tag{4.5.2}$$

From the derived Cartesian square

$$\begin{array}{ccc} E & \xrightarrow{\beta} & \mathrm{Fix}(D) \\ \downarrow{\scriptstyle\gamma} & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

we get that $\mathbf{L}_\beta \cong \gamma^* \mathbf{L}_f$. Denote both Y_0 and Y_1 by Y, and Z_0 and Z_1 by Z. From the derived Cartesian square

$$\begin{array}{ccc} \mathrm{Fix}(C) & \xrightarrow{\alpha} & E \\ \downarrow{\scriptstyle j} & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/Z}} & Y \times_Z Y \end{array}$$

we conclude that $\mathbf{L}_\alpha \cong j^* \mathbf{L}_{\Delta_{Y/Z}} \cong j^* \mathbf{L}_{f_1}[1]$. Combining these observations with (4.5.2), we get an exact triangle

$$\alpha^* \gamma^* \mathbf{L}_f \to \mathbf{L}_{\mathrm{Fix}(f)} \to j^* \mathbf{L}_{f_1}[1]. \tag{4.5.3}$$

Since $f_0 = f_1 : Y \to Z$ is smooth, we see that $\alpha^* \gamma^* \mathbf{L}_{f_1}[1]$ is a perfect complex in degrees ≥ -1. Since f is quasi-smooth, we see that $j^* \mathbf{L}_f$ is a perfect complex in degrees ≥ -1. By (4.5.3), we conclude that $\mathbf{L}_{\mathrm{Fix}(f)}$ is a perfect complex in degrees ≥ -1, i.e., $\mathrm{Fix}(f)$ is quasi-smooth. The relative dimension calculation of $\mathrm{Fix}(f)$ also follows from the exact triangle (4.5.3). $\square$

By Lemma 4.5.3, the pullback map on Borel-Moore homology

$$\mathrm{Fix}(f)^* : \mathrm{H}^{\mathrm{BM}}_{2i}(\mathrm{Fix}(D)) \to \mathrm{H}^{\mathrm{BM}}_{2i+2\delta}(\mathrm{Fix}(C))$$

is defined. It is induced from the map

$$\mathrm{Fix}(f)^* \mathbf{D}_{\mathrm{Fix}(D)} \xrightarrow{[\mathrm{Fix}(f)]} \mathrm{Fix}(f)^! \mathbf{D}_{\mathrm{Fix}(D)} \langle -\delta \rangle = \mathbf{D}_{\mathrm{Fix}(C)} \langle -\delta \rangle \tag{4.5.4}$$

by taking global sections.

Proposition 4.5.4. *With notation as in* (4.3.1) *and* (4.5.1), *assume that f_1 is smooth and f is quasi-smooth. Let $\delta = d(f) - d(f_1)$. Let $\mathfrak{d} \in \mathrm{Corr}_D(\mathcal{K}, \mathcal{K}\langle -i \rangle)$. Then we have*

$$\mathrm{Tr}(f^* \mathfrak{d}) = \mathrm{Fix}(f)^* \,\mathrm{Tr}(\mathfrak{d}) \in \mathrm{H}^{\mathrm{BM}}_{2i+2\delta}(\mathrm{Fix}(C)).$$

The proof of Proposition 4.5.4 is long, and will be given over the course of the next subsection.

4.6. **Proof of Proposition 4.5.4.** We begin with some preliminary technical observations.

4.6.1. *Another construction of the trace of a cohomological correspondence.* Let notation be as in §4.1. We fix an identification $Y_0 \cong Y_1 =: Y$.

The following alternative formulation of the trace will be useful. By Observation 4.1.1, we may interpret a cohomological correspondence $\mathfrak{c}\colon c_0^*\mathcal{K} \to c_1^!\mathcal{K}\langle -i\rangle$ as an element of $\mathrm{H}^{-2i}(C, c^!(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})(-i))$. We have a map

$$\mathrm{H}^*(C, c^!(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})) \to \mathrm{H}^*(\mathrm{Fix}(C), (\Delta')^* c^!(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})). \tag{4.6.1}$$

Then the base change transformation $(\Delta')^* c^! \xrightarrow{\diamond} (c')^! \Delta^*$ for the Cartesian square (4.1.2) induces a map

$$\mathrm{H}^*(\mathrm{Fix}(C), (\Delta')^* c^!(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})) \xrightarrow{\diamond} \mathrm{H}^*(\mathrm{Fix}(C), (c')^! \Delta^*(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})). \tag{4.6.2}$$

Applying the trace map $\Delta^*(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K}) \cong \mathbf{D}_Y(\mathcal{K}) \otimes \mathcal{K} \to \mathbf{D}_Y$, we get a map

$$\begin{aligned} \mathrm{H}^{-2i}(\mathrm{Fix}(C), (c')^! \Delta^*(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})(-i)) &\to \mathrm{H}^{-2i}(\mathrm{Fix}(C), (c')^! \mathbf{D}_Y(-i)) \\ &\cong \mathrm{H}^{-2i}(\mathrm{Fix}(C), \mathbf{D}_{\mathrm{Fix}(C)}(-i)) =: \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C)). \end{aligned} \tag{4.6.3}$$

Lemma 4.6.1. *The map* $\mathrm{Tr}\colon \mathrm{Corr}_C(\mathcal{K}, \mathcal{K}\langle -i\rangle) \cong \mathrm{H}^{-2i}(C, c^!(\mathbf{D}_Y(\mathcal{K}) \boxtimes \mathcal{K})(-i)) \to \mathrm{H}_{2i}^{\mathrm{BM}}(\mathrm{Fix}(C))$ *coincides with the composition of the maps* (4.6.1), (4.6.2), *and* (4.6.3).

Proof. In the proof we denote all pull-pull base change transformations (3.5.1) induced by Cartesian squares with the symbol $\diamond$.

Comparing the definitions, the Lemma amounts to the commutativity of the diagram

$$\begin{array}{ccccc} c^! & \xrightarrow{\mathrm{unit}(\Delta')} & \Delta'_*(\Delta')^* c^! & \xrightarrow{\diamond} & \Delta'_*(c')^! \Delta^* \\ \| & & & & \| \\ c^! & \xrightarrow{\mathrm{unit}(\Delta)} & c^! \Delta_* \Delta^* & \xrightarrow{\diamond} & \Delta'_*(c')^! \Delta^* \end{array} \tag{4.6.4}$$

By definition, the morphism $(\Delta')^* c^! \xrightarrow{\diamond} (c')^! \Delta^*$ is the composition

$$(\Delta')^* c^! \xrightarrow{\mathrm{unit}(\Delta)} (\Delta')^* c^! \Delta_* \Delta^* \xrightarrow{\diamond} (\Delta')^* \Delta'_* c^! \Delta^* \xrightarrow{\mathrm{counit}(\Delta')} c^! \Delta^*.$$

Therefore, the top row of (4.6.4) fits into a commutative diagram

$$\begin{array}{ccccc} c^! & \xrightarrow{\mathrm{unit}(\Delta')} & \Delta'_*(\Delta')^* c^! & \xrightarrow{\diamond} & \Delta'_*(c')^! \Delta^* \\ \| & & \downarrow{\scriptstyle \mathrm{unit}(\Delta)} & & \uparrow{\scriptstyle \mathrm{counit}(\Delta')} \\ c^! & \xrightarrow{\mathrm{unit}(\Delta') \circ \mathrm{unit}(\Delta)} & \Delta'_*(\Delta')^* c^! \Delta_* \Delta^* & \xrightarrow{\diamond} & \Delta'_*(\Delta')^* \Delta'_* c^! \Delta^* \end{array} \tag{4.6.5}$$

Similarly we have a commutative diagram

$$\begin{array}{ccc} c^! \Delta_* \Delta^* & \xrightarrow{\diamond} & \Delta'_*(c')^! \Delta^* \\ \downarrow{\scriptstyle \mathrm{unit}(\Delta')} & & \downarrow{\scriptstyle \mathrm{unit}(\Delta')} \\ \Delta'_*(\Delta')^* c^! \Delta_* \Delta^* & \xrightarrow{\diamond} & (\Delta')_*(\Delta')^*(\Delta'_*) c^! \Delta^* \end{array} \tag{4.6.6}$$

Putting together (4.6.5) and (4.6.6), we get a commutative diagram

$$\begin{array}{ccccc} c^! & \xrightarrow{\mathrm{unit}(\Delta')} & \Delta'_*(\Delta')^* c^! & \xrightarrow{\diamond} & \Delta'_*(c')^! \Delta^* \\ \| & & \downarrow{\scriptstyle \mathrm{unit}(\Delta)} & & \uparrow{\scriptstyle \mathrm{counit}(\Delta')} \\ c^! & \xrightarrow{\mathrm{unit}(\Delta') \circ \mathrm{unit}(\Delta)} & \Delta'_*(\Delta')^* c^! \Delta_* \Delta^* & \xrightarrow{\diamond} & \Delta'_*(\Delta')^* \Delta'_* c^! \Delta^* \\ \| & & \uparrow{\scriptstyle \mathrm{unit}(\Delta')} & & \uparrow{\scriptstyle \mathrm{unit}(\Delta')} \\ c^! & \xrightarrow{\mathrm{unit}(\Delta)} & c^! \Delta_* \Delta^* & \xrightarrow{\diamond} & (\Delta'_*)(c')^! \Delta^* \end{array}$$

whose upper and lower rows agree with those in (4.6.4). It remains to verify that the vertical composition on the right column, $\mathrm{counit}(\Delta') \circ \mathrm{unit}(\Delta')$, is the identity map. But this is one of the axioms of an adjunction. $\square$

4.6.2. *Alternative description of pullback cohomological correspondence.* Recall the setup of Proposition 4.5.4: suppose we are given commutative diagram of derived Artin stacks

$$\begin{array}{ccccc} Y_0 & \xleftarrow{c_0} & C & \xrightarrow{c_1} & Y_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ Z_0 & \xleftarrow{d_0} & D & \xrightarrow{d_1} & Z_1 \end{array}$$

such that f_1 is smooth and f is quasi-smooth with $d(f) - d(f_1) = \delta$. We suppose a fixed identification of $f_0 \colon Y_0 \to Z_0$ with $f_1 \colon Y_1 \to Z_1$.

Since f_1 is smooth and f is quasi-smooth, we may use Example 3.5.2 to give an alternative description of the pullback map f^* on cohomological correspondences: given $\mathfrak{d} : d_0^* \mathcal{K}_0 \to d_1^! \mathcal{K}_1$ (where $K_i \in D(Z_i)$), $f^* \mathfrak{d}$ is the composition

$$c_0^* f_0^* \mathcal{K}_0 = f^* d_0^* \mathcal{K}_0 \xrightarrow{f^* \mathfrak{d}} f^* d_1^! \mathcal{K}_1 \xrightarrow{[f]} f^! d_1^! \mathcal{K}_1 \langle -d(f) \rangle = c_1^! f_1^! \mathcal{K}_1 \langle -d(f) \rangle \overset{[f_1]}{\cong} c_1^! f_1^* \mathcal{K}_1 \langle -\delta \rangle.$$

By Observation 4.1.1, we have

$$\begin{aligned} \mathrm{Corr}_D(\mathcal{K}_0, \mathcal{K}_1) &= \mathrm{H}^0(D, d^!(\mathbf{D}_{Z_0}(\mathcal{K}_0) \boxtimes \mathcal{K}_1)), && (4.6.7) \\ \mathrm{Corr}_C(f_0^* \mathcal{K}_0, f_1^* \mathcal{K}_1 \langle -\delta \rangle) &= \mathrm{H}^0(C, c^!(\mathbf{D}_{Y_0}(f_0^* \mathcal{K}_0) \boxtimes f_1^* \mathcal{K}_1 \langle -\delta \rangle)). && (4.6.8) \end{aligned}$$

Consider the commutative square

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ Y_0 \times Y_1 & \xrightarrow{f_0 \times f_1} & Z_0 \times Z_1 \end{array}$$

Since $f_0 \times f_1$ is smooth and f is quasi-smooth, this square is pullable (cf. Example 3.1.3). The base change map (cf. §3.5) in this situation is defined and reads

$$f^* d^! \xrightarrow{\triangle} c^! (f_0 \times f_1)^* \langle -d(f) + d(f_0) + d(f_1) \rangle.$$

Using the identifications (4.6.7) and (4.6.8), the map $f^* : \mathrm{Corr}_D(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{Corr}_C(f_0^* \mathcal{K}_0, f_1^* \mathcal{K}_1 \langle -\delta \rangle)$ is induced by the following map upon taking H^0:

$$\begin{aligned} f^* d^! (\mathbf{D} \mathcal{K}_0 \boxtimes \mathcal{K}_1) &\xrightarrow{\triangle} c^! (f_0^* \mathbf{D} \mathcal{K}_0 \boxtimes f_1^* \mathcal{K}_1) \langle -d(f) + d(f_0) + d(f_1) \rangle && (4.6.9) \\ &\cong c^! (\mathbf{D}(f_0^! \mathcal{K}_0) \boxtimes f_1^* \mathcal{K}_1) \langle -d(f) + d(f_0) + d(f_1) \rangle \overset{[f_0]}{\cong} c^! (\mathbf{D}(f_0^* \mathcal{K}_0) \boxtimes f_1^* \mathcal{K}_1) \langle -\delta \rangle. \end{aligned}$$

4.6.3. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Fix}(C) & \xrightarrow{i_Y} & C & \xrightarrow{f} & D \\ \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ Y_0 & \xrightarrow{\Delta_Y} & Y_0 \times Y_1 & \xrightarrow{f_0 \times f_1} & Z_0 \times Z_1 \end{array} \qquad (4.6.10)$$

The left square is derived Cartesian and the right is pullable. Therefore by Lemma 3.5.3, the outer square is also pullable. By Proposition 3.5.4, the pull-pull base change map for the outer square is the composition of the base change maps for the two inner squares

$$i_Y^* f^* d^! \xrightarrow{i_Y^* \triangle} i_Y^* c^! (f_0 \times f_1)^* \langle -\delta + d(f_0) \rangle \xrightarrow{\diamond (f_0 \times f_1)^*} j_Y^! \Delta_Y^* (f_0 \times f_1)^* \langle -\delta + d(f_0) \rangle. \qquad (4.6.11)$$

On the other hand, consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Fix}(C) & \xrightarrow{\mathrm{Fix}(f)} & \mathrm{Fix}(D) & \xrightarrow{i_Z} & D \\ \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle j_Z} & & \downarrow{\scriptstyle d} \\ Y_0 & \xrightarrow{f_0} & Z_0 & \xrightarrow{\Delta_Z} & Z_0 \times Z_1 \end{array} \qquad (4.6.12)$$

where the right square is derived Cartesian and the left square is pullable, since f_0 is smooth and $\mathrm{Fix}(f)$ is quasi-smooth by Lemma 4.5.3. Again by Proposition 3.5.4, the pull-pull base change map for the outer square is the composition of the base change maps for the two inner squares

$$\mathrm{Fix}(f)^* i_Z^* d^! \xrightarrow{\mathrm{Fix}(f)^*\diamond} \mathrm{Fix}(f)^* j_Z^! \Delta_Z^* \xrightarrow{\vartriangle \Delta_Z^*} j_Y^! f_0^* \Delta_Z^* \langle -\delta + d(f_0)\rangle \cong j_Y^! f_0^! \Delta_Z^* \langle -\delta \rangle. \tag{4.6.13}$$

Since the outer squares of both diagrams (4.6.10) and (4.6.12) are the same, Proposition 3.5.4 implies that both (4.6.11) and (4.6.13) give the base change map for the same square. We thus get a commutative diagram

$$\begin{array}{ccccc} i_Y^* f^* d^! & \xrightarrow{i_Y^* \vartriangle} & i_Y^* c^!(f_0 \times f_1)^* \langle -\delta + d(f_0)\rangle & \xrightarrow{\diamond (f_0\times f_1)^*} & j_Y^! \Delta_Y^* (f_0 \times f_1)^* \langle -\delta + d(f_0)\rangle \\ \| & & & & \| \\ \mathrm{Fix}(f)^* i_Z^* d^! & \xrightarrow{\mathrm{Fix}(f)^*\diamond} & \mathrm{Fix}(f)^* j_Z^! \Delta_Z^* & \xrightarrow{\vartriangle \Delta_Z^*} & j_Y^! f_0^* \Delta_Z^* \langle -\delta + d(f_0)\rangle \end{array} \tag{4.6.14}$$

4.6.4. *Completion of the proof.* Now we may complete the proof of Proposition 4.5.4.

Consider applying (4.6.11) to an element of the form $\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1 \in D(Z_0 \times Z_1)$: we get a map

$$\begin{aligned} i_Y^* f^* d^! (\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1) &\to i_Y^* c^! (f_0^* \mathbf{D}(\mathcal{K}_0) \boxtimes f_1^* \mathcal{K}_1)\langle -\delta + d(f_0)\rangle && (4.6.15)\\ &\to j_Y^! \Delta_Y^* (f_0^* \mathbf{D}(\mathcal{K}_0) \boxtimes f_1^* \mathcal{K}_1)\langle -\delta + d(f_0)\rangle && (4.6.16)\\ &\cong j_Y^! (\mathbf{D}(f_0^* \mathcal{K}_0) \otimes f_1^* \mathcal{K}_1 \langle -\delta \rangle). && (4.6.17) \end{aligned}$$

Here we use $f_0^* \mathbf{D}\langle d(f_0)\rangle \cong f_0^! \mathbf{D} \cong \mathbf{D} f_0^*$. As we have observed in §4.6.2, for a cohomological correspondence $\mathfrak{d} \in \mathrm{Corr}_D(\mathcal{K}_0, \mathcal{K}_1)$, viewed as a global section of $d^!(\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1)$, $f^*\mathfrak{d}$ is the image of $\mathfrak{d}$ under the map on H^0 induced by the base change map (4.6.9), which is the first step in (4.6.15).

Suppose now $\mathcal{K}_1 = \mathcal{K}_0\langle -i\rangle$ (and recall that $f_0 = f_1$). Then we may compose (4.6.11) with the map $\mathrm{Tr}_{Y_0} \colon \mathbf{D}_{Y_0}(f_0^*\mathcal{K}_0) \otimes (f_1^* \mathcal{K}_0 \langle -i-\delta\rangle) \to \mathbf{D}_{Y_0}\langle -i-\delta\rangle$ to get a sequence of maps

$$i_Y^* f^* d^! (\mathbf{D}\mathcal{K}_0 \boxtimes \mathcal{K}_0\langle -i\rangle) \xrightarrow{(4.6.15)} j_Y^! (\mathbf{D}_{Y_0}(f_0^* \mathcal{K}_0) \otimes (f_1^* \mathcal{K}_0 \langle -i-\delta\rangle)) \xrightarrow{\mathrm{Tr}_{Y_0}} j_Y^! \mathbf{D}_{Y_0}\langle -i-\delta\rangle \cong \mathbf{D}_{\mathrm{Fix}(C)}\langle -i-\delta\rangle. \tag{4.6.18}$$

Then by Lemma 4.6.1 and the preceding paragraph, the image of $i_Y^* f^*\mathfrak{d}$ under (4.6.18) is $\mathrm{Tr}(f^*\mathfrak{d}) \in \mathrm{H}^{\mathrm{BM}}_{2i+2\delta}(\mathrm{Fix}(C))$.

Next consider applying (4.6.13) to $\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1 \in D(Z_0 \times Z_1)$: we get a map

$$\mathrm{Fix}(f)^* i_Z^* d^! (\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1) \to \mathrm{Fix}(f)^* j_Z^! (\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_1) \to j_Y^! f_0^! (\mathbf{D}(\mathcal{K}_0) \otimes \mathcal{K}_1)\langle -\delta\rangle. \tag{4.6.19}$$

Suppose now $\mathcal{K}_1 = \mathcal{K}_0\langle -i\rangle$. Then we may compose (4.6.13) with the map $\mathrm{Tr}_{Z_0} \colon \mathbf{D}(\mathcal{K}_0) \otimes \mathcal{K}_0\langle -i\rangle \to \mathbf{D}_{Z_0}\langle -i\rangle$ to get a commutative diagram

$$\begin{array}{ccccc} \mathrm{Fix}(f)^* i_Z^* d^! (\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_0\langle -i\rangle) & \longrightarrow & \mathrm{Fix}(f)^* j_Z^! (\mathbf{D}\mathcal{K}_0 \boxtimes \mathcal{K}_0\langle -i\rangle) & \longrightarrow & j_Y^! f_0^! (\mathbf{D}(\mathcal{K}_0) \otimes \mathcal{K}_0\langle -i-\delta\rangle) \\ & & \Big\downarrow{\scriptstyle \mathrm{Fix}(f)^* j_Z^! \mathrm{Tr}_{Z_0}} & & \Big\downarrow{\scriptstyle j_Y^! f_0^! \mathrm{Tr}_{Z_0}} \\ & & \mathrm{Fix}(f)^* j_Z^! \mathbf{D}_{Z_0}\langle -i\rangle & \longrightarrow & j_Y^! f_0^! \mathbf{D}_{Z_0}\langle -i-\delta\rangle \\ & & \| & & \| \\ & & \mathrm{Fix}(f)^* \mathbf{D}_{\mathrm{Fix}(D)}\langle -i\rangle & \xrightarrow{(4.5.4)} & \mathbf{D}_{\mathrm{Fix}(C)}\langle -i-\delta\rangle. \end{array} \tag{4.6.20}$$

Again we view the cohomological correspondence $\mathfrak{d} \in \mathrm{Corr}_D(\mathcal{K}_0, \mathcal{K}_0\langle -i\rangle)$ as a global section of $d^!(\mathbf{D}(\mathcal{K}_0) \boxtimes \mathcal{K}_0\langle -i\rangle)$. By Lemma 4.6.1, the image of this global section under $i_Z^* \mathrm{Tr}_{Z_0}$ is $\mathrm{Tr}(\mathfrak{d}) \in \mathrm{H}^{\mathrm{BM}}_{2i}(\mathrm{Fix}(D))$. Since the bottom map induces $\mathrm{Fix}(f)^*$ on Borel-Moore homology, we get that the image of $\mathrm{Fix}(f)^* i_Z^* \mathfrak{d}$ in the lower right corner of (4.6.20) is $\mathrm{Fix}(f)^* \mathrm{Tr}(\mathfrak{d}) \in \mathrm{H}^{\mathrm{BM}}_{2i+2\delta}(\mathrm{Fix}(C))$.

Finally, the commutativity of (4.6.14) establishes that (4.6.18) agrees with the upper-left to lower-right composition of (4.6.20), so then $\mathrm{Tr}(f^*\mathfrak{d}) = \mathrm{Fix}(f)^* \mathrm{Tr}(\mathfrak{d}) \in \mathrm{H}^{\mathrm{BM}}_{2i+2\delta}(\mathrm{Fix}(C))$, as desired. □

4.7. **Derived fundamental class as a trace.** Let notation be as in §4.1, and set $Y := Y_0 = Y_1$.

Proposition 4.7.1. *Assume that Y is smooth and c_1 is quasi-smooth. Then we have*

$$\mathrm{Tr}(\mathfrak{c}_Y) = [\mathrm{Fix}(C)] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Fix}(C)).$$

Remark 4.7.2. This result almost appears as [Ols15, Theorem 1.7], but we need more generality than is treated there. We will see that the proof is almost a trivial application of Proposition 4.5.4.

Proof. Consider the map of correspondences

$$\begin{array}{ccccc} Y & \xleftarrow{c_0} & C & \xrightarrow{c_1} & Y \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ \mathrm{pt} & \longleftarrow & \mathrm{pt} & \longrightarrow & \mathrm{pt} \end{array}$$

On the bottom row we have the trivial correspondence $\mathfrak{c}_{\mathrm{pt}} = \mathrm{Id} \in \mathrm{Corr}_{\mathrm{pt}}(\mathbf{Q}_\ell, \mathbf{Q}_\ell)$. By assumption π_1 is smooth and c_1 is quasi-smooth, so π is quasi-smooth, hence the diagram is right pullable, and it is immediate from the definitions that

$$\pi^* \mathfrak{c}_{\mathrm{pt}} = \mathfrak{c}_Y \in \mathrm{Corr}_C(\mathbf{Q}_{\ell,Y}, \mathbf{Q}_{\ell,Y}\langle -d(c_1)\rangle).$$

Then Proposition 4.5.4 implies that

$$\mathrm{Tr}(\pi^* \mathfrak{c}_{\mathrm{pt}}) = \mathrm{Fix}(\pi)^* \,\mathrm{Tr}(\mathfrak{c}_{\mathrm{pt}}) = \mathrm{Fix}(\pi)^*[\mathrm{pt}] = [\mathrm{Fix}(C)] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Fix}(C)).$$

□

Corollary 4.7.3. *In the setup of §4.2, suppose Y is smooth over $\mathbf{F}_q$ and c_1 is quasi-smooth. Then we have*

$$\mathrm{Tr}^{\mathrm{Sht}}_C(\mathfrak{c}_Y) = [\mathrm{Sht}(C)] \in \mathrm{H}^{\mathrm{BM}}_{2d(c_1)}(\mathrm{Sht}(C)).$$

4.8. **Cohomological co-correspondences.** Later when we study how cohomological correspondences interact with Fourier duality, we will naturally encounter variants of the above constructions in terms of what we call *co-correspondences* and *cohomological co-correspondences*.

4.8.1. *Definitions.* A *co-correspondence* between derived Artin stacks Y_0 and Y_1 is a diagram [6]

$$\begin{array}{ccccc} Y_0 & & & & Y_1 \\ & \searrow^{c_1'} & & \swarrow^{c_0'} & \\ & & C' & & \end{array} \tag{4.8.1}$$

We define a *cohomological co-correspondence* from $\mathcal{K}_0 \in D(Y_0)$ to $\mathcal{K}_1 \in D(Y_1)$ to be an element of $\mathrm{Hom}_{C'}(c'_{1!}\mathcal{K}_0, c'_{0*}\mathcal{K}_1)$. Let

$$\mathrm{CoCorr}_{C'}(\mathcal{K}_0, \mathcal{K}_1) := \mathrm{Hom}_{C'}(c'_{1!}\mathcal{K}_0, c'_{0*}\mathcal{K}_1). \tag{4.8.2}$$

To see the relation between cohomological correspondences and co-correspondences, suppose we have a Cartesian square

$$\begin{array}{ccccc} & & C^\flat & & \\ & \swarrow^{c_0} & & \searrow^{c_1} & \\ Y_0 & & & & Y_1 \\ & \searrow^{c_1'} & & \swarrow^{c_0'} & \\ & & C^\sharp & & \end{array} \tag{4.8.3}$$

Then for $\mathcal{K}_0 \in D(Y_0)$ and $\mathcal{K}_1 \in D(Y_1)$, there is a canonical isomorphism of vector spaces

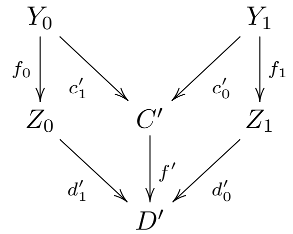
$$\gamma_C : \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \xrightarrow{\sim} \mathrm{CoCorr}_{C^\sharp}(\mathcal{K}_0, \mathcal{K}_1) \tag{4.8.4}$$

given by adjunctions and proper base change

$$\mathrm{Hom}_{C^\flat}(c_0^*\mathcal{K}_0, c_1^!\mathcal{K}_1) \cong \mathrm{Hom}_{Y_1}(c_{1!}c_0^*\mathcal{K}_0, \mathcal{K}_1) \cong \mathrm{Hom}_{Y_1}((c_0')^* c'_{1!}\mathcal{K}_0, \mathcal{K}_1) \cong \mathrm{Hom}_{C^\sharp}(c'_{1!}\mathcal{K}_0, c'_{0*}\mathcal{K}_1).$$

[6] We deliberately denote the map $Y_0 \to C'$ by c_1'. This is not a typo. This will be justified in the situation when we can complete the diagram into one of the form (4.8.3), in which we follow the convention that the names of arrows at opposite sides of a square differ by a prime.

4.8.2. *Pushforward and pullback of cohomological co-correspondences.* We have dual notions of pushability and pullability for co-correspondences. Consider a morphism of co-correspondences, i.e., a commutative diagram



(4.8.5)

Definition 4.8.1. The diagram of co-correspondences (4.8.5) is called *left pullable*, if the square with vertices (Y_0, C', Z_0, D') is pullable in the sense of Definition 3.1.1. We define the defect $\delta_{f'}$ to be the defect of the square (Y_0, C', Z_0, D').

Similarly, (4.8.5) is called *right pushable*, if the square with vertices (Y_1, C', Z_1, D') is pushable in the sense of Definition 3.1.1.

When (4.8.5) is left pullable, we have a pullback map of cohomological co-correspondences (where $\mathcal{K}_i \in D(Z_i)$)

$$(f')^* : \mathrm{CoCorr}_{D'}(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{CoCorr}_{C'}(f_0^*\mathcal{K}_0, f_1^*\mathcal{K}_1\langle -\delta_{f'}\rangle) \tag{4.8.6}$$

defined as follows. For $\mathfrak{c}' : d'_{1!}\mathcal{K}_1 \to d'_{0*}\mathcal{K}_0$, we define $(f')^*(\mathfrak{c}')$ as the composition

$$c'_{1!}f_0^*\mathcal{K}_0 \xrightarrow{\triangle} (f')^*d'_{1!}\mathcal{K}_0\langle -\delta_{f'}\rangle \xrightarrow{(f')^*(\mathfrak{c}')} (f')^*d'_{0*}\mathcal{K}_1\langle -\delta_{f'}\rangle \to c'_{0*}f_1^*\mathcal{K}_1\langle -\delta_{f'}\rangle \tag{4.8.7}$$

where the first map is the natural transformation (3.5.4).

When (4.8.5) is right pushable, we have a pushforward map of cohomological co-correspondences (where $\mathcal{K}_i \in D(Y_i)$)

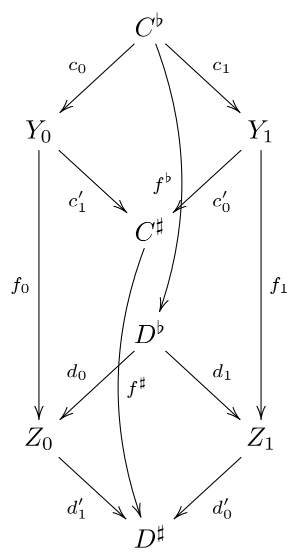
$$f'_! : \mathrm{CoCorr}_{C'}(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{CoCorr}_{D'}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) \tag{4.8.8}$$

defined as follows. For $\mathfrak{c}' : c'_{1!}\mathcal{K}_0 \to c'_{0*}\mathcal{K}_1$, we define $f'_!(\mathfrak{c}')$ as the composition

$$d'_{1!}f_{0!}\mathcal{K}_0 \cong f'_!c'_{1!}\mathcal{K}_0 \xrightarrow{f'_!(\mathfrak{c}')} f'_!c'_{0*}\mathcal{K}_1 \to d'_{0*}f_{1!}\mathcal{K}_1 \tag{4.8.9}$$

where the last map is the push-push natural transformation (3.3.3).

4.8.3. *Compatibility.* Suppose we are given a commutative diagram



(4.8.10)

where the top and bottom diamonds are derived Cartesian. We view $f^\flat : C^\flat \to D^\flat$ as a map of correspondences, and $f^\sharp : C^\sharp \to D^\sharp$ as a map of co-correspondences.

Proposition 4.8.2. *In the above situation,*

(1) *If $f^\sharp$ is left pullable, then $f^\flat$ is right pullable, and $\delta_{f^\sharp} = \delta_{f^\flat}$. In this case, we have a commutative diagram for any $\mathcal{K}_0 \in D(Z_0)$ and $\mathcal{K}_1 \in D(Z_1)$*

$$\begin{array}{ccc} \mathrm{CoCorr}_{D^\sharp}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\gamma_D} & \mathrm{Corr}_{D^\flat}(\mathcal{K}_0, \mathcal{K}_1) \\ \downarrow{\scriptstyle (f^\sharp)^*} & & \downarrow{\scriptstyle (f^\flat)^*} \\ \mathrm{CoCorr}_{C^\sharp}(f_0^*\mathcal{K}_0, f_1^*\mathcal{K}_1\langle -\delta_{f^\sharp}\rangle) & \xrightarrow{\gamma_C} & \mathrm{Corr}_{C^\flat}(f_0^*\mathcal{K}_0, f_1^*\mathcal{K}_1\langle -\delta_{f^\flat}\rangle) \end{array} \tag{4.8.11}$$

(2) *If $f^\sharp$ is right pushable, then $f^\flat$ is left pushable. In this case, we have a commutative diagram for any $\mathcal{K}_0 \in D(Y_0)$ and $\mathcal{K}_1 \in D(Y_1)$*

$$\begin{array}{ccc} \mathrm{CoCorr}_{C^\sharp}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\gamma_C} & \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \\ \downarrow{\scriptstyle f^\sharp_!} & & \downarrow{\scriptstyle f^\flat_!} \\ \mathrm{CoCorr}_{D^\sharp}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) & \xrightarrow{\gamma_D} & \mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) \end{array} \tag{4.8.12}$$

Proof. (1) Using that the top diamond in (4.8.10) is derived Cartesian, the fact that the square $(Y_0, C^\sharp, Z_0, D^\sharp)$ is pullable implies that the square $(C^\flat, Y_1, Z_0, D^\sharp)$ is pullable with the same defect. Using that the bottom diamond is derived Cartesian, we have $Y_1 \times_{Z_1} D^\flat \xrightarrow{\sim} Y_1 \times_{D^\sharp} Z_0$, from which we see that the square $(C^\flat, Y_1, D^\flat, Z_1)$ is pullable with the same defect as $(C^\flat, Y_1, Z_0, D^\sharp)$.

We prove the commutativity of the diagram (4.8.11). Let $\mathfrak{c}' : d'_{1!}\mathcal{K}_0 \to d'_{0*}\mathcal{K}_1$ be an element in $\mathrm{CoCorr}_{D^\sharp}(\mathcal{K}_0, \mathcal{K}_1)$. Consider the following diagram

$$\begin{array}{ccccccc} c_{1!}c_0^*f_0^*\mathcal{K}_0 & \xrightarrow{\diamond} & (c'_0)^*c'_{1!}f_0^*\mathcal{K}_0 & \xrightarrow{(c'_0)^*\triangle} & (c'_0)^*(f^\sharp)^*d'_{1!}\mathcal{K}_0\langle -\delta_{f^\sharp}\rangle & \xrightarrow{(c'_0)^*(f^\sharp)^*\mathfrak{c}'} & (c'_0)^*(f^\sharp)^*d'_{0*}\mathcal{K}_1\langle -\delta_{f^\sharp}\rangle \\ \| & & & & \| & & \downarrow \\ c_{1!}(f^\flat)^*d_0^*\mathcal{K}_0 & \xrightarrow{\triangle d_0^*} & f_1^*d_{1!}d_0^*\mathcal{K}_0\langle -\delta_{f^\flat}\rangle & \xrightarrow{\diamond} & f_1^*(d'_0)^*d'_{1!}\mathcal{K}_0\langle -\delta_{f^\flat}\rangle & \xrightarrow{f_1^*\mathfrak{c}''} & f_1^*\mathcal{K}_1\langle -\delta_{f^\flat}\rangle \end{array} \tag{4.8.13}$$

The right vertical map comes from the identity $(c'_0)^*(f^\sharp)^* = f_1^*(d'_0)^*$ by adjunction. The map $\mathfrak{c}'' : (d'_0)^*d'_{1!}\mathcal{K}_0 \to \mathcal{K}_1$ that appears on the bottom row is obtained from $\mathfrak{c}'$ by adjunction.

In the above diagram, the two ways of getting from $c_{1!}c_0^*f_0^*\mathcal{K}_0$ to $f_1^*\mathcal{K}_1\langle -\delta_{f^\flat}\rangle$, by going right and down, and by going down and right, are obtained from $\gamma_C(f^\sharp)^*(\mathfrak{c}')$ and $(f^\flat)^*\gamma_D(\mathfrak{c}')$ by adjunction respectively. Therefore it suffices to show that the above diagram is commutative. The equality in the middle divides the diagram into two parts. The right square is tautologically commutative. The top and bottom rows of the left part both give base change maps for the square $(C^\flat, Y_1, Z_0, D^\sharp)$

$$c_{1!}(f_0 \circ c_0)^* = c_{1!}(d_0 \circ f^\flat)^* \to (d'_0 \circ f_1)^*d'_{1!}\langle -\delta_{f^\flat}\rangle = (f^\sharp \circ c'_0)^*d'_{1!}\langle -\delta_{f^\sharp}\rangle \tag{4.8.14}$$

by decomposing it in two different ways

$$\begin{array}{ccc} C^\flat & \xrightarrow{c_1} & Y_1 \\ \downarrow{\scriptstyle c_0} & & \downarrow{\scriptstyle c'_0} \\ Y_0 & \xrightarrow{c'_1} & C^\sharp \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f^\sharp} \\ Z_0 & \xrightarrow{d'_1} & D^\sharp \end{array} \qquad \begin{array}{ccc} C^\flat & \xrightarrow{c_1} & Y_1 \\ \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle f_1} \\ D^\flat & \xrightarrow{d_1} & Z_1 \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d'_0} \\ Z_0 & \xrightarrow{d'_1} & D^\sharp \end{array} \tag{4.8.15}$$

The two base change maps agree by Proposition 3.2.3.

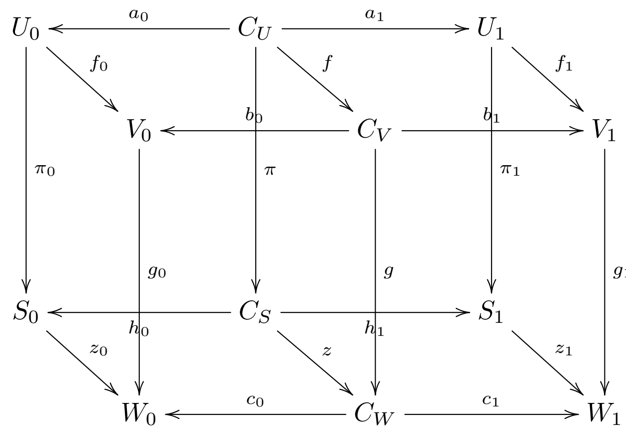
(2) The proof is similar. We omit it as the statement is not used in the sequel. □

5. Base change for cohomological correspondences

In this section we formulate and prove the *Base Change Theorem* for cohomological correspondences (Theorem 5.1.3). This will be used in Step (4) of the outline in §2.4.

5.1. **The setup.**

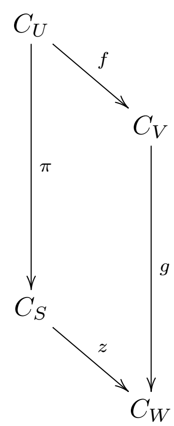
5.1.1. *Correspondences.* Suppose we are given a commutative diagram of derived Artin stacks



(5.1.1)

satisfying the following conditions:

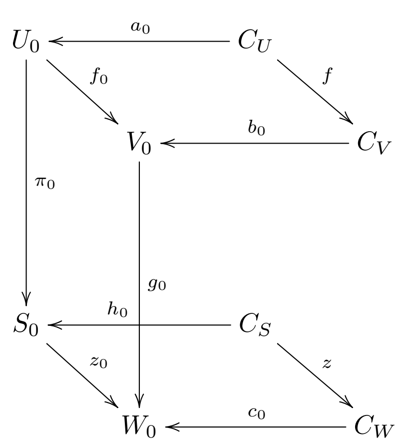
(1) The middle vertical parallelogram



(5.1.2)

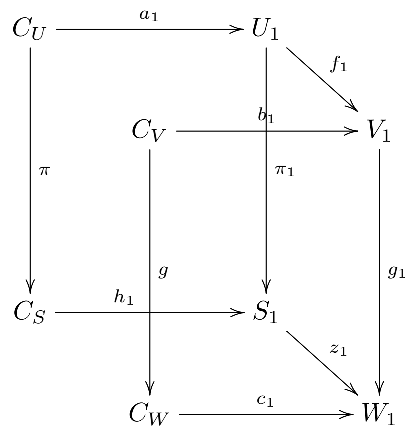
is derived Cartesian.

(2) The three squares in the following diagram are pushable



(5.1.3)

(3) The three squares in the following diagram are pullable



(5.1.4)

Moreover, the right square (U_1, V_1, S_1, W_1) above has defect zero.

We view C_S as a correspondence between S_0 and S_1, and similarly for C_U, C_V and C_W.

5.1.2. *Push and pull of cohomological correspondences.* Let $\mathcal{K}_i \in D(S_i)$ for $i = 0, 1$. Let $\mathfrak{s} \in \mathrm{Corr}_{C_S}(\mathcal{K}_0, \mathcal{K}_1)$.

Consider the back face of the diagram (5.1.1), viewed as a map of correspondences $\pi : C_U \to C_S$:

$$\begin{array}{ccccc} U_0 & \xleftarrow{a_0} & C_U & \xrightarrow{a_1} & U_1 \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ S_0 & \xleftarrow{h_0} & C_S & \xrightarrow{h_1} & S_1 \end{array} \tag{5.1.5}$$

By assumption, this map of correspondences is right pullable. Therefore, by §4.4, the map

$$\pi^* : \mathrm{Corr}_{C_S}(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{Corr}_{C_U}(\pi_0^*\mathcal{K}_0, \pi_1^*\mathcal{K}_{1\langle -\delta_\pi \rangle}) \tag{5.1.6}$$

is defined (where the defect δ_π is defined in Definition 4.3.1).

Consider the top face of the diagram (5.1.1), viewed as a map of correspondences $f : C_U \to C_V$:

$$\begin{array}{ccccc} U_0 & \xleftarrow{a_0} & C_U & \xrightarrow{a_1} & U_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ V_0 & \xleftarrow{b_0} & C_V & \xrightarrow{b_1} & V_1 \end{array} \tag{5.1.7}$$

By assumption, this map of correspondences is left pushable. Therefore, by §4.3, the map

$$f_! : \mathrm{Corr}_{C_U}(\pi_0^*\mathcal{K}_0, \pi_1^*\mathcal{K}_{1\langle -\delta_\pi \rangle}) \to \mathrm{Corr}_{C_V}(f_{0!}\pi_0^*\mathcal{K}_0, f_{1!}\pi_1^*\mathcal{K}_{1\langle -\delta_\pi \rangle}) \tag{5.1.8}$$

is defined.

The composition of the two maps give an element

$$f_!\pi^*(\mathfrak{s}) \in \mathrm{Corr}_{C_V}(f_{0!}\pi_0^*\mathcal{K}_0, f_{1!}\pi_1^*\mathcal{K}_{1\langle -\delta_\pi \rangle}). \tag{5.1.9}$$

Similarly, the bottom face of the diagram (5.1.1)

$$\begin{array}{ccccc} S_0 & \xleftarrow{h_0} & C_S & \xrightarrow{h_1} & S_1 \\ \downarrow{\scriptstyle z_0} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle z_1} \\ W_0 & \xleftarrow{c_0} & C_W & \xrightarrow{c_1} & W_1 \end{array} \tag{5.1.10}$$

is left pushable and the front face

$$\begin{array}{ccccc} V_0 & \xleftarrow{b_0} & C_V & \xrightarrow{b_1} & V_1 \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_1} \\ W_0 & \xleftarrow{c_0} & C_W & \xrightarrow{c_1} & W_1 \end{array} \tag{5.1.11}$$

is right pullable. Therefore the cohomological correspondence

$$g^*z_!(\mathfrak{s}) \in \mathrm{Corr}_{C_V}(g_0^*z_{0!}\mathcal{K}_0, g_1^*z_{1!}\mathcal{K}_1\langle -\delta_g\rangle) \tag{5.1.12}$$

is defined.

5.1.3. *Matching source and target.* We will now formulate a Base Change Theorem for cohomological correspondences, which is the main result of this section. The Base Change Theorem asserts that, referring to the diagram (5.1.1), for any $\mathfrak{s} \in \mathrm{Corr}_{C_S}(\mathcal{K}_0, \mathcal{K}_1)$, the cohomological correspondences $f_!\pi^*(\mathfrak{s})$ and $g^*z_!(\mathfrak{s})$ on C_V "agree". To make sense of it, we need to relate the source and targets of the respective cohomological correspondences.

By assumption, the square (U_0, V_0, S_0, W_0) in (5.1.1) is pushable. We get a base change map

$$g_0^*z_{0!} \xrightarrow{\triangledown} f_{0!}\pi_0^* : D(S_0) \to D(V_0). \tag{5.1.13}$$

By assumption, the square (U_1, V_1, S_1, W_1) in (5.1.1) is pullable with defect zero. We get a pull-pull map

$$\pi_1^*z_1^! \xrightarrow{\vartriangle} f_1^!g_1^* : D(W_1) \to D(U_1). \tag{5.1.14}$$

By adjunction (cf. Remark 3.5.1), it gives a base change map

$$f_{1!}\pi_1^* \to g_1^*z_{1!} : D(S_1) \to D(V_1). \tag{5.1.15}$$

Lemma 5.1.1. *We have an equality of defects (Definition 4.3.1) $\delta_\pi = \delta_g$.*

Proof. By Lemma 3.5.3, both defects are equal to the defect of the pullable square (C_U, V_1, C_S, W_1), using that both (C_U, C_V, C_S, C_W) and (U_1, V_1, S_1, W_1) have defect zero. $\square$

Example 5.1.2. A special case (which will be our case of interest) is when both (U_0, V_0, S_0, W_0) and (U_1, V_1, S_1, W_1) are derived Cartesian. In this case, the sources and targets of $f_!\pi^*(\mathfrak{s})$ and $g^*z_!(\mathfrak{s})$ are matched by the proper base change isomorphisms

$$f_{0!}\pi_0^*\mathcal{K}_0 \xrightarrow{\diamond} g_0^*z_{0!}\mathcal{K}_0, \quad f_{1!}\pi_1^*\mathcal{K}_1\langle -\delta_\pi\rangle \xrightarrow{\diamond} g_1^*z_{1!}\mathcal{K}_1\langle -\delta_g\rangle. \tag{5.1.16}$$

Theorem 5.1.3 (Base Change Theorem for cohomological correspondences)**.** *Under the assumptions in §5.1.1, for any $\mathfrak{s} \in \mathrm{Corr}_{C_S}(\mathcal{K}_0, \mathcal{K}_1)$, the following diagram is commutative*

$$\begin{array}{ccc} g_0^*z_{0!}\mathcal{K}_0 & \xrightarrow{g^*z_!(\mathfrak{s})} & g_1^*z_{1!}\mathcal{K}_1\langle -\delta_g\rangle \\ \downarrow{\scriptstyle (5.1.13)} & & \uparrow{\scriptstyle (5.1.15)} \\ f_{0!}\pi_0^*\mathcal{K}_0 & \xrightarrow{f_!\pi^*(\mathfrak{s})} & f_{1!}\pi_1^*\mathcal{K}_1\langle -\delta_\pi\rangle \end{array} \tag{5.1.17}$$

Here we use Lemma 5.1.1 to match the twists.

In particular, when both (U_0, V_0, S_0, W_0) and (U_1, V_1, S_1, W_1) are derived Cartesian, we have an equality of cohomological correspondences on C_V

$$f_!\pi^*(\mathfrak{s}) = g^*z_!(\mathfrak{s}) \tag{5.1.18}$$

under the isomorphisms (5.1.16).

Remark 5.1.4. In application to the modularity theorem, we are interested in the special case where both (U_0, V_0, S_0, W_0) and (U_1, V_1, S_1, W_1) are derived Cartesian, h_1 is quasi-smooth, $\mathcal{K}_0 = \mathbf{Q}_{\ell,S_0}$, $\mathcal{K}_1 = \mathbf{Q}_{\ell,S_1}\langle -d(h_1)\rangle$, and the map $\mathfrak{s} : h_0^*\mathcal{K}_0 = \mathbf{Q}_{\ell,C_S} \to h_1^!\mathcal{K}_1 = h_1^!\mathbf{Q}_{\ell,S_1}\langle -d(h_1)\rangle$ is given by the relative fundamental class $[h_1]$.

5.2. **Proof of Theorem 5.1.3.** Unravelling the constructions of $g^*z_!(\mathfrak{s})$ and $f_!\pi^*(\mathfrak{s})$, they appear as the top and bottom rows of the following diagram

$$\begin{array}{ccccccccccc} b_0^*g_0^*z_{0!}\mathcal{K}_0 & = & g^*c_0^*z_{0!}\mathcal{K}_0 & \xrightarrow{g^*\triangledown} & g^*z_!h_0^*\mathcal{K}_0 & \xrightarrow{g^*z_!\mathfrak{s}} & g^*z_!h_1^!\mathcal{K}_1 & \longrightarrow & g^*c_1^!z_{1!}\mathcal{K}_1 & \xrightarrow{\vartriangle z_{1!}} & b_1^!g_1^*z_{1!}\mathcal{K}_1\langle -\delta_g\rangle \\ \downarrow{\scriptstyle b_0^*\triangledown} & & & & \downarrow{\scriptstyle \diamond h_0^*} & & \uparrow{\scriptstyle \diamond h_1^!} & & & & \uparrow{\scriptstyle b_1^!\vartriangle} \\ b_0^*f_{0!}\pi_0^*\mathcal{K}_0 & \xrightarrow{\triangledown\pi_0^*} & f_!a_0^*\pi_0^*\mathcal{K}_0 & = & f_!\pi^*h_0^*\mathcal{K}_0 & \xrightarrow{f_!\pi^*\mathfrak{s}} & f_!\pi^*h_1^!\mathcal{K}_1 & \xrightarrow{f_!\vartriangle} & f_!a_1^!\pi_1^*\mathcal{K}_1\langle -\delta_\pi\rangle & \longrightarrow & b_1^!f_{1!}\pi_1^*\mathcal{K}_1\langle -\delta_\pi\rangle \end{array} \tag{5.2.1}$$

Here, the arrows marked by $\triangledown$ are the base change maps obtained from a pushable square as in §3.2; the arrows marked by $\vartriangle$ are the base change maps obtained from a pullable square as in §3.5, and the arrows marked by $\diamond$ are the proper base change isomorphisms, which are special cases of both $\triangledown$ and $\vartriangle$. The unmarked arrows are the tautological base change maps from commutative squares.

To prove the theorem, we need to check that all three rectangles in (5.2.1) commute. The middle square is clearly commutative. Below we check separately that the left and right sleeves commute.

5.2.1. *Left sleeve.* We need to show that the diagram of natural transformations

$$\begin{CD} b_0^*g_0^*z_{0!} @= g^*c_0^*z_{0!} @>{g^*\triangledown}>> g^*z_!h_0^* \\ @V{b_0^*\triangledown}VV @. @VV{\triangledown h_0^*}V \\ b_0^*f_{0!}\pi_0^* @>{\triangledown\pi_0^*}>> f_!a_0^*\pi_0^* @= f_!\pi^*h_0^* \end{CD} \tag{5.2.2}$$

is commutative. Here we have replaced the $\diamond$ with the $\triangledown$ in the right vertical arrow because the proper base change isomorphism $g^*z_! \xrightarrow{\sim} f_!\pi^*$ is a special case of the $\triangledown$ map from a pushable square.

Let

$$\alpha_0 := \pi_0 \circ a_0 = h_0 \circ \pi : C_U \to S_0, \tag{5.2.3}$$

$$\beta_0 := g_0 \circ b_0 = c_0 \circ g : C_V \to W_0. \tag{5.2.4}$$

The left cube in the diagram (5.1.1) provides two decompositions of the commutative square

$$\begin{CD} C_U @>{f}>> C_V \\ @V{\alpha_0}VV @VV{\beta_0}V \\ S_0 @>{z_0}>> W_0 \end{CD} \tag{5.2.5}$$

The first is

$$\begin{CD} C_U @>{f}>> C_V \\ @V{a_0}VV @VV{b_0}V \\ U_0 @>{f_0}>> V_0 \\ @V{\pi_0}VV @VV{g_0}V \\ S_0 @>{z_0}>> W_0 \end{CD} \tag{5.2.6}$$

in which both the upper and lower squares are pushable. By Proposition 3.2.3, the natural transformation $\triangledown$ for the square (5.2.5) agrees with the composition

$$\beta_0^*z_{0!} = b_0^*g_0^*z_{0!} \xrightarrow{b_0^*\triangledown} b_0^*f_{0!}\pi_0^* \xrightarrow{\triangledown\pi_0^*} f_!a_0^*\pi_0^* = f_!\alpha_0^* \, . \tag{5.2.7}$$

This is the lower composition of the diagram (5.2.2).

The second decomposition of (5.2.5) is

$$\begin{CD} C_U @>{f}>> C_V \\ @V{\pi}VV @VV{g}V \\ C_S @>{z}>> C_W \\ @V{h_0}VV @VV{c_0}V \\ S_0 @>{z_0}>> W_0 \end{CD} \tag{5.2.8}$$

in which both the upper and lower squares are pushable. By Proposition 3.2.3 again, the natural transformation $\triangledown$ for the square (5.2.5) agrees with the composition

$$\beta_0^*z_{0!} = g^*c_0^*z_{0!} \xrightarrow{g^*\triangledown} g^*z_!h_0^* \xrightarrow{\triangledown h_0^*} f_!\pi^*h_0^* = f_!\alpha_0^*. \tag{5.2.9}$$

This is the upper composition of the diagram (5.2.2). Therefore both compositions in (5.2.2) computes the same base change map $\triangledown : \beta_0^*z_{0!} \to f_!\alpha_0^*$. This proves that (5.2.2) is commutative.

5.2.2. *Right sleeve.* We need to show that the diagram of natural transformations

$$\begin{CD}
g^* z_! h_1^! @>>> g^* c_1^! z_{1!} @>{\triangle z_{1!}}>> b_1^! g_1^* z_{1!}\langle -\delta_g\rangle \\
@A{\triangle h_1^!}AA @. @A{b_1^! \triangle}AA \\
f_! \pi^* h_1^! @>{f_! \triangle}>> f_! a_1^! \pi_1^* \langle -\delta_\pi\rangle @>>> b_1^! f_{1!} \pi_1^* \langle -\delta_\pi\rangle
\end{CD} \tag{5.2.10}$$

is commutative. Here we have replaced the $\diamond$ with the $\triangle$ in the left vertical arrow because the proper base change isomorphism $f_! \pi^* \xrightarrow{\sim} g^* z_!$ is a special case of the $\triangle$ map from a pullable square.

Let

$$\alpha_1 := f_1 \circ a_1 = b_1 \circ f : C_U \to V_1, \tag{5.2.11}$$

$$\beta_1 := z_1 \circ h_1 = c_1 \circ z : C_S \to W_1. \tag{5.2.12}$$

The right cube in the diagram (5.1.1) provides two decompositions of the commutative square

$$\begin{CD}
C_U @>{\pi}>> C_S \\
@V{\alpha_1}VV @VV{\beta_1}V \\
V_1 @>{z_1}>> W_1
\end{CD} \tag{5.2.13}$$

The first is

$$\begin{CD}
C_U @>{\pi}>> C_S \\
@V{a_1}VV @VV{h_1}V \\
U_1 @>{\pi_1}>> S_1 \\
@V{f_1}VV @VV{z_1}V \\
V_1 @>{g_1}>> W_1
\end{CD} \tag{5.2.14}$$

in which both the upper and lower squares are pullable. By Proposition 3.5.4, the natural transformation $\triangle$ for the square (5.2.13) agrees with the composition

$$\pi^* \beta_1^! = \pi^* h_1^! z_1^! \xrightarrow{\triangle z_1^!} a_1^! \pi_1^* z_1^! \langle -\delta_\pi\rangle \xrightarrow{a_1^! \triangle} a_1^! f_1^! g_1^* \langle -\delta_\pi\rangle = \alpha_1^! g_1^* \langle -\delta_\pi\rangle \tag{5.2.15}$$

The second decomposition of (5.2.13) is

$$\begin{CD}
C_U @>{\pi}>> C_S \\
@V{f}VV @VV{z}V \\
C_V @>{g}>> C_W \\
@V{b_1}VV @VV{c_1}V \\
V_1 @>{g_1}>> W_1
\end{CD} \tag{5.2.16}$$

in which both the upper and lower squares are pullable. By Proposition 3.5.4 again, the natural transformation $\triangle$ for the square (5.2.13) agrees with the composition

$$\pi^* \beta_1^! = \pi^* z^! c_1^! \xrightarrow{\triangle c_1^!} f^! g^* c_1^! \xrightarrow{f^! \triangle} f^! b_1^! g_1^* = \alpha_1^! g_1^* \langle -\delta_g\rangle \tag{5.2.17}$$

Combining the two expressions of $\triangle\colon \pi^* \beta_1^! \to \alpha_1^! g_1^* \langle -\delta_g\rangle = \alpha_1^! g_1^* \langle -\delta_\pi\rangle$, we get a commutative diagram

$$\begin{CD}
\pi^* z_1^! c_1^! @>{\triangle c_1^!}>> f^! g^* c_1^! @>{f^! \triangle}>> f^! b_1^! g_1^* \langle -\delta_g\rangle \\
@| @. @| \\
\pi^* h_1^! z_1^! @>{\triangle z_1^!}>> a_r^! \pi_1^* z_1^! \langle -\delta_\pi\rangle @>{a_1^! \triangle}>> a_1^! f_1^! g_1^* \langle -\delta_\pi\rangle
\end{CD} \tag{5.2.18}$$

Compare (5.2.18) with (5.2.10). Starting in both diagrams from the lower left corner, going upward then turning right to arrive at the upper right corner, we see the two maps

$$\pi^* h_1^! z_1^! \to f^! b_1^! g_1^* \langle -\delta_g \rangle, \tag{5.2.19}$$

$$f_! \pi^* h_1^! \to b_1^! g_1^* z_{1!} \langle -\delta_g \rangle, \tag{5.2.20}$$

are related by adjunctions $(f_!, f^!)$ and $(z_{1!}, z_1^!)$. Similarly, starting in both diagrams from the lower left corner, going right and then turning upward to arrive at the upper right corner, we get two maps of the above shape that are again related by adjunctions. Since (5.2.18) is commutative, we conclude that (5.2.10) is also commutative. This completes the proof of Theorem 5.1.3. □

Part 2. Generalities on Fourier transform

6. Derived Fourier analysis

In this section we introduce a package of results that constitute what we call "derived Fourier analysis", because it occurs on generalization of vector bundles that we call *derived vector bundles*. These are the "total spaces" of perfect complexes, generalizing how vector bundles are the "total spaces" of locally free coherent sheaves. Then we generalize the Deligne-Laumon Fourier transform for ℓ-adic sheaves from vector bundles to derived vector bundles; we call this the *derived Fourier transform*. This theory of "derived Fourier analysis" is needed to lift the function-theoretic Fourier analysis in the proof for $r = 0$ to the level of sheaves. We establish several properties of the derived Fourier transform generalizing familiar ones, deferring most of the proofs to Appendix A.

6.1. **Derived Fourier transform.** Let S be a derived Artin stack, and $E \to S$ a vector bundle.

For $\widehat{E}$ the linear dual of E, we have the tautological evaluation pairing $\mathrm{ev}\colon E \times_S \widehat{E} \to \mathbf{A}^1$.

Let $\psi\colon \mathbf{F}_q \to \mathbf{Q}_\ell^\times$ be a non-trivial additive character and $\mathcal{L}_\psi$ be the corresponding Artin-Schreier sheaf on $\mathbf{A}^1$.

The *Deligne-Laumon Fourier transform* is the functor

$$\mathrm{FT}_E^\psi\colon D(E) \to D(E)$$

given by (following [Lau87, Définition 1.2.1.1] in our normalizations)

$$\mathcal{K} \mapsto \mathrm{pr}_{1!}(\mathrm{pr}_0^* \mathcal{K} \otimes \mathrm{ev}^* \mathcal{L}_\psi)[r]$$

where $r := \mathrm{rank}(E)$ is the rank of E, and the maps are as in the diagram

$$\begin{array}{ccccc} & & E \times_S \widehat{E} & \xrightarrow{\mathrm{ev}} & \mathbf{A}^1 \\ & \swarrow^{\mathrm{pr}_0} & & \searrow^{\mathrm{pr}_1} & \\ E & & & & \widehat{E} \end{array}$$

We will extend the Deligne-Laumon Fourier transform to certain "derived linear spaces" that we call *derived vector bundles*, generalizing vector bundles.

6.1.1. *Derived vector bundles.* Let S be a derived Artin stack. There is a functor Tot_S from the category $\mathrm{Perf}(S)$ of perfect complexes on S to the category of derived stacks over S, which extends the usual construction of a vector bundle from a locally free coherent sheaf. As far as we know, the construction is due to Toën and is documented in [Toe14, p.200-201] (and essentially goes back at least to [Toe06]); *however, be warned that Toën's convention differs from ours: what he calls* $\mathrm{Tot}_S(\mathcal{E})$ *is what we would call* $\mathrm{Tot}_S(\mathcal{E}^*)$. (The construction of Tot_S also appears in [Kha19], who agrees with Toën's convention and therefore disagrees with ours.)

The elegant definition from [Toe14, p. 201] explains that as a functor from (derived) S-schemes to anima, $\mathrm{Tot}_S(\mathcal{E})$ sends $u\colon T \to S$ to $\mathrm{Map}_{\mathrm{QCoh}(T)}(\mathcal{O}, u^*\mathcal{E})$, taking into account the reversal of convention as mentioned above. Here $\mathrm{Map}_{\mathrm{QCoh}(T)}$ invokes the enrichment of $\mathrm{QCoh}(T)$ over anima.

We will spell out the meaning of this definition in perhaps more familiar terms. We start by explicating $\mathrm{Tot}_S(\mathcal{E})$ in the special case where $\mathcal{E} \in \mathrm{Perf}(S)$ has tor-amplitude in $(-\infty, 0]$ (sometimes referred to as $\mathcal{E}$ being *connective*). Then $\mathrm{Tot}_S(\mathcal{E})$ represents the functor given by (derived) global sections, which at the level of 0-cells assigns to a derived affine test scheme $T \to S$ the sections $R\Gamma(T, \mathcal{E}|_T)$ viewed as an animated abelian group.

Next we explain how to generalize the construction of the preceding paragraph to general $\mathcal{E} \in \mathrm{Perf}(S)$. It is immediate from the definition that the construction $\mathcal{E} \mapsto \mathrm{Tot}_S(\mathcal{E})$, defined so far on connective $\mathcal{E}$, preserves limits (if they exist as connective complexes), hence sends exact triangles $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3$ to derived Cartesian squares

$$\begin{array}{ccc} \mathrm{Tot}_S(\mathcal{E}_1) & \longrightarrow & \mathrm{Tot}_S(\mathcal{E}_2) \\ \downarrow & & \downarrow \\ \mathrm{Tot}_S(0) & \longrightarrow & \mathrm{Tot}_S(\mathcal{E}_3) \end{array}$$

Note that $\mathrm{Tot}_S(0) \cong S$; we call the map $\mathrm{Tot}_S(0) \to \mathrm{Tot}_S(\mathcal{E})$ the *zero-section.* We extend the construction of $\mathrm{Tot}_S(-)$ to all $\mathcal{E} \in \mathrm{Perf}(S)$ by this condition. Concretely, $\mathrm{Tot}_S(\mathcal{E}[-1])$ is the derived self-intersection[7] of the zero-section of $\mathrm{Tot}_S(\mathcal{E})$,

$$\begin{array}{ccc} \mathrm{Tot}_S(\mathcal{E}[-1]) & \longrightarrow & \mathrm{Tot}_S(0) \\ \downarrow & & \downarrow \\ \mathrm{Tot}_S(0) & \longrightarrow & \mathrm{Tot}_S(\mathcal{E}) \end{array}$$

and for a general $\mathcal{E} \in \mathrm{Perf}(S)$ there exists some d such that $\mathcal{E}[d]$ is connective; then $\mathrm{Tot}_S(\mathcal{E})$ is obtained from $\mathrm{Tot}_S(\mathcal{E}[d])$ by iterating the procedure of forming derived self-intersection of the zero-section d times.

Example 6.1.1. Suppose $\mathcal{E}$ has tor-amplitude in $[0, \infty)$ (sometimes referred to as $\mathcal{E}$ being *co-connective*). Then $\mathcal{E}^*$ has tor-amplitude in $(-\infty, 0]$, and in particular is an animated $\mathcal{O}_S$-module. The forgetful functor from animated $\mathcal{O}_S$-algebras to animated $\mathcal{O}_S$-modules has a left adjoint, the *derived symmetric algebra* functor $\mathrm{Sym}_S^\bullet$ (see for example [Lur19, §25.2.2]). Then $\mathrm{Tot}_S(\mathcal{E})$ is the relative spectrum of $\mathrm{Sym}_S^\bullet(\mathcal{E}^*)$.

For $\mathcal{E} \in \mathrm{Perf}(S)$, we will call $E := \mathrm{Tot}_S(\mathcal{E})$ the *derived vector bundle* associated to E. The *virtual rank* of E, still denoted $\mathrm{rank}(E)$, is the locally constant function on S given by $s \mapsto \chi(\mathcal{E}_s)$, the Euler characteristic of the fiber of $\mathcal{E}$ at a geometric point s. **In general, for perfect complexes denoted with calligraphic letters such as $\mathcal{E}, \mathcal{E}'$, etc., the corresponding roman letters such as E, E', etc. denote their associated total spaces.**

The map $0 \to \mathcal{E}$ equips E with a *zero-section*

$$z_E \colon S \to E.$$

We caution that z_E is *not* necessarily a closed embedding – it is a closed embedding exactly when E comes from a perfect complex $\mathcal{E}$ with tor-amplitude in $[0, \infty)$.

Also, the map $\mathcal{E} \to 0$ equips E with a *projection*

$$\pi_E \colon E \to S.$$

We caution π_E is *not* necessarily representable in derived schemes – it is representable exactly when E comes from a perfect complex $\mathcal{E}$ with tor-amplitude in $[0, \infty)$.

Remark 6.1.2. If the perfect complex $\mathcal{E}$ has tor-amplitude in $(-\infty, 0]$, then the morphism $E \to S$ is represented by stacks which are *classical* in the sense of being isomorphic to their classical truncations. If $\mathcal{E}$ has tor-amplitude in $[0, \infty)$ then the morphism $E \to S$ is represented by derived schemes. Therefore, duality of derived vector bundles interchanges "stackiness" with "derivedness".

6.1.2. *The ℓ-adic Fourier transform for derived vector bundles.* Let S be a derived Artin stack, $\mathcal{E} \in \mathrm{Perf}(S)$. For $\mathcal{E}^* \in \mathrm{Perf}(S)$ the linear dual of $\mathcal{E}$, we have a tautological evaluation pairing $\mathcal{E} \times \mathcal{E}^* \to \mathcal{O}_S$. Setting $E := \mathrm{Tot}_S(\mathcal{E})$ and $\widehat{E} := \mathrm{Tot}_S(\mathcal{E}^*)$, this induces on total spaces a map

$$\mathrm{ev} \colon E \times_S \widehat{E} \to \mathbf{A}^1.$$

The Fourier transform

$$\mathrm{FT}_E^\psi \colon D(E) \to D(\widehat{E})$$

[7] Although we call this an "intersection", the zero-section will typically not be a closed embedding.

is defined as

$$\mathcal{K} \mapsto \mathrm{pr}_{1!}(\mathrm{pr}_0^*(\mathcal{K}) \otimes \mathrm{ev}^* \mathcal{L}_\psi)[r]$$

where $r := \mathrm{rank}(E)$ is the virtual rank of $E \to S$ (which is constant on each connected component of S, so the shift $[r]$ makes sense over each connected component of S), and the maps are as in the diagram

$$\begin{array}{ccccc} & & E \times_S \widehat{E} & \xrightarrow{\mathrm{ev}} & \mathbf{A}^1 \\ & \swarrow^{\mathrm{pr}_0} & & \searrow^{\mathrm{pr}_1} & \\ E & & & & \widehat{E} \end{array}$$

This extends the Deligne-Laumon Fourier transform for ℓ-adic sheaves for vector bundles, which corresponds to the case where $\mathcal{E}$ is a locally free coherent *sheaf* (concentrated in degree 0). When the additive character ψ is understood, we will simply omit it from the notation.

6.2. **Properties of the derived Fourier transform.** We now tabulate some basic properties of the derived Fourier transform. Below we let r be the virtual rank of $E \to S$. The non-trivial proofs are all found in Appendix A. Below we use the adjective "canonical" to describe an isomorphism whose construction does not depend on any auxiliary choices; we emphasize this because the proofs require considering, at intermediate stages, natural isomorphisms that a priori depend on auxiliary choices (but are ultimately seen to be independent of such choices a posteriori).

6.2.1. *Fourier transform of Gaussians.* Suppose $h_E \colon E \xrightarrow{\sim} \widehat{E}$ is a symmetric isomorphism. This induces:

- a quadratic form $q \colon E \to \mathbf{A}^1$ given by $q(e) := \langle e, h_E(e) \rangle$, and
- a quadratic form $\widehat{q} \colon \widehat{E} \to \mathbf{A}^1$ given by $q(\widehat{e}) := \langle h_E^{-1}(\widehat{e}), \widehat{e} \rangle$.

Then one has a canonical isomorphism

$$[2]^* \mathrm{FT}_E(q^* \mathcal{L}_\psi) \cong (-\widehat{q})^* \mathcal{L}_\psi \otimes (\pi_{\widehat{E}}^* \pi_{E!} q^* \mathcal{L}_\psi[r]).$$

The proof is the same as for classical vector bundles, which is found in [Lau87, Proposition 1.2.3.3].

6.2.2. *Base change.* Let $h \colon \widetilde{S} \to S$. For a derived vector bundle $E \to S$, let $\widetilde{E} \to E$ be its base change along h. So we have derived Cartesian squares

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{h^E} & E \\ \downarrow & & \downarrow \\ \widetilde{S} & \xrightarrow{h} & S \end{array} \qquad \begin{array}{ccc} \widetilde{\widehat{E}} & \xrightarrow{h^{\widehat{E}}} & \widehat{E} \\ \downarrow & & \downarrow \\ \widetilde{S} & \xrightarrow{h} & S \end{array}$$

Then there are canonical natural isomorphisms of functors $D(E) \to D(\widetilde{\widehat{E}})$

$$\mathrm{FT}_{\widetilde{E}} \circ (h^E)^* \cong (h^{\widehat{E}})^* \circ \mathrm{FT}_E \tag{6.2.1}$$

$$\mathrm{FT}_{\widetilde{E}} \circ (h^E)^! \cong (h^{\widehat{E}})^! \circ \mathrm{FT}_E \tag{6.2.2}$$

and canonical natural isomorphisms of functors $D(\widetilde{E}) \to D(\widehat{E})$

$$\mathrm{FT}_E \circ (h^E)_! \cong (h^{\widehat{E}})_! \circ \mathrm{FT}_{\widetilde{E}} \tag{6.2.3}$$

$$\mathrm{FT}_E \circ (h^E)_* \cong (h^{\widehat{E}})_* \circ \mathrm{FT}_{\widetilde{E}} . \tag{6.2.4}$$

The isomorphisms (6.2.1) and (6.2.3) follow directly from proper base change. The other natural isomorphisms will be constructed in §A.3.4.

6.2.3. *Involutivity.* There is a canonical natural isomorphism $\mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E \cong [-1]^*(-r)$ of functors $D(E) \to D(E)$, where $[-1]$ is multiplication by -1 on E.

Remark 6.2.1. The construction of this natural isomorphism appears to be signficantly more involved than in the situation of the Deligne-Laumon Fourier transform, and occupies much of Appendix A.

6.2.4. *Functoriality.* Let $f\colon E' \to E$ be a linear map of derived vector bundles having virtual ranks r', r respectively. This induces a morphism $\widehat{f}\colon \widehat{E} \to \widehat{E}'$ of dual derived bundles. Then we have canonical natural isomorphisms of functors $D(E') \to D(\widehat{E})$:

(1) $\widehat{f}^* \circ \mathrm{FT}_{E'} \cong \mathrm{FT}_E \circ f_![r'-r]$,
(2) $\widehat{f}^! \circ \mathrm{FT}_{E'} \cong \mathrm{FT}_E \circ f_*[r-r'](r-r')$,

and canonical natural isomorphisms of functors $D(E) \to D(\widehat{E}')$:

(3) $\mathrm{FT}_{E'} \circ f^* \cong \widehat{f}_! \circ \mathrm{FT}_E[r-r'](r-r')$,
(4) $\mathrm{FT}_{E'} \circ f^! \cong \widehat{f}_* \circ \mathrm{FT}_E[r'-r]$.

We record for convenience that in the Fourier dual coordinates, the previous two isomorphisms become natural isomorphisms of functors $D(\widehat{E}') \to D(E)$:

(5) $\mathrm{FT}_{\widehat{E}} \circ \widehat{f}^* \cong f_! \circ \mathrm{FT}_{\widehat{E}'}[r'-r](r'-r)$,
(6) $\mathrm{FT}_{\widehat{E}} \circ \widehat{f}^! \cong f_* \circ \mathrm{FT}_{\widehat{E}'}[r-r']$.

Moreover, we shall see in §A.2.6 that by construction, the natural isomorphisms above intertwine the adjunction $(f_!, f^!)$ with the adjunction $(\widehat{f}^*, \widehat{f}_*)$ (up to shift and twist), and the adjunction (f^*, f_*) with the adjunction $(\widehat{f}_!, \widehat{f}^!)$ (up to shift and twist). In particular, FT_E sends the unit and counit

$$\mathrm{Id} \to f^! f_! \qquad f_! f^! \to \mathrm{Id}$$

to the unit and the counit[8]

$$\mathrm{FT}_E \to \widehat{f}_* \widehat{f}^* \, \mathrm{FT}_E \qquad \widehat{f}^* \widehat{f}_* \, \mathrm{FT}_E \to \mathrm{FT}_E$$

under the above identifications, and similarly for the other adjunction.

Definition 6.2.2 (The delta-sheaf)**.** For a derived vector bundle $E \to S$, recall that $z_E\colon S \to E$ is the zero-section, which may not be a closed embedding. We define $\delta_E := z_{E!}\mathbf{Q}_{\ell,S}$.

Example 6.2.3 (Delta-constant duality)**.** Suppose $E \to S$ is a derived vector bundle of rank r. Then natural transformation (1) gives an isomorphism

$$\mathrm{FT}_E(\delta_E) \cong \mathbf{Q}_{\ell,\widehat{E}}[r]$$

and natural transformation (3) gives an isomorphism

$$\mathrm{FT}_{\widehat{E}} \, \mathbf{Q}_{\ell,\widehat{E}} \cong \delta_E[-r](-r).$$

6.2.5. *Verdier duality.* Letting $\mathbf{D}_E$ (resp. $\mathbf{D}_{\widehat{E}}$) denote the Verdier duality functor on E (resp. $\widehat{E}$), there is a canonical isomorphism naturally in $\mathcal{K} \in D(E)$

$$\mathbf{D}_{\widehat{E}}(\mathrm{FT}^{\psi}_E(\mathcal{K})) \cong \mathrm{FT}^{\psi^{-1}}_E(\mathbf{D}_E(\mathcal{K}))(r).$$

6.2.6. *Convolutions.* For $\mathcal{K}_0, \mathcal{K}_1 \in D(E)$, we write

$$\mathcal{K}_0 \star \mathcal{K}_1 = +_!(\mathrm{pr}_0^* \mathcal{K}_0 \otimes \mathrm{pr}_1^* \mathcal{K}_1)$$

where maps are as in the diagram

$$\begin{array}{ccccc} & & E \times_S E & \xrightarrow{+} & E \\ & \swarrow^{\mathrm{pr}_0} & & \searrow^{\mathrm{pr}_1} & \\ E & & & & E \end{array}$$

For $r = \mathrm{rank}(E)$, there is a canonical natural isomorphism

$$\mathrm{FT}_E(\mathcal{K}_0 \star \mathcal{K}_1) \cong \mathrm{FT}_E(\mathcal{K}_0) \otimes \mathrm{FT}_E(\mathcal{K}_1)[-r]$$

which is constructed formally from the functorialities of §6.2.4.

[8]In some normalizations of the involutivity isomorphism, a sign would appear here. We have set up our formalism so that no sign issues intervene here.

6.2.7. *Plancherel formula.* There is a canonical natural isomorphism

$$\pi_{\widehat{E}!}(\mathrm{FT}_E(\mathcal{K}_1) \otimes \mathrm{FT}_E(\mathcal{K}_2)) \cong \pi_{E!}(\mathcal{K}_1 \otimes [-1]^* \mathcal{K}_2)(-r).$$

This is obtained by writing the LHS as $\pi_{\widehat{E}!}\Delta^* \mathrm{FT}_E(\mathrm{pr}_0^*(\mathcal{K}_0) \boxtimes \mathrm{pr}_1^*(\mathcal{K}_1))$ and then applying the functorialities of §6.2.4 plus proper base change.

6.3. **Proper base change.** We will need the compatibility of the Fourier transform with proper base change, at least under a "global presentation" hypothesis.

6.3.1. *Globally presented derived vector bundles.* We introduce the following definition for technical reasons:

Definition 6.3.1. A perfect complex $\mathcal{E} \in \mathrm{Perf}(S)$ is *globally presented* if it is quasi-isomorphic to a bounded complex of vector bundles on S,

$$\mathcal{E} \cong (\ldots \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \ldots).$$

By definition of $\mathrm{Perf}(S)$ such a presentation exists Zariski-locally on S, but we are asking for its existence globally.

We say that the associated derived vector bundle $E = \mathrm{Tot}_S(\mathcal{E})$ is *globally presented* if $\mathcal{E}$ is globally presented.

We say that a map $f \colon E' \to E$ of derived vector bundles lying over $h \colon S' \to S$ is *globally presented* if there exist global presentations $\mathcal{E}^\bullet$ for E and $(\mathcal{E}^\bullet)'$ for E' and f is induced by a map of complexes $(\mathcal{E}^\bullet)' \to h^*\mathcal{E}^\bullet$.

More generally, we say that a diagram of derived vector bundles is *globally presented* if there exist global presentations for all derived vector bundles such that all maps between derived vector bundles are induced by maps of these presentations.

Example 6.3.2. Any diagram of classical vector bundles is globally presented.

The role of the notion of global presentation is the following. Certain proofs (deferred to Appendix A) towards the results already mentioned in this section rely on the notion of global presentation at intermediate stages. Furthermore, in the statement of Proposition 6.3.3 below, we impose a global presentation assumption. Various results in later sections depend on Proposition 6.3.3 and will therefore also require a global presentation assumption. We expect that this assumption is not actually necessary, but it provides a "shortcut" for the proof, ultimately because a globally presented map can be factored into the composition of a closed embedding and a smooth map (Lemma A.4.1).

6.3.2. *Compatibility with proper base change.* Consider a Cartesian square of globally presented derived vector bundles, along with the dual Cartesian square

$$\begin{array}{ccccc} & & B & & \\ & \swarrow^{g'} & & \searrow^{f'} & \\ A & & & & D \\ & \searrow_{f} & & \swarrow_{g} & \\ & & C & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & \widehat{C} & & \\ & \swarrow^{\widehat{f}} & & \searrow^{\widehat{g}} & \\ \widehat{A} & & & & \widehat{D} \\ & \searrow_{\widehat{g}'} & & \swarrow_{\widehat{f}'} & \\ & & \widehat{B} & & \end{array}$$

Then proper base change gives natural isomorphisms

$$g^* f_! \cong (f')_!(g')^* \qquad \text{and} \qquad \widehat{g}_! \widehat{f}^* \cong (\widehat{f}')^* \widehat{g}'_! \tag{6.3.1}$$

Let $d = d(f), \delta := d(g)$. According to §6.2.4, there are natural isomorphisms

$$\widehat{g}_! \widehat{f}^* \mathrm{FT}_A \cong \mathrm{FT}_D\, g^* f_! [d+\delta](\delta) \qquad \text{and} \qquad (\widehat{f}')^* \widehat{g}'_! \mathrm{FT}_A \cong \mathrm{FT}_D\, f'_!(g')^*[d+\delta](\delta). \tag{6.3.2}$$

Proposition 6.3.3. *Assume that f and g are globally presented (in particular, A, C, D are globally presented). Then the diagram*

$$\begin{array}{ccc} \widehat{g}_! \widehat{f}^* \mathrm{FT}_A & \xrightarrow{\sim} & \mathrm{FT}_D\, g^* f_! [d+\delta](\delta) \\ \downarrow\wr & & \downarrow\wr \\ (\widehat{f}')^* \widehat{g}'_! \mathrm{FT}_A & \xrightarrow{\sim} & \mathrm{FT}_D\, f'_!(g')^*[d+\delta](\delta) \end{array} \tag{6.3.3}$$

commutes, where the identifications are as in (6.3.1) *and* (6.3.2)*.*

This innocuous-looking statement turns out to be rather involved to prove, so the proof will be deferred to §A (see Proposition A.4.3). As discussed above, we believe that the technical assumption that f, g are globally presented is an artefact of the proof.

6.4. **Fourier transform of the Gysin map.** Let $f\colon E' \to E$ be a quasi-smooth morphism of derived vector bundles over S, which is equivalent to $\mathrm{cone}(\mathcal{E}' \to \mathcal{E})$ being locally represented by a complex of vector bundles in degrees ≤ 0. Then f has a relative fundamental class $[f]$, which induces a Gysin natural transformation $f^* \to f^!\langle -d(f)\rangle$, as explained in §3.4. Dualizing, this is equivalent to the condition that the dual map $\widehat{f}\colon \widehat{E} \to \widehat{E'}$ is separated (or equivalently in this case, representable by derived schemes), and therefore has a "forget supports" natural transformation $\mathrm{can}(\widehat{f})\colon \widehat{f}_! \to \widehat{f}_*$.

Example 6.4.1. If E and E' are classical vector bundles over S, then the map f is automatically LCI (and therefore quasi-smooth). Indeed, the graph of f provides a factorization

$$E' \hookrightarrow E \times_S E' \xrightarrow{\mathrm{pr}_1} E$$

which is a composition of a regular embedding and a smooth morphism.

We need the following identification of the Fourier transform of the Gysin map. After some inquiries, we found that this statement was unknown to experts even in the case where f is a map of classical vector bundles, hence automatically LCI by Example 6.4.1.

Proposition 6.4.2. *Let $f\colon E' \to E$ be a globally presented quasi-smooth map of derived vector bundles and let $\widehat{f}\colon \widehat{E} \to \widehat{E}'$ be the dual map to $f\colon E' \to E$. Then the diagram of functors $D(E) \to D(\widehat{E}')$*

$$\begin{array}{ccc} \widehat{f}_! \, \mathrm{FT}_E & \xrightarrow[\sim]{\mathrm{can}(\widehat{f})} & \widehat{f}_* \, \mathrm{FT}_E \\ \downarrow{\sim} & & \downarrow{\sim} \\ \mathrm{FT}_{E'} \, f^*[d(f)](d(f)) & \xrightarrow[\sim]{[f]} & \mathrm{FT}_{E'} \, f^![-d(f)] \end{array} \tag{6.4.1}$$

commutes.

Remark 6.4.3. The significance of Proposition 6.4.2 is to describe the derived (relative) fundamental class $[f]$ in terms of classical geometry *in the Fourier dual space.*

The proof of Proposition 6.4.2 is rather lengthy. We will begin with several reductions.

6.4.1. *Reduction to smooth derived vector bundles.* We begin by reducing Proposition 6.4.2 to the case where E and E' are both smooth (but still potentially stacky) over S.

Lemma 6.4.4. *We can find $\mathcal{F}, \mathcal{F}'$ with tor-amplitude in* $(-\infty, 0]$ *and a quasi-smooth map $g\colon F' \to F$ fitting into a derived Cartesian square*

$$\begin{array}{ccc} E' & \xrightarrow{h} & F' \\ \downarrow{f} & & \downarrow{g} \\ E & \xrightarrow{h} & F \end{array} \tag{6.4.2}$$

Proof. We have by assumption that f is represented by a map of complexes of vector bundles

$$\begin{array}{ccccccc} \dots & \longrightarrow & (\mathcal{E}')^{m-1} & \longrightarrow & (\mathcal{E}')^m & \longrightarrow & 0 \\ \downarrow & & \downarrow{f_{m-1}} & & \downarrow{f_m} & & \\ \dots & \longrightarrow & \mathcal{E}^{m-1} & \longrightarrow & \mathcal{E}^m & \longrightarrow & 0 \end{array} \tag{6.4.3}$$

We induct on the statement: as long as $m \geq 1$ and f is quasi-smooth, any such diagram is up to homotopy equivalence pulled back from one in which both rows are complexes of vector bundles which vanish in degrees at least m (in both rows).

To prove this, we will replace (6.4.3) via homotopy equivalences by a map of complexes for which $f_m = \mathrm{Id}$, for then

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & (\mathcal{E}')^{<m} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{<m}} \\ \mathcal{E} & \longrightarrow & \mathcal{E}^{<m} \end{array}$$

is a pullback square, where $(\dots)^{<m}$ refers to the naive truncation, and we may take $g\colon \mathcal{F}' \to \mathcal{F}$ to be $f_{<m}\colon (\mathcal{E}')^{<m} \to \mathcal{E}^{<m}$. Indeed, the assumption that f is quasi-smooth implies that it induces an isomorphism on $H^{\geq 2}$ and a surjection on H^1, so by the assumption that $m \geq 1$ the map $\mathcal{E}^{m-1} \oplus (\mathcal{E}')^m \xrightarrow{d+f_m} \mathcal{E}^m$ is surjective. We may replace $(\mathcal{E}')^{m-1} \xrightarrow{d'} (\mathcal{E}')^m$ by $\mathcal{E}^{m-1} \oplus (\mathcal{E}')^{m-1} \xrightarrow{\mathrm{Id}\oplus d'} \mathcal{E}^{m-1} \oplus (\mathcal{E}')^m$, and replace the diagram (6.4.3) by

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{E}^{m-1} \oplus (\mathcal{E}')^{m-1} & \xrightarrow{\mathrm{Id}\oplus d'} & \mathcal{E}^{m-1} \oplus (\mathcal{E}')^m & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}+f_{m-1}} & & \downarrow{\scriptstyle d+f_m} & & \\ \dots & \longrightarrow & \mathcal{E}^{m-1} & \xrightarrow{\quad d \quad} & \mathcal{E}^m & \longrightarrow & 0 \end{array}$$

Then we may replace the part $\mathcal{E}^{m-1} \xrightarrow{d} \mathcal{E}^m$ by its pullback along the surjection $f_m\colon (\mathcal{E}')^m \to \mathcal{E}^m$. The map in degree m is now the identity map, as desired. ☐

Lemma 6.4.5. *If Proposition 6.4.2 holds for the map $g\colon F' \to F$ in the right column of* (6.4.2)*, then it holds for the map $f\colon E' \to E$.*

Proof. By the base change property for relative fundamental classes [Kha19, Theorem 3.13], we have $h^*[g] = [f]$, meaning that the following diagram commutes:

$$\begin{array}{ccc} h^* g^* \mathbf{Q}_{\ell,F} & \xrightarrow{h^*[g]} & h^* g^! \mathbf{Q}_{\ell,F}\langle -d(g)\rangle \\ \| & & \downarrow{\scriptstyle \diamond} \\ f^* i^* \mathbf{Q}_{\ell,F} & \xrightarrow{[f]} & f^! h^* \mathbf{Q}_{\ell,F}\langle -d(f)\rangle \end{array}$$

We are granted that $\mathrm{FT}_{F'}([g]) = \mathrm{can}(\widehat{g})$. Then applying $\mathrm{FT}_{E'}$ to this commutative diagram, using §6.2.4, we have that

$$\mathrm{FT}_{E'}([f]) = \mathrm{FT}_{E'}(h^*[g])) = \widehat{h}_! \,\mathrm{FT}_{F'}([g]) = \widehat{h}_! \,\mathrm{can}(\widehat{g}) = \mathrm{can}(\widehat{f})$$

where the last equality used Lemma 3.3.1 (note that both maps $\widehat{g}$ and $\widehat{f}$ are separated and locally of finite type). ☐

Hence we have reduced the proof of Proposition 6.4.2 to the case where E' and E are smooth over S, and in the rest of the argument we will assume this to be the case.

6.4.2. *Equivalence of formulations.* Recall from Example 6.2.3 that for $r := \mathrm{rank}(E)$, we have $\mathrm{FT}(\mathbf{Q}_{\ell,E}) \cong \delta_{\widehat{E}}[-r](-r)$. Therefore, a special case of Proposition 6.4.2 is the following Lemma.

Lemma 6.4.6. *The relative fundamental class $f^*\mathbf{Q}_{\ell,E} \xrightarrow{[f]} f^!\mathbf{Q}_{\ell,E}\langle -d(f)\rangle$ is sent by $\mathrm{FT}_{E'}$ to*

$$\widehat{f}_! \delta_{\widehat{E}'}[-r'](-r') \xrightarrow{\mathrm{can}(\widehat{f})} \widehat{f}_* \delta_{\widehat{E}'}[-r'](-r').$$

However, the converse is also true, at least under the given assumptions.

Lemma 6.4.7. *Let E', E be derived vector bundles smooth over S, and $f\colon E' \to E$ a quasi-smooth and globally presented. If Lemma 6.4.6 holds for f, then Proposition 6.4.2 holds for f.*

Proof. Indeed, let $\mathcal{K} \in D(E)$. Then $f^*\mathcal{K} \to f^!\mathcal{K}\langle -d(f)\rangle$ is the composition

$$f^*\mathcal{K} = f^*\mathcal{K} \otimes f^*\mathbf{Q}_{\ell,E} \xrightarrow{\mathrm{Id}\otimes[f]} f^*\mathcal{K} \otimes f^!\mathbf{Q}_{\ell,E}\langle -d(f)\rangle \xrightarrow{\diamond} f^!(\mathcal{K} \otimes \mathbf{Q}_{\ell,E})\langle -d(f)\rangle = f^!\mathcal{K}\langle -d(f)\rangle \qquad (6.4.4)$$

where we recall that the second arrow is the base change map for the Cartesian square

$$\begin{array}{ccc} E' & \xrightarrow{(\mathrm{Id},f)} & E' \times E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times \mathrm{Id}} \\ E & \xrightarrow{\Delta} & E \times E \end{array}$$

Note that a global presentation for f induces a global presentation for this diagram.

Let us factor (6.4.4) into two halves:

$$f^*\mathcal{K} = f^*\mathcal{K} \otimes f^*\mathbf{Q}_{\ell,E} \xrightarrow{\mathrm{Id}\otimes[f]} f^*\mathcal{K} \otimes f^!\mathbf{Q}_{\ell,E}\langle -d(f)\rangle \tag{6.4.5}$$

and

$$f^*\mathcal{K} \otimes f^!\mathbf{Q}_{\ell,E}\langle -d(f)\rangle \xrightarrow{\diamond} f^!(\mathcal{K} \otimes \mathbf{Q}_{\ell,E})\langle -d(f)\rangle = f^!\mathcal{K}\langle -d(f)\rangle. \tag{6.4.6}$$

Abbreviate $\widehat{\mathcal{K}} := \mathrm{FT}_E(\mathcal{K})$. By hypothesis, the first half (6.4.5) is sent (up to shift and twist by $r - r'$) by $\mathrm{FT}_{E'}$ to

$$\widehat{f}_!\widehat{\mathcal{K}} = \widehat{f}_!\widehat{\mathcal{K}} \star \widehat{f}_!\delta_{\widehat{E}'} \xrightarrow{\mathrm{Id}\star \mathrm{can}(\widehat{f})} \widehat{f}_!\widehat{\mathcal{K}} \star \widehat{f}_*\delta_{\widehat{E}'}.$$

Applying Proposition 6.3.3, we see that the second half (6.4.6) is sent (up to shift and twist by $r - r'$) by $\mathrm{FT}_{E'}$ to

$$\widehat{f}_!\widehat{\mathcal{K}} \star \widehat{f}_*\delta_{\widehat{E}} \xrightarrow{\diamond} \widehat{f}_*(\widehat{\mathcal{K}} \star \delta_{\widehat{E}}) = \widehat{f}_*\widehat{\mathcal{K}}$$

where the arrow comes from base change for the Cartesian square

$$\begin{array}{ccc} \widehat{E} \times \widehat{E} & \xrightarrow{+} & \widehat{E} \\ \downarrow{\scriptstyle \mathrm{Id}\times\widehat{f}} & & \downarrow{\scriptstyle \widehat{f}} \\ \widehat{E} \times \widehat{E}' & \xrightarrow{\widehat{f}+\mathrm{Id}} & \widehat{E}' \end{array}$$

which has a global presentation induced by that of f.

To complete the proof, we need to show that the above composition agrees with the "forget supports" map for $\widehat{f}$. This follows from the compatibility statement in Lemma 3.3.1 that the following diagram commutes:

$$\begin{array}{ccc} \widehat{f}_!\widehat{\mathcal{K}} \star \widehat{f}_!\delta_{\widehat{E}} & \xrightarrow{\sim} & \widehat{f}_!\widehat{\mathcal{K}} \star \widehat{f}_*\delta_{\widehat{E}} \\ \| & & \downarrow \\ \widehat{f}_!(\widehat{\mathcal{K}} \star \delta_{\widehat{E}}) & \xrightarrow{\mathrm{can}(\widehat{f})} & \widehat{f}_*(\widehat{\mathcal{K}} \star \delta_{\widehat{E}}) \end{array}$$

where the top horizontal isomorphism is justified by the chain of identifications (using that $z_{\widehat{E}}$ and $z_{\widehat{E}'}$ are closed embeddings because $\mathcal{E}, \mathcal{E}'$ are connective by assumption)

$$\widehat{f}_!\delta_{\widehat{E}} = \widehat{f}_! z_{\widehat{E}!}\mathbf{Q}_{\ell,S} = z_{\widehat{E}'!}\mathbf{Q}_{\ell,S} = z_{\widehat{E}'*}\mathbf{Q}_{\ell,S} = \widehat{f}_* z_{\widehat{E}*}\mathbf{Q}_{\ell,S} = \widehat{f}_*\delta_{\widehat{E}}. \tag{6.4.7}$$

Indeed, tracing the diagram along the left and bottom gives $\mathrm{can}(\widehat{f})$, while tracing along the top and right gives (up to shift and twist by $r - r'$) the natural transformation $\mathrm{FT}_{E'}([f])$. This concludes the proof. □

6.4.3. *Proposition 6.4.2.* We now complete the proof of Proposition 6.4.2 by establishing Lemma 6.4.6. By Lemma 6.4.4 and Lemma 6.4.5, we may assume that E', E are smooth over S. Then the map f is LCI, hence $f^!\mathbf{Q}_{\ell,E}\langle -d(f)\rangle$ is isomorphic to $f^*\mathbf{Q}_{\ell,E} \cong \mathbf{Q}_{\ell,E'}$. Note that the space $\mathrm{Hom}_{E'}(f^*\mathbf{Q}_{\ell,E}, f^!\mathbf{Q}_{\ell,E}\langle -d(f)\rangle)$ then identifies with $\mathrm{H}^0(E'; \mathbf{Q}_\ell) \cong \mathrm{H}^0(S; \mathbf{Q}_\ell)$.

Similarly we see $\widehat{f}_!\delta_{\widehat{E}} \cong \delta_{\widehat{E}'} \cong \widehat{f}_*\delta_{\widehat{E}}$ and $\mathrm{Hom}_{\widehat{E}'}(\widehat{f}_!\delta_{\widehat{E}'}, \widehat{f}_*\delta_{\widehat{E}'}) \cong \mathrm{H}^0(\widehat{E}'; \mathbf{Q}_\ell) \cong \mathrm{H}^0(S; \mathbf{Q}_\ell)$. Of course, this also follows formally from the previous paragraph, since FT_E is an equivalence of categories.

By examining each connected component at a time, we reduce to the case where S is connected. Then $\mathrm{H}^0(S; \mathbf{Q}_\ell) \cong \mathbf{Q}_\ell$, so the two maps in question differ by some scalar. We want to verify that this scalar is 1; it suffices to check this after pulling back to a single point of S, since Fourier transform is compatible with pullback on the base. Thus we reduce to the case where S is a point.

For $f^!$, the six-functor formalism is developed (cf. [Ver67]) so that for a smooth morphism f, the shriek pullback $f^!$ is *equal* to $f^*\langle d(f)\rangle$. Hence if $f\colon E' \to E$ is surjective, then the Gysin map is *the* identity map, and since $\widehat{f}$ is a closed embedding the forget supports map $\widehat{f}_! \to \widehat{f}_*$ is *the* identity as well. So Proposition 6.4.2 is evident in this case.

By factoring $f \colon E' \to E$ as the composition of a linear smooth map and a linear closed embedding (cf. Lemma A.4.1), we may assume that f is a closed embedding (since the statement of Proposition 6.4.2 is compatible with compositions). Since S is a point, f necessarily has a splitting $\pi \colon E \to E'$, which is necessarily smooth. Then since $\pi \circ f = \mathrm{Id}$, we have

$$\mathbf{Q}_{\ell,E'} = f^*\pi^*\mathbf{Q}_{\ell,E'} \xrightarrow{[\pi]} f^*\pi^!\mathbf{Q}_{\ell,E'}\langle -d(\pi)\rangle \xrightarrow{[f]} f^!\pi^!\mathbf{Q}_{\ell,E'}\langle -d(\pi)\rangle = \mathbf{Q}_{\ell,E'}.$$

Since π is smooth, $[\pi]$ is the identity map. The composition $[\pi \circ f] = [\mathrm{Id}]$ is also the identity map, so we deduce that with respect to the identifications $f^*\pi^!\mathbf{Q}_{\ell,E'}\langle -d(\pi)\rangle = \mathbf{Q}_{\ell,E'}$ and $f^!\pi^!\mathbf{Q}_{\ell,E'}\langle -d(\pi)\rangle = \mathbf{Q}_{\ell,E}$ induced by the equalities in the equation above, $[f]$ is *the* identity map.

Similarly, we have

$$\delta_{\widehat{E'}} = \widehat{f}_! z_{\widehat{E}!}\mathbf{Q}_{\ell,S} \xrightarrow{\mathrm{can}(z_{\widehat{E}})} \widehat{f}_! z_{\widehat{E}*}\mathbf{Q}_{\ell,S} \xrightarrow{\mathrm{can}(\widehat{f})} \widehat{f}_* z_{\widehat{E}*}\mathbf{Q}_{\ell,S} = \delta_{\widehat{E}'}.$$

Since $z_{\widehat{E}}$ is a closed embedding by the hypothesis that $\mathcal{E}$ is connective, $\mathrm{can}(z_{\widehat{E}})$ is the identity map. The composition in the above diagram is $\mathrm{can}(\widehat{f} \circ z_{\widehat{E}})$, which is also the identity map because $\widehat{f} \circ z_{\widehat{E}} = z_{\widehat{E}'}$ is also a closed embedding. Therefore, $\mathrm{can}(\widehat{f})$ is also *the* identity map.

□

7. Fourier analysis of cohomological correspondences

In this section we study how derived Fourier transform interacts with cohomological correspondences. This provides the main content towards Step (5) of the outline §2.4.

7.1. **Fourier transform of cohomological correspondences.** In §4.8 we defined the notion of cohomological co-correspondence. These arise naturally as the Fourier transforms of cohomological correspondences, as we now explain.

7.1.1. *Over the same base.* Suppose we have a Cartesian square of derived vector bundles over a base S,

$$\begin{array}{ccccc} & & C^\flat & & \\ & \swarrow^{p_0} & & \searrow^{p_1} & \\ E_0 & & & & E_1 \\ & \searrow_{q_0} & & \swarrow_{q_1} & \\ & & C^\sharp & & \end{array} \tag{7.1.1}$$

The collection of dual derived bundles forms a Cartesian square of vector bundles over S:

$$\begin{array}{ccccc} & & \widehat{C^\sharp} & & \\ & \swarrow^{\widehat{q}_0} & & \searrow^{\widehat{q}_1} & \\ \widehat{E}_0 & & & & \widehat{E}_1 \\ & \searrow_{\widehat{p}_0} & & \swarrow_{\widehat{p}_1} & \\ & & \widehat{C^\flat} & & \end{array}$$

We may apply §6.2.4 to see that the cohomological correspondence $p_0^*\mathcal{K}_0 \to p_1^!\mathcal{K}_1$ is taken by $\mathrm{FT}_{C^\flat}$ to a cohomological co-correspondence on $\widehat{C^\flat}$

$$\widehat{p}_{0!}\,\mathrm{FT}_{E_0}(\mathcal{K}_0)[-d(p_0)](-d(p_0)) \to \widehat{p}_{1*}\,\mathrm{FT}_{E_1}(\mathcal{K}_1)[d(p_1)].$$

This may be converted as in (4.8.4) to a cohomological correspondence on $\widehat{C^\sharp}$

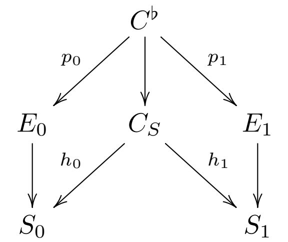
$$\widehat{q}_0^*\,\mathrm{FT}_{E_0}(\mathcal{K}_0) \to \widehat{q}_1^!\,\mathrm{FT}_{E_1}(\mathcal{K}_1)[d(p_0)+d(p_1)](d(p_0))$$

which we call $\mathrm{FT}_{C^\flat}(\mathfrak{c})$. The construction $\mathfrak{c} \mapsto \mathrm{FT}_{C^\flat}(\mathfrak{c})$ defines an isomorphism of vector spaces

$$\mathrm{FT}_{C^\flat} : \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \xrightarrow{\sim} \mathrm{Corr}_{\widehat{C^\sharp}}(\mathrm{FT}_{E_0}(\mathcal{K}_0), \mathrm{FT}_{E_1}(\mathcal{K}_1)[d(p_0)+d(p_1)](d(p_0))). \tag{7.1.2}$$

7.1.2. *Varying the base.* In §7.1.1 we explained that cohomological correspondences on certain correspondences of derived vector bundles over a base S could be Fourier transformed to a dual correspondence.

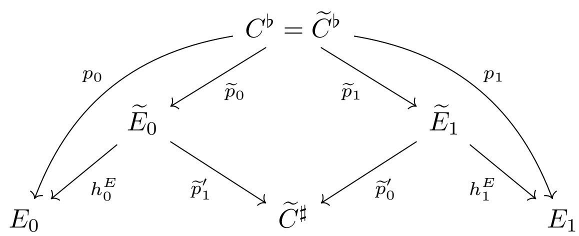
We will define the Fourier transform of a cohomological correspondence in a slightly more general situation, where the base of the derived vector bundles is also permitted to change. Suppose we are given a map of correspondence



(7.1.3)

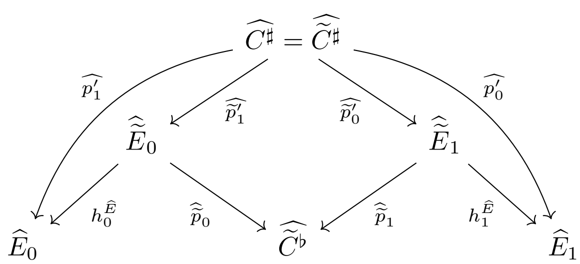
where $E_0, C^\flat$ and E_1 are derived vector bundles on S_0, C_S and S_1 respectively. Assume the maps p_0 and p_1 are linear.

Let $\widetilde{E}_0$ and $\widetilde{E}_1$ be the pullbacks of E_0 and E_1 to C_S via h_i. We can canonically extend the correspondence $E_0 \xleftarrow{p_0} C^\flat \xrightarrow{p_1} E_1$ into a commutative diagram



Here $\widetilde{C}^\sharp$ is defined to be the pushout of the correspondence of vector bundles $\widetilde{E}_0 \xleftarrow{\widetilde{p}_0} C^\flat \xrightarrow{\widetilde{p}_1} \widetilde{E}_1$, so that $\widetilde{C}^\sharp$ is also a derived vector bundle over C_S, and the inner diamond is derived Cartesian. When we view $\widetilde{C}^\flat$ as a correspondence between E_0 and E_1, we denote it by $C^\flat$; when we view it as a correspondence between $\widetilde{E}_0$ and $\widetilde{E}_1$, we denote it by $\widetilde{C}^\flat$.

Taking dual derived vector bundles we get a diagram



Again, when we view $\widehat{\widetilde{C}^\sharp}$ as a correspondence between $\widehat{E}_0$ and $\widehat{E}_1$, we denote it by $\widehat{C}^\sharp$.

For $\mathcal{K}_i \in D(E_i)$, $i = 0, 1$, we define an isomorphism of vector spaces

$$\mathrm{FT}_{C^\flat} : \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \xrightarrow{\sim} \mathrm{Corr}_{\widehat{C}^\sharp}(\mathrm{FT}_{E_0}(\mathcal{K}_0), \mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(\widetilde{p}_0)+d(\widetilde{p}_1)](d(\widetilde{p}_0))}) \qquad (7.1.4)$$

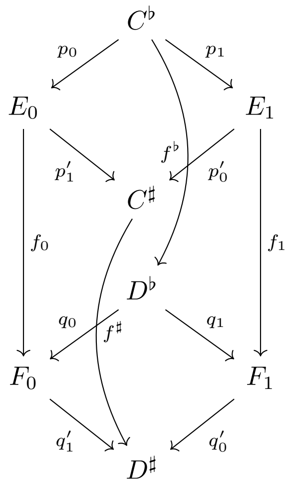
as the composition of isomorphisms

$$\begin{aligned}
\mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \quad &= \quad \mathrm{Corr}_{\widetilde{C}^\flat}((h_0^E)^*\mathcal{K}_0, (h_1^E)^!\mathcal{K}_1) \\
&\xrightarrow{\mathrm{FT}_{\widetilde{C}^\flat}} \quad \mathrm{Corr}_{\widehat{\widetilde{C}^\sharp}}(\mathrm{FT}_{\widetilde{E}_0}((h_0^E)^*\mathcal{K}_0), \mathrm{FT}_{\widetilde{E}_1}((h_1^E)^!\mathcal{K}_1)_{[d(\widetilde{p}_0)+d(\widetilde{p}_1)](d(\widetilde{p}_0))}) \\
&\cong \quad \mathrm{Corr}_{\widehat{\widetilde{C}^\sharp}}((h_0^{\widehat{E}})^*\,\mathrm{FT}_{E_0}(\mathcal{K}_0), (h_1^{\widehat{E}})^!\,\mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(\widetilde{p}_0)+d(\widetilde{p}_1)](d(\widetilde{p}_0))}) \\
&= \quad \mathrm{Corr}_{\widehat{C}^\sharp}(\mathrm{FT}_{E_0}(\mathcal{K}_0), \mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(\widetilde{p}_0)+d(\widetilde{p}_1)](d(\widetilde{p}_0))}).
\end{aligned}$$

Here we have used §6.2.2 in the second to last isomorphism.

7.2. **Functoriality.** We state and prove functorial properties of the Fourier transform of cohomological correspondences constructed in §7.1.1 and more generally in §7.1.2.

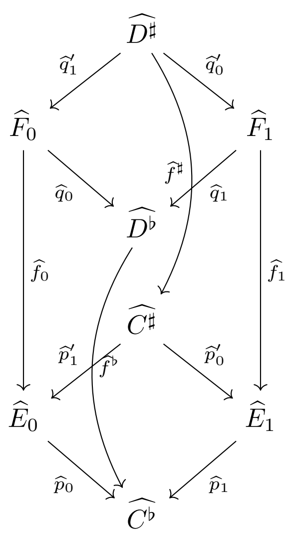
7.2.1. *Functoriality over the same base.* Suppose we have a commutative diagram



(7.2.1)

of derived vector bundles over S, where the top and bottom diamonds are derived Cartesian and all maps are linear.

The dual diagram to (7.2.1) is



(7.2.2)

Lemma 7.2.1. *In the above setup, the following are equivalent:*

1. *The map of correspondences $f^\flat : C^\flat \to D^\flat$ is left pushable;*
2. *The map of co-correspondences $f^\sharp : C^\sharp \to D^\sharp$ is right pushable;*
3. *The map of co-correspondences $\widehat{f^\flat} : \widehat{D^\flat} \to \widehat{C^\flat}$ is left pullable;*
4. *The map of correspondences $\widehat{f^\sharp} : \widehat{D^\sharp} \to \widehat{C^\sharp}$ is right pullable.*

Moreover, when (3) and (4) hold, we have $\delta_{\widehat{f^\flat}} = \delta_{\widehat{f^\sharp}}$.

Proof. Using that the bottom diamond in (7.2.1) is derived Cartesian, we see that the following diagram is derived Cartesian

$$\begin{array}{ccc} C^\flat & \xrightarrow{c^\flat} & E_0 \times_{F_0} D^\flat \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_1' \times_{q_1'} q_1} \\ E_1 & \xrightarrow{e_1} & C^\sharp \times_{D^\sharp} F_1 \end{array} \tag{7.2.3}$$

Now (1) $\iff$ $c^\flat$ is a closed embedding of derived vector bundles $\iff$ e_1 is a closed embedding of derived vector bundles (by the above derived Cartesian diagram) $\iff$ (2). This proves (1) $\iff$ (2). The same argument shows (3) $\iff$ (4) and that $\delta_{\widehat{f^\flat}} = \delta_{\widehat{f^\sharp}}$.

It remains to show that (1) $\iff$ (3). Let $\mathcal{E}_i, \mathcal{F}_i, \mathcal{C}^\flat, \cdots$ be perfect complexes over S whose total spaces are $E_i, F_i, C^\flat, \cdots$. We name the maps between these perfect complexes by the same name of the induced map between their total spaces. Let P be the derived fiber of the linear map $(p_0, f^\flat): C^\flat \to E_0 \times_{F_0} D^\flat$ over the zero section 0_S; let Q be the derived fiber of $(\widehat{q}_0, \widehat{f}_0): \widehat{F}_0 \to \widehat{D^\flat} \times_{\widehat{C^\flat}} \widehat{E}_0$ over the zero section 0_S.

Note that P is the total space of the perfect complex $\mathcal{P}$ over S that is obtained by taking the total complex of the double complex

$$\mathcal{C}^\flat \xrightarrow{(p_0, f^\flat)} \mathcal{E}_0 \oplus \mathcal{D}^\flat \xrightarrow{f_0 - q_0} \mathcal{F}_0. \tag{7.2.4}$$

Here $\mathcal{C}^\flat$ is placed in the original degrees, and the other terms are shifted accordingly. Then (1) is equivalent to

(1') $\mathcal{P}$ is locally represented by a complex of vector bundles in degrees ≥ 1.

Similarly, Q is the total space of the perfect complex $\mathcal{Q}$ over S that is obtained by taking the total complex of the double complex

$$\widehat{\mathcal{F}}_0 \xrightarrow{(\widehat{q}_0, \widehat{f}_0)} \widehat{\mathcal{D}^\flat} \oplus \widehat{\mathcal{E}}_0 \xrightarrow{\widehat{f}^\flat - \widehat{p}_0} \widehat{\mathcal{C}}^\flat. \tag{7.2.5}$$

Here $\widehat{\mathcal{F}}_0$ is placed in the original degrees, and the other terms are shifted accordingly. Then (3) is equivalent to

(3') $\mathcal{Q}$ is locally represented by a complex of vector bundles in degrees ≤ 1.

Observe that $\mathcal{Q}$ is quasi-isomorphic to $\mathcal{P}^*[-2]$. Therefore (1') $\iff$ (3'). This proves (1) $\iff$ (3). □

Assume that $f^\flat: C^\flat \to D^\flat$ is left pushable. Then for $\mathcal{K}_i \in D(E_i)$ $(i = 0, 1)$, the pushforward map

$$(f^\flat)_!: \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) \tag{7.2.6}$$

is defined. By Lemma 7.2.1, the map of correspondences $\widehat{f^\sharp}: \widehat{D^\sharp} \to \widehat{C^\sharp}$ is right pullable. Hence for $\mathcal{L}_i \in D(\widehat{E}_i)$ $(i = 0, 1)$, the pullback map

$$(\widehat{f^\sharp})^*: \mathrm{Corr}_{\widehat{C^\sharp}}(\mathcal{L}_0, \mathcal{L}_1) \to \mathrm{Corr}_{\widehat{D^\sharp}}(\widehat{f}_0^* \mathcal{L}_0, \widehat{f}_1^* \mathcal{L}_1 \langle -\delta_{\widehat{f^\sharp}} \rangle) \tag{7.2.7}$$

is defined.

On the other hand, applying Fourier transform to $f_{0!}\mathcal{K}_0$ and $f_{1!}\mathcal{K}_1$, we have by §6.2.4

$$\mathrm{FT}_{F_0}(f_{0!}\mathcal{K}_0) \cong \widehat{f}_0^* \, \mathrm{FT}_{E_0}(\mathcal{K}_0)[-d(f_0)], \quad \mathrm{FT}_{F_1}(f_{1!}\mathcal{K}_1) \cong \widehat{f}_1^* \, \mathrm{FT}_{E_1}(\mathcal{K}_1)[-d(f_1)]. \tag{7.2.8}$$

Lemma 7.2.2. *We have*

(1) $\delta_{\widehat{f^\sharp}} = d(p_0) - d(q_0)$.

(2) $d(p_1) - d(q_1) + d(f_1) = d(p_0) - d(q_0) + d(f_0)$.

Proof. (1) The proof of Lemma 7.2.1 shows that $\delta_{\widehat{f^\sharp}}$ is equal to the relative dimension of $C^\flat \to E_0 \times_{F_0} D^\flat$, which is $d(p_0) - d(q_0)$.

(2) Both sides are equal to $d(f^\flat)$. □

Proposition 7.2.3. *Assume that diagram* (7.2.1) *is globally presented.*

(1) Suppose the map of correspondences $f^\flat: C^\flat \to D^\flat$ is left pushable. Let $\mathcal{K}_i \in D(E_i)$ for $i = 0, 1$. Then the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\mathrm{FT}_{C^\flat}} & \mathrm{Corr}_{\widehat{C^\sharp}}(\mathrm{FT}_{E_0}(\mathcal{K}_0), \mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(p_0)+d(p_1)](d(p_0))}) \\
\downarrow{\scriptstyle (f^\flat)_!} & & \downarrow{\scriptstyle (\widehat{f^\sharp})^*} \\
\mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) & \xrightarrow{\mathbb{T}_{[d(f_0)]}\,\mathrm{FT}_{D^\flat}} & \mathrm{Corr}_{\widehat{D^\sharp}}(\widehat{f}_0^* \, \mathrm{FT}_{E_0}(\mathcal{K}_0), \widehat{f}_1^* \, \mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(q_0)+d(q_1)+d(f_0)-d(f_1)](d(q_0))})
\end{array} \tag{7.2.9}$$

Here we use Lemma 7.2.2 to match the differences of the twists that appear in the right vertical map with $\langle -\delta_{\widehat{f^\sharp}}\rangle$*, which is the correct twist for* $(\widehat{f^\sharp})^*$*.*

(2) Suppose the map of correspondences $f^\flat : C^\flat \to D^\flat$ *is right pullable. Let* $\mathcal{K}_i \in D(F_i)$ *for* $i = 0, 1$*. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathrm{Corr}_{D^\flat}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\quad \mathrm{FT}_{D^\flat} \quad} & \mathrm{Corr}_{\widehat{D^\sharp}}(\mathrm{FT}_{F_0}(\mathcal{K}_0), \mathrm{FT}_{F_1}(\mathcal{K}_1)_{[d(q_0)+d(q_1)](d(q_0))}) \\
\big\downarrow{\scriptstyle (f^\flat)^*} & & \big\downarrow{\scriptstyle (\widehat{f^\sharp})_!} \\
\mathrm{Corr}_{C^\flat}(f_0^*\mathcal{K}_0, f_1^*\mathcal{K}_1) & \xrightarrow{\mathbb{T}_{[d(f_0)](d(f_0))}\,\mathrm{FT}_{C^\flat}} & \mathrm{Corr}_{\widehat{C^\sharp}}(\widehat{f}_{0!}\,\mathrm{FT}_{F_0}(\mathcal{K}_0), \widehat{f}_{1!}\,\mathrm{FT}_{F_1}(\mathcal{K}_1)_{[d(q_0)+d(q_1)](d(q_0))})
\end{array}$$

Proof. (1) Below, to shorten notation, we write $\widehat{\mathcal{K}}_i := \mathrm{FT}_{E_i}(\mathcal{K}_i)$. By definition, $\mathrm{FT}_{C^\flat}$ is the composition of two isomorphisms

$$\mathrm{FT}_{C^\flat} : \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \xrightarrow{\mathrm{FT}'_{C^\flat}} \mathrm{CoCorr}_{\widehat{C^\flat}}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{K}}_{1[d(p_0)+d(p_1)](d(p_0))}) \xrightarrow{\gamma_C} \mathrm{Corr}_{\widehat{C^\sharp}}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{K}}_{1[d(p_0)+d(p_1)](d(p_0))}). \tag{7.2.10}$$

Similar remarks apply to $\mathrm{FT}_{D^\flat}$. Therefore it suffices to prove the commutativity of the following two diagrams separately

$$\begin{array}{ccc}
\mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\quad \mathrm{FT}'_{C^\flat} \quad} & \mathrm{CoCorr}_{\widehat{C^\flat}}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{K}}_{1[d(p_0)+d(p_1)](d(p_0))}) \\
\big\downarrow{\scriptstyle (f^\flat)_!} & & \big\downarrow{\scriptstyle (\widehat{f}^\flat)^*} \\
\mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) & \xrightarrow{\mathbb{T}_{[d(f_0)]}\,\mathrm{FT}'_{D^\flat}} & \mathrm{CoCorr}_{\widehat{D^\flat}}(\widehat{f}_0^*\widehat{\mathcal{K}}_0, \widehat{f}_1^*\widehat{\mathcal{K}}_{1[d(p'_0)+d(p'_1)+d(f_0)-d(f_1)](d(p'_0))})
\end{array} \tag{7.2.11}$$

and

$$\begin{array}{ccc}
\mathrm{CoCorr}_{\widehat{C^\flat}}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{K}}_1) & \xrightarrow{\gamma_C} & \mathrm{Corr}_{\widehat{C^\sharp}}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{K}}_1) \\
\big\downarrow{\scriptstyle (\widehat{f}^\flat)^*} & & \big\downarrow{\scriptstyle (\widehat{f}^\sharp)^*} \\
\mathrm{CoCorr}_{\widehat{D^\flat}}(\widehat{f}_0^*\widehat{\mathcal{K}}_0, \widehat{f}_1^*\widehat{\mathcal{K}}_1) & \xrightarrow{\gamma_D} & \mathrm{Corr}_{\widehat{D^\sharp}}(\widehat{f}_0^*\widehat{\mathcal{K}}_0, \widehat{f}_1^*\widehat{\mathcal{K}}_1)
\end{array} \tag{7.2.12}$$

Here the pullback map of co-correspondences $(\widehat{f}^\flat)^*$ is defined in §4.8.2. The commutativity of the diagram (7.2.12) is proved in Proposition 4.8.2.

It remains to show that (7.2.11) is commutative. Since the statement does not involve $C^\sharp$ and $D^\sharp$, we will omit the superscript $\flat$ from the notations and denote $C^\flat, D^\flat$ by C and D. For $\mathfrak{c} \in \mathrm{Corr}_C(\mathcal{K}_0, \mathcal{K}_1)$, denoting $\widehat{\mathcal{K}}_i = \mathrm{FT}_{E_i}(\mathcal{K}_i)$, we have to show the commutativity of the outer square of the diagram

$$\begin{array}{ccccccc}
\mathrm{FT}(q_0^* f_{0!}\mathcal{K}_0) & \xrightarrow{\nabla} & \mathrm{FT}(f_! p_0^*\mathcal{K}_0) & \xrightarrow{\mathrm{FT}(f_!\,\mathfrak{c})} & \mathrm{FT}(f_! p_1^!\mathcal{K}_1) & \longrightarrow & \mathrm{FT}(q_1^! f_{1!}\mathcal{K}_1) \\
\wr\big| & & \wr\big| & & \wr\big| & & \wr\big| \\
\widehat{q}_{0!}\widehat{f}_0^*\widehat{\mathcal{K}}_0[?](?) & \xrightarrow{\Delta} & \widehat{f}^*\widehat{p}_{0!}\widehat{\mathcal{K}}_0[?](?) & \xrightarrow{\widehat{f}^*\,\mathrm{FT}(\mathfrak{c})} & \widehat{f}^*\widehat{p}_{1*}\widehat{\mathcal{K}}_1[?](?) & \longrightarrow & \widehat{q}_{1*}\widehat{f}_1^*\widehat{\mathcal{K}}_1[?](?)
\end{array} \tag{7.2.13}$$

Here the arrows marked by ∇ and Δ are the base change maps attached to the pushable square (C, E_0, D, F_0) and the pullable square $(\widehat{F}_0, \widehat{D}, \widehat{E}_0, \widehat{C})$. The unmarked arrows are induced by the natural transformation $f_! p_1^! \to q_1^! f_{1!}$ attached to the square (C, E_1, D, F_1) and the natural transformation $\widehat{f}^*\widehat{p}_{1*} \to \widehat{q}_{1*}\widehat{f}_1^*$ attached to the dual square. The vertical isomorphisms are from §6.2.4. We have omitted the shifts and twists in the bottom row.

The middle square above is commutative by the naturality of the isomorphisms in §6.2.4. The right square is commutative: write $f_! p_1^! \to q_1^! f_{1!}$ as the composition

$$f_! p_1^! \xrightarrow{\text{unit}} f_! p_1^! f_1^! f_{1!} = f_! f^! q_1^! f_{1!} \xrightarrow{\text{counit}} q_1^! f_{1!} \tag{7.2.14}$$

Therefore it suffices to note that FT transforms the unit map $\mathrm{Id} \to f_1^! f_{1!}$ (resp. counit map $f_! f^! \to \mathrm{Id}$) to the unit map $\mathrm{Id} \to \widehat{f}_*\widehat{f}^*$ (resp. the counit map $\widehat{f}^*\widehat{f}_* \to \mathrm{Id}$), as explained in §6.2.4.

It remains to show that the left square in (7.2.13) commutes. Let $C^\natural = E_0 \times_{F_0} D$, with the induced map $c : C \to C^\natural$. The square (C, E_0, D, F_0) can be decomposed as the composition of two squares of derived vector bundles over S

$$\begin{array}{ccc} E_0 & \xleftarrow{p_0} & C \\ \| & & \downarrow{\scriptstyle c} \\ E_0 & \xleftarrow{p_0^\natural} & C^\natural \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ F_0 & \xleftarrow{q_0} & D \end{array} \tag{7.2.15}$$

where c is proper by assumption, and the bottom square is derived Cartesian by definition. Note that the assumptions guarantee that (7.2.15) is globally presented. Using the compatibility of the base change maps with composition of squares proved in Proposition 3.2.3 and Proposition 3.5.4, we reduce to showing the commutativity of the left square in (7.2.13) separately for the two squares in (7.2.15), i.e., for two special cases:

(1) The square (C, E_0, D, F_0) is derived Cartesian (hence so is its dual square).
(2) The map $f_0 = \mathrm{Id} : E_0 \to F_0 = E_0$ is the identity map, and f is proper (i.e., a closed embedding of derived vector bundles; dually $\widehat{f}$ is smooth).

The first case follows from Proposition 6.3.3 (proved in Proposition A.4.3), which applies because (7.2.15) is globally presented. In the second case, we reduce to showing that the outer square in the following diagram is commutative

$$\begin{array}{ccccccc} \mathrm{FT}\, q_0^* & \xrightarrow{\text{unit}} & \mathrm{FT}\, f_* f^* q_0^* & = & \mathrm{FT}\, f_* p_0^* & \xleftarrow{\cong} & \mathrm{FT}\, f_! p_0^* \\ \wr\| & & \wr\| & & \wr\| & & \wr\| \\ \widehat{q}_{0!}\, \mathrm{FT}\,[?](?) & \xrightarrow{\text{unit}} & \widehat{f}^! \widehat{f}_! \widehat{q}_{0!}\, \mathrm{FT}\,[?](?) & = & \widehat{f}^! \widehat{p}_{0!}\, \mathrm{FT}\,[?](?) & \xleftarrow{[\widehat{f}]} & \widehat{f}^* \widehat{p}_{0!}\, \mathrm{FT}\,[?](?)\langle d(\widehat{f})\rangle \end{array} \tag{7.2.16}$$

Here $? = -d(q_0)$. (The global presentability assumption is used here to produce the first and second vertical maps, since q_0 is not assumed to be smooth or a closed embedding.) Now the left and middle squares commutes because FT takes the unit to unit. The right square commutes by Proposition 6.4.2.

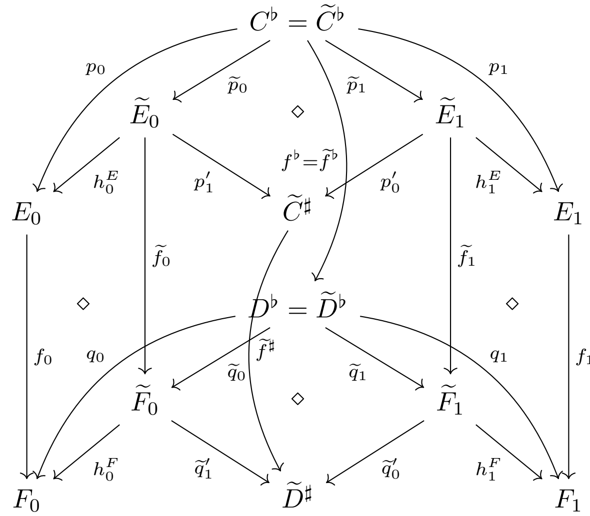
(2) An analogous argument can be applied. Alternatively, (2) follows from (1) using the near-involutivity of FT. □

7.2.2. *Functoriality over varying bases.* We next extend the preceding discussion to the situation where the base space may vary. Consider a diagram of maps of correspondences

$$\begin{array}{ccccc} & & C^\flat & & \\ & \swarrow{\scriptstyle p_0} & \downarrow{\scriptstyle f^\flat} & \searrow{\scriptstyle p_1} & \\ E_0 & & D^\flat & & E_1 \\ {\scriptstyle f_0}\downarrow & \swarrow{\scriptstyle q_0} & \downarrow & \searrow{\scriptstyle q_1} & \downarrow{\scriptstyle f_1} \\ F_0 & & C_S & & F_1 \\ \downarrow & \swarrow{\scriptstyle h_0} & & \searrow{\scriptstyle h_1} & \downarrow \\ S_0 & & & & S_1 \end{array} \tag{7.2.17}$$

where E_i and F_i are derived vector bundles over S_i (for $i = 0, 1$), and $C^\flat$ and $D^\flat$ are derived vector bundles over C_S. All maps between derived vector bundles are assumed to be linear.

Let $\widetilde{E}_i \to C_S$, $\widetilde{F}_i \to C_S$ and $\widetilde{f}_i : \widetilde{E}_i \to \widetilde{F}_i$ be the base changes of E_i, F_i and f_i along $h_i : C_S \to S_i$. Using the discussion in §7.1.2, we can canonically extend the upper part of the diagram (7.2.17) into a commutative diagram



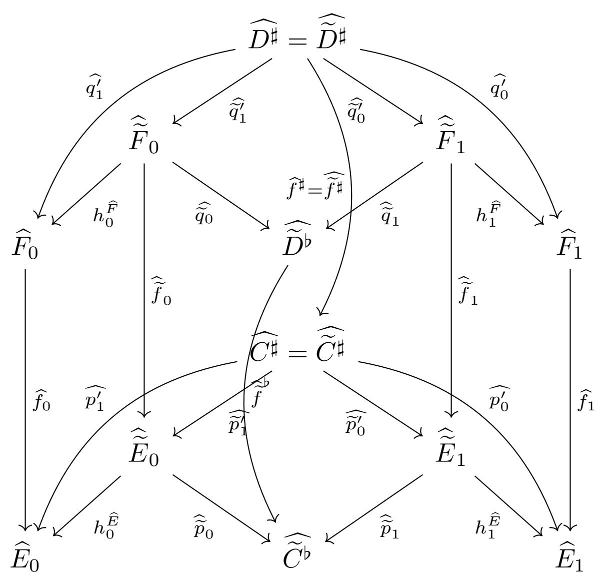
(7.2.18)

where the squares labeled by $\diamond$ are derived Cartesian.

Since the leftmost parallelogram is derived Cartesian, the square $(C^\flat, E_0, D^\flat, F_0)$ is pushable if and only if the square $(\widetilde{C}^\flat, \widetilde{E}_0, \widetilde{D}^\flat, \widetilde{F}_0)$ is pushable. In other words, the morphism $f^\flat : C^\flat \to D^\flat$ of correspondences is left pushable if and only if the morphism $\widetilde{f}^\flat : \widetilde{C}^\flat \to \widetilde{D}^\flat$ of correspondences is left pushable. When any of these equivalent conditions holds, we have a pushforward map

$$f_!^\flat : \mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) \to \mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1). \tag{7.2.19}$$

7.2.3. The dual diagram to (7.2.18) is:



(7.2.20)

Since the rightmost parallelogram is derived Cartesian, the square $(\widehat{D^\sharp}, \widehat{F}_1, \widehat{C^\sharp}, \widehat{E}_1)$ is pullable if and only if the square $(\widehat{\widetilde{D}^\sharp}, \widehat{\widetilde{F}}_1, \widehat{\widetilde{C}^\sharp}, \widehat{\widetilde{E}}_1)$ is pullable. In other words, the morphism $\widehat{f^\sharp} : \widehat{D^\sharp} \to \widehat{C^\sharp}$ of correspondences is right pullable if and only if the morphism $\widehat{\widetilde{f}^\sharp} : \widehat{\widetilde{D}^\sharp} \to \widehat{\widetilde{C}^\sharp}$ of correspondences is right pullable. When any of these equivalent conditions holds, we have a pullback map

$$(\widehat{f^\sharp})^* : \mathrm{Corr}_{\widehat{C^\sharp}}(\mathcal{L}_0, \mathcal{L}_1) \to \mathrm{Corr}_{\widehat{D^\sharp}}(\widehat{f}_0^*\mathcal{L}_0, \widehat{f}_1^*\mathcal{L}_1\langle -\delta_{\widehat{f^\sharp}}\rangle). \tag{7.2.21}$$

Moreover, by Lemma 7.2.1, $f^\flat$ is left pushable if and only if $\widehat{f^\sharp}$ is right pullable.

Proposition 7.2.4. *Assume the diagram (7.2.20) is globally presented.*

(1) Suppose the map of correspondences $f^\flat : C^\flat \to D^\flat$ is left pushable. Let $\mathcal{K}_i \in D(E_i)$ for $i = 0, 1$. Then the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\mathrm{FT}_{C^\flat}} & \mathrm{Corr}_{\widehat{C^\sharp}}(\mathrm{FT}_{E_0}(\mathcal{K}_0), \mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(\widetilde{p}_0)+d(\widetilde{p}_1)](d(\widetilde{p}_0))}) \\
\big\downarrow (f^\flat)_! & & \big\downarrow (\widehat{f^\sharp})^* \\
\mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) & \xrightarrow{\mathbb{T}_{[d(f_0)]}\,\mathrm{FT}_{D^\flat}} & \mathrm{Corr}_{\widehat{D^\sharp}}(\widehat{f}_0^*\,\mathrm{FT}_{E_0}(\mathcal{K}_0), \widehat{f}_1^*\,\mathrm{FT}_{E_1}(\mathcal{K}_1)_{[d(\widetilde{q}_0)+d(\widetilde{q}_1)+d(f_0)-d(f_1)](d(\widetilde{q}_0))})
\end{array} \tag{7.2.22}$$

Here we use Lemma 7.2.2 to match the differences of the twists that appear in the right vertical map with $\langle -\delta_{\widehat{f^\sharp}}\rangle$, which is the correct twist for $(\widehat{f^\sharp})^*$.

(2) Suppose the map of correspondences $f^\flat : C^\flat \to D^\flat$ is right pullable. Let $\mathcal{K}_i \in D(F_i)$ for $i = 0, 1$. Then the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Corr}_{D^\flat}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\mathrm{FT}_{D^\flat}} & \mathrm{Corr}_{\widehat{D^\sharp}}(\mathrm{FT}_{F_0}(\mathcal{K}_0), \mathrm{FT}_{F_1}(\mathcal{K}_1)_{[d(\widetilde{q}_0)+d(\widetilde{q}_1)](d(\widetilde{q}_0))}) \\
\big\downarrow (f^\flat)^* & & \big\downarrow (\widehat{f^\sharp})_! \\
\mathrm{Corr}_{C^\flat}(f_0^*\mathcal{K}_0, f_1^*\mathcal{K}_1) & \xrightarrow{\mathbb{T}_{[d(f_0)](d(f_0))}\,\mathrm{FT}_{C^\flat}} & \mathrm{Corr}_{\widehat{C^\sharp}}(\widehat{f}_{0!}\,\mathrm{FT}_{F_0}(\mathcal{K}_0), \widehat{f}_{1!}\,\mathrm{FT}_{F_1}(\mathcal{K}_1)_{[d(\widetilde{q}_0)+d(\widetilde{q}_1)](d(\widetilde{q}_0))})
\end{array}$$

Proof. (1) Let $\widetilde{\mathcal{K}}_0 = (h_0^E)^*\mathcal{K}_0$ and $\widetilde{\mathcal{K}}_1 = (h_1^E)^!\mathcal{K}_1$. Using the definition of $\mathrm{FT}_{C^\flat}$ and $\mathrm{FT}_{D^\flat}$ from §7.1.2, we decompose the square (7.2.22) into three squares

$$\begin{array}{ccc}
\mathrm{Corr}_{C^\flat}(\mathcal{K}_0, \mathcal{K}_1) & \xrightarrow{\sim} & \mathrm{Corr}_{\widetilde{C}^\flat}(\widetilde{\mathcal{K}}_0, \widetilde{\mathcal{K}}_1) \\
\big\downarrow (f^\flat)_! & & \big\downarrow (\widetilde{f}^\flat)_! \\
\mathrm{Corr}_{D^\flat}(f_{0!}\mathcal{K}_0, f_{1!}\mathcal{K}_1) & \xrightarrow{\sim} & \mathrm{Corr}_{\widetilde{D}^\flat}(\widetilde{f}_{0!}\widetilde{\mathcal{K}}_0, \widetilde{f}_{1!}\widetilde{\mathcal{K}}_1)
\end{array} \tag{7.2.23}$$

$$\begin{array}{ccc}
\mathrm{Corr}_{\widetilde{C}^\flat}(\widetilde{\mathcal{K}}_0, \widetilde{\mathcal{K}}_1) & \xrightarrow{\mathrm{FT}_{\widetilde{C}^\flat}} & \mathrm{Corr}_{\widehat{\widetilde{C}^\sharp}}(\mathrm{FT}(\widetilde{\mathcal{K}}_0), \mathrm{FT}(\widetilde{\mathcal{K}}_1)_{[d(\widetilde{p}_0)+d(\widetilde{p}_1)](d(\widetilde{p}_0))}) \\
\big\downarrow (\widetilde{f}^\flat)_! & & \big\downarrow (\widehat{\widetilde{f}^\sharp})^* \\
\mathrm{Corr}_{\widetilde{D}^\flat}(\widetilde{f}_{0!}\widetilde{\mathcal{K}}_0, \widetilde{f}_{1!}\widetilde{\mathcal{K}}_1) & \xrightarrow{\mathbb{T}_{[d(f_0)]}\,\mathrm{FT}_{\widetilde{D}^\flat}} & \mathrm{Corr}_{\widehat{\widetilde{D}^\sharp}}((\widehat{\widetilde{f}}_0)^*\,\mathrm{FT}(\widetilde{\mathcal{K}}_0), (\widehat{\widetilde{f}}_1)^*\,\mathrm{FT}(\widetilde{\mathcal{K}}_1)_{[d(\widetilde{q}_0)+d(\widetilde{q}_1)+d(f_0)-d(f_1)](d(\widetilde{q}_0))})
\end{array} \tag{7.2.24}$$

and

$$\begin{array}{ccc}
\mathrm{Corr}_{\widehat{\widetilde{C}^\sharp}}(\mathrm{FT}(\widetilde{\mathcal{K}}_0), \mathrm{FT}(\widetilde{\mathcal{K}}_1)) & \xrightarrow{\sim} & \mathrm{Corr}_{\widehat{C^\sharp}}(\mathrm{FT}(\mathcal{K}_0), \mathrm{FT}(\mathcal{K}_1)) \\
\big\downarrow (\widehat{\widetilde{f}^\sharp})^* & & \big\downarrow (\widehat{f^\sharp})^* \\
\mathrm{Corr}_{\widehat{\widetilde{D}^\sharp}}((\widehat{\widetilde{f}}_0)^*\,\mathrm{FT}(\widetilde{\mathcal{K}}_0), (\widehat{\widetilde{f}}_1)^*\,\mathrm{FT}(\widetilde{\mathcal{K}}_1)) & \xrightarrow{\sim} & \mathrm{Corr}_{\widehat{D^\sharp}}(\widehat{f}_0^*\,\mathrm{FT}(\mathcal{K}_0), \widehat{f}_1^*\,\mathrm{FT}(\mathcal{K}_1))
\end{array} \tag{7.2.25}$$

The commutativity of (7.2.24) is proved in Proposition 7.2.3.

Let us prove the commutativity of (7.2.23). After unraveling definitions, the non-obvious part is to show the commutativity of the following diagram of functors $D(E_0) \to D(D^\flat)$

$$\begin{array}{ccccc}
\widetilde{q}_0^*(h_0^F)^* f_{0!} & \xrightarrow{\widetilde{q}_0^{*}\diamond} & \widetilde{q}_0^*\widetilde{f}_{0!}(h_0^E)^* & \xrightarrow{\nabla (h_0^E)^*} & f_!^\flat \widetilde{p}_0^*(h_0^E)^* \\
\Vert & & & & \Vert \\
q_0^* f_{0!} & & \xrightarrow{\nabla} & & f_!^\flat p_0^*
\end{array} \tag{7.2.26}$$

This follows from Proposition 3.2.4 applied to the two squares

$$\begin{array}{ccccc} C^\flat & \xrightarrow{\widetilde{p}_0} & \widetilde{E}_0 & \xrightarrow{h_0^E} & E_0 \\ \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle \widetilde{f}_0} & & \downarrow{\scriptstyle f_0} \\ D^\flat & \xrightarrow{\widetilde{q}_0} & \widetilde{F}_0 & \xrightarrow{h_0^F} & F_0 \end{array} \tag{7.2.27}$$

Let us prove the commutativity of (7.2.25). After unraveling definitions, the non-obvious part is to show the commutativity of the following diagram of functors $D(E_0) \to D(D^\flat)$

$$\begin{array}{ccccc} (\widehat{f}^\sharp)^*(\widehat{\widetilde{p}'_0})^!(h_1^{\widehat{E}})^! & \xrightarrow{\Delta(h_1^{\widehat{E}})^!} & (\widehat{\widetilde{q}'_0})^!(\widehat{\widetilde{f}}_1)^*(h_1^{\widehat{E}})^! & \xrightarrow{(\widehat{\widetilde{q}'_0})^!\diamond} & (\widehat{\widetilde{q}'_0})^!(h_1^{\widehat{F}})^!(\widehat{f}_1)^* \\ \| & & & & \| \\ (\widehat{f}^\sharp)^*(\widehat{p'_0})^! & & \xrightarrow{\qquad\qquad\Delta\qquad\qquad} & & (\widehat{q'_0})^!(\widehat{f}_1)^* \end{array} \tag{7.2.28}$$

This follows from Proposition 3.5.5 applied to the two squares

$$\begin{array}{ccccc} \widehat{D}^\sharp & \xrightarrow{\widehat{\widetilde{q}'_0}} & \widehat{\widetilde{F}_1} & \xrightarrow{h_1^{\widehat{F}}} & \widehat{F}_1 \\ \downarrow{\scriptstyle \widehat{f}^\sharp} & & \downarrow{\scriptstyle \widehat{\widetilde{f}}_1} & & \downarrow{\scriptstyle \widehat{f}_1} \\ \widehat{C}^\sharp & \xrightarrow{\widehat{\widetilde{p}'_0}} & \widehat{\widetilde{E}_1} & \xrightarrow{h_1^{\widehat{E}}} & \widehat{E}_1 \end{array} \tag{7.2.29}$$

(2) The proof is similar. Alternatively, it follows from (1) using the near-involutivity of FT.

□

8. Arithmetic Fourier transform

In this section we introduce an "arithmetic" variant of the Fourier transform, which will be used to do Fourier analysis on the Borel-Moore homology of moduli spaces for shtuka-type objects.

$$\begin{array}{ccc} \mathrm{Sht}_V^r & \longrightarrow & \mathrm{Hk}_V^\flat \\ \downarrow & & \downarrow{\scriptstyle (b_0,b_r)} \\ V & \xrightarrow{(\mathrm{Frob},\mathrm{Id})} & V \times V \end{array}$$

When specialized to $r = 0$, the arithmetic Fourier transform recovers the finite Fourier transforms used in §2.3 to prove modularity for $r = 0$.

8.1. The general setup.

Let T be a derived Artin stack and $Y \to T$ be an étale locally free $\mathbf{F}_q$-vector space bundle of finite rank. In particular, $Y \to T$ is representable (in schemes) and finite étale. Let $\widehat{Y} \to T$ be the dual $\mathbf{F}_q$-vector space, i.e., at the level of étale sheaves over T we have

$$\widehat{Y} \mathcal{H}om_{\mathbf{F}_q}(Y, \underline{\mathbf{F}}_q).$$

Note that $\widehat{\widehat{Y}} \cong Y$.

Then we have an "evaluation" map

$$\mathrm{ev} \colon Y \times_T \widehat{Y} \to \underline{\mathbf{F}_q},$$

where $\underline{\mathbf{F}_q}$ is the set $\mathbf{F}_q$ viewed as a discrete scheme. Consider the diagram

$$\begin{array}{ccccc} & & Y \times_T \widehat{Y} \xrightarrow{\text{ev}} \underline{\mathbf{F}_q} & & \\ & \swarrow^{\text{pr}_0} & & \searrow^{\text{pr}_1} & \\ Y & & & & \widehat{Y} \\ & \searrow_{\pi} & & \swarrow_{\widehat{\pi}} & \\ & & T & & \end{array}$$

Definition 8.1.1. Let ψ be a nontrivial additive character of $\underline{\mathbf{F}_q}$. Let d be the rank of Y as an $\mathbf{F}_q$-vector space over T. We define the *arithmetic Fourier transform (with respect to ψ)* to be the map

$$\mathrm{FT}_Y^{\mathrm{arith},\psi} \colon \mathrm{H}_*^{\mathrm{BM}}(Y) \to \mathrm{H}_*^{\mathrm{BM}}(\widehat{Y})$$

given by

$$\alpha \mapsto (-1)^d \operatorname{pr}_{1!}(\operatorname{pr}_0^*(\alpha) \cdot \operatorname{ev}^* \psi).$$

Here, we used that $\mathrm{H}_*^{\mathrm{BM}}(-)$ is a module over $\mathrm{H}^0(-;\mathbf{Q}_{\ell,})$, or in other words, we can multiply Borel-Moore homology classes by locally constant functions. More generally, $\mathrm{H}_*^{\mathrm{BM}}(-)$ is a module over $\mathrm{H}^{2*}(-;\mathbf{Q}_\ell(*))$.

Similarly, we define the *arithmetic Fourier transform (on cohomology)* to be the map

$$\mathrm{FT}^{\mathrm{arith},\psi} \colon \mathrm{H}^*(Y;\mathbf{Q}_{\ell,Y}) \to \mathrm{H}^*(\widehat{Y};\mathbf{Q}_{\ell,Y})$$

given by

$$\alpha \mapsto (-1)^d \operatorname{pr}_{1!}(\operatorname{pr}_0^*(\alpha) \cdot \operatorname{ev}^* \psi).$$

When ψ is understood, we will suppress it from the notation, writing $\mathrm{FT}^{\mathrm{arith}} = \mathrm{FT}^{\mathrm{arith},\psi}$.

8.2. **Basic properties.** We establish some basic properties of the arithmetic Fourier transform, parallel to those of the usual finite Fourier transform (§2.3.7).

Lemma 8.2.1 (Plancherel property)**.** *Let $\alpha_1 \in \mathrm{H}_{2i}^{\mathrm{BM}}(Y)$ and $\beta_2 \in \mathrm{H}^{2j}(\widehat{Y};\mathbf{Q}_{\ell,\widehat{Y}}(j))$. Then*

$$\pi_!(\alpha_1 \cdot \mathrm{FT}^{\mathrm{arith}}(\beta_2)) = \widehat{\pi}_!(\mathrm{FT}^{\mathrm{arith}}(\alpha_1) \cdot \beta_2) \in \mathrm{H}_{2i-2j}^{\mathrm{BM}}(T).$$

Proof. We have

$$\begin{aligned} \pi_!(\alpha_1 \cdot \mathrm{FT}^{\mathrm{arith}}(\beta_2)) &= (-1)^d \pi_!(\alpha_1 \cdot \operatorname{pr}_{0!}(\operatorname{pr}_1^* \beta_2 \cdot \operatorname{ev}^* \psi)) \\ &= (-1)^d \pi_! \operatorname{pr}_{0!}(\operatorname{pr}_0^* \alpha_1 \cdot \operatorname{pr}_1^* \beta_2 \cdot \operatorname{ev}^* \psi) \\ &= (-1)^d \widehat{\pi}_! \operatorname{pr}_{1!}(\operatorname{pr}_0^* \alpha_1 \cdot \operatorname{pr}_1^* \beta_2 \cdot \operatorname{ev}^* \psi) \\ &= (-1)^d \widehat{\pi}_!(\operatorname{pr}_{1!}(\operatorname{pr}_0^* \alpha_1 \cdot \operatorname{ev}^* \psi) \cdot \beta_2) \\ &= \widehat{\pi}_!(\mathrm{FT}^{\mathrm{arith}}(\alpha_1) \cdot \beta_2). \end{aligned}$$

□

Next we examine compatibility of the arithmetic Fourier transform with base change. Let $\varphi \colon T' \to T$ be a proper map. Let $Y' \to T'$ the pullback of $Y \to T$ and $\widehat{Y}' \to T'$ the pullback of $\widehat{Y} \to T$. So we have Cartesian squares

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ T' & \xrightarrow{\varphi} & T \end{array} \qquad\qquad \begin{array}{ccc} \widehat{Y}' & \longrightarrow & \widehat{Y} \\ \downarrow & & \downarrow \\ T' & \xrightarrow{\varphi} & T \end{array}$$

The properness of φ ensures the existence of maps $\varphi_! \colon \mathrm{H}_*^{\mathrm{BM}}(Y') \to \mathrm{H}_*^{\mathrm{BM}}(Y)$ and $\varphi_! \colon \mathrm{H}_*^{\mathrm{BM}}(\widehat{Y}') \to \mathrm{H}_*^{\mathrm{BM}}(\widehat{Y})$.

Lemma 8.2.2. *Let $\varphi \colon T' \to T$ be proper and maintain the above notation. Then we have $\mathrm{FT}^{\mathrm{arith}} \circ \varphi_! = \varphi_! \circ \mathrm{FT}^{\mathrm{arith}}$ as maps $\mathrm{H}_*^{\mathrm{BM}}(Y') \to \mathrm{H}_*^{\mathrm{BM}}(\widehat{Y})$.*

Proof. The pushforward $\varphi_!$ satisfies base change against: smooth pullback, proper pushforward, and tensoring with $\mathrm{H}^0(-,\mathbf{Q}_\ell)$, and $\mathrm{FT}^{\mathrm{arith}}$ is a composition of such operations. □

Now suppose that $\varphi\colon T' \to T$ is quasi-smooth. Let $Y' \to T'$ the pullback of $Y \to T$ and $\widehat{Y}' \to T'$ the pullback of $\widehat{Y} \to T$. The quasi-smoothness of φ ensures the existence of maps $\varphi^*\colon \mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_{*+2d(\varphi)}(Y')$ and $\varphi^*\colon \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y}) \to \mathrm{H}^{\mathrm{BM}}_{*+2d(\varphi)}(\widehat{Y}')$.

Lemma 8.2.3. *Let $\varphi\colon T' \to T$ be quasi-smooth and maintain the above notation. Then we have $\mathrm{FT}^{\mathrm{arith}} \circ \varphi^* = \varphi^* \circ \mathrm{FT}^{\mathrm{arith}}$ as maps $\mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_{*+2d(\varphi)}(\widehat{Y}')$.*

Proof. The pullback φ^* satisfies base change against: smooth pullback, proper pushforward, and tensoring with $\mathrm{H}^*(-, \mathbf{Q}_\ell)$, and $\mathrm{FT}^{\mathrm{arith}}$ is a composition of such operations. □

Lemma 8.2.4 (Involutivity)**.** *We have $\mathrm{FT}^{\mathrm{arith}}_{\widehat{Y}} \circ \mathrm{FT}^{\mathrm{arith}}_{Y} = q^d[-1]^*$, where $[-1]$ is multiplication by -1 on $\widehat{Y}$ using its $\mathbf{F}_q$-vector space structure over T.*

Proof. First suppose that it $Y \to T$ is a *split* étale $\mathbf{F}_q$-vector space over T, i.e., there exists a (finite-dimensional) $\mathbf{F}_q$-vector space Y_0 such that $Y = Y_0 \times T$. In this case, $\mathrm{H}^{\mathrm{BM}}_*(Y) = \mathrm{H}^{\mathrm{BM}}_*(T) \otimes_{\mathbf{Q}_\ell} \mathbf{Q}_\ell[Y_0]$ and the arithmetic Fourier transform simplifies to the identity on $\mathrm{H}^{\mathrm{BM}}_*(T)$ tensored with the usual Fourier transform on $\mathbf{Q}_\ell[Y_0]$. Therefore, the identity follows as for the usual finite Fourier transform (§2.3.7).

Now, since $Y \to T$ is finite étale, it is split by some finite étale pullback $\varphi\colon T' \to T$. Letting $Y' \to T'$ be the pullback of $Y \to T$ along φ, the previous paragraph shows that

$$\mathrm{FT}^{\mathrm{arith}}_{\widehat{Y}'} \circ \mathrm{FT}^{\mathrm{arith}}_{Y'} = q^d[-1]^*. \tag{8.2.1}$$

According to Lemma 8.2.2 and Lemma 8.2.3, $\mathrm{FT}^{\mathrm{arith}}_{\widehat{Y}'} \circ\varphi^* = \varphi^* \circ \mathrm{FT}^{\mathrm{arith}}_{\widehat{Y}}$ and $\mathrm{FT}^{\mathrm{arith}}_{Y'} \circ\varphi^* = \varphi^* \circ \mathrm{FT}^{\mathrm{arith}}_{Y}$, so composing (8.2.1) with φ^* shows that

$$\varphi^* \, \mathrm{FT}^{\mathrm{arith}}_{\widehat{Y}} \circ \mathrm{FT}^{\mathrm{arith}}_{Y} = \varphi^* q^d[-1]^*.$$

Now apply $\varphi_!$. Since $\varphi_! \circ \varphi^*$ is multiplication by $\deg \varphi$ (which is invertible), we get the desired equation. □

Let $\varphi\colon Y' \to Y$ be an $\mathbf{F}_q$-linear map of $\mathbf{F}_q$-vector spaces over T. In particular, φ is finite étale, so that we have maps

$$\varphi^*\colon \mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_*(Y'),$$
$$\varphi_!\colon \mathrm{H}^{\mathrm{BM}}_*(Y') \to \mathrm{H}^{\mathrm{BM}}_*(Y).$$

Let $\widehat{\varphi}\colon \widehat{Y} \to \widehat{Y}'$ be the dual map. As $\widehat{\varphi}$ is also finite étale, we have maps

$$\widehat{\varphi}^*\colon \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y}') \to \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y}),$$
$$\widehat{\varphi}_!\colon \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y}) \to \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y}').$$

Proposition 8.2.5 (Functoriality for linear maps)**.** *Keep the above notation. Let d, d' be the ranks of Y, Y' as $\mathbf{F}_q$-vector spaces over T. Then we have the following identities.*

(1) $\mathrm{FT}^{\mathrm{arith}}_{Y} \circ\varphi_! = (-1)^{d-d'} \widehat{\varphi}^* \circ \mathrm{FT}^{\mathrm{arith}}_{Y'}$ *as maps* $\mathrm{H}^{\mathrm{BM}}_*(Y') \to \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y})$.
(2) $\mathrm{FT}^{\mathrm{arith}}_{Y'} \circ\varphi^* = (-1)^{d'-d} q^{d'-d} \widehat{\varphi}_! \circ \mathrm{FT}^{\mathrm{arith}}_{Y}$ *as maps* $\mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_*(\widehat{Y}')$.

Proof. First suppose that it $Y \to T$ is a *split* étale $\mathbf{F}_q$-vector space over T, i.e., there exists a (finite-dimensional) $\mathbf{F}_q$-vector space Y_0 such that $Y = Y_0 \times T$. In this case, $\mathrm{H}^{\mathrm{BM}}_*(Y) = \mathrm{H}^{\mathrm{BM}}_*(T) \otimes_{\mathbf{Q}_\ell} \mathbf{Q}_\ell[Y_0]$ and the arithmetic Fourier transform simplifies to the identity on $\mathrm{H}^{\mathrm{BM}}_*(T)$ tensored with the usual Fourier transform on $\mathbf{Q}_\ell[Y_0]$. Therefore, the identity follows as for the usual finite Fourier transform (§2.3.7).

In the general case, Y (therefore also $\widehat{Y}$) is split by some finite étale base change $T' \to T$. Then the proof reduces to that in the split case, as in the proof of Lemma 8.2.4. □

8.3. **Compatibility with sheaf-theoretic Fourier transform.** Let S be a derived stack and $p : E \to S$ be a vector bundle (note that we deliberately do not allow more general derived vector bundles here). Suppose $c = (c_0, c_1) : C \to S \times S$ is a correspondence and we are given an isomorphism of vector bundles over C

$$\iota : c_0^* E \simeq c_1^* E. \tag{8.3.1}$$

Let C_E be the total space of c_0^*E and of c_1^*E, identified via ι. Let $e_i : C_E \simeq c_i^*E \to E$ be the projection, for $i = 0, 1$. Then we get a correspondence $e = (e_0, e_1) : C_E \to E \times E$ that fits into a commutative diagram

$$\begin{array}{ccccc} E & \xleftarrow{e_0} & C_E & \xrightarrow{e_1} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_C} & & \downarrow{\scriptstyle p} \\ S & \xleftarrow{c_0} & C & \xrightarrow{c_1} & S \end{array} \tag{8.3.2}$$

such that both squares are Cartesian.

The above data induces a correspondence $\widehat{e} : C_{\widehat{E}} \to \widehat{E} \times \widehat{E}$ by passing to the dual vector bundles. Let $\widehat{p} : \widehat{E} \to S$ and $p_{\widehat{C}} : C_{\widehat{E}} \to \widehat{C}$ be the projections.

Let $\mathcal{K} \in D_c^b(E)$ and $\mathfrak{c} : e_0^*\mathcal{K} \to e_1^!\mathcal{K}\langle -n\rangle$ be a cohomological correspondence. Applying Fourier transforms, using that Fourier transform commutes with arbitrary $*$ and $!$ base change (§6.2.2), $\mathfrak{c}$ induces a cohomological correspondence of $\mathrm{FT}_E(\mathcal{K})$ as the composition:

$$\mathrm{FT}_{C_E}(\mathfrak{c}) : \widehat{e}_0^* \,\mathrm{FT}_E(\mathcal{K}) \simeq \mathrm{FT}_{C_E}(e_0^*\mathcal{K}) \xrightarrow{\mathrm{FT}_{C_E}(\mathfrak{c})} \mathrm{FT}_{C_E}(e_1^!\mathcal{K}\langle -n\rangle) \simeq \widehat{e}_1^* \,\mathrm{FT}_E(\mathcal{K})\langle -n\rangle. \tag{8.3.3}$$

Consider the map $c^{(1)} = (\mathrm{Frob}_S \circ c_0, c_1) : C \to S \times S$. This makes C into a self-correspondence of S in a different way, which we denote by $C^{(1)}$. Similarly, we define $C_E^{(1)}$ (a self-correspondence of E) and $C_{\widehat{E}}^{(1)}$ (a self-correspondence of $\widehat{E}$). Recall notation

$$\mathrm{Sht}(C) = \mathrm{Fix}(C^{(1)}), \quad \mathrm{Sht}(C_E) = \mathrm{Fix}(C_E^{(1)}), \quad \mathrm{Sht}(C_{\widehat{E}}) = \mathrm{Fix}(C_{\widehat{E}}^{(1)}). \tag{8.3.4}$$

Lemma 8.3.1. *The projections*

$$\pi : \mathrm{Sht}(C_E) \to \mathrm{Sht}(C), \quad \widehat{\pi} : \mathrm{Sht}(C_{\widehat{E}}) \to \mathrm{Sht}(C) \tag{8.3.5}$$

are relative $\mathbf{F}_q$*-vector spaces over* $\mathrm{Sht}(C)$ *that are dual to each other.*

Proof. Evident. □

Let $\mathcal{K} \in D(E)$. Then $\mathcal{K}$ is equipped with a canonical Weil structure $\mathrm{Frob}_E^* \mathcal{K} \simeq \mathcal{K}$. A cohomological correspondence $\mathfrak{c} : e_0^*\mathcal{K} \to e_1^!\mathcal{K}\langle -n\rangle$ induces a cohomological correspondence $\mathfrak{c}^{(1)}$ supported on $C_E^{(1)}$:

$$\mathfrak{c}^{(1)} : e_0^* \,\mathrm{Frob}_E^* \mathcal{K} \simeq e_0^*\mathcal{K} \xrightarrow{\mathfrak{c}} e_1^!\mathcal{K}\langle -n\rangle. \tag{8.3.6}$$

Taking trace we get

$$\mathrm{Tr}(\mathfrak{c}^{(1)}) \in \mathrm{H}_{2n}^{\mathrm{BM}}(\mathrm{Sht}_E). \tag{8.3.7}$$

Similarly, we have the cohomological correspondence $\mathrm{FT}_{C_E}(\mathfrak{c})^{(1)}$ of $\mathrm{FT}_E(\mathcal{K})$ supported on $C_{\widehat{E}}^{(1)}$, and its trace

$$\mathrm{Tr}^{\mathrm{Sht}}(\mathrm{FT}_{C_E}(\mathfrak{c})) := \mathrm{Tr}(\mathrm{FT}_{C_E}(\mathfrak{c})^{(1)}) \in \mathrm{H}_{2n}^{\mathrm{BM}}(\mathrm{Sht}(C_{\widehat{E}})). \tag{8.3.8}$$

Theorem 8.3.2. *In the above situation, we have*

$$\mathrm{Tr}^{\mathrm{Sht}}(\mathrm{FT}_{C_E}(\mathfrak{c})) = \mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}(C_E)}(\mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c})) \in \mathrm{H}_{2n}^{\mathrm{BM}}(\mathrm{Sht}(C_{\widehat{E}})). \tag{8.3.9}$$

Proof. It is easy to see that

$$\mathrm{FT}_{C_E}(\mathfrak{c}^{(1)}) = \mathrm{FT}_{C_E}(\mathfrak{c})^{(1)}. \tag{8.3.10}$$

Therefore we need to show

$$\mathrm{Tr}(\mathrm{FT}_{C_E}(\mathfrak{c}^{(1)})) = \mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}(C_E)}(\mathrm{Tr}(\mathfrak{c}^{(1)})). \tag{8.3.11}$$

Consider the following diagram of correspondences (correspondences are written horizontally and morphisms between correspondences are written vertically)

$$\begin{array}{ccccc} E & \xleftarrow{\mathrm{Frob}\circ e_0} & C_E^{(1)} & \xrightarrow{e_1} & E \\ \uparrow{\scriptstyle \mathrm{pr}_1} & & \uparrow{\scriptstyle \mathrm{pr}_1} & & \uparrow{\scriptstyle \mathrm{pr}_1} \\ E\times_S \widehat{E} & \xleftarrow{\mathrm{Frob}\circ d_0} & C^{(1)}_{E\times_S\widehat{E}} & \xrightarrow{d_1} & E\times_S\widehat{E} \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \widehat{E} & \xleftarrow{\mathrm{Frob}\circ \widehat{e}_0} & C^{(1)}_{\widehat{E}} & \xrightarrow{\widehat{e}_1} & \widehat{E} \end{array} \tag{8.3.12}$$

Note that all squares are étale topologically Cartesian. The Fourier transform $\mathrm{FT}(\mathfrak{c}^{(1)})$ is the composition of three operations

$$\mathrm{FT}(\mathfrak{c}^{(1)}) = \mathrm{pr}_{2!}(t_\psi(\mathrm{pr}_1^*(\mathfrak{c}^{(1)}))). \tag{8.3.13}$$

Here, for a cohomological correspondence $\mathfrak{d} : d_0^*\mathcal{F} \to d_1^!\mathcal{F}\langle -n\rangle$, $t_\psi(\mathfrak{d})$ means the cohomological correspondence

$$t_\psi(\mathfrak{d}) : d_0^*(\mathcal{F}\otimes \mathrm{ev}^*\mathcal{L}_\psi) \simeq d_0^*\mathcal{F}\otimes \widetilde{\mathrm{ev}}^*\mathcal{L}_\psi \xrightarrow{\mathfrak{d}\otimes\mathrm{Id}} d_1^!\mathcal{F}\langle -n\rangle \otimes \widetilde{\mathrm{ev}}^*\mathcal{L}_\psi \simeq d_1^!(\mathcal{F}\otimes\mathrm{ev}^*\mathcal{L}_\psi)\langle -n\rangle \tag{8.3.14}$$

where $\mathrm{ev} : E\times_S \widehat{E} \to \mathbf{G}_a$ and $\widetilde{\mathrm{ev}} : C_{E\times_S\widehat{E}} = C_E \times_C C_{\widehat{E}} \to \mathbf{G}_a$ are the tautological evaluation pairings.

On the other hand, let

$$\mathrm{pr}_1^{\mathrm{Sht}} : \mathrm{Sht}(C_{E\times_S\widehat{E}}) = \mathrm{Sht}(C_E)\times_{\mathrm{Sht}(C)} \mathrm{Sht}(C_{\widehat{E}}) \to \mathrm{Sht}(C_E)$$

$$\mathrm{pr}_2^{\mathrm{Sht}} : \mathrm{Sht}(C_{E\times_S\widehat{E}}) = \mathrm{Sht}(C_E)\times_{\mathrm{Sht}(C)} \mathrm{Sht}(C_{\widehat{E}}) \to \mathrm{Sht}(C_{\widehat{E}})$$

be the projections to the two factors (the maps are finite étale). Then $\mathrm{FT}^{\mathrm{arith}}$ is the composition $\mathrm{pr}_{2!}^{\mathrm{Sht}} \circ m_\psi \circ \mathrm{pr}_1^{\mathrm{Sht}\,*}$, where m_ψ means multiplying the Borel-Moore classes on $\mathrm{Sht}(C_E)\times_{\mathrm{Sht}(C)}\mathrm{Sht}(C_{\widehat{E}})$ by the function $\mathrm{ev}^{\mathrm{Sht}\,*}\psi$, where $\mathrm{ev}^{\mathrm{Sht}} : \mathrm{Sht}(C_E)\times_{\mathrm{Sht}(C)}\mathrm{Sht}(C_{\widehat{E}}) \to \mathbf{F}_q$ is the evaluation pairing.

By Proposition 4.5.4, $\mathrm{Tr}(\mathrm{pr}_1^*(\mathfrak{c}^{(1)})) = \mathrm{pr}_1^{\mathrm{Sht}\,*}\mathrm{Tr}(\mathfrak{c}^{(1)})$. It is also clear that taking trace intertwines t_ψ and m_ψ. Hence

$$\mathrm{Tr}(t_\psi\,\mathrm{pr}_1^*(\mathfrak{c}^{(1)})) = m_\psi\,\mathrm{pr}_1^{\mathrm{Sht}\,*}(\mathrm{Tr}(\mathfrak{c}^{(1)})). \tag{8.3.15}$$

Let $\mathcal{F} = \mathrm{pr}_1^*\mathcal{K}\otimes\mathrm{ev}^*\mathcal{L}_\psi \in D(E\times_S\widehat{E})$, and $\mathfrak{d} = t_\psi\,\mathrm{pr}_1^*(\mathfrak{c}) : d_0^*\mathcal{F} \to d_1^!\mathcal{F}\langle -n\rangle$ be the cohomological correspondence of $\mathcal{F}$ supported on $C_{E\times_S\widehat{E}}$. It remains to show

$$\mathrm{Tr}(\mathrm{pr}_{2!}(\mathfrak{d}^{(1)})) = \mathrm{pr}_{2!}^{\mathrm{Sht}}\,\mathrm{Tr}(\mathfrak{d}^{(1)}). \tag{8.3.16}$$

This follows by applying Lemma 8.3.3 below to the map of correspondences $\mathrm{pr}_2 : C_{E\times_S\widehat{E}} \to C_{\widehat{E}}$ (viewing $E\times_S\widehat{E}$ as a vector bundle over $\widehat{E}$). □

Lemma 8.3.3. *Consider the situation* (8.3.2). *Let* $\mathcal{K}\in D(E)$ *and* $\mathfrak{c}$ *be a cohomological correspondence* $\mathfrak{c} : e_0^*\mathcal{K} \to e_1^!\mathcal{K}\langle -n\rangle$. *Let* $p^{\mathrm{Sht}} : \mathrm{Sht}(C_E) \to \mathrm{Sht}(C)$ *be the induced map on fixed points of* $C_E^{(1)}$ *and* $C^{(1)}$, *which is a relative* $\mathbf{F}_q$*-vector space (in particular a finite morphism). Then*

$$\mathrm{Tr}(p_{C!}(\mathfrak{c})) = p_!^{\mathrm{Sht}}\,\mathrm{Tr}(\mathfrak{c}^{(1)}) \in \mathrm{H}_{2n}^{\mathrm{BM}}(\mathrm{Sht}(C)). \tag{8.3.17}$$

Proof. Consider the projective bundle $\overline{E} = \mathbf{P}(E\oplus\mathcal{O}) \to S$ that contains E as an open substack. Similarly let $C_{\overline{E}} = \mathbf{P}(c_0^*E\oplus\mathcal{O}) \cong \mathbf{P}(c_1^*E\oplus\mathcal{O}) \to C$ be the pullback projective bundle over C, via either c_0 or c_1. Then $C_{\overline{E}}$ is a self-correspondence of $\overline{E}$ with a proper map to C:

$$\begin{array}{ccccc} \overline{E} & \xleftarrow{\overline{e}_0} & C_{\overline{E}} & \xrightarrow{\overline{e}_1} & \overline{E} \\ \downarrow{\scriptstyle \overline{p}} & & \downarrow{\scriptstyle \overline{p}_C} & & \downarrow{\scriptstyle \overline{p}} \\ S & \xleftarrow{c_0} & C & \xrightarrow{c_1} & S \end{array} \tag{8.3.18}$$

Let $E_\infty = \overline{E} - E$ be the divisor at infinity, which is isomorphic to $\mathbf{P}(E)$. Similarly define $C_{E_\infty} = C_{\overline{E}} - C_E$, which is a self-correspondence of E_∞. We define $C_{\overline{E}}^{(1)}$ and $C_{E_\infty}^{(1)}$ by composing the first projections by the Frobenius. For the fixed points, we have an open and closed decomposition

$$\mathrm{Sht}(C_{\overline{E}}) = \mathrm{Sht}(C_E) \coprod \mathrm{Sht}(C_{E_\infty}). \tag{8.3.19}$$

Fiberwise over $\mathrm{Sht}(C)$, this decomposition takes the form $\mathbf{P}^N(\mathbf{F}_q) \simeq \mathbf{F}_q^N \coprod \mathbf{P}^{N-1}(\mathbf{F}_q)$, where N is the rank of E.

Let $j : E \hookrightarrow \overline{E}$ and $j_C : C_E \hookrightarrow C_{\overline{E}}$ be the open inclusions. The map of correspondences

$$\begin{array}{ccccc} E & \xleftarrow{e_0} & C_E & \xrightarrow{e_1} & E \\ \Big\downarrow j & & \Big\downarrow j_C & & \Big\downarrow j \\ \overline{E} & \xleftarrow{\overline{e}_0} & C_{\overline{E}} & \xrightarrow{\overline{e}_1} & \overline{E} \end{array}$$

has both squares Cartesian, so it is left pushable. Therefore we have the pushforward cohomological correspondence

$$\overline{\mathfrak{c}} := j_{C!}\mathfrak{c} \in \mathrm{Corr}_{C_{\overline{E}}}(j_!\mathcal{K}, j_!\mathcal{K}\langle -n \rangle). \tag{8.3.20}$$

Since $\overline{p}$ and $\overline{p}_C$ are proper, we have by Proposition 4.5.1,

$$\mathrm{Tr}(p_{C!}(\mathfrak{c}^{(1)})) = \mathrm{Tr}(\overline{p}_{C!}j_{C!}(\mathfrak{c}^{(1)})) = \mathrm{Tr}(\overline{p}_{C!}(\overline{\mathfrak{c}}^{(1)})) = \overline{p}_!^{\mathrm{Sht}}\,\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)}) \tag{8.3.21}$$

where $\overline{p}^{\mathrm{Sht}} : \mathrm{Sht}(C_{\overline{E}}) \to \mathrm{Sht}(C)$ is the obvious map. On the other hand, since j_C is an open embedding, we have by Proposition 4.5.4 that $\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)})|_{\mathrm{Sht}(C_E)} = \mathrm{Tr}(\mathfrak{c}^{(1)}) \in \mathrm{H}_{2n}^{\mathrm{BM}}(\mathrm{Sht}(C_E))$, therefore

$$p_!^{\mathrm{Sht}}(\mathrm{Tr}(\mathfrak{c}^{(1)})) = p_!^{\mathrm{Sht}}(\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)})|_{\mathrm{Sht}(C_E)}). \tag{8.3.22}$$

It remains to show that

$$\overline{p}_!^{\mathrm{Sht}}\,\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)}) = p_!^{\mathrm{Sht}}(\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)})|_{\mathrm{Sht}(C_E)}), \tag{8.3.23}$$

which would follow from the vanishing

$$\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)})|_{\mathrm{Sht}(C_{E_\infty})} = 0 \in \mathrm{H}_{2n}^{\mathrm{BM}}(\mathrm{Sht}(C_{E_\infty})). \tag{8.3.24}$$

It is clear that E_∞ is invariant under the correspondence $C_{\overline{E}}$ because $\overline{e}_1^{-1}(E_\infty) = C_{E\infty}$ which maps to E_∞ by $\overline{e}_0$. Therefore E_∞ is also invariant under $C_{\overline{E}}^{(1)}$. More importantly, $C_{\overline{E}}^{(1)}$ is contracting near E_∞ in the sense of Varshavsky [Var07, Definition 2.1.1(b)]. Indeed, let $\mathcal{I}_\infty$ be the defining ideal of E_∞ inside $\overline{E}$. Then both $e_0^*\mathcal{I}_\infty$ and $e_1^*\mathcal{I}_\infty$ are the defining ideal of C_{E_∞} inside C_E. Therefore, $\mathrm{Frob}^*(e_0^*\mathcal{I}_\infty) = (e_0^*\mathcal{I}_\infty)^q = (e_1^*\mathcal{I}_\infty)^q$, proving the contractibility of $C_{\overline{E}}^{(1)}$ near E_∞.

Now we can apply [Var07, Theorem 2.1.3][9] to the correspondence $C_{\overline{E}}^{(1)}$, the closed substack $Z = E_\infty \subset E$, and each connected component β of $\beta = \mathrm{Sht}(C_{E_\infty})$ as an open-closed substack of the fixed point locus of $C_{\overline{E}}^{(1)}$. Since $C_{\overline{E}}^{(1)}$ is contracting near E_∞, we conclude that the β-part of the trace

$$\mathrm{Tr}(\overline{\mathfrak{c}}^{(1)})|_\beta = \mathrm{Tr}(\overline{\mathfrak{c}}^{(1)}|_{E_\infty})|_\beta \in \mathrm{H}_{2n}^{\mathrm{BM}}(\beta). \tag{8.3.25}$$

Here $\overline{\mathfrak{c}}^{(1)}|_{E_\infty}$ is the restriction of the cohomological correspondence $\overline{\mathfrak{c}}^{(1)}$ as defined in [Var07, 1.5.6(a)], which is a cohomological correspondence of $(j_!\mathcal{K})|_{E_\infty} = 0$. Therefore the right side above is zero, hence $\mathrm{Tr}_\beta(\overline{\mathfrak{c}}^{(1)}) = 0$ for any connected component of $\mathrm{Sht}(C_{E_\infty})$. This proves (8.3.24) and finishes the proof of the lemma. □

[9]Varshavsky's proof is written in the setting of schemes (and without the shift by n). However, his proof goes through verbatim (even with a shift by n) for (higher) Artin stacks once one knows the compatibility of specialization and trace in the setting of (higher) Artin stacks, which is established in [LZ22, §3].

Part 3. Modularity

9. Modularity for cohomological correspondences

In this section we carry out Steps (2), (3), and (5) of the proof outline from §2.4.

First, in §9.1 we set up the derived vector bundles U, V, and W that were promised in Steps (2) and (3) of the outline in §2.4 and the Hecke correspondences for them. Switching the role of $\mathcal{E}_1$ and $\mathcal{E}_2$, in §9.2 we define the derived vector bundles $U^\perp, \widehat{V}, W^\perp$, etc. which were promised in Step (1) of the outline. The main result of this section is Theorem 9.4.1, showing that the Fourier transform on $\mathrm{Hk}_V^\flat$ of the cohomological correspondence $f_! \mathfrak{c}_U$ agrees (up to shift and twist) with the parallel cohomological construction obtained by interchanging $\mathcal{E}_1$ and $\mathcal{E}_2$. The proof of Theorem 9.4.1 uses the general results proved in §5 and §7. We regard Theorem 9.4.1 as an incarnation of modularity at the level of cohomological correspondences – the modularity of the higher theta series will be extracted from it by taking a trace in the sense of the sheaf-cycle correspondence – which explains the title of the section.

9.1. The stacks U, V, W and their Hecke correspondences.

We begin by defining various spaces of interest.

Suppose we are given a short exact sequence of coherent sheaves on X':

$$0 \to \sigma^* \mathcal{E}_2 \to \mathcal{E}_1^* \to \widetilde{Q}_1 \to 0 \tag{9.1.1}$$

where $\mathcal{E}_1, \mathcal{E}_2$ are vector bundles and $\widetilde{Q}_1$ is torsion.

9.1.1. *Definition of U, V and W.* Below we denote S for a Harder-Narasimhan truncation $\mathrm{Bun}_{U(n)}^{\leq \mu}$. (The reason for this truncation is to guarantee global presentability; see §9.3.) We define several derived vector bundles over S. Let $\mathcal{F}_{\mathrm{univ}}$ be the universal Hermitian bundle over $X' \times S$. Let R be any animated $\mathbf{F}_q$-algebra.

- Define $\mathcal{U} := \underline{\mathrm{RHom}(\mathcal{F}_{\mathrm{univ}}^*, \mathcal{E}_1^*)}$ to be the perfect complex on S whose pullback to an R-point $\mathcal{F} \in S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \mathcal{E}_1^* \otimes R)$ regarded as an animated R-module. Let $U := \mathrm{Tot}_S(\mathcal{U})$ be the associated derived vector bundle over S.
- Define $\mathcal{V} := \underline{\mathrm{RHom}(\mathcal{F}_{\mathrm{univ}}^*, \widetilde{Q}_1)}$ to be the perfect complex on S whose pullback to an R-point $\mathcal{F} \in S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \widetilde{Q}_1 \otimes R)$ regarded as an animated R-module. Let $V := \mathrm{Tot}_S(\mathcal{V})$ be the associated derived vector bundle over S. Since $\mathrm{Ext}^1_{X'_R}(\mathcal{F}^*, \widetilde{Q}_1 \otimes R) = 0$ as $\widetilde{Q}_1$ is torsion, $\mathcal{V}$ is in fact a locally free coherent sheaf (concentrated in degree 0) on S so that $V \to S$ is actually a classical vector bundle. This fact is not important for this section, although it plays a role in later sections.
- Define $\mathcal{W} := \underline{\mathrm{RHom}(\mathcal{F}_{\mathrm{univ}}^*, \sigma^* \mathcal{E}_2[1])}$ to be the perfect complex on S whose pullback to an R-point $\mathcal{F} \in S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \sigma^* \mathcal{E}_2[1] \otimes R)$ regarded as an animated R-module. Let $W := \mathrm{Tot}_S(\mathcal{W})$ be the associated derived vector bundle over S.

From (9.1.1) we get an exact triangle of sheaves on X',

$$\mathcal{E}_1^* \to \widetilde{Q}_1 \to \sigma^* \mathcal{E}_2[1] \tag{9.1.2}$$

which induces the exact triangle in $\mathrm{Perf}(S)$,

$$\mathcal{U} \to \mathcal{V} \to \mathcal{W}. \tag{9.1.3}$$

Forming total spaces, this is equivalent to the derived Cartesian square of derived vector bundles over S:

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ S & \xrightarrow{z} & W \end{array} \tag{9.1.4}$$

where $z : S \to W$ denotes the zero section.

Remark 9.1.1. By [FYZ21b, §5.7], U is isomorphic to the derived pullback of the derived Hitchin stack "$\mathscr{M}_{\mathrm{GL}(m)', U(n)} \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)}$" from [FYZ21b, §5] along the map $\{\mathcal{E}_1\} \times S \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)}$.

9.1.2. *Hecke stacks.* Recall the Hecke stack $\mathrm{Hk}^r_{U(n)}$ from [FYZ21b, §5.4]. The R-points of $\mathrm{Hk}^r_{U(n)}$ are diagrams

$$\begin{array}{ccccccccc} & \mathcal{F}^\flat_{1/2} & & & \cdots & & \mathcal{F}^\flat_{r-1/2} & & \\ \swarrow & & \searrow & & & \swarrow & & \searrow & \\ \mathcal{F}_0 & \dashrightarrow & \mathcal{F}_1 & \dashrightarrow & \cdots & \dashrightarrow & \mathcal{F}_{r-1} & \dashrightarrow & \mathcal{F}_r \end{array} \tag{9.1.5}$$

where each $\mathcal{F}_i \in \mathrm{Bun}_{U(n)}(R)$, and each $\mathcal{F}^\flat_{1/2}$ is a rank n vector bundle on X'_R, satisfying some conditions (for example, the maps are injective). We shall abbreviate such diagrams as $(\mathcal{F}_\star) \in \mathrm{Hk}^r_{U(n)}(R)$. There are maps

$$h_i \colon \mathrm{Hk}^r_{U(n)} \to \mathrm{Bun}_{U(n)}, \quad i = 0, \ldots, r \tag{9.1.6}$$

projecting to the datum of $\mathcal{F}_i$, as well as $\mathrm{pr} \colon \mathrm{Hk}^r_{U(n)} \to (X')^r$ projecting to the "legs" (r-tuple of points on X' at which the dashed maps have poles).

We define the open substack $\mathrm{Hk}^r_S \subset \mathrm{Hk}^r_{\mathscr{G}}$ as

$$\mathrm{Hk}^r_S := h_0^{-1}(S) \cap h_1^{-1}(S) \cap \ldots \cap h_r^{-1}(S).$$

Therefore the maps (9.1.6) restrict to give

$$h_i \colon \mathrm{Hk}^r_S \to S, \quad i = 0, \ldots, r.$$

We will at some points find it convenient to distinguish the different copies of S, so we will also sometimes use S_i to denote a copy of S and write $h_i \colon \mathrm{Hk}^r_S \to S_i$.

9.1.3. *Hecke stacks for U, V and W.* Given a diagram $(\mathcal{F}_\star) \in \mathrm{Hk}^r_S(R)$, define $\mathcal{F}^\flat_\bullet$ to be the perfect complex on X'_R in degrees 0 and 1

$$(\mathcal{F}^\flat_{1/2} \oplus \ldots \oplus \mathcal{F}^\flat_{r-1/2}) \to (\mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_{r-1}),$$

where the map sends $(s_{1/2}, \cdots, s_{r-1/2})$ to $(s_{1/2} - s_{3/2}, \cdots, s_{r-3/2} - s_{r-1/2})$ (using the solid arrows in (9.1.5) to identify $\mathcal{F}^\flat_{i-1/2}$ as subsheaves of $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$). Note that $\mathcal{F}^\flat_\bullet$ may have non-trivial cohomology sheaf in both degrees 0 and 1.

Define $\mathcal{F}^{\flat *}_\bullet$ to be the $\mathcal{O}_{X'_R}$-linear dual of $\mathcal{F}^\flat_\bullet$, i.e., the cone of the dual morphism (in degrees -1 and 0)

$$(\mathcal{F}^*_1 \oplus \ldots \oplus \mathcal{F}^*_{r-1}) \to ((\mathcal{F}^\flat_{1/2})^* \oplus \ldots \oplus (\mathcal{F}^\flat_{r-1/2})^*).$$

Note that $\mathcal{F}^{\flat *}_\bullet$ is a coherent *sheaf* on X'_R concentrated in degree 0 which may not be locally free.

We have a natural map of perfect complexes on X'_R

$$\mathfrak{p}_i : \mathcal{F}^\flat_\bullet \to \mathcal{F}_i, \quad i = 0, 1, \cdots, r$$

that is the composition of the projection to $\mathcal{F}^\flat_{i-1/2}$ and the natural inclusion $\mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathcal{F}_i$ when $i > 0$, and the composition of the projection to $\mathcal{F}^\flat_{i+1/2}$ and the natural inclusion $\mathcal{F}^\flat_{i+1/2} \hookrightarrow \mathcal{F}_i$ when $i < r$. Both constructions give the same map up to explicit chain homotopy when $0 < i < r$. Dualizing $\mathfrak{p}^*_i$ we get a map of coherent sheaves (in degree 0) on X'_R

$$\mathfrak{p}^*_i : \mathcal{F}^*_i \to \mathcal{F}^{\flat *}_\bullet. \tag{9.1.7}$$

As $\mathcal{F}_\star$ varies in Hk^r_S, the construction $\mathcal{F}_\star \mapsto \mathcal{F}^{\flat *}_\bullet$ gives a coherent sheaf $\mathcal{F}^{\flat *}_{\mathrm{univ},\bullet}$ over $X' \times \mathrm{Hk}^r_S$.

We now construct various spaces over Hk^r_S.

- Define $\mathcal{U}^\flat_{\mathrm{Hk}} := \underline{\mathrm{RHom}(\mathcal{F}^{\flat *}_{\mathrm{univ},\bullet}, \mathcal{E}^*_1)}$ to be the perfect complex on Hk^r_S whose pullback to an R-point $(\mathcal{F}_\star) \in \mathrm{Hk}^r_S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^{\flat *}_\bullet, \mathcal{E}^*_1 \otimes R)$ regarded as an animated R-module. Let $\mathrm{Hk}^\flat_U := \mathrm{Tot}_{\mathrm{Hk}^r_S}(\mathcal{U}^\flat_{\mathrm{Hk}})$ be the associated derived vector bundle over Hk^r_S.
- Define $\mathcal{V}^\flat_{\mathrm{Hk}} := \underline{\mathrm{RHom}(\mathcal{F}^{\flat *}_{\mathrm{univ},\bullet}, \widetilde{Q}_1)}$ to be the perfect complex on S whose pullback to an R-point $(\mathcal{F}_\star) \in \mathrm{Hk}^r_S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^{\flat *}_\bullet, \widetilde{Q}_1 \otimes R)$ regarded as an animated R-module. Let $\mathrm{Hk}^\flat_V = \mathrm{Tot}_{\mathrm{Hk}^r_S}(\mathcal{V}^\flat_{\mathrm{Hk}})$ be the associated derived vector bundle over Hk^r_S.
- Define $\mathcal{W}^\flat_{\mathrm{Hk}} := \underline{\mathrm{RHom}(\mathcal{F}^{\flat *}_{\mathrm{univ},\bullet}, \sigma^*\mathcal{E}_2[1])}$ to be the perfect complex on Hk^r_S whose pullback to an R-point $(\mathcal{F}_\star) \in \mathrm{Hk}^r_S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^{\flat *}_\bullet, \sigma^*\mathcal{E}_2[1] \otimes R)$ regarded as an animated R-module. Let $\mathrm{Hk}^\flat_W := \mathrm{Tot}_{\mathrm{Hk}^r_S}(\mathcal{W}^\flat_{\mathrm{Hk}})$ be the associated derived vector bundle over Hk^r_S.

From (9.1.2), we get an exact triangle of perfect complexes on Hk_S^r,

$$\mathcal{U}_{\mathrm{Hk}}^\flat \to \mathcal{V}_{\mathrm{Hk}}^\flat \to \mathcal{W}_{\mathrm{Hk}}^\flat. \tag{9.1.8}$$

At the level of total spaces, this induces maps $\mathrm{Hk}_U^\flat \xrightarrow{f} \mathrm{Hk}_V^\flat \xrightarrow{g} \mathrm{Hk}_W^\flat$ fitting into a derived Cartesian square of derived vector bundles over Hk_S^r:

$$\begin{array}{ccc} \mathrm{Hk}_U^\flat & \xrightarrow{f} & \mathrm{Hk}_V^\flat \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle g} \\ \mathrm{Hk}_S^r & \xrightarrow{z} & \mathrm{Hk}_W^\flat \end{array} \tag{9.1.9}$$

Remark 9.1.2. By [FYZ21b, §5.7], $\mathrm{Hk}_U^\flat$ is isomorphic to the derived pullback of the derived Hecke stack from [FYZ21b, §5], "$\mathrm{Hk}^r_{\mathscr{M}_{\mathrm{GL}(m)',U(n)}} \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Hk}^r_{U(n)}$" along the map $\{\mathcal{E}_1\} \times \mathrm{Hk}_S^r \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Hk}^r_{U(n)}$.

9.1.4. *Geometric properties.* For each $i = 0, 1, \ldots, r$, let

- $\widetilde{U}_i \to \mathrm{Hk}_S^r$ be the pullback of $U \to S$ along $h_i \colon \mathrm{Hk}_S^r \to S$.
- $\widetilde{V}_i \to \mathrm{Hk}_S^r$ be the pullback of $V \to S$ along $h_i \colon \mathrm{Hk}_S^r \to S$.
- $\widetilde{W}_i \to \mathrm{Hk}_S^r$ be the pullback of $W \to S$ along $h_i \colon \mathrm{Hk}_S^r \to S$.

Let

$$h_i^U : \widetilde{U}_i \to U, \quad h_i^V : \widetilde{V}_i \to V, \quad h_i^W : \widetilde{W}_i \to W, \tag{9.1.10}$$

The maps (9.1.7) induce natural maps of perfect complexes on Hk_S^r

$$\mathcal{U}_{\mathrm{Hk}}^\flat \to h_i^*\mathcal{U}, \quad \mathcal{V}_{\mathrm{Hk}}^\flat \to h_i^*\mathcal{V}, \quad \mathcal{W}_{\mathrm{Hk}}^\flat \to h_i^*\mathcal{W}.$$

At the level of total spaces, these induce maps of derived vector bundles over Hk_S^r for $i = 0, \ldots, r$:

$$\widetilde{a}_i : \mathrm{Hk}_U^\flat \to \widetilde{U}_i, \quad \widetilde{b}_i : \mathrm{Hk}_V^\flat \to \widetilde{V}_i, \quad \widetilde{c}_i : \mathrm{Hk}_W^\flat \to \widetilde{W}_i. \tag{9.1.11}$$

Composing these maps with $h_i^?$, we get maps for $i = 0, \cdots, r$

$$a_i : \mathrm{Hk}_U^\flat \to U_i, \quad b_i : \mathrm{Hk}_V^\flat \to V_i, \quad c_i : \mathrm{Hk}_W^\flat \to W_i. \tag{9.1.12}$$

Lemma 9.1.3. *We have the following properties of the morphisms in* (9.1.11).

(1) Each morphism $\widetilde{a}_i$ is a quasi-smooth closed embedding.
(2) Each morphism $\widetilde{b}_i$ is quasi-smooth and separated (in particular, representable in derived schemes).
(3) Each morphism $\widetilde{c}_i$ is a smooth vector bundle.

Proof. In all cases, the quasi-smoothness follows from the fact that the maps are induced by the dual of (9.1.7). Indeed, for $(\mathcal{F}_\star) \in \mathrm{Hk}_S^r(R)$, write

$$T_i := \mathrm{coker}(\mathcal{F}_i^* \to \mathcal{F}_\bullet^{\flat *}),$$

a torsion sheaf on X'_R (in degree 0). Then the relative tangent complex of $\widetilde{a}_i$ at any R-point of $\mathrm{Hk}_U^\flat$ over $(\mathcal{F}_\star)$ is $\mathrm{RHom}(T_i, \mathcal{E}_1^* \otimes R)$. Let $T_{i,\mathrm{univ}}$ be the universal version of T_i over $X' \times \mathrm{Hk}_S^r$, and form the perfect complex $\underline{\mathrm{RHom}(T_{i,\mathrm{univ}}, \mathcal{E}_1^*)}$ on Hk_S^r. Then the relative tangent complex of $\widetilde{a}_i$ is the pullback from Hk_S^r of $\underline{\mathrm{RHom}(T_{i,\mathrm{univ}}, \mathcal{E}_1^*)}$, which is represented by a locally free coherent sheaf in degree 1. This implies that $\widetilde{a}_i$ is a closed embedding.

The analysis of $\widetilde{b}_i$ is similar, except that its tangent complex is the pullback of $\underline{\mathrm{RHom}(T_{i,\mathrm{univ}}, \widetilde{Q}_1)}$, which is locally represented by a complex of locally free coherent sheaves in degrees $[0, 1]$.

The analysis of $\widetilde{c}_i$ is similar, except that its tangent complex is the pullback of $\underline{\mathrm{RHom}(T_{i,\mathrm{univ}}, \sigma^*\mathcal{E}_2[1])}$, which is locally represented by a locally free coherent sheaf in degree 0, which implies that $\widetilde{c}_i$ is a smooth vector bundle. □

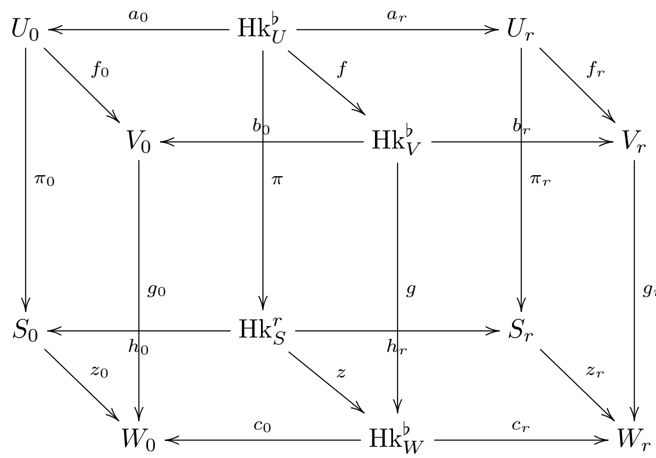
By the argument of [FYZ21a, Lemma 6.9], each map $h_i \colon \mathrm{Hk}_S^r \to S$ is smooth. Since the maps $h_i^?$ in (9.1.10) are all base changed from h_i, we get the following Corollary.

Corollary 9.1.4. *We have the following properties of the morphisms in* (9.1.12).

(1) Each morphism a_i is quasi-smooth.
(2) Each morphism b_i is quasi-smooth and representable in derived schemes.

(3) *Each morphism c_i is smooth.*

Corollary 9.1.5. *The diagram*



satisfies the conditions in §5.1.1. Here, z_i and z are the inclusions of zero sections, and π_i and π are the natural projections.

Proof. We first check all maps in the above diagram are representable in derived schemes (this is abbreviated "schematic" below) and separated.

- The horizontal maps. The maps h_i are separated and schematic. By Lemma 9.1.3, the maps $\widetilde{a}_i, \widetilde{b}_i$ and $\widetilde{c}_i$ are separated and schematic. Therefore the same is true for $a_i = h_i^U \circ \widetilde{a}_i, b_i = h_i^V \circ \widetilde{b}_i$ and $c_i = h_i^W \circ \widetilde{c}_i$.
- The maps f_i, f and z_i, z. Since W_i is a classical (i.e., isomorphic to its classical truncation) vector bundle stack over S_i, z_i is separated and schematic. The same is true for z. By the Cartesian diagrams (9.1.4) and (9.1.9), we see that f_i and f are also separated and schematic.
- The maps π_i, π and g_i, g. The map g_i is given by the linear map $\mathcal{V} \to \mathcal{W}$ of perfect complexes on S whose cone is $\mathcal{U}[1]$. To show g_i and π_i are separated and schematic, it suffices to observe that $\mathcal{U}$ is concentrated in degrees ≥ 0. Similarly, the fact that $\mathcal{U}^\flat_{\mathrm{Hk}}$ is concentrated in degrees ≥ 0 implies that g and π are schematic and separated.

The vertical squares (U_0, V_0, S_0, W_0), $(\mathrm{Hk}^\flat_U, \mathrm{Hk}^\flat_V, \mathrm{Hk}^r_S, \mathrm{Hk}^\flat_W)$ and (U_r, V_r, S_r, W_r) are derived Cartesian by (9.1.4) and (9.1.9). It remains to check the pushability and pullability of various squares, which will all be seen to be cases of Example 3.1.2 or Example 3.1.3.

(1) The square $(\mathrm{Hk}^\flat_U, U_0, \mathrm{Hk}^\flat_V, V_0)$ is pushable. For this it suffices to base change all relevant spaces to Hk^r_S and show instead that

$$\begin{CD} \widetilde{U}_0 @<{\widetilde{a}_0}<< \mathrm{Hk}^\flat_U \\ @V{\widetilde{f}_0}VV @VV{f}V \\ \widetilde{V}_0 @<{\widetilde{b}_0}<< \mathrm{Hk}^\flat_V \end{CD}$$

is pushable. This follows from the fact that $\widetilde{a}_0$ is proper (Lemma 9.1.3(1)) and $\widetilde{b}_0$ is separated.

(2) The square $(\mathrm{Hk}^r_S, S_0, \mathrm{Hk}^\flat_W, W_0)$ is pushable. After base change to Hk^r_S, it suffices to show that

$$\begin{CD} \mathrm{Hk}^r_S @= \mathrm{Hk}^r_S \\ @V{\widetilde{z}_0}VV @VV{z}V \\ \widetilde{W}_0 @<{\widetilde{c}_0}<< \mathrm{Hk}^\flat_W \end{CD}$$

is pushable. This follows from the fact that $\mathrm{Id}_{\mathrm{Hk}^r_S}$ is proper and $\widetilde{c}_0$ is separated.

(3) The square $(\mathrm{Hk}_U^\flat, U_r, \mathrm{Hk}_S^r, S_r)$ is pullable. It suffices to check that

$$\begin{array}{ccc} \mathrm{Hk}_U^\flat & \xrightarrow{\widetilde{a}_r} & \widetilde{U}_r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \widetilde{\pi}_r} \\ \mathrm{Hk}_S^r & = & \mathrm{Hk}_S^r \end{array}$$

is pullable. This follows from the fact that $\widetilde{a}_r$ is quasi-smooth and (Lemma 9.1.3(1)) and Id smooth.

(4) The square $(\mathrm{Hk}_V^\flat, V_r, \mathrm{Hk}_W^r, W_r)$ is pullable. It suffices to check that

$$\begin{array}{ccc} \mathrm{Hk}_V^\flat & \xrightarrow{\widetilde{b}_r} & \widetilde{V}_r \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \widetilde{g}_r} \\ \mathrm{Hk}_W^\flat & \xrightarrow{\widetilde{c}_r} & \widetilde{W}_r \end{array}$$

is pullable. This follows from the fact that $\widetilde{c}_r$ is smooth (Lemma 9.1.3(3)) and $\widetilde{b}_r$ is quasi-smooth (Lemma 9.1.3(2)).

□

Lemma 9.1.6. *For $i = 0, \dots, r$, we have $d(\widetilde{a}_i) + d(\widetilde{c}_i) = d(\widetilde{b}_i)$.*

Proof. We have

$$d(\widetilde{a}_i) = \mathrm{rank}(\mathcal{U}_{\mathrm{Hk}}^\flat) - \mathrm{rank}(\mathcal{U})$$
$$d(\widetilde{b}_i) = \mathrm{rank}(\mathcal{V}_{\mathrm{Hk}}^\flat) - \mathrm{rank}(\mathcal{V})$$
$$d(\widetilde{c}_i) = \mathrm{rank}(\mathcal{W}_{\mathrm{Hk}}^\flat) - \mathrm{rank}(\mathcal{W}).$$

By (9.1.3) and (9.1.8), we get the result.

□

9.2. **The stacks $U^\perp, \widehat{V}, W^\perp$ and their Hecke correspondences.** Dualizing the sequence (9.1.1) and applying σ^*, we get a short exact sequence

$$0 \to \sigma^*\mathcal{E}_1 \to \mathcal{E}_2^* \to \widetilde{Q}_2 \to 0 \tag{9.2.1}$$

where $\widetilde{Q}_2 = \sigma^*\widetilde{Q}_1^* := \sigma^*\mathcal{E}xt^1(\widetilde{Q}_1, \mathcal{O}_{X'})$.

9.2.1. *Definition of $U^\perp, \widehat{V}$ and $W^\perp$.* We apply the same constructions in §9.1 to get derived vector bundles $U^\perp, V'$ and $W^\perp$ over $S = \mathrm{Bun}_{U(n)}^{\le \mu}$ that fit into a derived Cartesian square

$$\begin{array}{ccc} U^\perp & \longrightarrow & V' \\ \downarrow & & \downarrow \\ S & \xrightarrow{z_{W^\perp}} & W^\perp \end{array} \tag{9.2.2}$$

To spell out the details, for any animated $\mathbf{F}_q$-algebra R,

- Define $\mathcal{U}^\perp := \underline{\mathrm{RHom}(\mathcal{F}_{\mathrm{univ}}^*, \mathcal{E}_2^*)}$ to be the perfect complex on S whose pullback to an R-point $\mathcal{F} \in S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \mathcal{E}_2^* \otimes R)$ regarded as an animated R-module. Its associated vector bundle is denoted $U^\perp \to S$.
- Define $\mathcal{V}' := \underline{\mathrm{RHom}(\mathcal{F}_{\mathrm{univ}}^*, \widetilde{Q}_2)}$ to be the perfect complex on S whose pullback to an R-point $\mathcal{F} \in S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \widetilde{Q}_2 \otimes R)$ regarded as an animated R-module. Since $\mathrm{Ext}^1_{X'_R}(\mathcal{F}^*, \widetilde{Q}_2 \otimes R) = 0$ as $\widetilde{Q}_2$ is torsion, $\mathcal{V}'$ is in fact a locally free coherent sheaf (concentrated in degree 0) on S. Its associated vector bundle is denoted $V' \to S$.
- Define $\mathcal{W}^\perp := \underline{\mathrm{RHom}(\mathcal{F}_{\mathrm{univ}}^*, \sigma^*\mathcal{E}_1[1])}$ to be the perfect complex on S whose pullback to an R-point $\mathcal{F} \in S(R)$ is naturally in R isomorphic to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \sigma^*\mathcal{E}_1[1] \otimes R)$ regarded as an animated R-module. Its associated vector bundle is denoted $W^\perp \to S$.

Lemma 9.2.1. *As vector bundles over S, Serre duality identifies V' with the dual vector bundle $\widehat{V}$ of V. Under this identification, $U^\perp$ is identified with $\widehat{W}$, and $W^\perp$ is identified with $\widehat{U}$, and the derived fiber sequence* (9.2.2) *is identified with the dual fiber square to* (9.1.4)

$$\begin{array}{ccc} \widehat{W} & \longrightarrow & \widehat{V} \\ \downarrow & & \downarrow \\ S & \xrightarrow{z_{\widehat{U}}} & \widehat{U} \end{array}$$

Proof. First we produce the identification $V' \cong \widehat{V}$. For any $(\mathcal{F}_\star) \in \mathrm{Hk}_S^r(R)$, we have $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \widetilde{Q}_1 \otimes R) \cong \mathrm{RHom}_{X'_R}(\widetilde{Q}_1^*[-1] \otimes R, \mathcal{F})$ as perfect complexes over R. By relative Serre duality, the latter is R-dual to $\mathrm{RHom}_{X'_R}(\mathcal{F}, \widetilde{Q}_1^* \otimes_{\mathcal{O}_{X'}} \omega_{X'} \otimes R)$. Using the Hermitian form $h_{\mathcal{F}} \colon \mathcal{F} \xrightarrow{\sim} \sigma^* \mathcal{F}^\vee$, we have

$$\begin{aligned} \mathrm{RHom}_{X'_R}(\mathcal{F}, \widetilde{Q}_1^* \otimes_{\mathcal{O}_{X'}} \omega_{X'} \otimes R) &\cong \mathrm{RHom}_{X'_R}(\sigma^*\mathcal{F}^\vee, \widetilde{Q}_1^* \otimes_{\mathcal{O}_{X'}} \omega_{X'} \otimes R) \\ &\cong \mathrm{RHom}_{X'_R}(\mathcal{F}^*, \sigma^* \widetilde{Q}_1^* \otimes R) = \mathrm{RHom}_{X'_R}(\mathcal{F}^*, \widetilde{Q}_2 \otimes R). \end{aligned}$$

(Note that all RHom appearing above are in fact concentrated in degree 0, hence identified with their respective Hom's.) This shows that $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \widetilde{Q}_1 \otimes R)$ is R-dual to $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \widetilde{Q}_2 \otimes R)$.

Under the above identifications, the exact triangle

$$\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}_1^*) \to \mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{Q}_1) \to \mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\mathcal{E}_2[1])$$

is dual to the exact triangle

$$\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}_2^*) \to \mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{Q}_2) \to \mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\mathcal{E}_1[1]).$$

On total spaces, this says that $U \to V \to W$ is dual to $U^\perp \to V' \to W^\perp$. □

From now on, we will identify V' with $\widehat{V}$ using Lemma 9.2.1.

9.2.2. *More Hecke stacks.* We let $\mathrm{Hk}^\flat_{U^\perp} \to \mathrm{Hk}^r_S$, and $\mathrm{Hk}^\flat_{W^\perp} \to \mathrm{Hk}^r_S$, be the derived vector bundles on Hk^r_S defined similarly to $\mathrm{Hk}^\flat_U \to \mathrm{Hk}^r_S$, and $\mathrm{Hk}^\flat_W \to \mathrm{Hk}^r_S$, respectively, but interchanging $\mathcal{E}_1$ and $\mathcal{E}_2$. We let $\mathrm{Hk}^\flat_{\widehat{V}} \to \mathrm{Hk}^r_S$ be the vector bundle defined similarly to $\mathrm{Hk}^\flat_V \to \mathrm{Hk}^r_S$, but replacing $\widetilde{Q}_1$ with $\widetilde{Q}_2$.

Given a diagram $(\mathcal{F}_\star) \in \mathrm{Hk}^r_S(R)$, define $\mathcal{F}^\sharp_\bullet$ to be the cokernel of the injective map of coherent sheaves on X'_R

$$\mathcal{F}^\sharp_\bullet := \mathrm{coker}\left(\mathcal{F}^\flat_{1/2} \oplus \ldots \oplus \mathcal{F}^\flat_{r-1/2} \to \mathcal{F}_0 \oplus \ldots \oplus \mathcal{F}_r\right),$$

where the map sends $(s_{1/2}, \cdots, s_{r-1/2})$ to $(-s_{1/2}, s_{1/2} - s_{3/2}, \cdots, s_{r-3/2} - s_{r-1/2}, s_{r-1/2})$ (using the solid arrows in (9.1.5) to identify $\mathcal{F}^\flat_{i-1/2}$ as subsheaves of $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$). Hence $\mathcal{F}^\sharp_\bullet$ is a coherent sheaf on X'_R concentrated in degree 0. Let $\mathcal{F}^{\sharp*}_\bullet$ be the $\mathcal{O}_{X'_R}$-linear dual of $\mathcal{F}^\sharp_\bullet$, i.e., $\mathcal{F}^{\sharp*}_\bullet$ is the perfect complex on X'_R in degrees 0 and 1,

$$(\mathcal{F}_0^* \oplus \ldots \oplus \mathcal{F}_r^*) \to ((\mathcal{F}^\flat_{1/2})^* \oplus \ldots \oplus (\mathcal{F}^\flat_{r-1/2})^*).$$

Note that the cohomology sheaves of $\mathcal{F}^{\sharp*}_\bullet$ may be nontrivial in both degrees 0 and 1.

Comparing with the definition of $\mathcal{F}^\flat_\bullet$ given in §9.1.3, we have an exact triangle in $\mathrm{Perf}(\mathrm{Hk}^r_S)$:

$$\mathcal{F}^\flat_\bullet \to (\mathcal{F}_0 \oplus \mathcal{F}_r) \to \mathcal{F}^\sharp_\bullet \to \mathcal{F}^\flat_\bullet[1] \tag{9.2.3}$$

We now construct the $\sharp$-version of the the Hecke stacks over Hk^r_S.

- Define a perfect complexes $\mathcal{U}^\sharp_{\mathrm{Hk}}, \mathcal{V}^\sharp_{\mathrm{Hk}}$ and $\mathcal{W}^\sharp_{\mathrm{Hk}}$ on Hk^r_S similarly to $\mathcal{U}^\flat_{\mathrm{Hk}}, \mathcal{V}^\flat_{\mathrm{Hk}}$ and $\mathcal{W}^\flat_{\mathrm{Hk}}$ respectively, replacing $\mathcal{F}^{\flat*}_\bullet$ with $\mathcal{F}^{\sharp*}_\bullet$. Let $\mathrm{Hk}^\sharp_U := \mathrm{Tot}_{\mathrm{Hk}^r_S}(\mathcal{U}^\sharp_{\mathrm{Hk}}), \mathrm{Hk}^\sharp_V := \mathrm{Tot}_{\mathrm{Hk}^r_S}(\mathcal{V}^\sharp_{\mathrm{Hk}})$ and $\mathrm{Hk}^\sharp_W = \mathrm{Tot}_{\mathrm{Hk}^r_S}(\mathcal{W}^\sharp_{\mathrm{Hk}})$ be the associated derived vector bundles over Hk^r_S.
- Switching the roles of $\mathcal{E}_1$ and $\mathcal{E}_2$ and replacing $\widetilde{Q}_1$ by $\widetilde{Q}_2$, we define analogously the derived vector bundles $\mathrm{Hk}^\sharp_{U^\perp}, \mathrm{Hk}^\sharp_{\widehat{V}}$ and $\mathrm{Hk}^\sharp_{W^\perp}$ over Hk^r_S.

9.2.3. The exact triangle (9.2.3) induces three exact triangles in $\mathrm{Perf}(\mathrm{Hk}^r_S)$:

$$\begin{aligned} \mathcal{U}^\flat_{\mathrm{Hk}} \to h_0^*\mathcal{U} \oplus h_r^*\mathcal{U} \to \mathcal{U}^\sharp_{\mathrm{Hk}} \to \mathcal{U}^\flat_{\mathrm{Hk}}[1] \\ \mathcal{V}^\flat_{\mathrm{Hk}} \to h_0^*\mathcal{V} \oplus h_r^*\mathcal{V} \to \mathcal{V}^\sharp_{\mathrm{Hk}} \to \mathcal{V}^\flat_{\mathrm{Hk}}[1] \\ \mathcal{W}^\flat_{\mathrm{Hk}} \to h_0^*\mathcal{W} \oplus h_r^*\mathcal{W} \to \mathcal{W}^\sharp_{\mathrm{Hk}} \to \mathcal{W}^\flat_{\mathrm{Hk}}[1], \end{aligned}$$

which induce three (derived) Cartesian squares of derived vector bundles over Hk^r_S:

$$\begin{array}{ccccc} & & \mathrm{Hk}^\flat_U & & \\ & \swarrow^{\widetilde{a}_0} & & \searrow^{\widetilde{a}_r} & \\ \widetilde{U}_0 & & & & \widetilde{U}_r \\ & \searrow_{\widetilde{a}'_r} & & \swarrow_{\widetilde{a}'_0} & \\ & & \mathrm{Hk}^\sharp_U & & \end{array} \qquad \begin{array}{ccccc} & & \mathrm{Hk}^\flat_V & & \\ & \swarrow^{\widetilde{b}_0} & & \searrow^{\widetilde{b}_r} & \\ \widetilde{V}_0 & & & & \widetilde{V}_r \\ & \searrow_{\widetilde{b}'_r} & & \swarrow_{\widetilde{b}'_0} & \\ & & \mathrm{Hk}^\sharp_V & & \end{array} \qquad \begin{array}{ccccc} & & \mathrm{Hk}^\flat_W & & \\ & \swarrow^{\widetilde{c}_0} & & \searrow^{\widetilde{c}_r} & \\ \widetilde{W}_0 & & & & \widetilde{W}_r \\ & \searrow_{\widetilde{c}'_r} & & \swarrow_{\widetilde{c}'_0} & \\ & & \mathrm{Hk}^\sharp_W & & \end{array} \tag{9.2.4}$$

Analogously, switching the roles of $\mathcal{E}_1$ and $\mathcal{E}_2$, and using the $\widetilde{Q}_2$ instead of $\widetilde{Q}_1$, the exact triangle (9.2.3) induces three (derived) Cartesian squares of derived vector bundles over Hk^r_S:

$$\begin{array}{ccccc} & & \mathrm{Hk}^\flat_{U^\perp} & & \\ & \swarrow^{\widetilde{a}_0^\perp} & & \searrow^{\widetilde{a}_r^\perp} & \\ \widetilde{U}_0^\perp & & & & \widetilde{U}_r^\perp \\ & \searrow_{(\widetilde{a}'_r)^\perp} & & \swarrow_{(\widetilde{a}'_0)^\perp} & \\ & & \mathrm{Hk}^\sharp_{U^\perp} & & \end{array} \qquad \begin{array}{ccccc} & & \mathrm{Hk}^\flat_{\widehat{V}} & & \\ & \swarrow^{\widetilde{\beta}_0} & & \searrow^{\widetilde{\beta}_r} & \\ \widetilde{\widehat{V}}_0 & & & & \widetilde{\widehat{V}}_r \\ & \searrow_{\widetilde{\beta}'_r} & & \swarrow_{\widetilde{\beta}'_0} & \\ & & \mathrm{Hk}^\sharp_{\widehat{V}} & & \end{array} \qquad \begin{array}{ccccc} & & \mathrm{Hk}^\flat_{W^\perp} & & \\ & \swarrow^{\widetilde{c}_0^\perp} & & \searrow^{\widetilde{c}_r^\perp} & \\ \widetilde{W}_0^\perp & & & & \widetilde{W}_r^\perp \\ & \searrow_{(\widetilde{c}'_r)^\perp} & & \swarrow_{(\widetilde{c}'_0)^\perp} & \\ & & \mathrm{Hk}^\sharp_{W^\perp} & & \end{array} \tag{9.2.5}$$

Lemma 9.2.2. *Let $\widehat{\mathrm{Hk}^\flat_U}, \widehat{\mathrm{Hk}^\flat_V}$ and $\widehat{\mathrm{Hk}^\flat_W}$ be the dual derived bundles to $\mathrm{Hk}^\flat_U, \mathrm{Hk}^\flat_V$ and $\mathrm{Hk}^\flat_W$ over Hk^r_S.*[10] *Then we have identifications*

(1) $\mathrm{Hk}^\sharp_{U^\perp} \cong \widehat{\mathrm{Hk}^\flat_W}$ *and* $\mathrm{Hk}^\flat_{U^\perp} \cong \widehat{\mathrm{Hk}^\sharp_W}$.
(2) $\mathrm{Hk}^\sharp_{\widehat{V}} \cong \widehat{\mathrm{Hk}^\flat_V}$ *and* $\mathrm{Hk}^\flat_{\widehat{V}} \cong \widehat{\mathrm{Hk}^\sharp_V}$.
(3) $\mathrm{Hk}^\sharp_{W^\perp} \cong \widehat{\mathrm{Hk}^\flat_U}$ *and* $\mathrm{Hk}^\flat_{W^\perp} \cong \widehat{\mathrm{Hk}^\sharp_U}$.

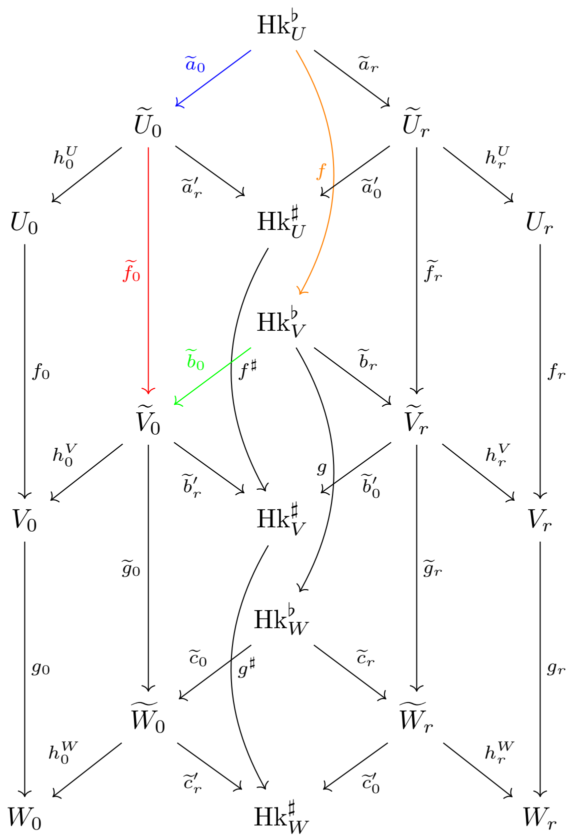
Moreover, under these identifications, the first (resp. second, resp. third) derived Cartesian square in (9.2.5) *is the dual to the third (resp. second, resp. first) derived Cartesian square in* (9.2.4). *In particular, we have*

$$\widetilde{\beta}_0 = \widehat{\widetilde{b}'_r}, \quad \widetilde{\beta}_r = \widehat{\widetilde{b}'_0}, \quad \widetilde{\beta}'_r = \widehat{\widetilde{b}_0}, \quad \widetilde{\beta}'_0 = \widehat{\widetilde{b}_r}.$$

Proof. Similar to Lemma 9.2.1. ☐

[10] Note $\widehat{\mathrm{Hk}^\flat_V}$ has a different meaning from $\mathrm{Hk}^\flat_{\widehat{V}}$.

9.2.4. *Summary.* Starting from (9.1.1), we defined a collection of spaces and maps as in the diagram below:

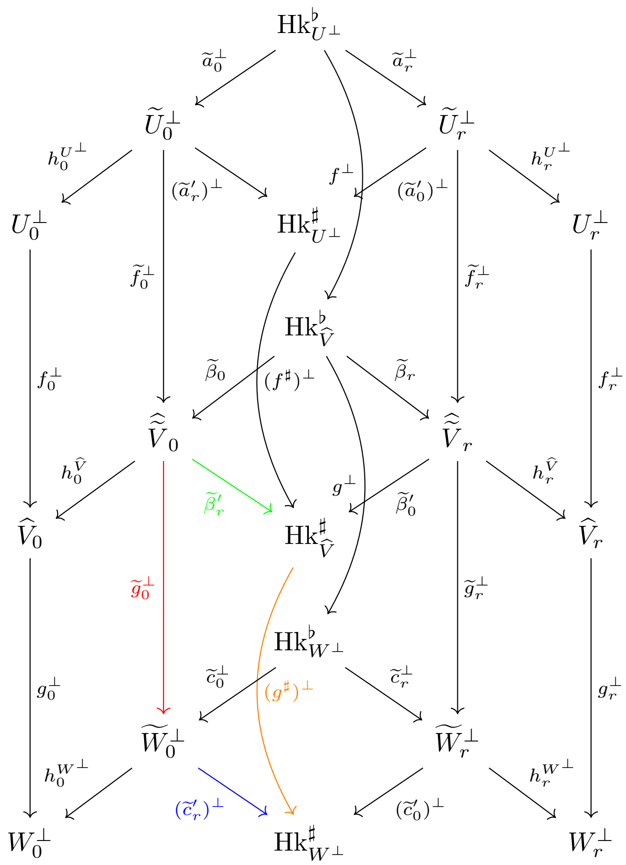


(9.2.6)

Here:

- The maps in the columns come from exact triangles of perfect complexes.
- The three diamonds in the middle are derived Cartesian.
- The four parallelograms on the left and right sides are derived Cartesian.

Starting from (9.2.1), we defined a collection of spaces and maps as in the diagram below:



(9.2.7)

Again:

- The maps in the columns come from exact triangles of perfect complexes.
- The three diamonds in the middle are derived Cartesian.
- The four parallelograms on the left and right sides are derived Cartesian.

Furthermore, (9.2.7) is the dual to the diagram (9.2.6). The duality exchanges U with $W^\perp$, V with $\widehat{V}$, and W with $U^\perp$, and exchanges $\flat$ and $\sharp$ superscripts. Sample examples of dual morphisms in (9.2.6) and (9.2.7) are colored with the same color.

9.3. **Global presentability.** We will want to apply the derived Fourier theory of §6 and §7 to the ensemble of spaces and maps in §9.2.4. In order to justify this we need to check that all the derived vector bundles are globally presented, and all the maps are globally presented. The reason for Harder-Narasimhan truncation (in this section) is to guarantee these properties.

The following observations are useful to perform this check.

(1) If $\mathcal{T}$ is a torsion coherent sheaf on X', then the perfect complex $\underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{T})$ on $\mathrm{Bun}^{\le\mu}_{U(n)}$ is a vector bundle.
(2) Given μ and any coherent sheaf $\mathcal{E}$ on X', for any divisor D on X' with $\deg D$ sufficiently large (depending on μ and $\mathcal{E}$), $\underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}(D))$ is a vector bundle.

Now, from the exact triangle in $\mathrm{Perf}(X')$,

$$\mathcal{E} \to \mathcal{E}(D) \to \mathcal{E}|_D(D)$$

we get an exact triangle of complexes in $\mathrm{Perf}(\mathrm{Bun}^{\le\mu}_{U(n)})$,

$$\underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}) \to \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}(D)) \to \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}|_D(D)). \tag{9.3.1}$$

By the observations above, the second and third terms in (9.3.1) are vector bundles, so this presents $\underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E})$ as a 2-term complex of vector bundles. Since the derived vector bundles $U_i, W_i, V_i, \widetilde{U}_i, \widetilde{W}_i, \widetilde{V}_i$ appearing in §9.1 are all instances of this construction, they are all globally presented.

Furthermore, if $\mathcal{E} \to \mathcal{E}'$ is a map of coherent sheaves on X', then for any divisor D on X' with $\deg D$ sufficiently large (depending on $\mu, \mathcal{E}, \mathcal{E}'$) the diagram

$$\begin{array}{ccccc} \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}) & \longrightarrow & \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}(D)) & \longrightarrow & \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}|_D(D)) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}') & \longrightarrow & \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}'(D)) & \longrightarrow & \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}'|_D(D)) \end{array}$$

gives a global presentation for $\underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}) \to \underline{\mathrm{RHom}}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}')$ as a map of complexes of vector bundles. A similar trick gives a global presentation for all the commutative quadrilaterals involving the U_i, V_i, W_i and S_i are globally presented.

Since the universal perfect complex $\mathcal{F}^\flat_\bullet$ used to construct $\mathcal{U}^\flat_{\mathrm{Hk}}, \mathcal{V}^\flat_{\mathrm{Hk}}, \mathcal{W}^\flat_{\mathrm{Hk}}$ is assembled out of the pullbacks of the $\mathcal{F}_{\mathrm{univ}}$ from the various projections to $\mathrm{Bun}^{\leq\mu}_{U(n)}$, all the maps between $\mathrm{Hk}^\flat_U, \mathrm{Hk}^\flat_V, \mathrm{Hk}^\flat_W$ and Hk^r_S are also globally presented (this implicitly includes the statement that the individual derived vector bundles are globally presented). The same applies to all the variants of these spaces and maps considered in §9.1.

We have now verified that the diagrams in (9.2.6) *and* (9.2.7) *are globally presented, so that we may apply the results of §7 to them.*

9.4. **Comparison of cohomological correspondences.** We refer to the diagram (9.2.6) and its Fourier dual, diagram (9.2.7).

By Corollary 9.1.4, the relative fundamental class of the quasi-smooth map a_r defines a cohomological correspondence

$$\mathfrak{c}_U = [a_r] \in \mathrm{Corr}_{\mathrm{Hk}^\flat_U}(\mathbf{Q}_{\ell,U_0}, \mathbf{Q}_{\ell,U_r}\langle -d(a_r)\rangle).$$

Similarly, the relative fundamental class of $a_r^\perp$ defines a cohomological correspondence

$$\mathfrak{c}_{U^\perp} = [a_r^\perp] \in \mathrm{Corr}_{\mathrm{Hk}^\flat_{U^\perp}}(\mathbf{Q}_{\ell,U_0^\perp}, \mathbf{Q}_{\ell,U_r^\perp}\langle -d(a_r^\perp)\rangle).$$

By Corollary 9.1.5, the pushforward of cohomological correspondences along the morphism of correspondences $f : \mathrm{Hk}^\flat_U \to \mathrm{Hk}^\flat_V$ is defined, giving

$$f_!(\mathfrak{c}_U) \in \mathrm{Corr}_{\mathrm{Hk}^\flat_V}(f_{0!}\mathbf{Q}_{\ell,U_0}, f_{r!}\mathbf{Q}_{\ell,U_r}\langle -d(a_r)\rangle).$$

Similarly,

$$f_!^\perp(\mathfrak{c}_{U^\perp}) \in \mathrm{Corr}_{\mathrm{Hk}^\flat_{\widehat{V}}}(f^\perp_{0!}\mathbf{Q}_{\ell,U_0^\perp}, f^\perp_{r!}\mathbf{Q}_{\ell,U_r^\perp}\langle -d(a_r^\perp)\rangle)$$

is defined.

Recall the notion of Fourier transform of cohomological correspondences from §7.1. We have

$$\mathrm{FT}_{\mathrm{Hk}^\flat_V}(f_!(\mathfrak{c}_U)) \in \mathrm{Corr}_{\mathrm{Hk}^\flat_{\widehat{V}}}(\mathrm{FT}_{V_0}(f_{0!}\mathbf{Q}_{\ell,U_0}), \mathrm{FT}_{V_r}(f_{r!}\mathbf{Q}_{\ell,U_r})[d(\widetilde{b}_0)+d(\widetilde{b}_r)](d(\widetilde{b}_0))\langle -d(a_r)\rangle). \tag{9.4.1}$$

Here we use that $\widehat{\mathrm{Hk}^\sharp_V} \cong \mathrm{Hk}^\flat_{\widehat{V}}$ (see Lemma 9.2.2). Note that $d(\widetilde{b}_0) = d(\widetilde{b}_r)$, therefore

$$[d(\widetilde{b}_0) + d(\widetilde{b}_r)](d(\widetilde{b}_0))\langle -d(a_r)\rangle = \langle d(\widetilde{b}_r) - d(a_r)\rangle.$$

Hence (9.4.1) simplifies to

$$\mathrm{FT}_{\mathrm{Hk}^\flat_V}(f_!(\mathfrak{c}_U)) \in \mathrm{Corr}_{\mathrm{Hk}^\flat_{\widehat{V}}}(\mathrm{FT}_{V_0}(f_{0!}\mathbf{Q}_{\ell,U_0}), \mathrm{FT}_{V_r}(f_{r!}\mathbf{Q}_{\ell,U_r})\langle d(\widetilde{b}_r) - d(a_r)\rangle). \tag{9.4.2}$$

Since $U^\perp$ is the orthogonal complement of U relative to V (in the derived sense), by §6.2.4 and Example 6.2.3, we have canonical isomorphisms for $i = 0, r$:

$$\mathrm{FT}_{V_i}(f_{i!}\mathbf{Q}_{\ell,U_i}) \cong f^\perp_{i!}\mathbf{Q}_{\ell,U_i^\perp}[\mathrm{rank}(\mathcal{V})]\langle -\mathrm{rank}(\mathcal{U})\rangle.$$

Note the shift and twist on the right side is the same for $i = 0$ and $i = r$. Therefore $\mathrm{FT}_{\mathrm{Hk}^\flat_V}(f_!(\mathfrak{c}_U))$ can also be viewed as an element in

$$\mathrm{Corr}_{\mathrm{Hk}^\flat_{\widehat{V}}}(f^\perp_{0!}\mathbf{Q}_{\ell,U_0^\perp}, f^\perp_{r!}\mathbf{Q}_{\ell,U_r^\perp}\langle d(\widetilde{b}_r) - d(a_r)\rangle).$$

On the other hand, by Lemma 9.2.2, $\widetilde{c}_r$ is dual to $(\widetilde{a}'_0)^{\perp}$, which has the same relative dimension as $\widetilde{a}_0^{\perp}$. Therefore, by Lemma 9.1.6, we have

$$d(\widetilde{b}_r) - d(\widetilde{a}_r) = d(\widetilde{c}_r) = -d(\widetilde{a}_0^{\perp}) = -d(\widetilde{a}_r^{\perp}).$$

Therefore

$$d(\widetilde{b}_r) - d(a_r) = d(\widetilde{b}_r) - d(\widetilde{a}_r) - d(h_r) = -d(\widetilde{a}_r^{\perp}) - d(h_r) = -d(a_r^{\perp}).$$

We can therefore view $\mathrm{FT}_{\mathrm{Hk}^{\flat}_{V}}(f_!(\mathfrak{c}_U))$ as an element in $\mathrm{Corr}_{\mathrm{Hk}^{\flat}_{\widehat{V}}}(f^{\perp}_{0!}\mathbf{Q}_{\ell,U_0^{\perp}}, f^{\perp}_{r!}\mathbf{Q}_{\ell,U_r^{\perp}}\langle -d(a_r^{\perp})\rangle)$.

The main result of this section is the following theorem.

Theorem 9.4.1. *Under the above notations, we have*

$$\mathbb{T}_{[d(f_0)+d(\pi_0)](d(\pi_0))}\,\mathrm{FT}_{\mathrm{Hk}^{\flat}_{V}}(f_!(\mathfrak{c}_U)) = f^{\perp}_!(\mathfrak{c}_{U^{\perp}})$$

as elements of $\mathrm{Corr}_{\mathrm{Hk}^{\flat}_{\widehat{V}}}(f^{\perp}_{0!}\mathbf{Q}_{\ell,U_0^{\perp}}, f^{\perp}_{r!}\mathbf{Q}_{\ell,U_r^{\perp}}\langle -d(a_r^{\perp})\rangle)$.

Proof. Let $\mathfrak{s} \in \mathrm{Corr}_{\mathrm{Hk}^r_S}(\mathbf{Q}_{\ell,S}, \mathbf{Q}_{\ell,S}\langle -d(h_r)\rangle)$ be given by the relative fundamental class of h_r

$$\mathfrak{s} = [h_r] : h_0^*\mathbf{Q}_{\ell,S} \to h_r^!\mathbf{Q}_{\ell,S}\langle -d(h_r)\rangle.$$

Recall the maps of correspondences

$$\pi : \mathrm{Hk}^{\flat}_U \to \mathrm{Hk}^r_S, \quad \pi^{\perp} : \mathrm{Hk}^{\flat}_{U^{\perp}} \to \mathrm{Hk}^r_S, \quad z^{\perp} : \mathrm{Hk}^r_S \to \mathrm{Hk}^{\flat}_{W^{\perp}}, \quad g^{\perp} : \mathrm{Hk}^{\flat}_{\widehat{V}} \to \mathrm{Hk}^{\flat}_{W^{\perp}}.$$

The theorem follows from a sequence of equalities of cohomological correspondences

$$\begin{aligned}\mathbb{T}_{[d(f_0)+d(\pi_0)](d(\pi_0))}\,\mathrm{FT}(f_!\,\mathfrak{c}_U) &\overset{(1)}{=} \mathbb{T}_{[d(f_0)+d(\pi_0)](d(\pi_0))}\,\mathrm{FT}(f_!\pi^*\mathfrak{s}) \\ &\overset{(2)}{=} (g^{\perp})^* z^{\perp}_!\,\mathrm{FT}(\mathfrak{s}) \overset{(3)}{=} (g^{\perp})^* z^{\perp}_!\mathfrak{s} \overset{(4)}{=} f^{\perp}_!(\pi^{\perp})^*\mathfrak{s} \overset{(5)}{=} f^{\perp}_!\,\mathfrak{c}_{U^{\perp}}.\end{aligned}$$

We explain the reason for each equality:

(1),(5) follow from the equalities

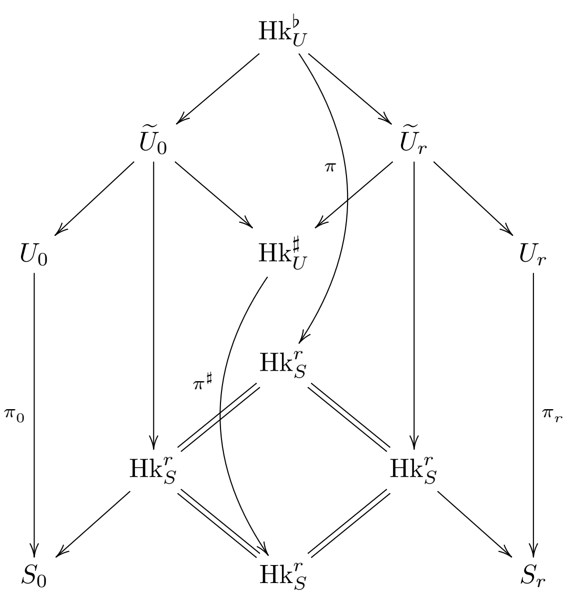
$$\pi^*\mathfrak{s} = \mathfrak{c}_U, \quad (\pi^{\perp})^*\mathfrak{s} = \mathfrak{c}_{U^{\perp}}$$

which will be proved in Lemma 9.4.2.

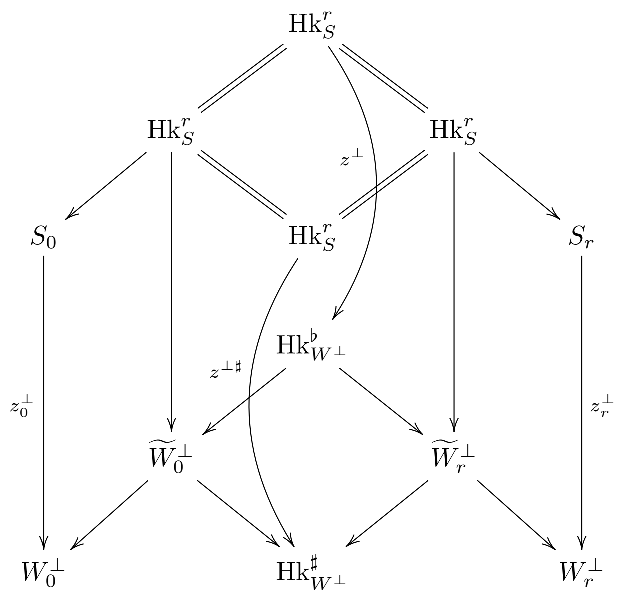
(2) involves two applications of Proposition 7.2.4, namely

$$\mathbb{T}_{[d(f_0)]}\,\mathrm{FT}\circ f_! = (g^{\perp})^*\circ\mathrm{FT}, \quad \mathbb{T}_{[d(\pi_0)](d(\pi_0))}\,\mathrm{FT}\circ\pi^* = z^{\perp}_!\circ\mathrm{FT}.$$

We used here that $g^{\perp\sharp} : \mathrm{Hk}^{\sharp}_{\widehat{V}} \to \mathrm{Hk}^{\sharp}_{W^{\perp}}$ is the dual of the map of co-correspondences $f : \mathrm{Hk}^{\flat}_U \to \mathrm{Hk}^{\flat}_V$, as summarized in the diagrams (9.2.6) and (9.2.7). Meanwhile, π is dual to $z^{\perp\sharp}$; in fact, the diagram



is dual to the diagram



(3) is the trivial equality $\mathfrak{s} = \mathrm{FT}_{\mathrm{Hk}_S^r}(\mathfrak{s})$.
(4) follows from Theorem 5.1.3 and Lemma 9.4.2. Note that we have verified in Corollary 9.1.5 that the hypotheses of Theorem 5.1.3 hold in this situation.

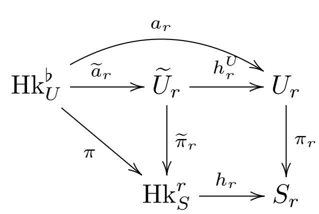
□

Lemma 9.4.2. *We have* $\pi^*\mathfrak{s} = \mathfrak{c}_U \in \mathrm{Corr}_{\mathrm{Hk}_U^\flat}(\mathbf{Q}_{\ell,U_0}, \mathbf{Q}_{\ell,U_r}\langle -d(a_r)\rangle)$.

Proof. Unravelling the definition, we need to show that the composition

$$\mathbf{Q}_{\ell,\mathrm{Hk}_U^\flat} \xrightarrow{\pi^*[h_r]} \pi^* h_r^! \mathbf{Q}_{\ell,S_r}\langle -d(h_r)\rangle \xrightarrow{\triangle} a_r^! \pi_r^* \mathbf{Q}_{\ell,S_r}\langle -d(a_r)\rangle$$

is equal to the relative fundamental class $[a_r]$. Here $\triangle$ is the pull-pull base change map attached to the pullable (outer) square



(9.4.3)

By construction, $\triangle$ is the composition of two steps

$$\pi^* h_r^! \mathbf{Q}_{\ell,S_r}\langle -d(h_r)\rangle = \widetilde{a}_r^* \widetilde{\pi}_r^* h_r^! \mathbf{Q}_{\ell,S_r}\langle -d(h_r)\rangle \xrightarrow{\diamond} \widetilde{a}_r^* (h_r^U)^! \pi_r^* \mathbf{Q}_{\ell,S_r}\langle -d(h_r)\rangle$$
$$\xrightarrow{[\widetilde{a}_r]} \widetilde{a}_r^! (h_r^U)^! \pi_r^* \mathbf{Q}_{\ell,S_r}\langle -d(\widetilde{a}_r) - d(h_r)\rangle = a_r^! \pi_r^* \mathbf{Q}_{\ell,S_r}\langle -d(a_r)\rangle.$$

As explained in §3.4, on the level of global sections the map $\diamond$ (adjoint to proper base change for the derived Cartesian square in (9.4.3)) sends the relative fundamental class of h_r to the relative fundamental class of h_r^U. Post-composing with $[\widetilde{a}_r]$, we get $[a_r]$. □

10. Proof of the generic modularity theorem

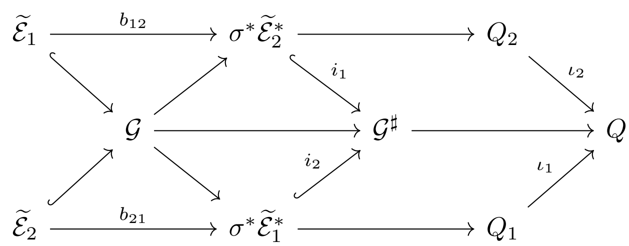
In this section we carry out Steps (6) and (7) of the proof outline from §2.4, thus completing the proof of Theorem 2.2.3.

10.1. **Transverse Lagrangians.** Let $\mathcal{G} \in \mathrm{Bun}_{GU^-(2m)}(k)$. For notational simplicity, we assume the similitude line bundle of $\mathcal{G}$ is trivial, therefore the skew-Hermitian form $h_{\mathcal{G}} : \mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^*$. Let $\widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2 \hookrightarrow \mathcal{G}$ be two transverse Lagrangian sub-bundles. The Modularity Conjecture asserts the equality of elements in $\mathrm{Ch}_{(n-m)r}(\mathrm{Sht}^r_{U(n)})_{\overline{\mathbf{Q}}_\ell}$

$$\widetilde{Z}^r_m(\mathcal{G}, \widetilde{\mathcal{E}}_1) \overset{?}{=} \widetilde{Z}^r_m(\mathcal{G}, \widetilde{\mathcal{E}}_2).$$

Our goal is to show that they have the same image in $\mathrm{H}^{\mathrm{BM}}_{2(n-m)r}(\mathrm{Sht}^{r,\leq\mu}_{U(n)} \times_{(X')^r} (X'-T)^r; \overline{\mathbf{Q}}_\ell)$, for any given Harder-Narasimhan polygon μ for $\mathrm{Bun}_{U(n)}$ and a finite set of closed points T depending only on $\mathcal{E}_1$ and $\mathcal{E}_2$. This will establish Theorem 2.2.3.

We reproduce the diagram (2.3.19) with $\mathcal{E}_i$ replaced by $\widetilde{\mathcal{E}}_i$:



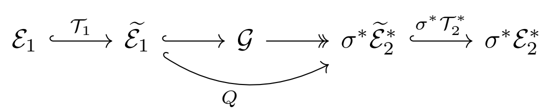
(10.1.1)

Here b_{12} is the composition

$$\widetilde{\mathcal{E}}_1 \to \mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^* \to \sigma^*\widetilde{\mathcal{E}}_2^*.$$

The map b_{21} is defined similarly, and the three rows are short exact. In particular, the maps ι_1 and ι_2 are isomorphisms of torsion sheaves on X'. As in §2.3.3, the duality between Q_1 and Q_2 equip Q with two Hermitian structures $h_{12} : Q \xrightarrow{\sim} \sigma^*Q^*$ and $h_{21} : Q \xrightarrow{\sim} \sigma^*Q^*$, related by $h_{12} = -h_{21}$.

For each $i \in \{1,2\}$ let $\mathcal{E}_i \hookrightarrow \widetilde{\mathcal{E}}_i$ be a sub-sheaf with cokernel a torsion coherent sheaf $\mathcal{T}_i$ on X'. Let $\mathcal{T}_i^* = \mathrm{RHom}(\mathcal{T}_i, \mathcal{O}_{X'})[1]$ be its dual torsion sheaf on X'. Therefore, $\widetilde{\mathcal{E}}_i$ is the saturation of $\mathcal{E}_i$ in $\mathcal{G}$, and in the diagram below



(10.1.2)

the arrows are labeled by their cokernels.

10.1.1. *Assumptions on $\mathcal{T}_1$ and $\mathcal{T}_2$.* Fix a Harder-Narasimhan polygon μ for $\mathrm{Bun}_{U(n)}$, and write $S = \mathrm{Bun}^{\leq\mu}_{U(n)}$ for the corresponding open substack of $\mathrm{Bun}_{U(n)}$.

We make the following assumptions:

(1) The supports $|Q|, |\mathcal{T}_1|, |\mathcal{T}_2|$ are disjoint after mapping to X.
(2) For all $\mathcal{F} \in S(\overline{k}) = \mathrm{Bun}^{\leq\mu}_{U(n)}(\overline{k})$ we have for $i = 1, 2$

$$\mathrm{Ext}^1_{X_{\overline{k}}}(\mathcal{F}^*, \mathcal{E}_i^*) = 0. \tag{10.1.3}$$

Note that by the dualities in (2.3.53), (10.1.3) is equivalent to

$$\mathrm{Hom}_{X_{\overline{k}}}(\mathcal{F}^*, \sigma^*\mathcal{E}_i) = 0 \tag{10.1.4}$$

for all $\mathcal{F} \in \mathrm{Bun}^{\leq\mu}_{U(n)}(\overline{k})$ and $i = 1, 2$.

Remark 10.1.1. Given any μ, the conditions (10.1.3), (10.1.4) can be arranged by taking $\mathcal{E}_i = \widetilde{\mathcal{E}}_i(-D_i)$ for sufficiently large divisors D_1, D_2 whose supports over X are disjoint from each other and from $|Q|$. By taking D_i to be a sufficiently large multiple of a single closed point, we may even arrange that the support $|\mathcal{T}_i|$ is a single closed point of X'.

Let

$$\widetilde{Q}_1 := Q^* \oplus \mathcal{T}_1^* \oplus \sigma^*\mathcal{T}_2, \tag{10.1.5}$$
$$\widetilde{Q}_2 := \sigma^*Q \oplus \sigma^*\mathcal{T}_1 \oplus \mathcal{T}_2^*. \tag{10.1.6}$$

From (10.1.2) and the disjointness assumption in §10.1.1, we have short exact sequences

$$0 \to \sigma^* \mathcal{E}_2 \to \mathcal{E}_1^* \to \widetilde{Q}_1 \to 0, \tag{10.1.7}$$

$$0 \to \sigma^* \mathcal{E}_1 \to \mathcal{E}_2^* \to \widetilde{Q}_2 \to 0 \tag{10.1.8}$$

Remark 10.1.2. The torsion coherent sheaves $\mathcal{T}_1, \mathcal{T}_2$ are auxiliary objects introduced for purely technical reasons (to ensure that certain spaces are smooth, and that certain maps are closed embeddings). They do not appear in §2.3, but are necessary when $r > 0$ in the argument as currently construed.

10.1.2. *Moduli spaces.* For $i \in \{0, r\}$ we define $U_i, \widetilde{U}_i, V_i, \widetilde{V}_i, W_i, \widetilde{W}_i$, etc. as in §9.

The vanishing assumption (10.1.3) implies that U, V and W are all classical vector bundles over S, and we have a short exact sequence of classical vector bundles over S

$$0 \to U \to V \to W \to 0. \tag{10.1.9}$$

Similarly, we have a short exact sequence of classical vector bundles over S

$$0 \to U^\perp \to \widehat{V} \to W^\perp \to 0. \tag{10.1.10}$$

10.1.3. *Open locus.* We denote

$$X^\circ = X - \nu(|Q| \cup |\mathcal{T}_1| \cup |\mathcal{T}_2|) = X - \nu(|\widetilde{Q}_1|) = X - \nu(|\widetilde{Q}_2|); \tag{10.1.11}$$

$$X'^\circ = \nu^{-1}(X^\circ). \tag{10.1.12}$$

For a stack $\mathcal{Y}$ over X°, we denote

$$\mathcal{Y}^\circ := \mathcal{Y} \times_{X^r} (X^\circ)^r. \tag{10.1.13}$$

In particular, $\mathrm{Hk}_S^{r,\circ} \subset \mathrm{Hk}_S^r$ denotes the open substack where the legs are all disjoint from $|\widetilde{Q}_1| \cup |\widetilde{Q}_2|$.

On $(X^\circ)^r$, the structure of Hk_V^r is simpler.

Lemma 10.1.3. *For any $0 \le i \le r$, the restriction $\widetilde{b}_i^\circ \colon \mathrm{Hk}_V^{\flat,\circ} \to \widetilde{V}_i^\circ$ of $\widetilde{b}_i$ and the restriction $\widetilde{b}_i'^\circ : \widetilde{V}_i^\circ \to \mathrm{Hk}_V^{\sharp,\circ}$ of $\widetilde{b}_i'$ are isomorphisms.*

Proof. Let $(\mathcal{F}_\star) \in \mathrm{Hk}_S^{r,\circ}(R)$. The projection $\widetilde{b}_i \colon \mathrm{Hk}_V^\flat \to \widetilde{V}_i$, when base changed over $\mathcal{F}_\star : \mathrm{Spec}\, R \to \mathrm{Hk}_S^r$, is induced by the injective map $\mathfrak{p}_i^* : \mathcal{F}_i^* \to \mathcal{F}_\bullet^{\flat *}$ in (9.1.7). The cokernel of $\mathfrak{p}_i^*$ is a torsion sheaf supported set-theoretically on the union of the legs of $\mathcal{F}_\star$, which by assumption are disjoint from $\widetilde{Q}_1$. Therefore $\mathfrak{p}_i^*$ restricts to an isomorphism in an open neighborhood of $|\widetilde{Q}_1| \times \mathrm{Spec}\, R \subset X'_R$, and hence induces an isomorphism of R-modules

$$\mathrm{RHom}_{X'_R}(\mathcal{F}_\bullet^{\flat *}, \widetilde{Q}_1 \otimes R) \xrightarrow{\sim} \mathrm{RHom}_{X'_R}(\mathcal{F}_i^*, \widetilde{Q}_1 \otimes R)$$

This being true for any R-point $\mathcal{F}_\star$ of $\mathrm{Hk}_S^{r,\circ}$, we conclude that $\widetilde{b}_i^\circ \colon \mathrm{Hk}_V^{\flat,\circ} \to \widetilde{V}_i^\circ$ is an isomorphism.

The argument for $\widetilde{b}_i'^\circ$ is similar. □

Corollary 10.1.4. *The projection map $\mathrm{Sht}_V^{r,\circ} \to \mathrm{Sht}_S^{r,\circ} = \mathrm{Sht}_{U(n)}^{r,\le\mu,\circ}$ is a relative finite-dimensional $\mathbf{F}_q$-vector space. Hence the theory of the arithmetic Fourier transform (§8) applies to it.*

10.2. **Calculation of traces.** Recall from Remark 9.1.1 that the spaces U_i from §5 can be viewed as the derived fiber of the derived Hitchin stack $\mathscr{M}_{H_1,H_2}$ from [FYZ21b, §5] over $\{\mathcal{E}_1\} \times S \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)}$, where $H_1 = \mathrm{GL}(m)'$ and $H_2 = U(n)$. Similarly, we explained in Remark 9.1.2 that $\mathrm{Hk}_U^\flat$ was an open substack of the derived fiber of the derived Hecke stack $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}}$ from [FYZ21b, §5] over $\{\mathcal{E}_1\} \times S \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)}$. Therefore, the derived fibered product

$$\begin{array}{ccc} \mathrm{Sht}_U^r & \longrightarrow & \mathrm{Hk}_U^\flat \\ \downarrow & & \downarrow{\scriptstyle (a_0,a_r)} \\ U_0 & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & U_0 \times U_r \end{array}$$

is equipped with an open embedding in $\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}$, and in particular is of virtual dimension $d(a_r)$. We then have two natural cycles in $\mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}_U^r)$:

(1) The intrinsic derived fundamental class $[\mathrm{Sht}^r_U] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^r_U)$, which agrees with the restriction of the intrinsic derived fundamental class $[\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}})$ along the étale map $\mathrm{Sht}^r_U \to \mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}$.
(2) The trace of the cohomological correspondence $\mathfrak{c}_U$, denoted $\mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}_U) \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^r_U)$, see §4.2. Here, $\mathrm{Tr}^{\mathrm{Sht}}$ is calculated using the Weil structure on $\mathfrak{c}_U$ coming from the canonical identification $\mathrm{Frob}^* \mathbf{Q}_{\ell,U} = \mathbf{Q}_{\ell,U}$.

We assemble the earlier results to calculate the trace of our cohomological correspondences. For any space ? over k, we equip $\mathbf{Q}_{\ell,?}$ with the natural Weil structure $\mathrm{Frob}^* \mathbf{Q}_{\ell,?} = \mathbf{Q}_{\ell,?}$. This equips all of our cohomological correspondences with a Weil structure.

The assumptions (10.1.3) imply that the maps $U_i \to S$ and $U_i^\perp \to S$ are smooth, hence U_i and $U_i^\perp$ are smooth. Then by Proposition 4.7.1 we have

$$\mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}_U) = [\mathrm{Sht}^r_U] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^r_U). \tag{10.2.1}$$

In particular, Sht^r_U is the open substack of $\mathrm{Sht}^r_{\mathcal{M}_{\mathcal{E}_1}}$, so it is quasi-smooth and $[\mathrm{Sht}^r_U]$ is the restriction of $[\mathrm{Sht}^r_{\mathcal{M}_{\mathcal{E}_1}}]$, which was called $[\mathcal{Z}^r_{\mathcal{E}_1}]$ in [FYZ21a].

Similarly, we have

$$\mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}_{U^\perp}) = [\mathrm{Sht}^r_{U^\perp}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r^\perp)}(\mathrm{Sht}^r_{U^\perp}), \tag{10.2.2}$$

where $\mathrm{Sht}^r_{U^\perp}$ is defined by the derived Cartesian square

$$\begin{array}{ccc} \mathrm{Sht}^r_{U^\perp} & \longrightarrow & \mathrm{Hk}^\flat_{U^\perp} \\ \big\downarrow & & \big\downarrow {\scriptstyle (a_0^\perp, a_r^\perp)} \\ U_0^\perp & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & U_0^\perp \times U_r^\perp \end{array}$$

Next, the assumptions (10.1.4) imply that the maps $f_i \colon U_i \to V_i$, $f \colon \mathrm{Hk}^\flat_U \to \mathrm{Hk}^\flat_V$, $f_i^\perp \colon U_i^\perp \to \widehat{V}_i$, and $f^\perp \colon \mathrm{Hk}^\flat_{U^\perp} \to \mathrm{Hk}^\flat_{\widehat{V}}$ are all closed embeddings, in particular proper. Therefore, by Proposition 4.5.1, we have

$$\mathrm{Tr}^{\mathrm{Sht}}(f_! \, \mathfrak{c}_U) = \mathrm{Sht}(f)_! \, \mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}_U) \overset{(10.2.1)}{=} \mathrm{Sht}(f)_! [\mathrm{Sht}^r_U] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^r_V), \tag{10.2.3}$$

where we write $\mathrm{Sht}(f) := \mathrm{Fix}(f^{(1)}) \colon \mathrm{Sht}^r_U \to \mathrm{Sht}^r_V$ for the map induced by taking fixed points of the twisted cohomological correspondence $\mathfrak{c}_U^{(1)}$, and similarly for other cohomological correspondences. We similarly have

$$\mathrm{Tr}^{\mathrm{Sht}}(f^\perp_! \, \mathfrak{c}_{U^\perp}) = \mathrm{Sht}(f^\perp)_! \, \mathrm{Tr}^{\mathrm{Sht}}(\mathfrak{c}_{U^\perp}) \overset{(10.2.2)}{=} \mathrm{Sht}(f^\perp)_! [\mathrm{Sht}^r_{U^\perp}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r^\perp)}(\mathrm{Sht}^r_{\widehat{V}}). \tag{10.2.4}$$

Recall from Corollary 10.1.4 that $\mathrm{Sht}^{r,\circ}_V \to \mathrm{Sht}^{r,\circ}_S$ is a relative $\mathbf{F}_q$-vector space. We now relate the cycle classes (10.2.3) and (10.2.4) under the arithmetic Fourier transform on $\mathrm{Sht}^{r,\circ}_V$ as defined in §8.

Theorem 10.2.1. *We have*

$$\mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_V}(\mathrm{Sht}(f)^\circ_! [\mathrm{Sht}^{r,\circ}_U]) = (-1)^{d(U/S)+d(f_0)} q^{d(U/S)} \cdot \mathrm{Sht}(f^\perp)^\circ_! [\mathrm{Sht}^{r,\circ}_{U^\perp}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_{\widehat{V}}).$$

Here $\mathrm{Sht}(f)^\circ : \mathrm{Sht}^{r,\circ}_U \to \mathrm{Sht}^{r,\circ}_V$ *is the restriction of* $\mathrm{Sht}(f)$*, and similarly for* $\mathrm{Sht}(f^\perp)^\circ$.

Remark 10.2.2. A priori, $\mathrm{Sht}(f^\perp)^\circ_! [\mathrm{Sht}^{r,\circ}_{U^\perp}]$ lies in $\mathrm{H}^{\mathrm{BM}}_{2d(a_r^\perp)}(\mathrm{Sht}^{r,\circ}_V)$. We note that $d(a_r) = d(a_r^\perp) = (n-m)r$, so the statement makes sense.

Proof. We apply Theorem 8.3.2 with $E = V$, $C_E = \mathrm{Hk}^{\flat,\circ}_V$ and $\mathfrak{c} = (f_! \, \mathfrak{c}_U)|_{\mathrm{Hk}^{\flat,\circ}_V}$. Then Theorem 8.3.2 tells us that

$$\mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_V}\left(\mathrm{Tr}^{\mathrm{Sht}}(f_! \, \mathfrak{c}_U)|_{\mathrm{Sht}^{r,\circ}_V}\right) = \left(\mathrm{Tr}^{\mathrm{Sht}} \, \mathrm{FT}_{\mathrm{Hk}^\flat_V}(f_! \, \mathfrak{c}_U)\right)|_{\mathrm{Sht}^{r,\circ}_{\widehat{V}}} \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_{\widehat{V}}). \tag{10.2.5}$$

By Theorem 9.4.1 we have

$$\mathrm{FT}_{\mathrm{Hk}^\flat_V}(f_! \, \mathfrak{c}_U) = \mathbb{T}_{[-d(U/S)-d(f_0)](-d(U/S))}(f^\perp)_! \, \mathfrak{c}_{U^\perp}$$

Putting this into (10.2.5) and then taking the trace, using (4.2.6), (10.2.3) and (10.2.4), yields the result. □

10.2.1. *Test functions.* We introduce some notation for functions on Sht_V and $\mathrm{Sht}_{\widehat{V}}$. The decompositions $\widetilde{Q}_1 := Q^* \oplus \mathcal{T}_1^* \oplus \sigma^*\mathcal{T}_2$ and $\widetilde{Q}_2 := \sigma^* Q \oplus \sigma^*\mathcal{T}_1 \oplus \mathcal{T}_2^*$ induce the following.

- A decomposition
$$V \cong V^{(0)} \times_S V^{(1)} \times_S V^{(2)},$$
where $V^{(0)} \to S$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, Q^*)}$, $V^{(1)} \to S$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{T}_1^*)}$, and $V^{(2)} \to S$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\mathcal{T}_2)}$.
- A decomposition
$$\widehat{V} \cong \widehat{V}^{(0)} \times_S \widehat{V}^{(1)} \times_S \widehat{V}^{(2)},$$
where $\widehat{V}^{(0)} \to S$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^* Q)}$, $\widehat{V}^{(1)} \to S$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\mathcal{T}_1)}$, and $V^{(2)} \to S$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{T}_2^*)}$. As the notation suggests, $\widehat{V}^{(i)} \to S$ is the dual bundle to $V^{(i)} \to S$. As in §2.3.3, the Hermitian structures h_{12} and h_{21} on Q induce two isomorphisms $V^{(0)} \cong \widehat{V}^{(0)}$, which are the negatives of each other.
- A decomposition
$$\mathrm{Hk}^{\flat}_V \cong \mathrm{Hk}^{\flat}_{V^{(0)}} \times_{\mathrm{Hk}^r_S} \mathrm{Hk}^{\flat}_{V^{(1)}} \times_{\mathrm{Hk}^r_S} \times \mathrm{Hk}^{\flat}_{V^{(2)}}$$
where $\mathrm{Hk}^{\flat}_{V^{(0)}}$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^{\flat *}_{\mathrm{univ},\bullet}, Q^*)}$, etc.
- A decomposition
$$\mathrm{Hk}^{\flat}_{\widehat{V}} \cong \mathrm{Hk}^{\flat}_{\widehat{V}^{(0)}} \times_{\mathrm{Hk}^r_S} \mathrm{Hk}^{\flat}_{\widehat{V}^{(1)}} \times_{\mathrm{Hk}^r_S} \mathrm{Hk}^{\flat}_{\widehat{V}^{(2)}},$$
where $\mathrm{Hk}^{\flat}_{\widehat{V}^{(0)}}$ is the vector bundle associated to $\underline{\mathrm{RHom}(\mathcal{F}^{\flat *}_{\mathrm{univ},\bullet}, \sigma^* Q)}$, etc. The Hermitian structures h_{12} and h_{21} on Q induce two isomorphisms $\mathrm{Hk}^{\flat}_{V^{(0)}} \cong \mathrm{Hk}^{\flat}_{\widehat{V}^{(0)}}$, which are the negatives of each other.

  By Lemma 10.1.3, we see that $\mathrm{Hk}^{\flat,\circ}_{\widehat{V}^{(i)}} \to \mathrm{Hk}^{r,\circ}_S$ is the dual bundle to $\mathrm{Hk}^{\flat,\circ}_{V^{(i)}} \to \mathrm{Hk}^{r,\circ}_S$.
- A decomposition
$$\mathrm{Sht}^r_V = \mathrm{Sht}^r_{V^{(0)}} \times_{\mathrm{Sht}^r_S} \mathrm{Sht}^r_{V^{(1)}} \times_{\mathrm{Sht}^r_S} \mathrm{Sht}^r_{V^{(2)}}$$
where $\mathrm{Sht}^r_{V^{(i)}}$ is defined by the derived fibered product
$$\begin{array}{ccc} \mathrm{Sht}^r_{V^{(i)}} & \longrightarrow & \mathrm{Hk}^{\flat}_{V^{(i)}} \\ \downarrow & & \downarrow {\scriptstyle (b_0^{(i)}, b_r^{(i)})} \\ V^{(i)} & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & V^{(i)} \times V^{(i)} \end{array}$$
- A decomposition
$$\mathrm{Sht}^r_{\widehat{V}} = \mathrm{Sht}^r_{\widehat{V}^{(0)}} \times_{\mathrm{Sht}^r_S} \mathrm{Sht}^r_{\widehat{V}^{(1)}} \times_{\mathrm{Sht}^r_S} \mathrm{Sht}^r_{\widehat{V}^{(2)}}$$
where $\mathrm{Sht}^r_{\widehat{V}^{(i)}}$ is defined by the derived fibered product
$$\begin{array}{ccc} \mathrm{Sht}^r_{\widehat{V}^{(i)}} & \longrightarrow & \mathrm{Hk}^{\flat}_{\widehat{V}^{(i)}} \\ \downarrow & & \downarrow {\scriptstyle (\widehat{a}_0^{(i)}, \widehat{a}_r^{(i)})} \\ \widehat{V}^{(i)} & \xrightarrow{(\mathrm{Id},\mathrm{Frob})} & \widehat{V}^{(i)} \times \widehat{V}^{(i)} \end{array}$$
We note that $\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(i)}}$ is dual to $\mathrm{Sht}^{r,\circ}_{V^{(i)}}$ as $\mathbf{F}_q$-vector spaces over $\mathrm{Sht}^{r,\circ}_S$ in the sense of §8.1. The Hermitian structures h_{12} and h_{21} on Q induce two isomorphisms $\mathrm{Sht}^r_{V^{(0)}} \cong \mathrm{Sht}^r_{\widehat{V}^{(0)}}$, which are the negatives of each other.

We denote $\mathfrak{q}_{12} \colon \mathrm{Sht}^r_{V^{(0)}} \to \underline{\mathbf{F}_q}$ and $\mathfrak{q}_{21} \colon \mathrm{Sht}^r_{V^{(0)}} \to \underline{\mathbf{F}_q}$ the two quadratic forms induced by h_{12} and h_{21}, respectively. Namely, $\mathfrak{q}_{12}$ is the composition
$$\mathfrak{q}_{12} : \mathrm{Sht}^r_{V^{(0)}} \xrightarrow{(\mathrm{Id}, h_{12})} \mathrm{Sht}^r_{V^{(0)}} \times_{\mathrm{Sht}^r_S} \mathrm{Sht}^r_{\widehat{V}^{(0)}} \to \underline{\mathbf{F}_q}, \tag{10.2.6}$$
and similarly for $\mathfrak{q}_{21}$. They satisfy $\mathfrak{q}_{12} = -\mathfrak{q}_{21}$.

- We let $\mathfrak{q}_{12}^*\psi$ be the pullback of ψ to $\mathrm{Sht}^r_{V^{(0)}}$ via $\mathfrak{q}_{12}$, and similarly for $\mathfrak{q}_{21}$. Abusing notation, we will also use the same notation $\mathfrak{q}_{12}^*\psi$ to denote its pullback to Sht^r_V and to $\mathrm{Sht}^{r,\circ}_{\widehat{V}}$. The meaning will be clear from context.

- We let $\delta_{\mathrm{Sht}^{r,\circ}_{V^{(i)}}}$ be the indicator function of the zero-section of the relative $\mathbf{F}_q$-vector space $\mathrm{Sht}^{r,\circ}_{V^{(i)}} \to \mathrm{Sht}^{r,\circ}_S$. Abusing notation, we will also use this same notation to denote its pullback to $\mathrm{Sht}^{r,\circ}_V$.
- We let $\mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(i)}}}$ be the constant function of $\mathrm{Sht}^{r,\circ}_{V^{(i)}}$ with value 1. Abusing notation, we will also use this same notation to denote its pullback to $\mathrm{Sht}^{r,\circ}_V$.
- We let $(\mathfrak{q}_{12}^*\psi \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(2)}}})$ denote the product of the above functions, viewed as a locally constant function on $\mathrm{Sht}^{r,\circ}_V$.
- We use similar notation on $\mathrm{Sht}^{r,\circ}_{\widehat{V}}$.

Lemma 10.2.3. *Let $d^{(i)}$ be the rank of $\mathrm{Sht}^{r,\circ}_{V^{(i)}}$ as an $\mathbf{F}_q$-vector space over $\mathrm{Sht}^{r,\circ}_S$. Then we have the following identities.*

$$\mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_{V^{(i)}}}(\delta_{\mathrm{Sht}^{r,\circ}_{V^{(i)}}}) = (-1)^{d^{(i)}}\mathbb{1}_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(i)}}} \tag{10.2.7}$$

$$\mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_{V^{(i)}}}(\mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(i)}}}) = (-1)^{d^{(i)}} q^{d^{(i)}}\delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(i)}}} \tag{10.2.8}$$

$$\mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_{V^{(0)}}}(\mathfrak{q}_{12}^*\psi) = (-1)^{d^{(0)}}[1/2]^*\mathfrak{q}_{12}^*[-1]^*\psi \cdot q^{d^{(0)}/2}\eta_{F'/F}(D_Q)^n \tag{10.2.9}$$

where we recall that $D_Q \in \mathrm{Div}(X)$ denotes the divisor of Q.

Proof. As in the proof of Lemma 8.2.4, we can reduce to the case where $\mathrm{Sht}^{r,\circ}_{V^{(0)}} \to \mathrm{Sht}^{r,\circ}_S$ is split, and then to the usual finite Fourier transform, which we handled in §2.3.7. □

Corollary 10.2.4. *Let $d = d^{(0)} + d^{(1)} + d^{(2)}$ be the rank of $\mathrm{Sht}^{r,\circ}_V$ as an $\mathbf{F}_q$-vector space over $\mathrm{Sht}^{r,\circ}_S$. Then we have*

$$\mathrm{FT}^{\mathrm{arith}}(\mathfrak{q}_{12}^*\psi \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(2)}}}) = (-1)^d q^{d^{(2)}+\frac{1}{2}d^{(0)}}\eta_{F'/F}(D_Q)^n \cdot ([1/2]^*\mathfrak{q}_{12}^*[-1]^*\psi \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(2)}}})$$

as functions on $\mathrm{Sht}^{r,\circ}_{\widehat{V}}$.

Proof. Multiply the equations (10.2.9), (10.2.8), (10.2.7) together. □

10.2.2. *Higher theta series.* Here we relate $[\mathrm{Sht}^{r,\circ}_U]$ and $[\mathrm{Sht}^{r,\circ}_{U^\perp}]$ to the special cycles $[\mathcal{Z}^{r,\circ}_{\widetilde{\mathcal{E}}_1}]$ and $[\mathcal{Z}^{r,\circ}_{\widetilde{\mathcal{E}}_2}]$, and then to the higher theta series associated to $\widetilde{\mathcal{E}}_1$ and $\widetilde{\mathcal{E}}_2$.

Recall that for $i \in \{1,2\}$ we had exact sequences of coherent sheaves on X',

$$\mathcal{E}_i \to \widetilde{\mathcal{E}}_i \to \mathcal{T}_i. \tag{10.2.10}$$

This induces an exact triangle in $\mathrm{Perf}(R)$ for any $\mathcal{F} \in \mathrm{Bun}_{U(n)}(R)$,

$$\mathrm{RHom}_{X'_R}(\widetilde{\mathcal{E}}_i, \mathcal{F}) \to \mathrm{RHom}_{X'_R}(\mathcal{E}_i, \mathcal{F}) \to \mathrm{RHom}_{X'_R}(\mathcal{T}_i, \mathcal{F}[1]). \tag{10.2.11}$$

By linear duality we have $\mathrm{RHom}_{X'_R}(\mathcal{T}_i, \mathcal{F}[1]) \cong \mathrm{RHom}_{X'_R}(\mathcal{F}^*, \mathcal{T}_i^*)$. Since $\mathcal{T}_i^*$ is a torsion sheaf, $\mathrm{RHom}_{X'_R}(\mathcal{F}^*, \mathcal{T}_i^*)$ is equivalent to a locally free coherent sheaf (concentrated in degree 0). Let $\mathcal{N}_{\mathcal{T}_i}$ be the total space of $\underline{\mathrm{RHom}(\mathcal{T}_i, \mathcal{F}_{\mathrm{univ}}[1])} \cong \underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{T}_i^*)}$, a vector bundle over $\mathrm{Bun}_{U(n)}$.

Below we refer to the notation of [FYZ21b, §5]: we will use the pair $(H_1, H_2) = (\mathrm{GL}(m)', U(n))$. For $i \in \{1,2\}$, we let

- $\mathcal{M}_{\mathcal{E}_i}$ be the derived fiber of the Hitchin stack $\mathcal{M}_{H_1,H_2}$ from [FYZ21b, Definition 5.14] over $\mathcal{E}_i \in \mathrm{Bun}_{H_1}(k)$, and
- $\mathrm{Hk}^r_{\mathcal{M}_{\mathcal{E}_i}}$ be the derived fiber of the derived Hitchin stack $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$ over $\mathcal{E}_i \in \mathrm{Bun}_{H_1}(k)$.

We define $\mathcal{M}_{\widetilde{\mathcal{E}}_i}$ and $\mathrm{Hk}^r_{\mathcal{M}_{\widetilde{\mathcal{E}}_i}}$ similarly.

Then the exact triangle (10.2.11) corresponds at the level of total spaces to a derived Cartesian square

$$\begin{array}{ccc} \mathcal{M}_{\widetilde{\mathcal{E}}_i} & \longrightarrow & \mathcal{M}_{\mathcal{E}_i} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{U(n)} & \xrightarrow{z} & \mathcal{N}_{\mathcal{T}_i} \end{array} \tag{10.2.12}$$

Let $\mathrm{Hk}^r_{\mathcal{N}_{\mathcal{T}_i}}$ be the total space of $\underline{\mathrm{RHom}(\mathcal{T}_i, \mathcal{F}^\flat_{\mathrm{univ},\bullet}[1])}$ on $\mathrm{Hk}^r_{U(n)}$. The exact triangle

$$\underline{\mathrm{RHom}(\widetilde{\mathcal{E}}_i, \mathcal{F}^\flat_{\mathrm{univ},\bullet})} \to \underline{\mathrm{RHom}(\mathcal{E}_i, \mathcal{F}^\flat_{\mathrm{univ},\bullet})} \to \underline{\mathrm{RHom}(\mathcal{T}_i, \mathcal{F}^\flat_{\mathrm{univ},\bullet}[1])}$$

corresponds at the level of total spaces to a derived Cartesian square

$$\begin{array}{ccc} \mathrm{Hk}^r_{\mathcal{M}_{\widetilde{\mathcal{E}}_i}} & \longrightarrow & \mathrm{Hk}^r_{\mathcal{M}_{\mathcal{E}_i}} \\ \downarrow & & \downarrow \\ \mathrm{Hk}^r_{U(n)} & \xrightarrow{z} & \mathrm{Hk}^r_{\mathcal{N}_{\mathcal{T}_i}} \end{array} \tag{10.2.13}$$

Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hk}^r_{\mathcal{M}_{\mathcal{E}_i}} & \xrightarrow{(h_0,h_r)} & \mathcal{M}_{\mathcal{E}_i} \times \mathcal{M}_{\mathcal{E}_i} & \xleftarrow{(\mathrm{Id},\mathrm{Frob})} & \mathcal{M}_{\mathcal{E}_i} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hk}^r_{\mathcal{N}_{\mathcal{T}_i}} & \xrightarrow{(h_0,h_r)} & \mathcal{N}_{\mathcal{T}_i} \times \mathcal{N}_{\mathcal{T}_i} & \xleftarrow{(\mathrm{Id},\mathrm{Frob})} & \mathcal{N}_{\mathcal{T}_i} \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hk}^r_{U(n)} & \xrightarrow{(h_0,h_r)} & \mathrm{Bun}_{U(n)} \times \mathrm{Bun}_{U(n)} & \xleftarrow{(\mathrm{Id},\mathrm{Frob})} & \mathrm{Bun}_{U(n)} \end{array} \tag{10.2.14}$$

By (10.2.12) and (10.2.13), the derived fibered product of the columns of (10.2.14) is

$$\mathrm{Hk}^r_{\mathcal{M}_{\widetilde{\mathcal{E}}_i}} \xrightarrow{(h_0,h_r)} \mathcal{M}_{\widetilde{\mathcal{E}}_i} \times \mathcal{M}_{\widetilde{\mathcal{E}}_i} \xleftarrow{(\mathrm{Id},\mathrm{Frob})} \mathcal{M}_{\widetilde{\mathcal{E}}_i} \tag{10.2.15}$$

and the derived fibered product of the rows of (10.2.14) is

$$\begin{array}{c} \mathrm{Sht}^r_{\mathcal{M}_{\mathcal{E}_i}} = \mathcal{Z}^r_{\mathcal{E}_i} \\ \downarrow \\ \mathrm{Sht}^r_{\mathcal{N}_{\mathcal{T}_i}} \\ \uparrow \\ \mathrm{Sht}^r_{U(n)} \end{array} \tag{10.2.16}$$

In turn, the derived fibered products of (10.2.15) and (10.2.16) are canonically identified by the same proof as for [YZ17, Lemma A.9]. This shows:

Corollary 10.2.5. *The commutative square*

$$\begin{array}{ccc} \mathcal{Z}^r_{\widetilde{\mathcal{E}}_i} & \longrightarrow & \mathcal{Z}^r_{\mathcal{E}_i} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^r_{U(n)} & \xrightarrow{0} & \mathrm{Sht}^r_{\mathcal{N}_{\mathcal{T}_i}} \end{array}$$

is derived Cartesian.

To compare with earlier objects:

- Restricting $\mathrm{Sht}^r_{\mathcal{N}_{\mathcal{T}_1}} \to \mathrm{Sht}^r_{U(n)}$ along the open embedding $\mathrm{Sht}^r_S \hookrightarrow \mathrm{Sht}^r_{U(n)}$ recovers $\mathrm{Sht}^r_{V^{(1)}} \to \mathrm{Sht}^r_S$, i.e., we have a derived Cartesian square

$$\begin{array}{ccc} \mathrm{Sht}^r_{V^{(1)}} & \xhookrightarrow{\text{open}} & \mathrm{Sht}^r_{\mathcal{N}_{\mathcal{T}_1}} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^r_S & \xhookrightarrow{\text{open}} & \mathrm{Sht}^r_{U(n)} \end{array} \tag{10.2.17}$$

Restricting $\mathrm{Sht}^r_{\mathcal{N}_{\mathcal{T}_2}} \to \mathrm{Sht}^r_{U(n)}$ along the open embedding $\mathrm{Sht}^r_S \hookrightarrow \mathrm{Sht}^r_{U(n)}$ recovers $\mathrm{Sht}^r_{\widehat{V}^{(2)}} \to \mathrm{Sht}^r_S$.

- Restricting $\mathcal{Z}^r_{\mathcal{E}_1} \to \mathrm{Sht}^r_{U(n)}$ along the open embedding $\mathrm{Sht}^r_S \hookrightarrow \mathrm{Sht}^r_{U(n)}$ recovers $\mathrm{Sht}^r_U \to \mathrm{Sht}^r_S$, and restricting $\mathcal{Z}^r_{\mathcal{E}_2} \to \mathrm{Sht}^r_{U(n)}$ along the open embedding $\mathrm{Sht}^r_S \hookrightarrow \mathrm{Sht}^r_{U(n)}$ recovers $\mathrm{Sht}^r_{U^\perp} \to \mathrm{Sht}^r_S$. In other words, we have derived Cartesian squares

$$\begin{array}{ccc} \mathrm{Sht}^r_U & \xhookrightarrow{\text{open}} & \mathcal{Z}^r_{\mathcal{E}_1} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^r_S & \xhookrightarrow{\text{open}} & \mathrm{Sht}^r_{U(n)} \end{array} \qquad \begin{array}{ccc} \mathrm{Sht}^r_{U^\perp} & \xhookrightarrow{\text{open}} & \mathcal{Z}^r_{\mathcal{E}_2} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^r_S & \xhookrightarrow{\text{open}} & \mathrm{Sht}^r_{U(n)} \end{array} \tag{10.2.18}$$

Abbreviate $\mathcal{Z}^{r,\le\mu}_{\widetilde{\mathcal{E}}_i}$ for $\mathcal{Z}^r_{\widetilde{\mathcal{E}}_i} \times_{\mathrm{Sht}^r_{U(n)}} \mathrm{Sht}^r_S$. We have derived Cartesian squares

$$\begin{array}{ccc} \mathcal{Z}^{r,\le\mu}_{\widetilde{\mathcal{E}}_1} & \hookrightarrow & \mathrm{Sht}^r_U \\ \downarrow & & \downarrow \\ \mathrm{Sht}^r_S & \xhookrightarrow{0} & \mathrm{Sht}^r_{V^{(1)}} \end{array} \qquad \begin{array}{ccc} \mathcal{Z}^{r,\le\mu}_{\widetilde{\mathcal{E}}_2} & \hookrightarrow & \mathrm{Sht}^r_{U^\perp} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^r_S & \xhookrightarrow{0} & \mathrm{Sht}^r_{\widehat{V}^{(2)}} \end{array} \tag{10.2.19}$$

By Corollary 10.1.4, $\mathrm{Sht}^{r,\circ}_{V^{(i)}} \to \mathrm{Sht}^{r,\circ}_S$ is finite étale, so the zero section $\mathrm{Sht}^{r,\circ}_S \xrightarrow{0} \mathrm{Sht}^{r,\circ}_{V^{(i)}}$ is open-closed, which by (10.2.19) implies that $\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_1} = \mathcal{Z}^r_{\widetilde{\mathcal{E}}_1}|_{\mathrm{Sht}^{r,\circ}_S} \hookrightarrow \mathrm{Sht}^{r,\circ}_U$ is also open-closed; similarly for $\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_2} \hookrightarrow \mathrm{Sht}^{r,\circ}_{U^\perp}$.

Corollary 10.2.6. *(1) Viewing $\delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \in \mathrm{H}^0(\mathrm{Sht}^{r,\circ}_V; \mathbf{Q}_\ell)$ and $\mathrm{Sht}(f)^\circ_![\mathrm{Sht}^{r,\circ}_U] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_V)$, we have*

$$\mathrm{Sht}(f)^\circ_![\mathrm{Sht}^{r,\circ}_U] \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} = [\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_1}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_V).$$

(2) Viewing $\delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(2)}}} \in \mathrm{H}^0(\mathrm{Sht}^{r,\circ}_{\widehat{V}}; \mathbf{Q}_\ell)$ and $\mathrm{Sht}(f^\perp)^\circ_![\mathrm{Sht}^{r,\circ}_{U^\perp}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r^\perp)}(\mathrm{Sht}^{r,\circ}_{V^\perp})$, we have

$$\mathrm{Sht}(f^\perp)^\circ_![\mathrm{Sht}^{r,\circ}_{U^\perp}] \cdot \delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(2)}}} = [\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_2}] \in \mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_{\widehat{V}}).$$

Lemma 10.2.7. *For $i = 1, 2$, let $\mathcal{A}_{\widetilde{\mathcal{E}}_i}$ be the Hitchin base as in* [FYZ21b, §3.3]. *For $a \in \mathcal{A}_{\widetilde{\mathcal{E}}_i}(k)$, let $\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_i}(a) = \mathcal{Z}^r_{\widetilde{\mathcal{E}}_i}(a) \times_{\mathrm{Sht}^r_{U(n)}} \mathrm{Sht}^{r,\circ}_S$.*

(1) We have an equality in $\mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_V)$:

$$\mathrm{Sht}(f)^\circ_![\mathrm{Sht}^{r,\circ}_U] \cdot (\mathfrak{q}^*_{12}\psi \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(2)}}}) = \sum_{a \in \mathcal{A}_{\widetilde{\mathcal{E}}_1}(k)} \psi(\langle e_{\mathcal{G},\mathcal{E}_1}, a\rangle)\, \mathrm{Sht}(f)^\circ_![\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_1}(a)].$$

(2) We have an equality in $\mathrm{H}^{\mathrm{BM}}_{2d(a_r)}(\mathrm{Sht}^{r,\circ}_{\widehat{V}})$:

$$\mathrm{Sht}(f^\perp)^\circ_![\mathrm{Sht}^{r,\circ}_{U^\perp}] \cdot (\mathfrak{q}^*_{21}\psi \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(1)}}} \cdot \delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(2)}}}) = \sum_{a \in \mathcal{A}_{\widetilde{\mathcal{E}}_2}(k)} \psi(\langle e_{\mathcal{G},\mathcal{E}_2}, a\rangle)\, \mathrm{Sht}(f^\perp)^\circ_![\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_2}(a)].$$

Proof. (1) By Corollary 10.2.6, we have

$$\mathrm{Sht}(f)^\circ_![\mathrm{Sht}^{r,\circ}_U] \cdot (\mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(0)}}} \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(2)}}}) = \sum_{a \in \mathcal{A}_{\widetilde{\mathcal{E}}_1}(k)} \mathrm{Sht}(f)^\circ_![\mathcal{Z}^{r,\le\mu,\circ}_{\widetilde{\mathcal{E}}_1}(a)].$$

Then observe that by (2.3.45), the function $\mathfrak{q}^*_{12}\psi$ on Sht^r_V coincides with the one sending $(\mathcal{F}_\star, t) \in \mathrm{Sht}^r_V(R)$ to $\psi(\langle e_{\mathcal{G},\mathcal{E}_1}, a(t)\rangle)$.

(2) Similar, using (2.3.46). □

10.3. **Conclusion of the proof of Theorem 2.2.3.** Let $d, d^{(i)}$, for $i \in \{0, 1, 2\}$, be as in §10.2.1. Note that $d^{(i)}$ is also the rank of $V^{(i)}$ as a vector bundle over S.

By Lemma 10.2.7 we have

$$\widetilde{Z}^r_m(\widetilde{\mathcal{E}}_1, \mathcal{G})|_{\mathrm{Sht}^{r,\circ}_S} = \chi(\det \widetilde{\mathcal{E}}_1) q^{n(\deg \widetilde{\mathcal{E}}_1 - \deg \omega_X)/2} \langle \mathrm{Sht}(f)^\circ_![\mathrm{Sht}^{r,\circ}_U], \mathfrak{q}^*_{12}\psi \cdot \delta_{\mathrm{Sht}^{r,\circ}_{V^{(1)}}} \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V^{(2)}}} \rangle \tag{10.3.1}$$

and

$$\widetilde{Z}^r_m(\widetilde{\mathcal{E}}_2, \mathcal{G})|_{\mathrm{Sht}^{r,\circ}_S} = \chi(\det \widetilde{\mathcal{E}}_2) q^{n(\deg \widetilde{\mathcal{E}}_2 - \deg \omega_X)/2} \langle \mathrm{Sht}(f^\perp)^\circ_![\mathrm{Sht}^{r,\circ}_{U^\perp}], \mathfrak{q}^*_{21}\psi \cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(1)}}} \cdot \delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}^{(2)}}} \rangle. \tag{10.3.2}$$

By the Plancherel formula of Lemma 8.2.1 and the near-involutivity of $\mathrm{FT}^{\mathrm{arith}}$ of Lemma 8.2.4, we have

$$\begin{aligned}&\langle \mathrm{Sht}(f)^{\circ}_{!}[\mathrm{Sht}^{r,\circ}_U], \mathfrak{q}_{12}^*\psi\cdot\delta_{\mathrm{Sht}^{r,\circ}_{V(1)}}\cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V(2)}}\rangle \\ =\ &\frac{1}{q^d}\langle \mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_V}(\mathrm{Sht}(f)^{\circ}_{!}[\mathrm{Sht}^{r,\circ}_U]), \mathrm{FT}^{\mathrm{arith}}_{\mathrm{Sht}^{r,\circ}_V}(\mathfrak{q}_{12}^*\psi\cdot\delta_{\mathrm{Sht}^{r,\circ}_{V(1)}}\cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{V(2)}})\rangle.\end{aligned} \tag{10.3.3}$$

Then we use Theorem 10.2.1 and Corollary 10.2.4 to rewrite the right side of (10.3.3) as

$$\begin{aligned}&\frac{1}{q^d}q^{d(U/S)}(-1)^{d(U/S)+d(f_0)}(-1)^d q^{d^{(2)}+\frac{1}{2}d^{(0)}}\eta_{F'/F}(D_Q)^n \\ &\cdot\langle \mathrm{Sht}(f^{\perp})^{\circ}_{!}[\mathrm{Sht}^{r,\circ}_{U^{\perp}}], [1/2]^*\mathfrak{q}_{12}^*[-1]^*\psi\cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{\widehat{V}(1)}}\cdot\delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}(2)}}\rangle.\end{aligned} \tag{10.3.4}$$

Since $\mathfrak{q}_{12} = -\mathfrak{q}_{21}$, we have $\mathfrak{q}_{12}^*[-1]^* = \mathfrak{q}_{21}^*$. Eliminating the $[1/2]^*$ does not affect the expression since $\mathrm{Sht}(f^{\perp})^{\circ}_{!}[\mathrm{Sht}^{r,\circ}_{U^{\perp}}]$ is invariant under the scaling $\mathbf{F}_q^{\times}$ action on $\mathrm{Sht}^{r,\circ}_V$. Then clearly (10.3.4) agrees with (10.3.2) up to sign and integral power of q. Therefore, it remains to check the sign and the exponent of q. The exponent of q in (10.3.4) is

$$-d + d(U/S) + d^{(2)} + \frac{1}{2}d^{(0)} = -d^{(1)} - d^{(0)} + d(U/S) + \frac{1}{2}d^{(0)}. \tag{10.3.5}$$

Recall that $d^{(1)} := \mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*, \mathcal{T}_1^*)})$. By the exact triangle of perfect complexes over S

$$\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{\mathcal{E}}_1^*)} \to \underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}_1^*)} \to \underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{T}_1^*)}$$

we have $-d^{(1)} = -\mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}_1^*)}) + \mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{\mathcal{E}}_1^*)})$ and $\mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \mathcal{E}_1^*)}) = d(U/S)$. Putting all this into (10.3.5) simplifies the exponent of q to

$$\mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{\mathcal{E}}_1^*)}) - d^{(0)} + \frac{1}{2}d^{(0)}. \tag{10.3.6}$$

By the exact triangle

$$\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\widetilde{\mathcal{E}}_2)} \to \underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{\mathcal{E}}_1^*)} \to \underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, Q^*)}$$

we have

$$\mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \widetilde{\mathcal{E}}_1^*)}) = \mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\widetilde{\mathcal{E}}_2)}) + \mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, Q^*)}). \tag{10.3.7}$$

Putting (10.3.7) into (10.3.5), noting that $\mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, Q^*)}) = d^{(0)}$, the exponent of q in (10.3.4) simplifies to

$$\mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\widetilde{\mathcal{E}}_2)}) + \frac{1}{2}d^{(0)}. \tag{10.3.8}$$

In §2.3.9 we exactly showed that

$$n(\deg\widetilde{\mathcal{E}}_1 - \deg\omega_X)/2 + \mathrm{rank}(\underline{\mathrm{RHom}(\mathcal{F}^*_{\mathrm{univ}}, \sigma^*\widetilde{\mathcal{E}}_2)}) + \frac{1}{2}d^{(0)} = n(\deg\widetilde{\mathcal{E}}_2 - \deg\omega_X)/2,$$

which after comparing (10.3.1), (10.3.2) shows that the exponents match! Hence we have established that

$$\widetilde{Z}^r_m(\widetilde{\mathcal{E}}_1, \mathcal{G}) = \pm\widetilde{Z}^r_m(\widetilde{\mathcal{E}}_2, \mathcal{G}).$$

Finally, we match the signs. We found above that

$$\begin{aligned}\widetilde{Z}^r_m(\widetilde{\mathcal{E}}_1, \mathcal{G})|_{\mathrm{Sht}^{r,\circ}_S} \ =\ &\chi(\det\widetilde{\mathcal{E}}_1)\eta_{F'/F}(D_Q)^n(-1)^{d(U/S)+d(f_0)}(-1)^d q^{?} \\ &\langle \mathrm{Sht}(f^{\perp})^{\circ}_{!}[\mathrm{Sht}^{r,\circ}_{U^{\perp}}], \mathfrak{q}_{21}^*\psi\cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{\widehat{V}(1)}}\cdot\delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}(2)}}\rangle\end{aligned} \tag{10.3.9}$$

while

$$\widetilde{Z}^r_m(\widetilde{\mathcal{E}}_2, \mathcal{G})|_{\mathrm{Sht}^{r,\circ}_S} = \chi(\det\widetilde{\mathcal{E}}_2)q^{?}\langle \mathrm{Sht}(f^{\perp})^{\circ}_{!}[\mathrm{Sht}^{r,\circ}_{U^{\perp}}], \mathfrak{q}_{21}^*\psi\cdot \mathbb{1}_{\mathrm{Sht}^{r,\circ}_{\widehat{V}(1)}}\cdot\delta_{\mathrm{Sht}^{r,\circ}_{\widehat{V}(2)}}\rangle, \tag{10.3.10}$$

the exponent "?" having been established to be the same in both expressions. Next, we note that $d(U/S) - d(f_0) = d(V/S) = d$, hence $d(U/S) + d(f_0) + d \equiv 0 \pmod 2$. Furthermore, in §2.3.9 we calculated that $\chi(\det\widetilde{\mathcal{E}}_1)\eta_{F'/F}(D_Q)^n = \chi(\det\widetilde{\mathcal{E}}_2)$. Therefore, (10.3.9) and (10.3.10) are equal. $\square$

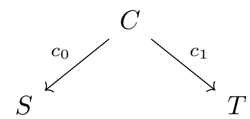
Appendix A. Derived Fourier analysis: proofs

This appendix justifies assertions made in §6 about the derived Fourier transform.

We remind the reader of the notational conventions introduced in §6 regarding derived vector bundles: for perfect complexes denoted with calligraphic letters such as $\mathcal{E}, \mathcal{E}'$, etc., the corresponding Roman letters such as E, E', etc. denote their associated total spaces.

A.1. **Preliminaries.** We record here some general facts about functors on sheaf categories induced by kernel sheaves. In particular, we introduce a configuration that we call a "butterfly", which induces natural transformations between such functors.

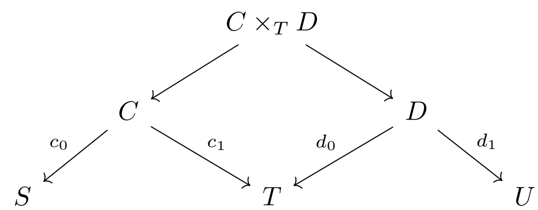
A.1.1. *Correspondences.* We begin with some preliminaries on correspondences. Given a correspondence



we associate the functor

$$\Phi_C := c_{1!}c_0^* \colon D(S) \to D(T).$$

Consider the fibered product of correspondences



Then by proper base change, we have a natural isomorphism of functors

$$\Phi_{C\times_T D} \cong \Phi_D \circ \Phi_C : D(S) \to D(U). \tag{A.1.1}$$

More generally, for any $\mathcal{K} \in D(C)$, we have a functor

$$\Phi_{C,\mathcal{K}} : c_{1!}(c_0^*(-) \otimes \mathcal{K}) \colon D(S) \to D(T) \tag{A.1.2}$$

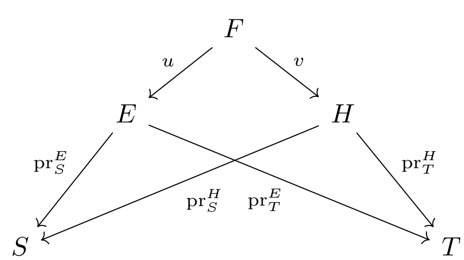
using $\mathcal{K}$ as the kernel sheaf. The functor considered in the previous paragraph corresponds to the special case $\mathcal{K} = \mathbf{Q}_{\ell,C}$. By proper base change, we have a natural isomorphism of functors $D(S) \to D(T)$

$$\Phi_{C,\mathcal{K}} \cong \Phi_{S\times T, c_!\mathcal{K}} \tag{A.1.3}$$

where $c = (c_0, c_1) : C \to S \times T$.

What is a general mechanism to construct natural transformations between such functors? For this we introduce the notion of a "butterfly", which is essentially a correspondence between correspondences.

A.1.2. *About butterflies.* Suppose we have a commutative diagram of derived stacks



(A.1.4)

where u is proper, v is quasi-smooth, and all morphisms are separated. We will refer to such a diagram as a *butterfly* (we codify this notion because many butterflies will come up). We also let $\mathrm{pr}_S^F \colon F \to S$ and $\mathrm{pr}_T^F \colon F \to T$ be the obvious maps. We will construct a natural transformation

$$\Phi_E = \mathrm{pr}_{T!}^E\,\mathrm{pr}_S^{E*} \xrightarrow{\bigstar} \Phi_{H}\langle -d(v)\rangle = \mathrm{pr}_{T!}^H\,\mathrm{pr}_S^{H*}\langle -d(v)\rangle \tag{A.1.5}$$

of functors $D(S) \to D(T)$, as the composition of the following maps.

(1) The unit for (u^*, u_*) induces

$$\mathrm{pr}^E_{T!}\,\mathrm{pr}^{E*}_S \to \mathrm{pr}^E_{T!}\,u_*u^*\,\mathrm{pr}^{E*}_S$$

(2) Since $u_! = u_*$ because u is proper, we have identifications

$$\mathrm{pr}^E_{T!}\,u_*u^*\,\mathrm{pr}^{E*}_S = \mathrm{pr}^E_{T!}\,u_!u^*\,\mathrm{pr}^{E*}_S = \mathrm{pr}^F_{T!}\,\mathrm{pr}^{F*}_S = \mathrm{pr}^H_{T!}\,v_!v^*\,\mathrm{pr}^{H*}_S\,.$$

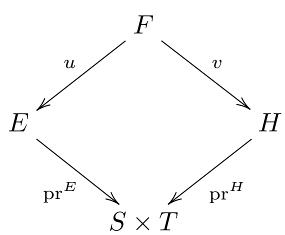
(3) Since v is quasi-smooth, its relative fundamental class induces (as explained in §3.4) a natural transformation $[v]\colon v^* \to v^!\langle -d(v)\rangle$, which gives a natural transformation

$$\mathrm{pr}^H_{T!}\,v_!v^*\,\mathrm{pr}^{H*}_S \to \mathrm{pr}^H_{T!}\,v_!v^!\langle -d(v)\rangle\,\mathrm{pr}^{H*}_S\,.$$

(4) The counit for $(v_!, v^!)$ gives a natural transformation

$$\mathrm{pr}^H_{T!}\,v_!v^!\langle -d(v)\rangle\,\mathrm{pr}^{H*}_S \to \mathrm{pr}^H_{T!}\,\mathrm{pr}^{H*}_S\,\langle -d(v)\rangle.$$

Remark A.1.1. Let $\mathrm{pr}^E = (\mathrm{pr}^E_S, \mathrm{pr}^E_T) : E \to S \times T$, and similarly define pr^H and pr^F. A butterfly of the form (A.1.4) can be viewed as a correspondence F between E and H over $S \times T$



(A.1.6)

The fundamental class of v gives a cohomological correspondence

$$\mathfrak{c}_v \in \mathrm{Corr}_F(\mathbf{Q}_{\ell,E}, \mathbf{Q}_{\ell,H}\langle -d(v)\rangle). \tag{A.1.7}$$

The assumption that u is proper implies that the map of correspondences $(\mathrm{pr}^E, \mathrm{pr}^F, \mathrm{pr}^H) : (E \leftarrow F \to H) \to (S \times T = S \times T = S \times T)$ is left pushable. Therefore

$$\begin{aligned} \mathrm{pr}^F_! : \mathrm{Corr}_F(\mathbf{Q}_{\ell,E}, \mathbf{Q}_{\ell,H}\langle -d(v)\rangle) &\to \mathrm{Corr}_{S\times T}(\mathrm{pr}^E_!\,\mathbf{Q}_{\ell,E}, \mathrm{pr}^H_!\,\mathbf{Q}_{\ell,H}\langle -d(v)\rangle) && \text{(A.1.8)} \\ &= \mathrm{Hom}_{S\times T}(\mathrm{pr}^E_!\,\mathbf{Q}_{\ell,E}, \mathrm{pr}^H_!\,\mathbf{Q}_{\ell,H}\langle -d(v)\rangle) && \text{(A.1.9)} \end{aligned}$$

is defined. In particular we get a map

$$\mathrm{pr}^F_!\,\mathfrak{c}_v : \mathrm{pr}^E_!\,\mathbf{Q}_{\ell,E} \to \mathrm{pr}^H_!\,\mathbf{Q}_{\ell,H}\langle -d(v)\rangle \tag{A.1.10}$$

which induces a natural transformation

$$\Phi_E = \Phi_{S\times T, \mathrm{pr}^E_!\,\mathbf{Q}_{\ell,E}} \to \Phi_{S\times T, \mathrm{pr}^H_!\,\mathbf{Q}_{\ell,H}\langle -d(v)\rangle} = \Phi_H\langle -d(v)\rangle. \tag{A.1.11}$$

Unwinding the definitions we see that this construction recovers the map $\bigstar$ in (A.1.5).

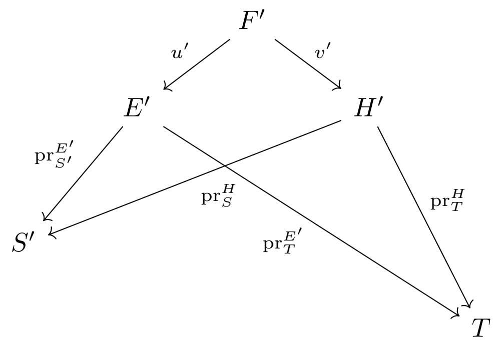
Next we establish some compatibilities between natural transformations induced by different butterflies.

A.1.3. *Butterfly and pushforward.* Consider the butterfly (A.1.4). Suppose we are given a map $f\colon S' \to S$. Then the construction in §A.1.2 gives a natural transformation

$$\Phi_E f_! \xrightarrow{\bigstar f_!} \Phi_H f_!\langle -d(v)\rangle \tag{A.1.12}$$

of functors $D(S') \to D(T)$.

On the other hand, we have another natural transformation of the same form coming from another butterfly. Consider the diagram where supscript $'$ means pullback along f.



This is also a butterfly (u' is proper and v' is quasi-smooth) and therefore gives another natural transformation

$$\Phi_{E'} \xrightarrow{\bigstar'} \Phi_{H'\langle -d(v')\rangle} = \Phi_{H'\langle -d(v)\rangle} \tag{A.1.13}$$

of functors $D(S') \to D(T)$.

The Cartesian squares

$$\begin{array}{ccc} S' & \xleftarrow{\mathrm{pr}^{E'}_{S'}} & E' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^E} \\ S & \xleftarrow[\mathrm{pr}^{E}_{S}]{} & E \end{array} \qquad \begin{array}{ccc} S' & \xleftarrow{\mathrm{pr}^{H'}_{S'}} & H' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^H} \\ S & \xleftarrow[\mathrm{pr}^{H}_{S}]{} & H \end{array}$$

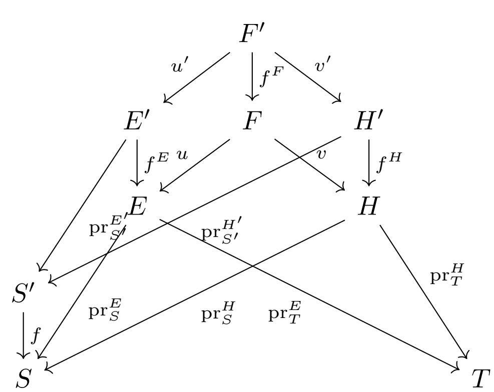
induce proper base change identifications of functors $D(S') \to D(T)$

$$\Phi_{E'} \cong \Phi_E f_!, \tag{A.1.14}$$

$$\Phi_{H'} \cong \Phi_H f_!. \tag{A.1.15}$$

Lemma A.1.2. *With respect to the identifications* (A.1.14) *and* (A.1.15)*, the two natural transformations* (A.1.12) *and* (A.1.13) *agree.*

Proof. Consider the diagram below, where all parallelograms are Cartesian.



This yields a diagram of natural transformations

$$
\begin{array}{ccccc}
\mathrm{pr}^{E}_{T!}\,\mathrm{pr}^{E*}_{S}\,f_! & \xrightarrow{\diamond_1} & \mathrm{pr}^{E}_{T!}\,f^{E}_!\,\mathrm{pr}^{E'*}_{S'} & = & \mathrm{pr}^{E'}_{T!}\,\mathrm{pr}^{E'*}_{S'} \\
\big\downarrow{\scriptstyle \mathrm{unit}(u)} & & \big\downarrow{\scriptstyle \mathrm{unit}(u)} & & \big\downarrow{\scriptstyle \mathrm{unit}(u')} \\
 & & & & \mathrm{pr}^{E'}_{T!}(u')_*(u')^*\,\mathrm{pr}^{E'*}_{S'} \\
 & & & & \| \\
\mathrm{pr}^{E}_{T!}\,u_*u^*\,\mathrm{pr}^{E*}_{S}\,f_! & \xrightarrow{\diamond_1} & \mathrm{pr}^{E}_{T!}\,u_*u^* f^{E}_!\,\mathrm{pr}^{E'*}_{S'} & \xrightarrow{\diamond_2} & \mathrm{pr}^{E}_{T!}\,u_* f^{F}_!(u')^*\,\mathrm{pr}^{E'*}_{S'} \\
\| & & & & \| \\
\mathrm{pr}^{F}_{T!}\,\mathrm{pr}^{F*}_{S}\,f_! & & \xrightarrow{\diamond_0} & & \mathrm{pr}^{F}_{T!}\,f^{F}_!\,\mathrm{pr}^{F'*}_{S'} \\
\| & & & & \| \\
\mathrm{pr}^{H}_{T!}\,v_!v^*\,\mathrm{pr}^{H*}_{S}\,f_! & \xrightarrow{\diamond_4} & \mathrm{pr}^{H}_{T!}\,v_!v^* f^{H}_!\,\mathrm{pr}^{H'*}_{S'} & \xrightarrow{\diamond_3} & \mathrm{pr}^{H}_{T!}\,v_! f^{F}_!(v')^*\,\mathrm{pr}^{H'*}_{S'} \\
\big\downarrow{\scriptstyle [v]} & & \big\downarrow{\scriptstyle [v]} & & \big\downarrow{\scriptstyle [v']} \\
\mathrm{pr}^{H}_{T!}\,v_!v^!\,\mathrm{pr}^{H*}_{S}\,f_!\langle -d(v)\rangle & \xrightarrow{\diamond_4} & \mathrm{pr}^{H}_{T!}\,v_!v^! f^{H}_!\,\mathrm{pr}^{H'*}_{S'}\langle -d(v)\rangle & \longleftarrow & \mathrm{pr}^{H}_{T!}\,v_! f^{F}_!(v')^!\,\mathrm{pr}^{H'*}_{S'}\langle -d(v)\rangle \\
 & & & & \| \\
\big\downarrow{\scriptstyle \mathrm{counit}(v)} & & \big\downarrow{\scriptstyle \mathrm{counit}(v)} & & \mathrm{pr}^{H}_{T!}\,f^{H}_! v'_!(v')^!\,\mathrm{pr}^{H'*}_{S'}\langle -d(v)\rangle \\
 & & & & \big\downarrow{\scriptstyle \mathrm{counit}(v')} \\
\mathrm{pr}^{H}_{T!}\,\mathrm{pr}^{H*}_{S}\,f_!\langle -d(v)\rangle & \xrightarrow{\diamond_4} & \mathrm{pr}^{H}_{T!}\,f^{H}_!\,\mathrm{pr}^{H'*}_{S'}\langle -d(v)\rangle & = & \mathrm{pr}^{H'}_{T!}\,\mathrm{pr}^{H'*}_{S'}\langle -d(v)\rangle
\end{array}
$$

The natural transformation (A.1.12) is the composition along the left column, while the natural transformation (A.1.13) is the composition along the right column. All the arrows labeled by $\diamond_?$ are isomorphisms,

(1) $\diamond_0$ is induced by the base change natural isomorphism for the Cartesian square

$$
\begin{array}{ccc}
S' & \longleftarrow & F' \\
\downarrow & & \downarrow \\
S & \longleftarrow & F
\end{array}
$$

(2) $\diamond_1$ is induced by the base change natural isomorphism for the Cartesian square

$$
\begin{array}{ccc}
S' & \longleftarrow & E' \\
\downarrow & & \downarrow \\
S & \longleftarrow & E
\end{array}
$$

(3) $\diamond_2$ is induced by the base change natural isomorphism for the Cartesian square

$$
\begin{array}{ccc}
E' & \longleftarrow & F' \\
\downarrow & & \downarrow \\
E & \longleftarrow & F
\end{array}
$$

(4) $\diamond_3$ is induced by the base change natural isomorphism for the Cartesian square

$$
\begin{array}{ccc}
H' & \longleftarrow & F' \\
\downarrow & & \downarrow \\
H & \longleftarrow & F
\end{array}
$$

(5) $\diamond_4$ is induced by the base change natural isomorphism for the Cartesian square

$$\begin{array}{ccc} S' & \longleftarrow & H' \\ \downarrow & & \downarrow \\ S & \longleftarrow & H \end{array}$$

It remains to show that the above diagram is commutative. The only non-obvious squares to check are the two wide rectangles in the middle. Their commutativity follows from the observation that the two diagrams

$$\begin{array}{ccccc} S' & \longleftarrow & E' & \longleftarrow & F' \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & E & \longleftarrow & F \end{array}$$

and

$$\begin{array}{ccccc} S' & \longleftarrow & H' & \longleftarrow & F' \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & H & \longleftarrow & F \end{array}$$

give two decompositions of the same Cartesian square

$$\begin{array}{ccc} S' & \longleftarrow & F' \\ \downarrow & & \downarrow \\ S & \longleftarrow & F \end{array}$$

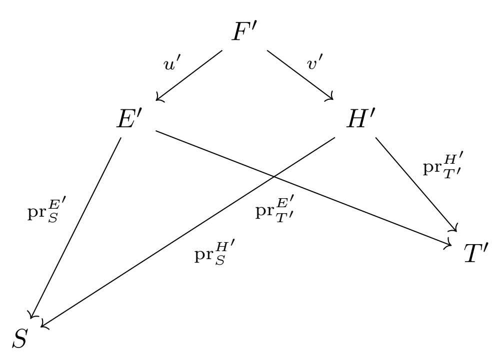
so that the composition of their base change natural isomorphisms agree. □

A.1.4. *Butterfly and pullback.* Next suppose we are given $g\colon T' \to T$. Then the construction in §A.1.2 gives a natural transformation

$$g^*\Phi_E = g^* \operatorname{pr}^E_{T!} \operatorname{pr}^{E*}_S \xrightarrow{g^*\bigstar} g^*\Phi_H\langle -d(v)\rangle = g^* \operatorname{pr}^H_{T!} \operatorname{pr}^{H*}_S \langle -d(v)\rangle \tag{A.1.16}$$

of functors $D(S) \to D(T')$.

On the other hand, we have another natural transformation of the same form coming from another butterfly. Consider the diagram where superscript $'$ means pullback along g,



This is also a butterfly (u' is proper and v' is quasi-smooth) and therefore gives another natural transformation

$$\Phi_{E'} \xrightarrow{\bigstar'} \Phi_{H'}\langle -d(v')\rangle = \Phi_{H'}\langle -d(v)\rangle \tag{A.1.17}$$

of functors $D(S) \to D(T')$.

The Cartesian squares

$$\begin{array}{ccc} E' & \longrightarrow & T' \\ \downarrow{\scriptstyle g^E} & & \downarrow{\scriptstyle g} \\ E & \longrightarrow & T \end{array} \qquad \begin{array}{ccc} H' & \longrightarrow & T' \\ \downarrow{\scriptstyle g^H} & & \downarrow{\scriptstyle g} \\ H & \longrightarrow & T \end{array}$$

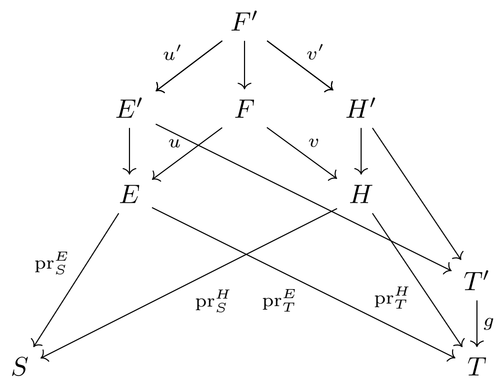
induce proper base change identifications of functors $D(S') \to D(T)$

$$\Phi_{E'} \cong g^* \Phi_E, \tag{A.1.18}$$

$$\Phi_{H'} \cong g^* \Phi_H. \tag{A.1.19}$$

Lemma A.1.3. *With respect to the identifications* (A.1.18) *and* (A.1.19), *the two natural transformations* (A.1.16) *and* (A.1.17) *agree.*

Proof. The proof is analogous to that for Lemma A.1.2, considering instead the diagram below, all of whose parallelograms are Cartesian.



□

A.2. **Functorialities.** The running notational conventions are now restored: E, E' are derived vector bundles over the base S. Let $f\colon E' \to E$ be a map of derived vector bundles over a derived stack S. Let r be the rank of E and r' be the rank of E'. We begin by establishing some of the easier functorialities for the derived Fourier transform.

A.2.1. $\mathrm{FT}\, f_!$ *versus* $\widehat{f}^*\,\mathrm{FT}$.

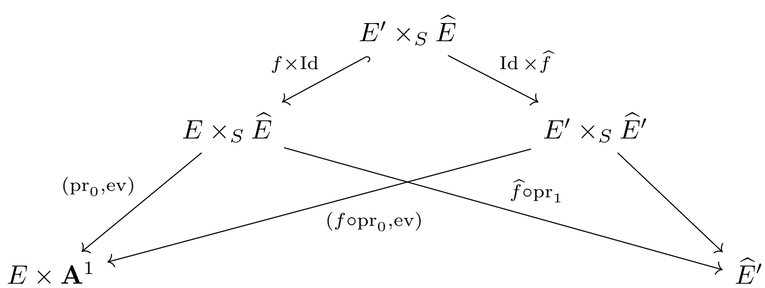
Lemma A.2.1. *We have a natural isomorphism of functors* $D(E') \to D(\widehat{E})$

$$\mathrm{FT}_E \circ f_![r'-r] \cong \widehat{f}^* \circ \mathrm{FT}_{E'}.$$

Proof. The proof is the same as for classical vector bundles, cf. [Lau87, Théorème 1.2.2.4]. (In fact the same proof works if we replace $\mathcal{L}_\psi$ by any $\mathcal{L} \in D(\mathbf{A}^1)$.) □

A.2.2. $\mathrm{FT}\, f^*$ *versus* $\widehat{f}_!\,\mathrm{FT}$*: special cases.* Assume that $f\colon E' \to E$ is either a closed embedding or is smooth. We explicate that since f is a map of derived vector bundles, f is a closed embedding if and only if $\mathbf{T}_f$ has tor-amplitude in $[1,\infty)$, and f is smooth if and only if $\mathbf{T}_f$ has tor-amplitude in $(-\infty, 0]$. In particular, f is a closed embedding if and only if $\widehat{f}$ is smooth.

Therefore, if f is a closed embedding then the diagram



is a butterfly. Composing its natural transformation ★ with the functor $D(E) \to D(E \times \mathbf{A}^1)$ given by $\mathcal{K} \mapsto \mathcal{K} \boxtimes \mathcal{L}_\psi$, it induces a natural transformation

$$\mathrm{FT}_{E'} \circ f^* \leftarrow \widehat{f}_! \circ \mathrm{FT}_E[r-r'](r-r'). \tag{A.2.1}$$

Lemma A.2.2. *If f is a closed embedding, then the natural transformation* (A.2.1) *is an isomorphism.*

Proof. Examining the definition of (A.2.1), and using that $\widehat{f}$ is smooth, it suffices to check that the natural map

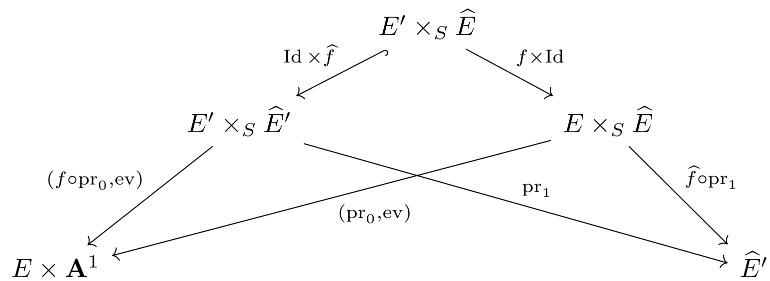
$$\widehat{f}_! \operatorname{pr}_{1!}(\operatorname{pr}_0^* \mathcal{K} \otimes \operatorname{ev}^* \mathcal{L}_\psi) \to \widehat{f}_! \operatorname{pr}_{1!} u_* u^*(\operatorname{pr}_0^* \mathcal{K} \otimes \operatorname{ev}^* \mathcal{L}_\psi)$$

is an isomorphism for all $\mathcal{K} \in D(E)$.

Let $U := E \setminus E'$ and $j \colon U \times_S \widehat{E} \hookrightarrow E \times_S \widehat{E}$ be the open complement of $u = f \times \mathrm{Id}$. Then it suffices to check that $\widehat{f}_! \operatorname{pr}_{1!} j_! j^*(\operatorname{pr}_0^* \mathcal{K} \otimes \operatorname{ev}^* \mathcal{L}_\psi) = 0$.

For this we can localize over stalks of the base S and thus assume that S is a geometric point. Let C be the cone of f; then $\widehat{C}$ is the (derived) fiber of $\widehat{f}$. The fiber of $\widehat{f}_! \operatorname{pr}_{1!}$ over $y \in \widehat{E}'$ is $E \times (\widehat{C} + y)$, so by proper base change the stalk of $\widehat{f}_! \operatorname{pr}_{1!} j_! j^*(\operatorname{pr}_0^* \mathcal{K} \otimes \operatorname{ev}^* \mathcal{L}_\psi)$ at y has cohomology groups $H_c^*(U \times (\widehat{C} + y), \operatorname{pr}_0^* \mathcal{K} \otimes \operatorname{ev}^* \mathcal{L}_\psi)$, which we want to show vanish. By the projection formula, it suffices to show that for the first projection map $\operatorname{pr}_0 \colon U \times (\widehat{C} + y) \to U$, we have $\operatorname{pr}_{0!} \operatorname{ev}^* \mathcal{L}_\psi = 0$. Indeed, by definition any geometric point $u \in U$ has non-zero image in C, so $\operatorname{ev}^* \mathcal{L}_\psi|_{\operatorname{pr}_0^{-1}(u)}$ is a non-trivial character sheaf, so its cohomology vanishes. ☐

If f is smooth, then $\widehat{f} \colon \widehat{E} \to \widehat{E}'$ is a closed embedding and the diagram



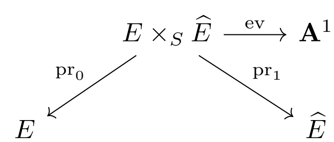
is a butterfly, so it induces a natural transformation

$$\operatorname{FT}_{E'} \circ f^* \to \widehat{f}_! \circ \operatorname{FT}_E[r - r'](r - r'). \tag{A.2.2}$$

A similar argument as for Lemma A.2.2 shows that

Lemma A.2.3. *If f is smooth, then the natural transformation* (A.2.2) *is an isomorphism.*

A.2.3. *The right adjoint of* FT. Let $'\operatorname{FT}_E^\psi \colon D(E) \to D(\widehat{E})$ be the functor $\mathcal{K} \mapsto \operatorname{pr}_{1*}(\operatorname{ev}^* \mathcal{L}_\psi \otimes \operatorname{pr}_0^!(\mathcal{K}))[-r]$, where maps are as in the diagram



By the compatibility of right adjoints with composition, $'\operatorname{FT}_{\widehat{E}}^{-\psi} \colon D(\widehat{E}) \to D(E)$ is the right adjoint of FT_E^ψ. When ψ is understood we abbreviate $'\operatorname{FT}_{\widehat{E}} :=' \operatorname{FT}_{\widehat{E}}^\psi$.

Taking the right adjoint of Lemma A.2.1 (and interchanging ψ with $-\psi$) gives a natural isomorphism

$$f^! \circ ('\operatorname{FT}_{\widehat{E}})[r - r'] \cong ('\operatorname{FT}_{\widehat{E}'}) \circ \widehat{f}_*. \tag{A.2.3}$$

This can also be proved directly by a similar argument as for Lemma A.2.1.

We are especially interested in the case where $\widehat{f}$ is a closed embedding (and dually f is smooth), in which case we can replace $\widehat{f}_*$ with $\widehat{f}_!$ and $f^!$ with $f^*\langle d(f)\rangle$ above.

Taking the right adjoint of (A.2.1) and (A.2.2) (and interchanging ψ with $-\psi$) gives a natural isomorphism

$$f_* \circ ('\operatorname{FT}_{\widehat{E}'})[r - r'](r - r') \cong ('\operatorname{FT}_{\widehat{E}}) \circ \widehat{f}^! \tag{A.2.4}$$

if f is smooth or a closed embedding. Alternatively, this isomorphism can be proved directly using analogous constructions to those in §A.1.2 and §A.2.2.

We are especially interested in the case where $\widehat{f}$ is smooth (and dually f is a closed embedding), in which case we can replace $\widehat{f}^!$ with $\widehat{f}^*\langle d(f)\rangle$ and f_* with $f_!$ above.

A.2.4. *Involutivity.* The key to the functoriality properties of the derived Fourier transform is the (near) involutivity property. This appears to be significantly more subtle to establish than in the situation for classical vector bundles. It will be done later in §A.3.

Lemma A.2.4. *Let E be a derived vector bundle over S of rank r. Recall δ_E denotes $z_! \mathbf{Q}_{\ell,S}$ where $z : S \to E$ is the zero section. Then any isomorphism*

$$\alpha_E : \delta_E \cong \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r) \tag{A.2.5}$$

determines an isomorphism of functors

$$\mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E \cong [-1]^*(-r). \tag{A.2.6}$$

Proof. Unravelling the definition of $\mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E$ and using proper base change, we see that it is the endofunctor of $D(E)$ given by the kernel sheaf $\mathrm{pr}_{02!}(\mathrm{ev}_{01} + \mathrm{ev}_{12})^* \mathcal{L}_\psi[2r] \in D(E \times E)$, where

- $\mathrm{pr}_{02} : E \times_S \widehat{E} \times_S E \to E \times_S E$ is the projection to the first and third factors, and
- $\mathrm{ev}_{01} + \mathrm{ev}_{12} : E \times_S \widehat{E} \times_S E \to \mathbf{A}^1$ is given by $(x, y, z) \mapsto \langle x + z, y\rangle$.

On the other hand, $[-1]^* : D(E) \to D(E)$ is given by the kernel sheaf $\Delta^-_! \mathbf{Q}_{\ell,E}$, where $\Delta^- : E \to E \times_S E$ is the anti-diagonal. To give an isomorphism (A.2.6), it therefore suffices to give an isomorphism of kernel sheaves

$$\mathrm{pr}_{02!}(\mathrm{ev}_{01} + \mathrm{ev}_{12})^* \mathcal{L}_\psi[2r] \cong \Delta^-_! \mathbf{Q}_{\ell,E}(-r). \tag{A.2.7}$$

If we let $a : E \times_S E \to E$ be the addition map, then proper base change for the Cartesian square

$$\begin{array}{ccc} E \times_S \widehat{E} \times_S E & \xrightarrow{(\mathrm{pr}_0 + \mathrm{pr}_2, \mathrm{pr}_1)} & E \times_S \widehat{E} \\ \downarrow{\scriptstyle \mathrm{pr}_{02}} & & \downarrow{\scriptstyle \mathrm{pr}_E} \\ E \times_S E & \xrightarrow{a} & E \end{array}$$

supplies an isomorphism

$$\mathrm{pr}_{02!}(\mathrm{ev}_{01} + \mathrm{ev}_{12})^* \mathcal{L}_\psi[2r] \cong a^* \,\mathrm{pr}_{E!}\, \mathrm{ev}^* \,\mathcal{L}_\psi[2r] \tag{A.2.8}$$

Also, proper base change for the Cartesian square

$$\begin{array}{ccc} E & \longrightarrow & S \\ \downarrow{\scriptstyle \Delta^-} & & \downarrow{\scriptstyle z} \\ E \times_S E & \xrightarrow{a} & E \end{array} \tag{A.2.9}$$

supplies an isomorphism

$$\Delta^-_! \mathbf{Q}_{\ell,E} \cong a^* \delta_E. \tag{A.2.10}$$

Now, an isomorphism $\alpha_E \colon \delta_E(-r) \cong \mathrm{pr}_{E!}\, \mathrm{ev}^* \,\mathcal{L}_\psi[2r]$ induces an isomorphism $a^*\delta_E(-r) \cong a^* \,\mathrm{pr}_{E!}\, \mathrm{ev}^* \,\mathcal{L}_\psi[2r]$. Combining this with (A.2.8) and (A.2.10), we get an isomorphism (A.2.7). Since the canonicity of (A.2.7) will be a recurring theme, we note that it only depended on α_E. □

Definition A.2.5. Let $E \to S$ be a derived vector bundle of virtual rank r. An *involutivity datum* on E a pair of natural isomorphisms

$$\eta_E \colon \mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E \xrightarrow{\sim} [-1]^*(-r) \quad \text{and} \quad \eta_{\widehat{E}} \colon \mathrm{FT}_E \circ \mathrm{FT}_{\widehat{E}} \xrightarrow{\sim} [-1]^*(-r)$$

such that the composition

$$\mathrm{FT}_E \xrightarrow{\mathrm{FT}_E \circ \eta_E^{-1}} \mathrm{FT}_E \circ \mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E[-1]^*(r) \xrightarrow{\eta_{\widehat{E}} \circ \mathrm{FT}_E} \mathrm{FT}_E$$

is multiplication by $(-1)^r$.

Remark A.2.6. We will later equip every derived vector bundle with a canonical involutivity datum, but the construction is rather circuitous; for example, it will involve first constructing involutivity data that (a priori seem to) depend on auxiliary choices.

A.2.5. *Self-adjointness.* In §A.2.3 we defined $'\mathrm{FT}_E$, and we explained that $'\mathrm{FT}_{\widehat{E}}^{-\psi}$ was the right adjoint to FT_E^{ψ}.

Lemma A.2.7. *Let $E \to S$ be a derived vector bundle of virtual rank r. Any involutivity datum $(\eta_E, \eta_{\widehat{E}})$ on E induces natural isomorphisms*

$$\mathrm{FT}_E \xrightarrow{\sim} ('\mathrm{FT}_E)(-r).$$

and

$$\mathrm{FT}_{\widehat{E}} \xrightarrow{\sim} ('\mathrm{FT}_{\widehat{E}})(-r).$$

Proof. The given data of η_E and $\eta_{\widehat{E}}$ show that $\mathrm{FT}_E^{\psi}(r)$ and $\mathrm{FT}_{\widehat{E}}^{-\psi}$ are inverses, and in particular adjoints. Since $'\mathrm{FT}_E$ was by definition the right adjoint of $\mathrm{FT}_{\widehat{E}}^{-\psi}(r)$, this induces the natural isomorphism $\mathrm{FT}_E \xrightarrow{\sim} ('\mathrm{FT}_E)(-r)$; the other isomorphism is obtained similarly. □

Example A.2.8 (Compatibility with Verdier duality)**.** Let $E \to S$ be a derived vector bundle of rank r and $\mathbf{D}_E$ (resp. $\mathbf{D}_{\widehat{E}}$) denote the Verdier duality functor on E (resp. $\widehat{E}$). We have

$$\mathbf{D}_{\widehat{E}} \circ \mathrm{FT}_E^{\psi} \cong ('\mathrm{FT}_E^{-\psi}) \circ \mathbf{D}_E.$$

By Lemma A.2.7, an involutivity datum for E equips FT_E and $\mathrm{FT}_{\widehat{E}}$ with natural isomorphisms

$$\mathbf{D}_{\widehat{E}} \circ \mathrm{FT}_E^{\psi} \cong \mathrm{FT}_E^{-\psi} \circ \mathbf{D}_E(\mathcal{K})(r).$$

A.2.6. *More functoriality.* Let $f \colon E' \to E$ be a morphism of derived vector bundles over S. Assume that E, E' are equipped with involutivity data. We may then produce the remaining natural isomorphisms claimed in §6.2.4.

Recall that we always had the natural isomorphism

$$\mathrm{FT}_E \circ f_![r'-r] \cong \widehat{f}^* \circ \mathrm{FT}_{E'} \tag{A.2.11}$$

without any assumptions. Taking right adjoints in (A.2.11), applying Lemma A.2.7, and relabeling terms gives

$$\widehat{f}^! \circ \mathrm{FT}_{E'} \cong \mathrm{FT}_E \circ f_*[r-r'](r-r').$$

Pre-composing (A.2.11) with $\mathrm{FT}_{\widehat{E}'}$ and post-composing with $\mathrm{FT}_{\widehat{E}}$, applying $(-1)^{\mathrm{rank}\, E}\eta_{\widehat{E}} \colon [-1]^*(-r) \xrightarrow{\sim} \mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E$ and $\eta_{\widehat{E}'} \colon \mathrm{FT}_{E'} \circ \mathrm{FT}_{\widehat{E}'} \xrightarrow{\sim} [-1]^*(-r')$, and re-labeling terms, gives the isomorphism

$$\mathrm{FT}_{E'} \circ f^* \cong \widehat{f}_! \circ \mathrm{FT}_E[r-r'](r-r'), \tag{A.2.12}$$

Taking right adjoints in (A.2.12), applying Lemma A.2.7, and relabeling terms gives

$$\mathrm{FT}_{E'} \circ f^! \cong \widehat{f}_* \circ \mathrm{FT}_E[r'-r].$$

A.3. **Involutivity.** We will construct a canonical involutivity datum for any derived vector bundle $E \to S$. According to Lemma A.2.4, it suffices to produce isomorphisms

$$\alpha_E \colon \delta_E \cong \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r), \quad \text{and} \quad \alpha_{\widehat{E}} \colon \delta_{\widehat{E}} \cong \mathrm{FT}_E(\mathbf{Q}_{\ell,E})[r](r).$$

A.3.1. *The case where S is a point.* Suppose S is a point (i.e., the spectrum of a field). We will produce canonical $\alpha_{\widehat{E}}$; the dual argument produces α_E. This analysis will be used later to prove involutivity in general, and may be illuminating in any case.

Since S is a point, $\mathcal{E}$ is quasi-isomorphic to a formal complex $\bigoplus_i \mathcal{E}^i[-i]$, with vanishing differentials. By the compatibility of FT with exterior tensor products (namely, that Fourier transform on a product of bundles takes an exterior tensor product of complexes to the exterior tensor product of the Fourier transforms of each factor), it suffices to a treat the case where $\mathcal{E} = \mathcal{E}^i[-i]$, so we assume this to be the case.

If $i = 0$, then $\alpha_{\widehat{E}}$ is the classical one of Laumon implicit in the proof of [Lau87, Théorème 1.2.2.1]. We will therefore focus on the case $i \neq 0$.

- If $i > 0$, then E has classical truncation S. Therefore, π_E^* induces an equivalence $D(S) \xrightarrow{\sim} D(E)$, whose inverse is its right adjoint π_{E*}. We then have $\pi_E^* = \pi_E^! \colon D(S) \xrightarrow{\sim} D(E)$ and $\pi_{E*} = \pi_{E!} \colon D(E) \xrightarrow{\sim} D(S)$. Since $\pi_E \circ z_E = \mathrm{Id}_S$, we deduce $z_{E*} = \pi_E^*$, $z_E^* = \pi_{E*}$, $z_{E!} = z_{E*}$, and $z_E^! = z_E^*$.

- If $i < 0$, then E is the i-fold iterated classifying stack of a product of copies of $\mathbf{G}_a$. Since $\mathbf{G}_a$ is a connected unipotent group scheme, π_E^* induces an equivalence $D(S) \xrightarrow{\sim} D(E)$, whose inverse is therefore π_{E*}. Since π_E is smooth of rank r, we have $\pi_E^! = \pi_E^* \langle r \rangle$ and $\pi_{E!} \cong \pi_{E*}\langle -r \rangle$. Since $\pi_E \circ z_E = \mathrm{Id}_S$, we deduce that $z_{E*} \cong \pi_E^*$, $z_E^* \cong \pi_{E*}$, $z_E^! \cong z_E^*\langle -r \rangle$, and $z_{E!} \cong z_{E*}\langle r \rangle$.

Below we will always use π_E^* to identify $D(S) \cong D(E)$ and $\pi_{\widehat{E}}^*$ to identify $D(S) \cong D(\widehat{E})$. We will therefore view FT_E as an endofunctor of $D(S)$.

- If $i > 0$, then under the above identifications FT_E is simply the endofunctor $[r]$ of $D(S)$, so $\mathrm{FT}_E(\mathbf{Q}_{\ell,E})$ identifies with $\mathbf{Q}_{\ell,S}[r] \in D(S)$. On the other hand, $\delta_{\widehat{E}} = z_{\widehat{E}!}\mathbf{Q}_{\ell,S}$ identifies with $\mathbf{Q}_{\ell,S}\langle r \rangle \in D(S)$. Thus we obtain $\alpha_{\widehat{E}}$.
- If $i < 0$, then under the above identifications FT_E is the endofunctor $[-r](-r)$ of $D(S)$, so $\mathrm{FT}_E(\mathbf{Q}_{\ell,E})$ identifies with $\mathbf{Q}_{\ell,S}[-r](-r) \in D(S)$. On the other hand, $\delta_{\widehat{E}} = z_{\widehat{E}!}\mathbf{Q}_{\ell,S}$ identifies with $\mathbf{Q}_{\ell,S}$. We take the $\alpha_{\widehat{E}}$ to be $(-1)^r$ times the obvious isomorphism.

A.3.2. *Bootstrapping from vector bundles.* Suppose $\mathcal{E} \in \mathrm{Perf}(S)$ admits a global presentation by a finite complex of vector bundles on S, say of the form

$$\ldots \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \ldots$$

We will use the following device to bootstrap from the theory for classical vector bundles, where the theory was established by Laumon [Lau87]. Consider the "stupid truncations"

$$\mathcal{E}^{\geq 0} = (\mathcal{E}^0 \to \mathcal{E}^1 \to \ldots)$$

and

$$\mathcal{E}^{\leq 0} = (\ldots \to \mathcal{E}^{-1} \to \mathcal{E}^0)$$

and let $E^{\geq 0}, E^{\leq 0}$ be the associated derived vector bundles. Then we have a pullback (and pushout) square in $\mathrm{Perf}(S)$

$$\begin{array}{ccc} \mathcal{E}^{\geq 0} & \longrightarrow & \mathcal{E}^0 \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}^{\leq 0} \end{array}$$

which induces a derived Cartesian square of total spaces

$$\begin{array}{ccc} E^{\geq 0} & \xhookrightarrow{i'} & E^0 \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ E & \xhookrightarrow{i} & E^{\leq 0} \end{array} \tag{A.3.1}$$

Note that $E^0 \to S$ is a classical vector bundle, while $E^{\leq 0} \to S$ is smooth (and represented by classical stacks), and $E^{\geq 0} \to S$ is separated (and representable in derived schemes). Here i is a closed embedding, so $i_! = i_* \colon D(E) \hookrightarrow D(E^{\geq 0})$ is fully faithful and similarly for i'. Also, p is smooth with acyclic geometric fibers, so $p^* \colon D(E^{\geq 0}) \hookrightarrow D(E^0)$ is fully faithful and similarly for p'. Hence we have a fully faithful embedding

$$p^* i_! \colon D(E) \hookrightarrow D(E^0). \tag{A.3.2}$$

A.3.3. *Global presentations.* We now establish involutivity for derived vector bundles E that admit a global presentation. Note that, by the definition of a perfect complex, any derived vector bundles admits a global presentation locally on the base S.

Lemma A.3.1. *Suppose E is a derived vector bundle.*
(1) Any global presentation $E^\bullet$ of E induces an involutivity datum on E.
(2) If S is a point, then the involutivity datum from (1) is naturally isomorphic to that from §A.3.1.

Proof. (1) As in (A.3.2) we have a fully faithful embedding $p^* i_! \colon D(E) \hookrightarrow D(E^0)$. We produce a natural isomorphism

$$p^* i_! \, \mathrm{FT}_{\widehat{E}} \circ \mathrm{FT}_E \cong p^* i_! [-1]^*(-r).$$

By construction the maps $p, \widehat{i}$ are smooth and the maps $i, \widehat{p}$ are closed embeddings, so that we may apply the functorialities in §A.2 to produce a sequence of natural isomorphisms:

$$\begin{aligned} p^* i_! \operatorname{FT}_{\widehat{E}} \operatorname{FT}_E &\cong p^* \operatorname{FT}_{\widehat{E^{\leq 0}}} \widehat{i}^* \operatorname{FT}_E (?)[?] \cong \operatorname{FT}_{\widehat{E^0}} \widehat{p}_! \widehat{i}^* \operatorname{FT}_E (?)[?] \\ &\cong \operatorname{FT}_{\widehat{E^0}} \widehat{p}_! \operatorname{FT}_{E^{\leq 0}} i_! (?)[?] \cong \operatorname{FT}_{\widehat{E^0}} \operatorname{FT}_{E_0} p^* i_! (?)[?] \end{aligned}$$

where the shifts and twists are suppressed. Now using the canonical involutivity datum $\eta_{E^0} \colon \operatorname{FT}_{\widehat{E^0}} \operatorname{FT}_{E_0} \xrightarrow{\sim} [-1]^*(-r_0)$ for $r_0 := \operatorname{rank}(E_0)$, and the full faithfulness of $p^* i_!$, this reflects to an isomorphism $\eta_E \colon \operatorname{FT}_{\widehat{E}} \operatorname{FT}_E \xrightarrow{\sim} [-1]^*(-r)$. The natural isomorphism $\eta_{\widehat{E}}$ is obtained by the same argument on the dual bundle. The sign condition for $\eta_E, \eta_{\widehat{E}}$ reduces to that for $\eta_{E_0}, \eta_{\widehat{E_0}}$, which is classical.

(2) By construction, the compatibility reduces to Lemma A.3.2 below. □

Lemma A.3.2. *Let $f \colon E' \to E$ a map of derived vector bundles.*

(1) Suppose f is a closed embedding. Then the diagram

$$\begin{array}{ccc} f_! \operatorname{FT}_{\widehat{E}'} \operatorname{FT}_{E'} & \xrightarrow[\sim]{\eta_{\widehat{E}'}} & f_![-1]^*(-r') \\ \downarrow \wr & & \| \\ \operatorname{FT}_{\widehat{E}} \widehat{f}^* \operatorname{FT}_E [r'-r](r'-r) & & \| \\ \downarrow \wr & & \| \\ \operatorname{FT}_{\widehat{E}'} \operatorname{FT}_{E'} f_! (r-r') & \xrightarrow[\eta_{\widehat{E}'}]{\sim} & f_![-1]^*(-r') \end{array} \tag{A.3.3}$$

commutes.

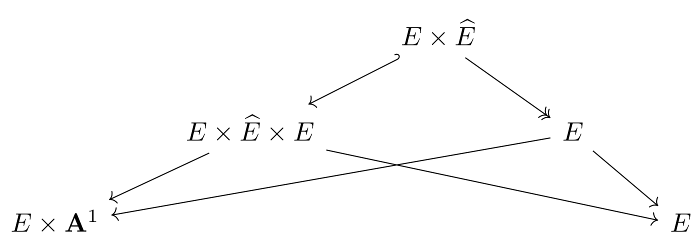
(2) Suppose f is smooth. Then the diagram

$$\begin{array}{ccc} f^* \operatorname{FT}_{\widehat{E}} \operatorname{FT}_E & \xrightarrow[\sim]{\eta_{\widehat{E}}} & f^*[-1]^*(-r) \\ \downarrow \wr & & \| \\ \operatorname{FT}_{\widehat{E}} \widehat{f}_! \operatorname{FT}_E [r'-r] & & \| \\ \downarrow \wr & & \| \\ \operatorname{FT}_{\widehat{E}'} \operatorname{FT}_{E'} f^* [r'-r](r'-r) & \xrightarrow[\eta_{\widehat{E}'}]{\sim} & f^*[-1]^*(-r') \end{array} \tag{A.3.4}$$

commutes.

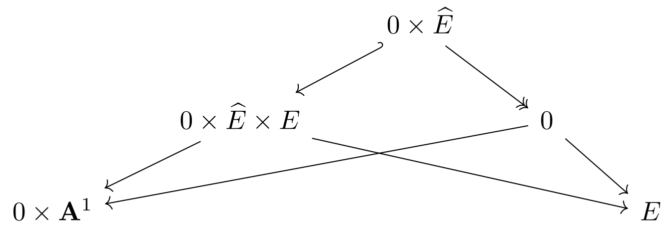
Proof. The two situations are similar so we just prove (1). The two paths are each given by an isomorphism of the respective kernel sheaves with the constant sheaf on the graph of f in $E' \times_S E$. Hence they differ by a scalar in $H^0(S, \mathbf{Q}_{\ell,S})$. We can compute this scalar locally, thus reducing to $S = \mathrm{pt}$ by base change. Then E and E' factor as a product of derived vector bundles concentrated in degree i, so we may reduce to the case where they are each in degree i. If $i \neq 0$, then the Lemma follows from tracing through the explicit descriptions in §A.3.1. We henceforth focus on the case where $i = 0$, so E and E' are classical vector bundles. Observe that we can compute the scalar in question on any non-zero object; we will take the object $\delta_{E'}$. This reduces to the case where $E' = 0$.

The isomorphism $\eta_E \colon \operatorname{FT}_{\widehat{E}} \operatorname{FT}_E \xrightarrow{\sim} [-1]^*(-r)$ is the map induced by the butterfly



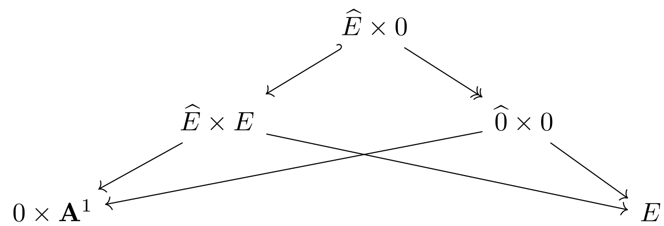
(A.3.5)

using the Artin-Schreier sheaf on $\mathbf{A}^1$. According to §A.1.3, the bottom row of (A.3.3) is then given by the butterfly obtained by pulling back along $f\colon 0 \hookrightarrow E$:



(A.3.6)

Tracing through §A.2.2, we find that the isomorphism $\mathrm{FT}_{\widehat{E}}\,\mathrm{FT}_E\, f_! \xrightarrow{\sim} f_![-1]^*(-r)$ given by going around the left and top of (A.3.3) is also given by a butterfly:



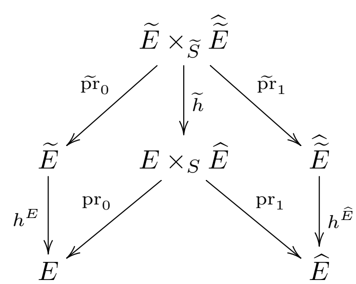
(A.3.7)

By inspection, this agrees with the butterfly (A.3.6). □

A.3.4. *Base change.* Consider the setup of §6.2.2 where we perform a base change $h : \widetilde{S} \to S$. We will prove the isomorphisms (6.2.1), (6.2.3), (6.2.4) and (6.2.2) *without any assumptions.* (At an intermediate stage the proof invokes Lemma A.3.1, and the proof is used subsequently to construct the canonical involutivity datum for general derived vector bundles, which explains why it appears here.)

The isomorphisms (6.2.1) and (6.2.3) follow directly from proper base change.

Now consider (6.2.4). Consider the commutative diagram



(A.3.8)

We have a natural transformation

$$\mathrm{FT}_E \circ h^E_* \to h^{\widehat{E}}_* \circ \mathrm{FT}_{\widetilde{E}} \tag{A.3.9}$$

as the composition (for any $\mathcal{K} \in D(\widetilde{E})$)

$$\mathrm{pr}_{1!}(\mathrm{pr}_0^*\, h^E_* \mathcal{K} \otimes \mathrm{ev}^*\, \mathcal{L}_\psi) \xrightarrow{(1)} \mathrm{pr}_{1!}(\widetilde{h}_* \widetilde{\mathrm{pr}}_0^* \mathcal{K} \otimes \mathrm{ev}^*\, \mathcal{L}_\psi) \cong \mathrm{pr}_{1!}\, \widetilde{h}_*(\widetilde{\mathrm{pr}}_0^* \mathcal{K} \otimes \widetilde{\mathrm{ev}}^* \mathcal{L}_\psi) \xrightarrow{(2)} h^{\widehat{E}}_* \widetilde{\mathrm{pr}}_{1!}(\widetilde{\mathrm{pr}}_0^* \mathcal{K} \otimes \widetilde{\mathrm{ev}}^* \mathcal{L}_\psi). \tag{A.3.10}$$

Here (1) is the natural transformation $\mathrm{pr}_0^*\, h^E_* \to \widetilde{h}_* \widetilde{\mathrm{pr}}_0^*$ coming from the left parallelogram in (A.3.8), and (2) is the natural transformation $\mathrm{pr}_{1!}\, \widetilde{h}_* \to h^{\widehat{E}}_* \widetilde{\mathrm{pr}}_{1!}$ obtained by adjunction from the proper base change isomorphism attached to the right parallelogram in (A.3.8). To check (A.3.9) is an isomorphism, one can work Zariski locally over S, hence reducing to the case where E admits a global presentation. In this case, using Lemma A.2.7, we may replace FT with ${}'\mathrm{FT}$ in (A.3.9) (up to a twist), which then is visibly an isomorphism by proper base change.

Now consider (6.2.2). We have a natural transformation

$$\mathrm{FT}_{\widetilde{E}} \circ (h^E)^! \to (h^{\widehat{E}})^! \circ \mathrm{FT}_E \tag{A.3.11}$$

as the composition (for any $\mathcal{K} \in D(E)$)

$$\widetilde{\mathrm{pr}}_{1!}(\widetilde{\mathrm{pr}}_0^*(h^E)^!\mathcal{K}\otimes\widetilde{\mathrm{ev}}^*\mathcal{L}_\psi) \xrightarrow{(1)} \widetilde{\mathrm{pr}}_{1!}(\widetilde{h}^!\,\mathrm{pr}_0^*\,\mathcal{K}\otimes\widetilde{\mathrm{ev}}^*\mathcal{L}_\psi) \cong \widetilde{\mathrm{pr}}_{1!}\widetilde{h}^!(\mathrm{pr}_0^*\,\mathcal{K}\otimes\mathrm{ev}^*\,\mathcal{L}_\psi) \xrightarrow{(2)} (h^{\widehat{E}})^!\,\mathrm{pr}_{1!}(\mathrm{pr}_0^*\,\mathcal{K}\otimes\mathrm{ev}^*\,\mathcal{L}_\psi). \tag{A.3.12}$$

Here (1) is the natural transformation $\widetilde{\mathrm{pr}}_0^*(h^E)^! \to \widetilde{h}^!\,\mathrm{pr}_0^*$ obtained by adjunction from the proper base change isomorphism attached to the left parallelogram in (A.3.8), and (2) is the natural transformation $\widetilde{\mathrm{pr}}_{1!}\widetilde{h}^! \to (h^{\widehat{E}})^!\,\mathrm{pr}_{1!}$ coming from the right parallelogram in (A.3.8). The proof that (A.3.11) is an isomorphism is similar to that of (A.3.9).

A.3.5. *Involutivity in general.* Finally we can produce the promised canonical involutivity datum for a general derived vector bundle.

Let $E \to S$ be a derived vector bundle of rank r. Since $\delta_E := z_!\mathbf{Q}_{\ell,S}$, producing a map

$$\alpha_E : \delta_E \to \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r). \tag{A.3.13}$$

is equivalent to producing the adjoint map

$$\alpha'_E : \mathbf{Q}_{\ell,S} \to z^!\,\mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r). \tag{A.3.14}$$

Lemma A.3.3. *If the map* (A.3.13) *is an isomorphism, then the adjoint map* (A.3.14) *is an isomorphism.*

Proof. The map α'_E is the composition

$$\mathbf{Q}_{\ell,S} \to z^!z_!\mathbf{Q}_{\ell,S} \xrightarrow{z^!\alpha_E} z^!\,\mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r).$$

Since α_E is an isomorphism by assumption, it suffices to see that the unit map $\mathrm{Id} \to z^!z_!$ is an isomorphism. Equipping E with the natural $\mathbf{G}_m$-action (by scaling), we will apply the "stacky contraction principle" of Drinfeld-Gaitsgory [DG15, Theorem C.5.3], which says that for $\pi\colon E \to S$, $\pi_!$ is right adjoint to $z_!$. (For the application of the theorem, note that linear maps between derived vector bundles are always safe, and that the proof written for D-modules applies just as well to ℓ-adic sheaves with the usual cosmetic adjustments.) Hence the unit map $\mathrm{Id} \to z^!z_!$ is identified with the unit map $\mathrm{Id} \to \pi_!z_! \cong \mathrm{Id}_{S!}$, and is then obviously a natural isomorphism. □

Lemma A.3.4. *Let E be a derived vector bundle over S of rank r. Then there is a canonical (i.e., independent of auxiliary choices) isomorphism*

$$\alpha_E : \delta_E \cong \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r). \tag{A.3.15}$$

Proof. We will first construction a canonical isomorphism

$$\alpha'_E : \mathbf{Q}_{\ell,S} \cong z^!\,\mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r). \tag{A.3.16}$$

Let $\mathcal{K} \in D(S)$ denote the right side. We claim that $\mathcal{K}$ is a *sheaf* concentrated in degree 0, and Zariski-locally isomorphic to the constant sheaf $\mathbf{Q}_{\ell,S}$. The claim can be checked Zariski-locally, so to prove it we may assume that E has a global presentation. Then Lemma A.3.1 equips E with an involutivity datum (which a priori depends on the choice of presentation), so §A.2.6 applies and then Example 6.2.3 gives an isomorphism

$$\alpha_E\colon \delta_E \xrightarrow{\sim} \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r).$$

Then by Lemma A.3.3, the adjoint map $\mathbf{Q}_{\ell,S} \xrightarrow{\sim} z^!\,\mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r)$ is an isomorphism. This establishes the claim.

We now return to the case of general E. In the proof of the claim, we saw that Zariski-locally on S we have global presentations of E, which induce local trivializations of $\mathcal{K}$. By Lemma A.3.1(2), these local trivializations of $\mathcal{K}$ agree on the stalk at any point of S. Since the claim implies in particular that $\mathcal{K}$ is a local system on S, the local trivializations therefore glue to give an isomorphism $\mathbf{Q}_{\ell,S} \cong \mathcal{K}$. (Note that we are gluing in the abelian category of local systems on S; it was crucial to first establish that $\mathcal{K}$ lies in this subcategory in order to be able to glue.) Furthermore, Lemma A.3.1(2) shows that the resulting α'_E is independent of any and all choices of local trivializations.

The canonical isomorphism α'_E then induces by adjunction a canonical map

$$\alpha_E : \delta_E \to \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r). \tag{A.3.17}$$

We claim that α_E is an isomorphism. This claim can be checked after base changing to a point of S. Hence we may assume that E admits a global presentation, and then by its construction, the map α_E agrees with the map induced by the map $\delta_E \to \mathrm{FT}_{\widehat{E}}(\mathbf{Q}_{\ell,\widehat{E}})[r](r)$ produced by the involutivity datum of Lemma A.3.1 and Example 6.2.3, which is an isomorphism. This finishes the proof. $\square$

Now we have established that all derived vector bundles have canonical involutivity data (not depending for example on global presentations). Henceforth we implicitly equip all derived vector bundles with said canonical involutivity data.

A.4. **Fourier transform and proper base change.** In this subsection we prove Proposition 6.3.3. The notational conventions of the subsection are now reset. We maintain the notation and setup of §6.3.2.

Lemma A.4.1. *Let $f\colon E' \to E$ be a globally presented map of derived vector bundles. Then f admits a factorization*

$$f\colon E' \xrightarrow{i} \widetilde{E} \xrightarrow{p} E$$

such that i is a closed embedding and p is smooth.

Proof. Let $K := \mathrm{Tot}_S(\mathcal{K})$ be the derived kernel of f. By the hypothesis, $\mathcal{K}$ admits a global presentation

$$\mathcal{K}^\bullet = \cdots \mathcal{K}^{-1} \to \mathcal{K}^0 \to \mathcal{K}^1 \to \cdots$$

(for example, we can take for $\mathcal{K}^\bullet$ the usual cone construction of f shifted by 1). Then we have an exact triangle for naive truncations

$$\mathcal{K}^{\geq 1} \to \mathcal{K} \to \mathcal{K}^{\leq 0}.$$

We take $\widetilde{\mathcal{E}} := \mathrm{cone}(\mathcal{K}^{\geq 1} \to \mathcal{E}')$, which then has a factorization

$$\mathcal{E}' \to \widetilde{\mathcal{E}} \to \mathrm{cone}(\mathcal{K} \to \mathcal{E}') \cong \mathcal{E}. \tag{A.4.1}$$

Note that the last isomorphism is in $\mathrm{Perf}(S)$ and we do not claim that it is represented by a map of the given presentations. Regardless, (A.4.1) induces a diagram of total spaces

$$E' \xrightarrow{i} \widetilde{E} \xrightarrow{p} E$$

whose composition is f, whose first map has derived kernel $\mathrm{Tot}_S(\mathcal{K}^{\geq 1})$ hence is a closed embedding, and whose second map has derived kernel $\mathrm{Tot}_S(\mathcal{K}^{\leq 0})$ hence is smooth. $\square$

A.4.1. *Formulation of the main statement.* We expand the formulation of the compatibility (6.3.3). Consider a Cartesian square of derived vector bundles:

$$\begin{array}{ccccc} & & B & & \\ & \swarrow^{g'} & & \searrow^{f'} & \\ A & & & & D \\ & \searrow_{f} & & \swarrow_{g} & \\ & & C & & \end{array} \tag{A.4.2}$$

where all maps are linear. Set $d := d(f)$, $\delta := d(g)$. Using §6.2.4 and proper base change, we have a hexagon of functors $D(A) \to D(\widehat{D})$:

$$\begin{array}{ccccc} \mathrm{FT}\, g^* f_![d+\delta](\delta) & \xleftarrow{\sim} & \mathrm{FT}\, f'_!(g')^*[d+\delta](\delta) & \xrightarrow{\sim} & (\widehat{f'})^*\,\mathrm{FT} \circ (g')^*[\delta](\delta) \\ \uparrow{\scriptstyle\sim} & & & & \uparrow{\scriptstyle\sim} \\ \widehat{g}_!\,\mathrm{FT}\, f_![d] & \xleftarrow{\sim} & \widehat{g}_!\widehat{f}^*\,\mathrm{FT} & \xrightarrow{\sim} & (\widehat{f'})^*\widehat{g}'_!\,\mathrm{FT} \end{array} \tag{A.4.3}$$

Lemma A.4.2. *Assume that each of f, g is either smooth or a closed embedding. Then diagram* (A.4.3) *commutes.*

Proposition A.4.3. *Assume that f and g are globally presented. Then diagram* (A.4.3) *commutes.*

Remark A.4.4. We emphasize that Lemma A.4.2 has a weaker hypothesis than Proposition A.4.3; in particular, it does *not* require maps f, g to be globally presented. This is important because when reducing Proposition A.4.3 to Lemma A.4.2, we may not be able to guarantee that intermediate diagrams are globally presented compatibly with the original global presentation of (A.4.2). Fortunately, because of the weaker hypothesis, there is no need to arrange such compatibility.

Proof of Proposition A.4.3, assuming Lemma A.4.2. By Lemma A.4.1, we may factor f as $A \xrightarrow{i} \widetilde{A} \xrightarrow{p} C$ where i is a closed embedding and p is smooth. This induces a factorization of the diagram (A.4.2) into a sequence of two Cartesian squares

$$\begin{array}{ccccc} & & B & & \\ & \overset{g'}{\swarrow} & \downarrow{\scriptstyle i'} & & \searrow{\scriptstyle f'} \\ A & & \widetilde{D} & & \\ & \overset{i}{\searrow} \quad \swarrow & & \searrow{\scriptstyle p'} & \\ & & \widetilde{A} & & D \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle g} & \\ & & C & & \end{array} \tag{A.4.4}$$

We now apply Lemma A.4.2 to the two inner Cartesian squares. Lemma A.4.2 is compatible with compositions of Cartesian squares, so the commutativity of the analogous hexagons to (A.4.3) for the two inner squares of (A.4.4) implies the commutativity of (A.4.3). ☐

It remains to prove Lemma A.4.2. We make some initial reductions. We claim that the result is immediate if f is a closed embedding, or if g is smooth. Indeed:

- If f is a closed embedding then $\widehat{f}$ is smooth and we may replace $f_!$ by f_* everywhere in (A.4.3). Applying adjunctions and Proposition 6.4.2, the commutativity of (A.4.3) is then equivalent to the commutativity of

$$\begin{array}{ccc} \mathrm{FT}\, f_* & \xrightarrow{\mathrm{unit}(g')} & \mathrm{FT}\, f_* g'_* (g')^* \\ \sim\uparrow & & \sim\uparrow \\ \widehat{f}^*\, \mathrm{FT}[d](d) & \xrightarrow{\mathrm{unit}(\widehat{g}')} & \widehat{f}^* \widehat{g}'^! \widehat{g}'_!\, \mathrm{FT}[d](d) \end{array}$$

  which in turn follows from the fact that FT preserves the unit of an adjunction, which is incorporated into our construction (§6.2.4).
- If g is smooth then $\widehat{g}$ is a closed embedding and we may replace $\widehat{g}_!$ by $\widehat{g}_*$ everywhere in (A.4.3), and a similar argument applies.

We may and do henceforth assume that f is smooth and g is a closed embedding.

A.4.2. *More general statement.* In fact, we can formulate a more general statement. The transform FT is defined with respect to $\mathcal{L}_\psi$ on $\mathbf{A}^1$ via pull-push in the diagram

$$A \times \mathbf{A}^1 \longleftarrow A \times_S \widehat{A} \longrightarrow \widehat{A}$$

More generally, define the functor $\mathrm{Conv}\colon D(A \times \mathbf{A}^1) \to D(\widehat{A})$ as $\mathrm{pr}_{1!}(\mathrm{pr}_0, \mathrm{ev})^*$ (see §A.1.1) for the diagram

$$A \times \mathbf{A}^1 \xleftarrow{(\mathrm{pr}_0, \mathrm{ev})} A \times_S \widehat{A} \xrightarrow{\mathrm{pr}_1} \widehat{A}.$$

Composing Conv with the functor $D(A) \to D(A \times \mathbf{A}^1)$ given by external product with the Artin-Schreier sheaf $\mathcal{L}_\psi$ recovers FT^ψ, up to shift.

We will construct a hexagon

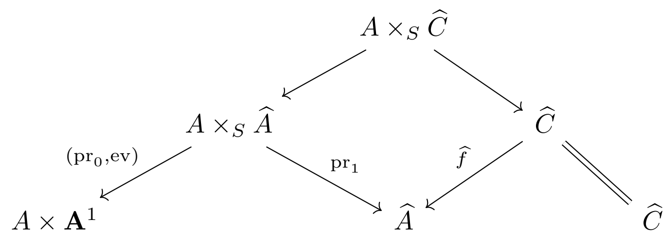
$$\begin{array}{ccccc} \mathrm{Conv}\, g^* f_! \langle\delta\rangle & \xleftarrow{\sim} & \mathrm{Conv}\, f'_!(g')^* \langle\delta\rangle & \xrightarrow{\sim} & (\widehat{f}')^*\, \mathrm{Conv}(g')^* \langle\delta\rangle \\ \uparrow & & & & \uparrow \\ \widehat{g}_!\, \mathrm{Conv}\, f_! & \xleftarrow{\sim} & \widehat{g}_! \widehat{f}^*\, \mathrm{Conv} & \xrightarrow{\sim} & (\widehat{f}')^* \widehat{g}'_!\, \mathrm{Conv} \end{array} \tag{A.4.5}$$

that recovers (A.4.3) in the above manner; note however that in general the vertical arrows are not isomorphisms. The top left and bottom right vertical arrows are the natural isomorphisms induced by proper base change. The other arrows are explained in §A.4.3 and §A.4.4 below.

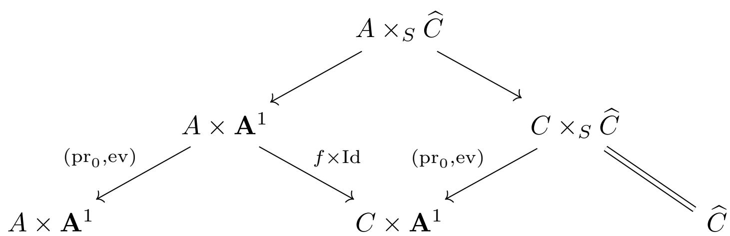
We are still assuming that g, g' are closed embeddings (so $\widehat{g}, \widehat{g}'$ are smooth) and f, f' are smooth (so $\widehat{f}, \widehat{f}'$ are closed embeddings).

A.4.3. For any $f\colon A \to C$, we will define a natural isomorphism $\widehat{f}^* \operatorname{Conv} \cong \operatorname{Conv} f_!$ of functors $D(A) \to D(\widehat{C})$. (We do not need the smoothness of f here.) *This supplies the bottom left and top right horizontal arrows of* (A.4.5).

- The functor $\widehat{f}^* \operatorname{Conv}$ is given by the correspondence



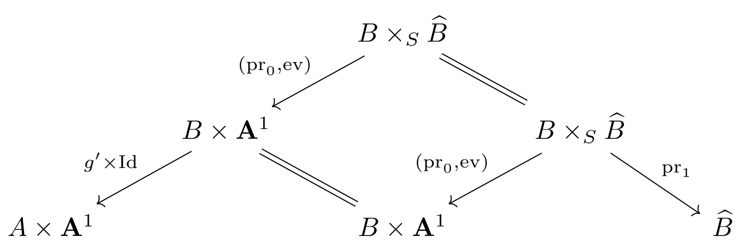
- The functor $\operatorname{Conv} f_!$ is given by the correspondence



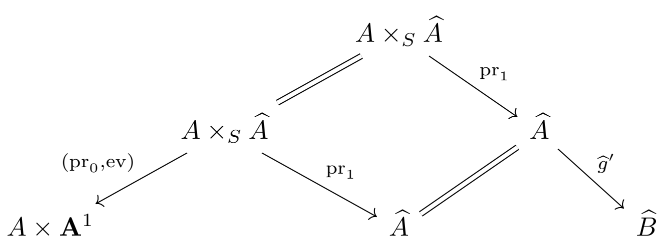
These correspondences between $A \times \mathbf{A}^1$ and $\widehat{C}$ visibly agree.

A.4.4. For any smooth $g'\colon B \to A$, we will define a natural transformation $\widehat{g}'_! \operatorname{Conv} \to \operatorname{Conv}(g')^*$ of functors $D(A \times \mathbf{A}^1) \to D(\widehat{B})$. (We *will* use the smoothness of g' here.) *This supplies the vertical arrows of* (A.4.5).

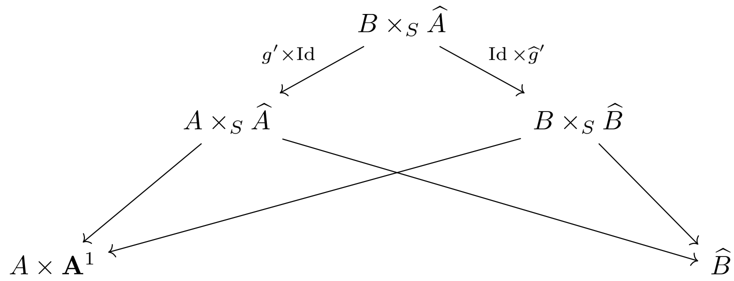
- The functor $\operatorname{Conv}(g')^*$ is given by the correspondence



- The functor $\widehat{g}'_! \operatorname{Conv}$ is given by the diagram



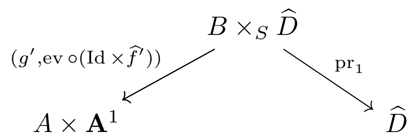
To compare these, we consider the diagram



where the maps are as above. This is a butterfly thanks to the assumption that g' is a closed embedding (so $\widehat{g}'$ is smooth), and therefore gives a natural transformation

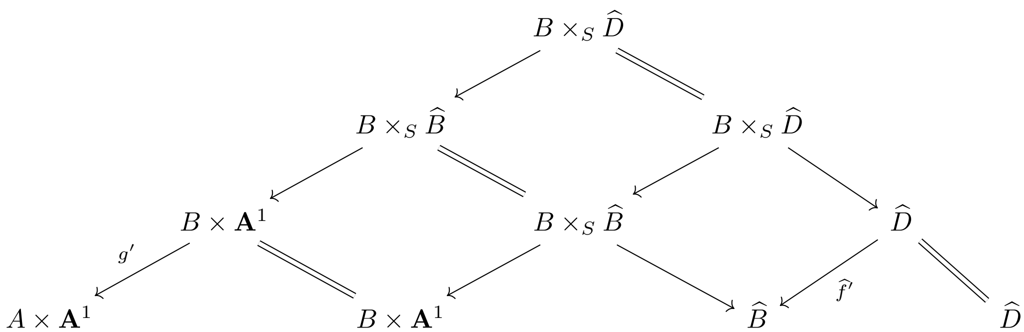
$$\widehat{g}'_! \operatorname{Conv} \xrightarrow{\bigstar} \operatorname{Conv}(g')^* \langle -d(\widehat{g}') \rangle = \operatorname{Conv}(g')^* \langle \delta \rangle.$$

A.4.5. Returning to the hexagon (A.4.5), the top row of natural isomorphisms all come from the correspondence

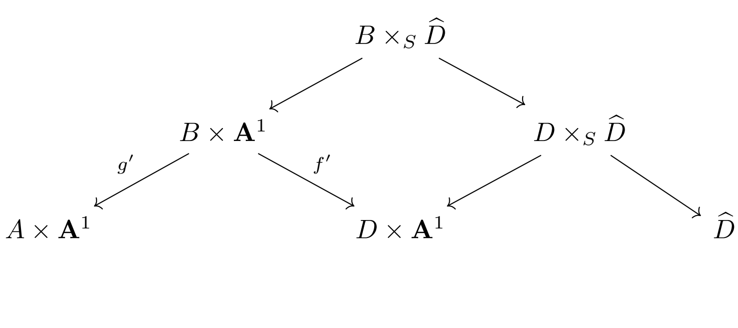


via §A.1.1, as seen in the diagrams below:

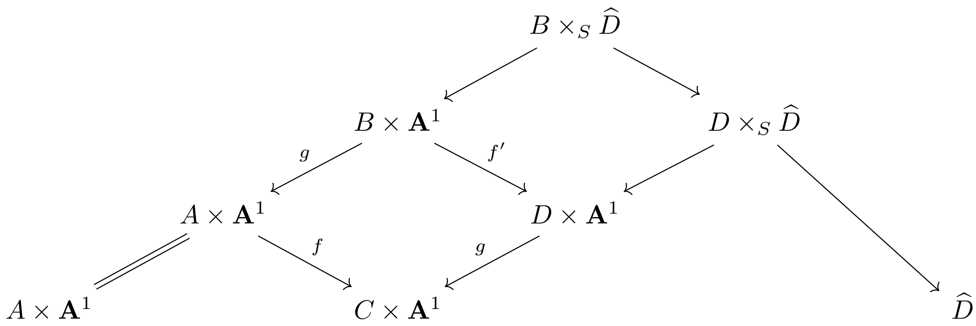
- $(\widehat{f}')^* \operatorname{Conv}(g')^*$ is induced via §A.1.1 by the fibered product of correspondences



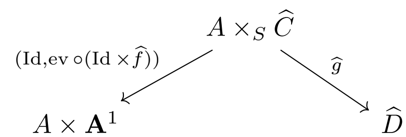
- $\operatorname{Conv} f'_!(g')^*$ is induced via §A.1.1 by the fibered product of correspondences



- $\operatorname{Conv} g^* f_!$ is induced via §A.1.1 by the fibered product of correspondences

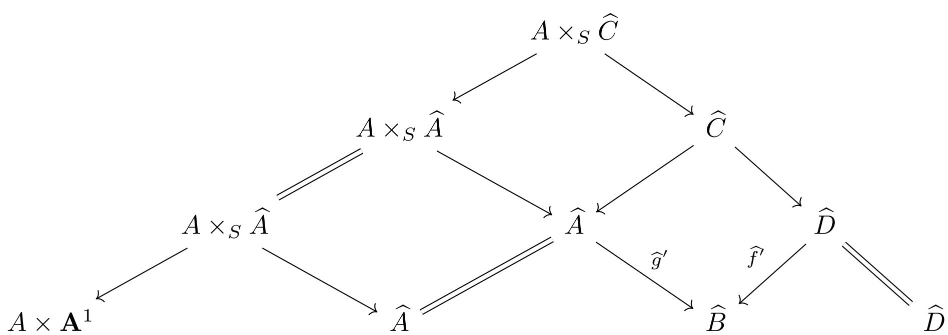


On the other hand, the bottom row of natural isomorphisms in the hexagon all come from the correspondence

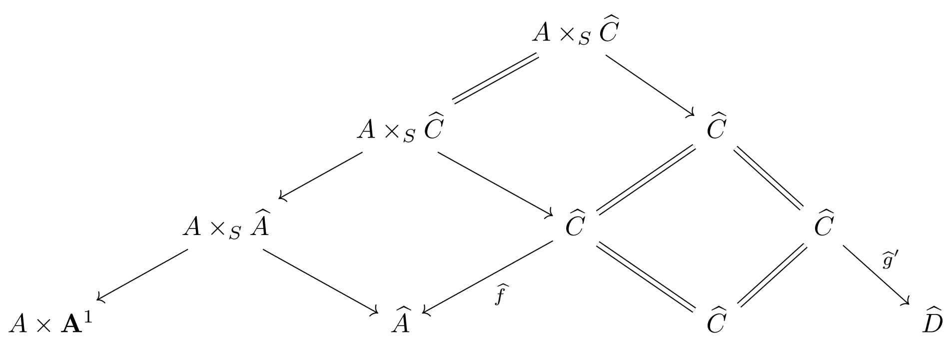


via §A.1.1, as seen in the diagrams below:

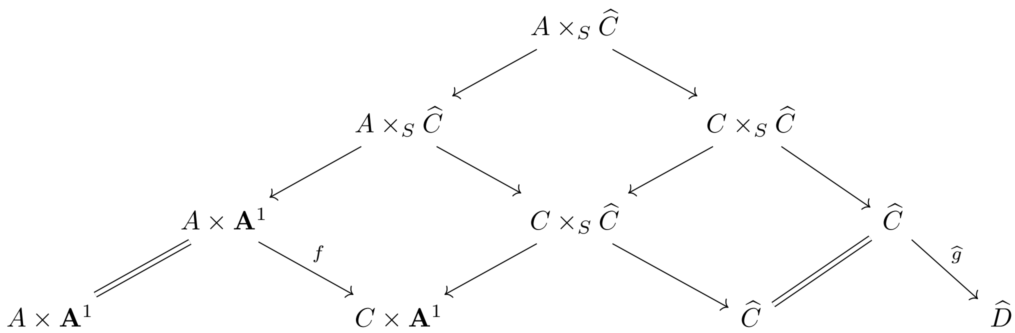
- $(\widehat{f'})^* (\widehat{g'})_! \operatorname{Conv}$ is induced via §A.1.1 by the fibered product of correspondences



- $\widehat{g}_! (\widehat{f})^* \operatorname{Conv}$ is induced via §A.1.1 by the fibered product of correspondences



- $\widehat{g}_! \operatorname{Conv} f_!$ is induced via §A.1.1 by the fibered product of correspondences



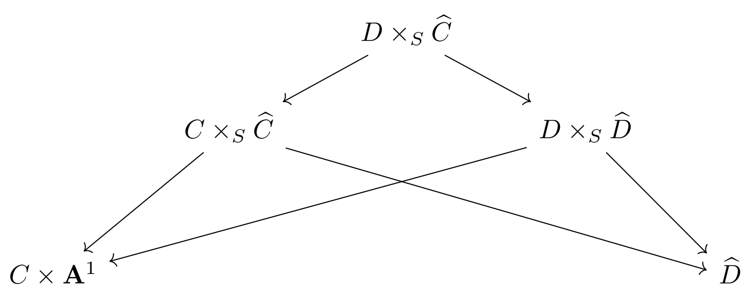
Since FT is (up to shift) the restriction of Conv to a particular subcategory, Proposition A.4.3 is implied by the following.

Proposition A.4.5. *Hexagon* (A.4.5) *commutes.*

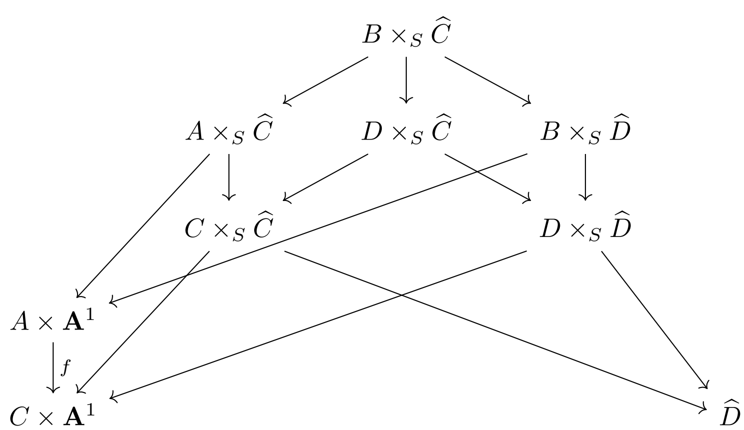
Proof. By the proceeding discussion, the natural transformation

$$\widehat{g}_! \operatorname{Conv} f_! \to \operatorname{Conv} g^* f_! \langle \delta \rangle$$

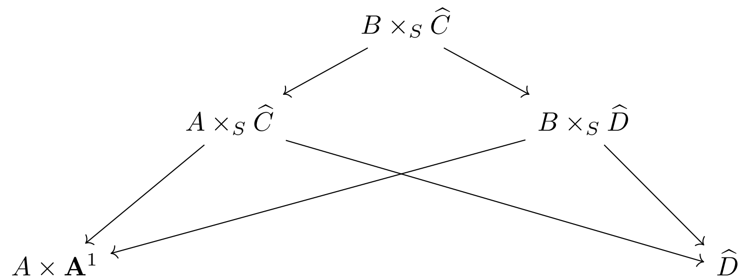
is the pre-composition of the natural transformation $\widehat{g}_! \operatorname{Conv} \xrightarrow{\bigstar} \operatorname{Conv} g^* \langle d(g) \rangle$ coming from the butterfly



with $f_!$. Computing the pullback of this butterfly along f gives the diagram



and Lemma A.1.2 identifies the resulting natural transformation with the one from the upper butterfly

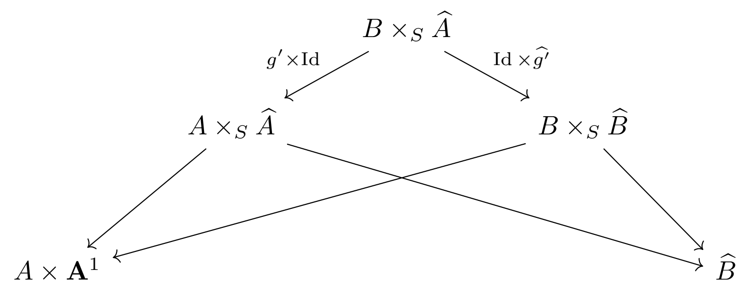


(A.4.6)

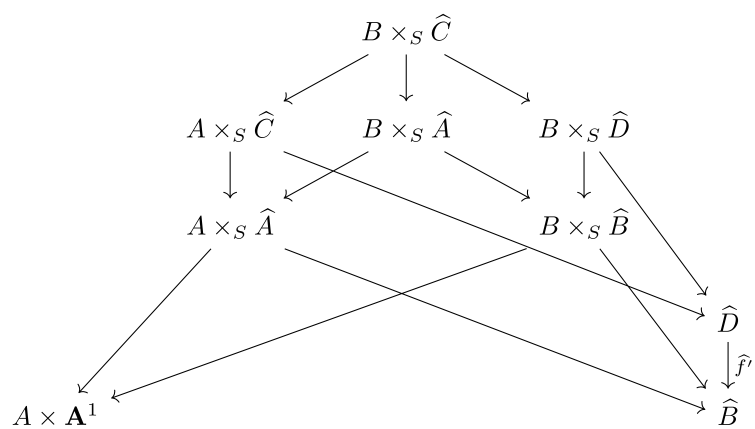
On the other hand, the natural transformation

$$(\widehat{f'})^* \widehat{g'}_! \operatorname{Conv} \to (\widehat{f'})^* \operatorname{Conv}(g')^* \langle \delta \rangle$$

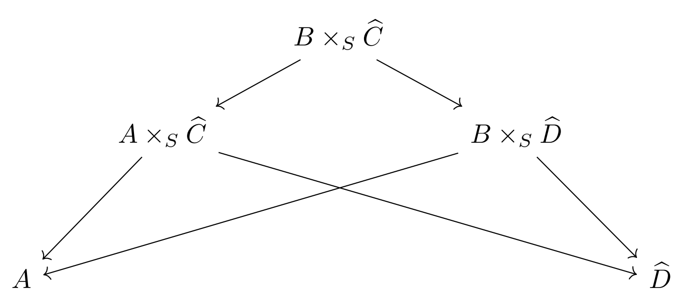
is the composition of $(\widehat{f'})^*$ with the natural transformation $\widehat{g'}_! \operatorname{Conv} \xrightarrow{\bigstar} \operatorname{Conv}(g')^* \langle \delta \rangle$ coming from the butterfly



Computing the pullback of this butterfly along $\widehat{f'}$ gives the diagram



and Lemma A.1.3 identifies the resulting natural transformation with the one from the upper butterfly



which is visibly the same butterfly as (A.4.6). ☐

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