MODULAR FUNCTORIALITY IN THE LOCAL LANGLANDS CORRESPONDENCE

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Abstract. We develop a theory of Smith–Treumann localization in the context of Fargues-Scholze's Geometrization of the Local Langlands Correspondence. We then apply this theory to prove some conjectures of Treumann-Venkatesh concerning mod-ℓ Local Langlands functoriality between a reductive group G and its fixed subgroup under an order ℓ automorphism. As another application, we explicitly calculate the Fargues-Scholze parameters of certain mod-ℓ toral representations.

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1. Introduction

1.1. **Modular Langlands functoriality.** Let E be a local field of residue characteristic p and G be a reductive group over E. For simplicity, we assume for now that G is split; beyond the Introduction, the text always treats general G. The Local Langlands Correspondence predicts, vaguely speaking, that the smooth representations of $G(E)$ over a field F are controlled by the *Langlands dual group* $\check{G}$ over F.

For example, letting $W_E \subset \mathrm{Gal}(E^s/E)$ be the Weil group of E, the Local Langlands Correspondence predicts that there should be a natural parametrization of irreducible smooth representations of $G(E)$ over F by "L-parameters", which are homomorphisms $W_E \to \check{G}(F)$ up to conjugacy. The existence of such a parametrization has surprising implications for representation theory: for example, a homomorphism $\check{\psi}\colon \check{H} \to \check{G}$ of dual groups induces an obvious map

$$\{L\text{-parameters for } \check{H}\} \xrightarrow{\check{\psi}_*} \{L\text{-parameters for } \check{G}\}$$

and therefore suggests some *Langlands functoriality* operation from (packets of) irreducible representations of $H(E)$ to (packets of) irreducible representations of $G(E)$. In practice, it is usually difficult to construct such operations directly, or to describe them explicitly in representation-theoretic terms.

In this paper we investigate Langlands functoriality for a specific class of dual homomorphisms $\check{\psi}$ that was identified by Treumann-Venkatesh in [TV16]. It considers the situation where the reductive group H arises as the fixed points of an order ℓ automorphism σ of another reductive group G, where ℓ is a prime different from p. Furthermore, we take k to be an algebraically closed field of characteristic ℓ. In this situation (and under some hypotheses), Treumann-Venkatesh constructed a dual homomorphism $\check{\psi}\colon \check{H} \to \check{G}$ over k, with a specific property in terms of the Satake isomorphism that we will discuss later.

We refer to this situation as *modular functoriality*, because its construction depends on special features of modular arithmetic (and this paper depends in turn on special features of modular representation theory). By contrast, Langlands' original conjectures were focused on the case where the coefficient field has characteristic zero. We note that in many examples the map $\check{\psi}$ does not lift to characteristic zero, and the resulting functoriality is truly specific to modular arithmetic.

The story extends to non-split groups. In that case, the dual group $\check{G}$ should be augmented to the L-group ${}^L G \cong \widehat{G} \rtimes W_E$, where the action of W_E on $\widehat{G}$ reflects the twisting of G over E relative to its split form. This generalization is necessary to treat some of the most interesting examples.

Example 1.1.1. A familiar example is cyclic base change, where H is a reductive group over E and $G = \mathrm{Res}_{E'/E}(H_{E'})$ for a cyclic ℓ-extension E'/E. Let σ be a generator of $\mathrm{Gal}(E'/E)$ acting on G in the natural way, so that $G^\sigma = H$. This particular example lifts to characteristic 0, but many do not; several examples are tabulated in the arXiv version of Treumann-Venkatesh's paper [TV14]. We also consider here some interesting examples that are ruled out by the hypotheses of [TV14, TV16]: a useful one is where σ is conjugation by a strongly regular element of $G(E)$, in which case H is a (not necessarily split!) maximal torus. Note that the work of Treumann-Venkatesh only treats examples where G is simply connected and H is semisimple.

1.2. **Conjectures of Treumann-Venkatesh.** We describe some conjectures, essentially due to Treumann-Venkatesh, concerning modular Langlands functoriality. We again restrict our attention to the case where G and H are split, for simplicity. We let $\mathscr{H}(G, K)$ be the spherical Hecke algebra of G (with respect to a hyperspecial maximal compact subgroup K stable under σ) with coefficients in k. Then the Satake isomorphism supplies a k-algebra isomorphism

$$\mathscr{H}(G, K) \cong \mathcal{O}(\check{G} /\!\!/ \check{G})$$

Under some technical assumptions, Treumann-Venkatesh construct a k-algebra homomorphism $\mathrm{br}\colon \mathscr{H}(G, K) \to \mathscr{H}(H, K^\sigma)$ which they call the *(normalized) Brauer homomorphism*. (The construction uses in an essential way the assumption that $\mathrm{char}\ k = \ell = \mathrm{ord}(\sigma)$.) This implies the existence of a commutative diagram

$$\begin{array}{ccc} \mathscr{H}(G, K) & \xrightarrow[\text{Satake}]{\sim} & \mathcal{O}(\check{G} /\!\!/ \check{G}) \\ \downarrow{\scriptstyle \mathrm{br}} & & \vdots \\ \mathscr{H}(H, K^\sigma) & \xrightarrow[\text{Satake}]{\sim} & \mathcal{O}(\check{H} /\!\!/ \check{H}) \end{array}$$

It is then natural to ask if the dashed map is induced by restriction along a homomorphism $\check{\psi}\colon \check{H} \to \check{G}$. If so, then $\check{\psi}$ will be called a "σ-dual homomorphism". One of the main theorems of Treumann-Venkatesh [TV16, §1.3] is that if G is simply connected and H is semisimple, then a σ-dual homomorphism exists, with three possible exceptions if G has type E_6. In [TV14, §1.4(iii)], Treumann-Venkatesh pose the open problem of constructing a σ-dual homomorphism in full generality. Our first theorem addresses this question in full generality.

Theorem 1.2.1. *A σ-dual homomorphism $\check{\psi}\colon \check{H} \to \check{G}$ exists.*

Example 1.2.2. For applications, there are important cases beyond the hypotheses of Treumann–Venkatesh. For example, in §10 we study a situation where H is a torus, while Treumann–Venkatesh assume that H is semisimple.

The construction of the σ-dual homomorphism in [TV16] is a tour de force: it invokes classification theorems to tabulate all examples of modular functoriality, and analyzes them case-by-case. By contrast, the proof of Theorem 1.2.1 is completely uniform across all cases, and makes no use of classification theorems. The idea is quite natural: instead of contemplating the dual groups, we try to *categorify* the Brauer homomorphism br to a *Brauer functor* $\widetilde{br}$ from $\mathrm{Sat}(\mathrm{Gr}_G; k)$ to $\mathrm{Sat}(\mathrm{Gr}_H; k)$. After equipping this Brauer functor with a Tannakian structure, we obtain the σ-dual homomorphism for free from the

Geometric Satake equivalence.[1]

$$\begin{array}{ccc} \mathrm{Sat}(\mathrm{Gr}_G; k) & \xrightarrow[\text{Geom. Satake}]{\sim} & \mathrm{Rep}_k(\check{G}) \\ \downarrow \widetilde{br} & & \vdots \\ \mathrm{Sat}(\mathrm{Gr}_H; k) & \xrightarrow[\text{Geom. Satake}]{\sim} & \mathrm{Rep}_k(\check{H}) \end{array}$$

Although this idea is simple, its implementation is quite involved and will be elaborated upon later.

We next turn to describe conjectures of Treumann-Venkatesh pertaining to the Local Langlands Correspondence. We write $\mathbf{Z}[\sigma]$ for the group ring of $\langle\sigma\rangle \cong \mathbf{Z}/\ell\mathbf{Z}$ and let $N := 1 + \sigma + \ldots + \sigma^{\ell-1} \in \mathbf{Z}[\sigma]$. If Π is a representation of $G(E) \rtimes \langle\sigma\rangle$, then its *Tate cohomology* groups $\mathrm{T}^j(\Pi)$, for $j \in \mathbf{Z}/2\mathbf{Z}$, are defined as

$$\mathrm{T}^0(\Pi) := \frac{\ker(1-\sigma \mid \Pi)}{\mathrm{Im}\,(N \mid \Pi)} \qquad \mathrm{T}^1(\Pi) := \frac{\ker(N \mid \Pi)}{\mathrm{Im}\,(1-\sigma \mid \Pi)} \tag{1.2.1}$$

and the $G(E)$-action on Π induces an $H(E)$-action on each $\mathrm{T}^j(\Pi)$.

Conjecture 1.2.3 ([TV16, Conjecture 6.3])**.** *Let Π be an irreducible smooth representation of $G(E)$ over an algebraically closed field k of characteristic ℓ, whose isomorphism class is fixed by σ, so that the $G(E)$-action on Π uniquely extends to a $G(E) \rtimes \langle\sigma\rangle$-action by* [TV16, Proposition 6.1]*. Then for each $j \in \mathbf{Z}/2\mathbf{Z}$:*

1. *(Admissibility Conjecture) $\mathrm{T}^j(\Pi)$ is admissible as a representation of $H(E)$.*
2. *(Functoriality Conjecture) The L-parameter of every irreducible $H(E)$-subquotient of $\mathrm{T}^j(\Pi)$ is sent by the σ-dual homomorphism $\check{\psi}$ to the L-parameter for $\Pi^{(\ell)} := \Pi \otimes_{k,\mathrm{Frob}_\ell} k$, the Frobenius twist of Π.*

Part (1) of the Conjecture has been proved by Dhar–Nadimpalli [DN25]. We will focus on Part (2). At the time of its formulation, the Local Langlands Correspondence was constructed only for certain families of groups, so the precise meaning of Conjecture 1.2.3(2) was left vague for general groups. We may now formulate a precise version for all G using the work of Fargues-Scholze [FS21]; this is what we will describe next.

Example 1.2.4. For $G = \mathrm{GL}_n$, Vignéras had constructed in [Vig01] the Local Langlands Correspondence over k (using the characteristic zero version due to Harris-Taylor [HT01]), many years before the work of Fargues-Scholze. This gives a precise meaning to Conjecture 1.2.3(2) when G and H are both general linear groups, and in the context of cyclic base change, special cases have been proven by other authors, as will be discussed more in §1.6. Even for this case, where the statements of our results can be formulated in classical terms, the proofs will utilize the work of Fargues-Scholze.

1.3. **The Fargues-Scholze correspondence.** An output of the work [FS21] of Fargues-Scholze is a map

$$\left\{\begin{array}{c}\text{irreducible admissible representations} \\ \Pi \text{ of } G(E) \text{ over } k\end{array}\right\}/\sim \longrightarrow \left\{\begin{array}{c}\text{semisimple } L\text{-parameters} \\ \rho_\Pi \colon W_E \to {}^L G(k)\end{array}\right\}/\sim \tag{1.3.1}$$

that we call the *Fargues-Scholze correspondence*. We refer to ρ_Π, which is most naturally regarded as an element of $\mathrm{H}^1(W_E; \widehat{G}(k))$, as the *Fargues-Scholze parameter* of Π. The map $\Pi \mapsto \rho_\Pi$ is expected to be the semi-simplification of "the" hypothetical Local Langlands Correspondence. For specific groups including tori and GL_n, the Local Langlands Correspondence has been constructed previously (by class field theory and by Harris-Taylor, respectively), and in these cases it is known that the Fargues-Scholze correspondence is compatible with the previous construction.

For most groups, the Fargues-Scholze correspondence is quite mysterious. For example, it is expected that (1.3.1) is surjective and has finite fibers, but proving this is wide open. For *regular supercuspidal*

[1] A subtle but important point found by Treumann-Venkatesh is that, when taking into account the full L-group, a σ-dual homomorphism may not exist with the "usual" notion of L-group due to Langlands. Treumann-Venkatesh suggest in [TV14, §7.8] that this problem might be repaired by instead using the "c-group", which is the variant of the L-group that naturally comes out of the geometric Satake equivalence [Zhu17a, Remark 5.5.11]. For non-archimedean local fields there is actually an isomorphism between the L-group and the c-group (possibly depending on a choice of square root of q in k), but for the most canonical statements one needs to use the c-group.

representations of quite general groups, Kaletha [Kal19] has prescribed explicit constructions for the L-parameters, which are strongly supported by the fact that the correspondence of Kaletha satisfies the expected endoscopic character relations [FKS23, Theorem 4.4.4]; while for general groups almost nothing is known about the Fargues-Scholze parameters of such representations. We remark that in the cases where the Fargues-Scholze correspondence has been explicated, there have been important geometric implications; an example is the work of Koshikawa [Kos21] towards torsion vanishing in the cohomology of Shimura varieties, where the relevant compatibility with the classical LLC is proven in [HKW22].

1.3.1. *The functoriality conjecture.* We prove the following result concerning modular functoriality in the Fargues-Scholze correspondence, which (at least away from small ℓ) we take as fulfilling Conjecture 1.2.3(2). Write ${}^L\psi\colon {}^LH \to {}^LG$ for the L-group form of the σ-dual homomorphism, ρ^{ss} for the semisimplification of an L-parameter ρ [FS21, Proposition VIII.3.2], and $\rho \sim \rho'$ if the L-parameters ρ, ρ' are conjugate by an element of $\widehat{G}(k)$.

Theorem 1.3.1. *Let Π be an irreducible smooth representation of $G(E)$ whose isomorphism class is fixed by σ, equipped with its unique extension to a $G(E) \rtimes \langle\sigma\rangle$-action by* [TV16, Proposition 6.1]. *Then for each $j \in \mathbf{Z}/2\mathbf{Z}$ and every $H(E)$-irreducible subquotient π of $\mathrm{T}^j(\Pi)$, we have $\rho_{\Pi^{(\ell)}} \sim ({}^L\psi \circ \rho_\pi)^{\mathrm{ss}}$.*

This result is deduced from Theorem 9.1.1 in Example 9.1.2. Theorem 9.1.1 treats "derived smooth representations" $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$. The derived version is more useful for calculations but less elementary to formulate, so we do not state it here.

Theorem 1.3.1 does not seem to be made any easier by assuming the full categorical conjectures of [FS21], or the conjectural compatibility with Kaletha's explicit Local Langlands Correspondence, or any other standard conjectures about the Local Langlands Correspondence that we know. On the other hand, Theorem 1.3.1 may well be useful for proving some of these other conjectures. For example, as an application of (the derived version of) Theorem 1.3.1, we calculate explicitly the Fargues-Scholze parameters of certain classes of representations.

1.3.2. *Explicit calculation of Fargues-Scholze parameters.* The basic idea is that if the L-parameter $\rho_\Pi\colon W_E \to {}^LG(k)$ admits some factorization

$$\begin{array}{ccc} W_E & \xrightarrow{\quad\rho_\Pi\quad} & {}^LG(k) \\ & \searrow \quad \nearrow_{{}^L\psi} & \\ & {}^LH(k) & \end{array} \tag{1.3.2}$$

through an L-parameter into ${}^LH(k)$ where H is a *torus* arising as the fixed points of some appropriate σ, then Theorem 1.3.1 identifies ρ_Π explicitly in terms of the L-parameter of $\mathrm{T}^j(\Pi)$, which is computable since the Fargues-Scholze correspondence is completely understood for tori. Note that many supercuspidal parameters factor through the L-group of *some* (not necessarily split) torus; for example, this is the case for regular supercuspidal parameters when G is tamely ramified and p does not divide the order of the Weyl group of G.

We apply this idea to the toral representations considered in work of Chan-Oi [CO]. They are generalizations of depth-zero supercuspidals, with an analogous construction but instead using "deeper-level Deligne-Lusztig representations" studied in work of Chan-Ivanov [CI21]; consequently, they include supercuspidal representations with arbitrarily high depth. The input for the (modular version of the) Chan-Oi construction is an elliptic unramified torus $T \subset G$ and a character $\theta\colon T(E) \to k^\times$.

Under the assumption that T contains a strongly regular element of order ℓ, we prove (Corollary 10.4.3) that some irreducible constituent in the Chan-Oi construction has Fargues-Scholze parameter of the form

$$W_E \xrightarrow{{}^L\theta} {}^LT(k) \xrightarrow{{}^Lj} {}^LG(k), \tag{1.3.3}$$

where ${}^Lj$ is the canonical embedding of the L-group of an unramified maximal torus. This is morally in accordance with the prediction of Kaletha's explicit Local Langlands Correspondence for regular supercuspidal representations in [Kal19]; we say "morally" because Kaletha does not consider modular

coefficients. Because the precise definition of a toral representation is complicated, we defer the precise formulation of our result to Theorem 10.4.2.

We emphasize that the simple appearance of (1.3.3) belies the intricacy of Kaletha's recipe, which for example involves twisting the most natural (from a representation-theoretic perspective) guess for the L-parameter by a subtle "twisting character". In the relatively simple example of epipelagic representations of unitary groups, these characters were explicated in [FRT20] and found to be already very complicated there. The geometric construction of Chan-Oi somehow bakes this twisting character into the geometry (see [CO, §8] for more discussion of this point) which is then reflected in our computation.

1.4. **Further results.** We mention some further results.

1.4.1. *Existence of functorial lifts.* The following Theorem guarantees the existence of functorial lifts along any σ-dual homomorphism.

Theorem 1.4.1. *Let π be an irreducible smooth representation of $H(E)$ over k, with Fargues-Scholze parameter $\rho_\pi \in \mathrm{H}^1(W_E; \widehat{H}(k))$. Then there exists an irreducible smooth representation Π of $G(E)$ over k with Fargues-Scholze parameter $\rho_\Pi \sim ({}^L\psi \circ \rho_\pi)^{\mathrm{ss}} \in \mathrm{H}^1(W_E; \widehat{G}(k))$.*

The more general version (allowing non-split groups) is Theorem 9.5.1. This version allows us to treat further examples such as the following.

Example 1.4.2 (Base change)**.** Taking H to be a reductive group over E and $G = \mathrm{Res}_{E'/E}(H_{E'})$ for a cyclic ℓ-extension E'/E and σ a generator of $\mathrm{Gal}(E'/E)$ acting in the natural way, Theorem 1.4.1 asserts the existence of base change along E'/E. In equal characteristic and for the Genestier-Lafforgue correspondence, this was established in [Fen24, Theorem 1.1].

Example 1.4.3. Take σ to be conjugation by a strongly regular element of $G(E)$ of order ℓ. Then H is a torus, so every $\rho \in \mathrm{H}^1(W_E; \widehat{H}(k))$ is realized as a Fargues-Scholze parameter. Theorem 9.5.1 therefore implies that every semisimple $\rho \in \mathrm{H}^1(W_E; \widehat{G}(k))$ which factors through ${}^L\psi$ is realized as a Fargues-Scholze parameter. It would be interesting to investigate which ${}^L\psi$ can arise as a σ-dual homomorphism, as this might help to show surjectivity of the Fargues-Scholze correspondence over k: recall that if p is not too small relative to G, then every supercuspidal L-parameter factors through the L-group of some torus, and we know that the Fargues-Scholze correspondence is surjective for tori.

1.4.2. *Functoriality for the Bernstein center.* Let $\mathfrak{Z}(G;k)$ be the Bernstein center of G with coefficients in k, and similarly for H. The Fargues-Scholze correspondence (1.3.1) is deduced from the construction of a k-algebra homomorphism

$$\mathrm{FS}_G \colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathfrak{Z}(G;k) \tag{1.4.1}$$

where $\mathrm{Exc}_k(W_E, \widehat{G})$ is the *excursion algebra* (over k).

Building on ideas of Treumann-Venkatesh, we construct a map of Bernstein centers

$$\mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G;k) \to \mathfrak{Z}(H;k) \tag{1.4.2}$$

which we call the *Treumann-Venkatesh homomorphism*. We show in Theorem 9.3.2 that the diagram (1.4.3) commutes.

$$\begin{array}{ccc} \mathrm{Exc}_k(W_E, \widehat{G}) & \xrightarrow{{}^L\psi^*} & \mathrm{Exc}_k(W_E, \widehat{H}) \\ {\scriptstyle \mathrm{FS}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{FS}_H} \\ \mathfrak{Z}(G;k) & \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} & \mathfrak{Z}(H;k) \end{array} \tag{1.4.3}$$

This shows that $\mathfrak{Z}_{\mathrm{TV}}$ realizes functoriality for the Bernstein center. In fact, it is the main ingredient in the proof of Theorem 1.4.1.

1.5. **Methods.** We will outline the ideas behind some of the proofs. In the sketch below, we use some abbreviated and simplified notation which does not match that of the main body of the text.

We begin by commenting on the proof of Theorem 1.2.1. As explained earlier, the proof is based on categorifying the Brauer homomorphism of Treumann-Venkatesh to a *Brauer functor*, from perverse sheaves on the affine Grassmannian Gr_G for G to perverse sheaves on the affine Grassmannian Gr_H for H, which should be Tannakian (i.e., additive, symmetric monoidal, and compatible with the fiber functor).

The construction of the Brauer homomorphism is based on the observation that the *restriction of σ-equivariant functions* from $G(E)/K$ to $H(E)/K^\sigma$ has the miraculous property of being compatible with convolution specifically in characteristic ℓ (the order of σ). The naive categorification of this operation would be restriction of sheaves from Gr_G to Gr_H, but the miracle does not (naively) persist to the level of sheaves. To overcome this, we use an operation called *Smith-Treumann localization*, which has been studied in other contexts [RW22, LL21], and combine it with the theory of the *Tate construction*.

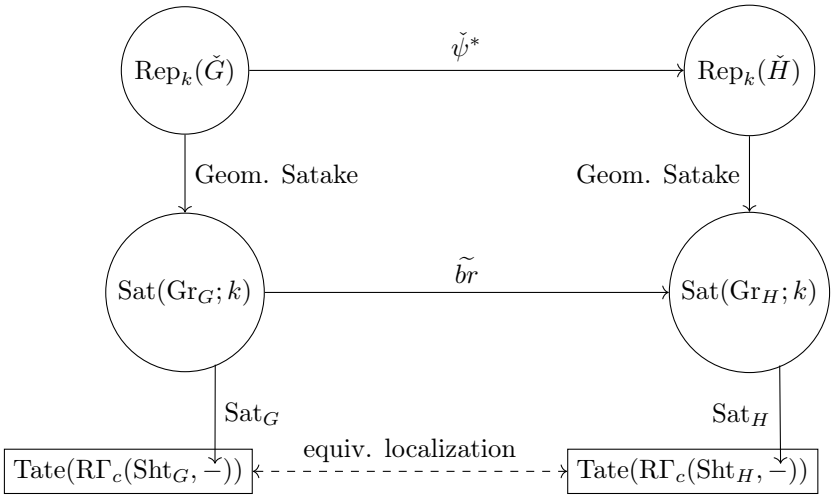

FIGURE 1. This cartoon (produced with the aid of ChatGPT-3.5 after much coaxing) depicts the Brauer functor $\widetilde{br}$ interfacing with the Tate cohomology of moduli of local shtukas and with the σ-dual homomorphism $\check{\psi}$.

Next we turn towards the proof of Theorem 1.3.1. For this we need to integrate the Brauer functor with the construction of the Fargues-Scholze correspondence for G and for H. The Fargues-Scholze correspondence for G can be obtained by constructing excursion operators on the cohomology of *moduli spaces* Sht_G *of local G-shtukas* with coefficients in Satake sheaves coming from Gr_G. We need to compare such cohomology groups to the ones obtained from applying the Brauer functor to get Satake sheaves on Gr_H, transporting them to the moduli spaces Sht_H of local H-shtukas, and then taking cohomology. This comparison is mediated by *equivariant localization* for Tate cohomology. Figure 1 depicts a cartoon of the strategy. From this comparison, we extract relations among certain excursion operators for $\check{G}$ and for $\check{H}$, which are ultimately used to establish Theorem 1.3.1.

1.6. **Related work.** We note some related work on the problems studied here.

1.6.1. *The functoriality conjecture.* We focus first on the "functoriality conjecture", Conjecture 1.2.3(2). Ronchetti [Ron16] studied Conjecture 1.2.3(2) in the special case of cyclic base change for GL_n, proving it for depth-zero cuspidal representations "of level zero and minimal-maximal type".

Dhar–Nadimpalli [ND24] studied Conjecture 1.2.3(2) in the special case of cyclic base change for GL_n, proving it for generic representations (i.e., those possessing mod-ℓ Whittaker models).

The earlier work of the author [Fen24] proved the statement analogous to Theorem 1.3.1 for the Genestier-Lafforgue correspondence (in equal characteristic) in the special case of cyclic base change, and in doing so introduced an early form of some ideas developed here. We point out several differences.

- The methods of [Fen24] were based on local-global compatibility, and were thus fundamentally restricted to the function field setting. Here our methods are purely local, and we encompass both mixed characteristic and equal characteristic local fields. To do this, we work in the context of p-adic geometry, which presents substantial new difficulties.
- The argument of [Fen24] was restricted to the case of cyclic base change, for both local and global reasons. On the *local* side, the σ-dual homomorphism is obvious for cyclic base change, and the construction of the Brauer functor in that case exploited some simplifying special features of cyclic base change; the present paper introduces more general and conceptual arguments to treat the local aspects in arbitrary generality. We still do not know how to generalize the *global* arguments even in the function field case, but fortunately they are irrelevant for the new approach.

The author's work [BFH$^+$24] with Böckle-Harris-Khare-Thorne explicates Conjecture 1.2.3(2) for the Genestier-Lafforgue correspondence in the special case of cyclic base change for toral supercuspidals, assuming several conjectures about torsion in the cohomology of deep-level Deligne-Lusztig varieties. The analogous results for the Fargues-Scholze correspondence are subsumed by the L-parameter calculations in §10.

1.6.2. *The σ-dual homomorphism.* The work [RW22] of Riche-Williamson, which gives a geometric proof of the linkage principle, is not logically related to problems we study here, but has some philosophical similarities. We apply Smith–Treumann localization from Gr_G to Gr_H coming from an automorphism of G, while Riche-Williamson apply (a slightly different form of) Smith–Treumann localization to a self-embedding of Gr_G coming from the action of $\mu_\ell \subset \mathbf{G}_m$ via loop rotation.

The paper [LL21] of Leslie-Lonergan also applies Smith–Treumann localization along this self-embedding, in an attempt to give a geometric construction of the Frobenius contraction functor on $\mathrm{Rep}_k(\check{G})$. Our work also has some common features to theirs.

1.7. **Organization of the paper.** The paper begins in earnest with §3, which develops Smith–Treumann localization for diamonds, and §4, which studies the relation of this theory to the Tate construction and Tate cohomology. In §5 we calculate the fixed point subspaces for various affine Grassmannian and shtuka spaces that will arise. Then we categorify the Treumann–Venkatesh theory of the Brauer homomorphism in §6. Using Tannakian reconstruction, we turn around to construct the σ-dual homomorphism and establish its compatibilities in §7. In §8 we interface this theory with cohomology of local shtukas and excursion operators. In §9 we deduce consequences for the Treumann–Venkatesh Conjecture. Finally, in §10 we illustrate some first applications to computing the Fargues–Scholze parameters of certain supercuspidal representations.

1.8. **AI methodology.** After the initial referee report was received in early 2026, various AI models were used to proofread the text for errors. The initially submitted version of this paper required an assumption that ℓ is sufficiently large (with effective bounds). Most recently, a review by GPT-6 uncovered a gap in a proof of a Lemma (no longer present in the manuscript), and suggested a fix for the gap that used the Tate construction. Upon contemplating this issue, the author realized that parts of the paper could be reworked to eliminate the assumptions on ℓ entirely. The author had been thinking about how to eliminate the technical assumptions on ℓ, for application to his sequel work with Sean Cotner based on these results.

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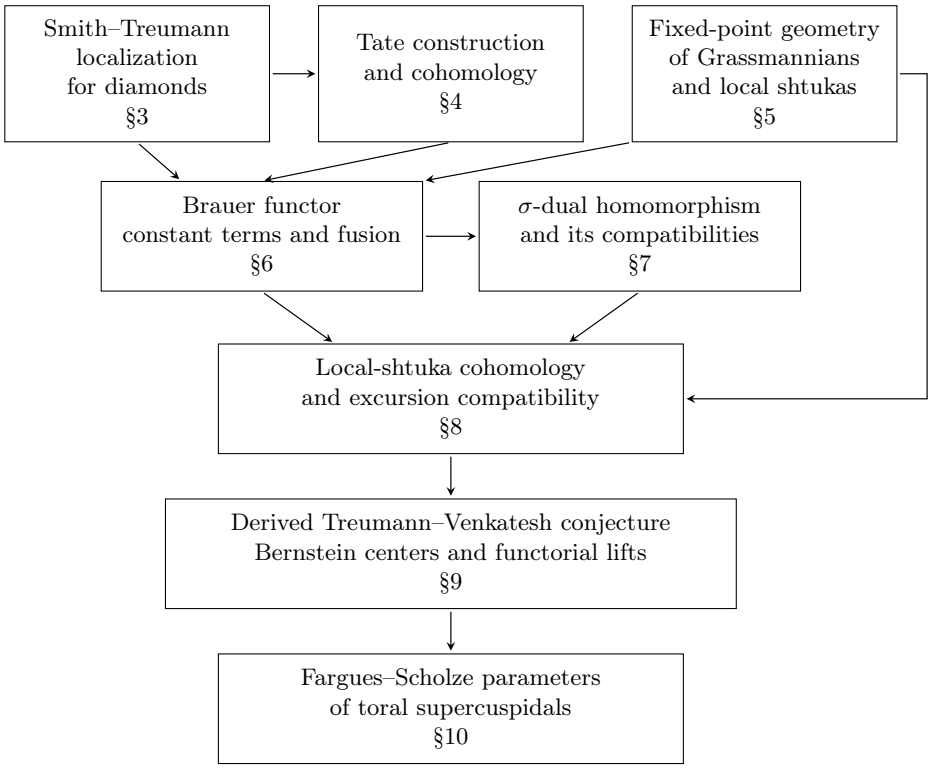

FIGURE 2. Leitfaden of the paper.

2. NOTATION

We fix a prime p.

2.1. **Local fields.** We let E be a local field of residue characteristic p, $\mathcal{O}_E$ its ring of integers, ϖ_E a uniformizer, and $\mathbf{F}_q = \mathcal{O}_E/\varpi_E$ its residue field.

Let $\breve{E}$ be the completion of the maximal unramified extension of E. For a reductive group G/E, $B(G)$ denotes the Kottwitz set of G, i.e., elements of $G(\breve{E})$ modulo Frobenius-conjugacy.

Let E^s be a separable closure of E and $W_E \subset \mathrm{Gal}(E^s/E)$ be the Weil group of E.

2.2. **The group Σ.** Throughout, ℓ denotes a prime number different from p and Σ denotes a finite cyclic group of order ℓ. The notation σ always denotes a generator of Σ; conversely, if an automorphism σ is constructed first then Σ always denotes the group generated by σ. Some constructions (e.g., Tate cohomology) are phrased in terms of σ, but ultimately all constructions are independent of the choice of σ.

We denote by N the element $1 + \sigma + \ldots + \sigma^{\ell-1} \in \mathbf{Z}[\Sigma]$.

For an object Y with an action of Σ, we denote by Y^σ or $\mathrm{Fix}(\sigma, Y)$ the σ-fixed points of Y.

2.3. **Reductive groups.** Throughout, G denotes a reductive group over E. Throughout, reductive groups are understood to be connected. When the notation H appears, it refers to a reductive group arising as the subgroup of G fixed by an action of Σ. We denote by $\iota\colon H \to G$ the tautological embedding, and we use the same notation for induced maps such as $H(E) \to G(E)$, $B(H) \to B(G)$, $\mathrm{Gr}_H \to \mathrm{Gr}_G$, etc.

For a split torus T, we denote by $X_*(T)$ and $X^*(T)$ the cocharacter and character groups of T, respectively.

We denote by $\mathrm{Rep}_k^{\mathrm{sm}} G(E)$ the category of smooth representations of $G(E)$ over k.

2.4. **Perfectoid spaces.** The notation (C, C^+) will always mean that C is an algebraically closed perfectoid field and $C^+ \subset C^\circ$ is an open integrally closed subring, where C° denotes the ring of power-bounded elements in C.

Following [FS21], we denote by $\mathrm{Perf} = \mathrm{Perf}_{\overline{\mathbf{F}}_p}$ the category of perfectoid spaces over $\overline{\mathbf{F}}_p$.

2.5. **Coefficients.** Throughout, k denotes an algebraically closed field of characteristic ℓ. We denote by $\mathbb{O} := W(k)$ the Witt vectors of k.

We write $\mathrm{Frob}_\ell \colon k \to k$ for the absolute Frobenius automorphism $x \mapsto x^\ell$.

We will use Λ to denote a coefficient ring which is finite over $\mathbb{O}$. In particular, Λ is ℓ-adically complete.

2.6. **Sheaves.** The six-functor formalism for sheaves of Λ-modules is developed in [Sch22] (the adic case is in §26 of *loc. cit.*).

At the request of the referee, we point out some subtleties in the construction of this sheaf theory. If Y is any small v-stack, then [Sch22] defines a derived category of étale sheaves $D_{\text{ét}}(Y; \Lambda)$. In general, this is defined by v-descent from strictly totally disconnected spaces, hence can be tricky to understand concretely. For discrete coefficients, if Y is a locally spatial diamond, then it has an étale site $Y_{\text{ét}}$ and [Sch22, Proposition 14.15] says that $D_{\text{ét}}(Y; \Lambda)$ is the left completion of $D(Y_{\text{ét}}; \Lambda)$. In particular, the bounded subcategories of the two are identified. For adic coefficients, we define $D_{\text{ét}}(Y; \Lambda)$ as a derived inverse limit from the torsion case.

Also, if Y is a locally spatial diamond then $Y_{\text{ét}}$ has enough points by [Sch22, Proposition 14.3], and geometric stalks form a conservative family.

Definition 2.6.1. We say that a map $f \colon Y \to Y'$ of small v-stacks on Perf is *shriekable* if it is compactifiable, representable in locally spatial diamonds, and has locally finite $\dim.\,\mathrm{trg}$. This is exactly the hypothesis in [Sch22, §1] for $\mathrm{R}f_! \colon D_{\text{ét}}(Y; \Lambda) \to D_{\text{ét}}(Y'; \Lambda)$ to exist, and under the same assumptions its right adjoint $\mathrm{R}f^! \colon D_{\text{ét}}(Y'; \Lambda) \to D_{\text{ét}}(Y; \Lambda)$ exists.

2.7. **Categories.** We fix some notation for abstract categorical constructions.

2.7.1. *Equivariant derived categories.* We begin with some discussion of equivariant derived categories. There are two settings of interest, abelian categories and triangulated categories, where the definitions need to be treated differently.

For a 1-category C with an action of a group Γ, we let $\mathsf{C}^{B\Gamma}$ denote the category of Γ-equivariant objects in C. For an ∞-category $\mathcal{C}$ with an action of a group Γ, we let $\mathcal{C}^{B\Gamma}$ denote the category of Γ-equivariant objects in $\mathcal{C}$. Our notation is not completely standard (one might also see $\mathsf{C}^{h\Gamma}$ or C^{Γ} used to denote the same notion in the literature), but follows [FS21, I.13]. We can give a precise formulation of the definitions in a somewhat uniform manner:

- A Γ-action on a 1-category C can be viewed as a pseudofunctor $\mathsf{F} \colon B\Gamma \to \mathsf{Cat}$ from the groupoid $B\Gamma$ to the 2-category of 1-categories Cat, carrying the neutral point of $B\Gamma$ to C, and we have
$$\mathsf{C}^{B\Gamma} := 2\text{-}\lim_{B\Gamma} \mathsf{F}.$$
- A Γ-action on an ∞-category $\mathcal{C}$ can be viewed as a functor $\mathcal{F} \colon B\Gamma \to \mathsf{Cat}_\infty$ from the ∞-groupoid $B\Gamma$ to the ∞-category of ∞-categories Cat_∞, carrying the neutral point of $B\Gamma$ to $\mathcal{C}$, and we have
$$\mathcal{C}^{B\Gamma} := \lim_{B\Gamma} \mathcal{F}.$$

Finally, if a triangulated category C is the homotopy category of a stable ∞-category $\mathcal{C}$, then we denote by $\mathsf{C}^{B\Gamma}$ the homotopy category of $\mathcal{C}^{B\Gamma}$. Often we abuse notation by suppressing the discussion of $\mathcal{C}$ when there is a natural choice; for example, if C is the triangulated derived category of sheaves on a site, then $\mathcal{C}$ is taken to be the stable ∞-category of sheaves on that site.

2.7.2. *Tensor products.* Let A be a ring and let C, D be A-linear finitely cocomplete[2] abelian categories. We abbreviate $\mathsf{C} \otimes_A \mathsf{D}$ for the (Kelly) tensor product of C and D relative to A [LF13, §2.4]. We note that this coincides with Deligne's tensor product if the latter exists, by [LF13, Theorem 1].

[2] i.e., admitting finite colimits.

3. Smith–Treumann localization for diamonds

In this section we develop for diamonds a form of *Smith–Treumann localization*, which refers to a type of sheaf-theoretic equivariant localization from a Σ-equivariant space to its Σ-fixed points. There is a loose analogy between Smith–Treumann localization and the perhaps more familiar hyperbolic localization for spaces with a $\mathbf{G}_m$-action.

The output of the theory looks similar to that for complex algebraic varieties or schemes (developed in [Tre19] and [RW22]). However, the details of the proofs are quite different, owing to the different behavior of étale sheaves on adic spaces. For example, the theory relies crucially on finiteness results, which in the case of schemes comes from quasi-compactness (among other things). On the other hand, non-quasicompactness is ubiquitous in p-adic geometry. For example, the complement of a closed subspace is usually not quasi-compact, hence the extension-by-zero from such a complement is not constructible; this example already illustrates that we will frequently need to contend with non-constructible sheaves.

The contents of this section are as follows. In §3.1 we codify the notion of "extra small" v-stacks. The adjective "extra small" is a slight strengthening of "small", and guarantees that we can approximate the v-stack by quasicompact subspaces. In §3.3 we establish bounds on tor-amplitude.

3.1. Extra small v-stacks.

Let ω_1 be the first uncountable cardinal. Recall that an ω_1-*cofiltered* inverse system I is one for which any functor $J \to I$, with J a countable category, extends to $J^{\triangleleft} \to I$, where the cone $J^{\triangleleft}$ consists of J plus an extra object with a unique map to each object of J.

Definition 3.1.1 (Extra small v-stacks)**.** We say that a v-stack Y is *extra small* if for any ω_1-cofiltered inverse system $\{S_i = \mathrm{Spa}(R_i, R_i^+)\}_{i \in I}$ of affinoid perfectoid spaces with inverse limit $S = \mathrm{Spa}(R, R^+)$, the natural map

$$\varinjlim_{i \in I} Y(S_i) \to Y(S)$$

is an isomorphism.

Definition 3.1.2 (Relative extra smallness)**.** A morphism $f \colon X \to Y$ of v-stacks is *relatively extra small* if, for every ω_1-cofiltered inverse system of affinoid perfectoid spaces $S_i \to Y$ with inverse limit $S \to Y$, the natural map

$$\varinjlim_{i} \mathrm{Hom}_Y(S_i, X) \longrightarrow \mathrm{Hom}_Y(S, X)$$

is an isomorphism.

Remark 3.1.3. It is clear from the definition that relative extra smallness is preserved by arbitrary base change.

Moreover, if X and Y are extra small, then every morphism $X \to Y$ is relatively extra small. Indeed, a map $S \to X$ factors through some S_i by extra smallness of X; after increasing i, it is easy to see that this factorization is as maps over Y.

The following Lemma justifies the terminology.

Lemma 3.1.4. *If a v-stack Y is extra small, then Y is small.*

Proof. The proof is contained in the first paragraph of [FS21, Proof of Proposition III.1.3]. □

Remark 3.1.5. Informally speaking, all the small v-stacks that have come up in our experience are also extra small. The remark below [FS21, Proposition III.1.3] suggests that all "reasonable" v-stacks are extra small.

Example 3.1.6. The last paragraph of the proof of [FS21, Proposition III.1.3] shows that Bun_G is extra small. By minor variations on this argument, the following v-stacks are also extra small.

1. The local Hecke stack $\mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1_X)^I}$ [FS21, Definition VI.1.6], and the global Hecke stack Hck^I_G [FS21, §IX.2].
2. The Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I}$ [FS21, Definition VI.1.8].
3. The moduli spaces of local shtukas $\mathrm{Sht}_{(G,b,\mu_\bullet),K}$ [SW20, Lecture 23].

It follows from the same proofs, or alternatively from Remark 3.1.3, that in all of these cases the projection morphisms to $(\mathrm{Div}^1_X)^I$ are relatively extra small.

The importance of extra smallness (for us) is to control the cohomological dimension of direct image along open embeddings. For quasi-compact open embeddings, the direct image is actually exact [Sch22, Lemma 21.13], but we will typically be contending with non-quasicompact open embeddings. Under hypotheses that spaces are extra small, we will be able to approximate open embeddings by quasicompact ones, as articulated in the following Lemma (pointed out by Peter Scholze).

Lemma 3.1.7. *Let Y be a locally spatial diamond admitting a countable open cover by spatial diamonds Y_n. Let $Z \hookrightarrow Y$ be a relatively extra small closed embedding. Then $U := Y \setminus Z$ is a countable union of quasicompact open subspaces.*

Proof. Let $Z_n := Z \cap Y_n$. If $Y_n \setminus Z_n$ is a countable union of quasicompact open subsets for each n, then

$$Y \setminus Z = \bigcup_n (Y_n \setminus Z_n)$$

is also a countable union of quasicompact open subsets. Therefore, renaming Y_n to Y and Z_n to Z, we may assume that Y is spatial and $Z \to Y$ is relatively extra small, with the goal of showing that $U := Y \setminus Z$ is a countable union of quasicompact open subsets.

By [Sch22, Proposition 11.24], Y admits a surjective universally open quasi-pro-étale cover by a strictly totally disconnected space. Since relative extra smallness is preserved by pullback, and such a cover carries quasicompact opens to quasicompact opens, we may pass to the cover and assume that Y is strictly totally disconnected. Since $|Z|$ is closed and generalizing, it is the inverse image of a closed subset of $\pi_0(Y)$, and $Y \setminus Z$ is a union of clopen subsets.

Consider the collection $\mathcal{Z}$ of closed sub-diamonds $Z_i \subset Y$ such that

- $Z_i \supset Z$, and
- $Y \setminus Z_i$ is a countable union of quasi-compact open subsets.

Viewing $\mathcal{Z}$ as a partially ordered set under inclusion, we claim that $\mathcal{Z}$ is ω_1-cofiltered. Indeed, if I is countable then we have that

$$Y \setminus \bigcap_{i \in I} Z_i = \bigcup_{i \in I} (Y \setminus Z_i)$$

is a countable union of countable unions of quasi-compact open subsets, so $\bigcap_{i \in I} Z_i \in \mathcal{Z}$. The poset $\mathcal{Z}$ is non-empty, since it contains Y.

There is an obvious map

$$Z \to \varprojlim_{Z_i \in \mathcal{Z}} Z_i, \tag{3.1.1}$$

which we claim is an isomorphism. Recall that closed sub-diamonds of Y are determined by their underlying topological spaces, as follows from [Sch22, Proposition 11.20] (the generalization of this statement to v-stacks is [AGLR22, Lemma 2.7]), so it suffices to check (3.1.1) at the level of topological spaces. We will show that

$$|\varprojlim_{Z_i \in \mathcal{Z}} Z_i| = \varprojlim_{Z_i \in \mathcal{Z}} |Z_i| = |Z|, \tag{3.1.2}$$

The first equality above follows from [Sch22, Lemma 11.22], since this is an inverse limit of spatial diamonds with qcqs transition maps (here the transition maps are closed immersions). To see the second equality, we note that any point of the open subspace $|Y \setminus Z|$ lies in a clopen subset thereof since Y is strictly totally disconnected. If $y \in |Y| \setminus |Z|$ and $V \subset |Y| \setminus |Z|$ is such a clopen neighborhood, then the closed subdiamond $Y \setminus V$ belongs to $\mathcal{Z}$ and does not contain y. This shows that $\varprojlim_{Z_i \in \mathcal{Z}} |Z_i| \subset |Z|$, and this containment is an equality because every Z_i contains Z.

Since Y is spatial, there exists [Sch22, Proposition 11.24] a quasi-pro-étale cover of Y by a strictly totally disconnected space, which is of the form

$$\mathrm{Spa}(B, B^+) \twoheadrightarrow Y \tag{3.1.3}$$

by [Sch22, Proposition 1.15]. Recall that a closed subspace of a strictly totally disconnected space is again strictly totally disconnected by [Sch22, Proposition 7.16], and then affinoid by [Sch22, Proposition 1.15].

Hence for each $Z_i \in \mathcal{Z}$, the cover (3.1.3) pulls back to a cover $\mathrm{Spa}(A_i, A_i^+) \twoheadrightarrow Z_i$ where $\mathrm{Spa}(A_i, A_i^+)$ is strictly totally disconnected. The system $\{\mathrm{Spa}(A_i, A_i^+)\}_{Z_i \in \mathcal{Z}}$ is ω_1-cofiltered, since it is the pullback of the ω_1-cofiltered system of closed subdiamonds $Z_i \subset Y$ along the cover (3.1.3). Then, letting $\mathrm{Spa}(A, A^+) := \varprojlim_{Z_i \in \mathcal{Z}} \mathrm{Spa}(A_i, A_i^+)$,[3] the composite map

$$\mathrm{Spa}(A, A^+) = \varprojlim_{Z_i \in \mathcal{Z}} \mathrm{Spa}(A_i, A_i^+) \to \varprojlim_{Z_i \in \mathcal{Z}} Z_i = Z$$

factors, as a map over Y, through some $g_i \colon \mathrm{Spa}(A_i, A_i^+) \to Z$ by relative extra smallness. The two pullbacks of g_i to $\mathrm{Spa}(A_i, A_i^+) \times_{Z_i} \mathrm{Spa}(A_i, A_i^+)$ have the same composite with $Z \hookrightarrow Y$ and hence agree, since $Z \hookrightarrow Y$ is a monomorphism. Consequently g_i descends along the v-cover $\mathrm{Spa}(A_i, A_i^+) \twoheadrightarrow Z_i$ to a map $Z_i \to Z$ over Y. As $Z \subseteq Z_i$ are closed subdiamonds of Y, this forces $Z_i = Z$. Hence $Z \in \mathcal{Z}$, which means by definition that $Y \setminus Z$ is a countable union of quasi-compact open subsets. □

Motivated by Lemma 3.1.7, we make the following definition.

Definition 3.1.8. We say that a locally spatial diamond Y is *(ω_0-)locally spatial* if Y can be written as a (countable) union $\bigcup_n Y_n$ of open spatial sub-diamonds. (Recall that $\omega_0 = \aleph_0$ is the cardinality of the natural numbers.) We say that a closed subdiamond $Z \hookrightarrow Y$ is *ω_0-locally extra small* if such a union can be arranged so that $Y_n \cap Z$ is extra small for each n.

We say that a morphism of v-stacks $f \colon Y' \to Y$ is *(ω_0-)locally spatial* if for any spatial diamond X and any map $X \to Y$, the fibered product $X \times_Y Y'$ is (ω_0-)locally spatial.

Example 3.1.9. Let $i \colon Z \hookrightarrow Y$ be a relatively extra small closed embedding of ω_0-locally spatial diamonds. Then the complementary open embedding $j \colon U \hookrightarrow Y$ is ω_0-locally spatial. Indeed, after any base change from a spatial diamond $X \to Y$, relative extra smallness is preserved and Lemma 3.1.7, applied with the single spatial open X, writes $X \setminus (X \times_Y Z)$ as a countable union of quasicompact open subdiamonds.

Definition 3.1.10. For $d \geq 0$, we say that a locally spatial diamond Y has *ℓ-cohomological dimension* $\leq d$ if every quasicompact open subdiamond $W \subseteq Y$ has cohomology on étale $\mathbf{F}_\ell$-sheaves of amplitude in $[0, d]$. We say that Y is *ℓ-bounded* if it has ℓ-cohomological dimension $\leq d$ for some $d < \infty$.

Example 3.1.11. The following statements are used to bound the ℓ-cohomological dimension in practice. Recall from [Sch22, Definition 21.1] that the *dimension* of a locally spectral space is the supremum of the lengths of chains of specializations. By [Sch22, Proposition 21.11], if Y is a spatial diamond of finite dimension, then

$$Y \text{ has } \ell\text{-cohomological dimension} \leq \dim Y + \sup_y \mathrm{cd}_\ell\, y$$

where y runs through maximal points of Y. According to [Sch22, Proposition 21.16], if $f \colon Y \to \mathrm{Spa}(C, C^+)$ is a map of locally spatial diamonds, then $\mathrm{cd}_\ell\, y \leq \dim.\,\mathrm{trg}\, f$ for all maximal points $y \in Y$. These estimates are uniform on quasicompact open subdiamonds: any such open $W \subseteq Y$ is spatial, satisfies $\dim W \leq \dim Y$, and the same bound on the cohomological dimensions of its maximal points applies.

Lemma 3.1.12. *Let A be a finite commutative $\mathbb{O}$-algebra. Let Y be an ω_0-locally spatial diamond with ℓ-cohomological dimension $\leq d$. Let $Z \hookrightarrow Y$ be a relatively extra small closed subdiamond with (open) complement $j \colon U \hookrightarrow Y$. Then $\mathrm{R}j_* \colon D_{\acute{e}t}(U; A) \to D_{\acute{e}t}(Y; A)$ has cohomological dimension $\leq d + 1$.*

Proof. The statement is local, so we may immediately reduce to the case that Y is a spatial diamond; the pullback of $Z \hookrightarrow Y$ remains relatively extra small.

By Lemma 3.1.7, there is a nested countable sequence of quasicompact open embeddings $j_n \colon U_n \hookrightarrow Y$ such that $U_n \subset U$ and $\varinjlim_n U_n = U$. Let $i_n \colon U_n \hookrightarrow U$ denote the inclusion, so that $j_n = j \circ i_n$. Hence we have

$$\mathrm{R}j_* \cong \mathrm{R}\varprojlim_n \mathrm{R}j_{n*} i_n^* \colon D_{\text{ét}}(U; A) \to D_{\text{ét}}(Y; A).$$

[3] So $A = \varinjlim A_i$, which we note is already complete since any Cauchy sequence already lies in some A_i by the ω_1-cofiltered condition.

By [Sch22, Lemma 21.13], the (derived) direct image along a *quasi-compact* open embedding is exact, hence has cohomological dimension zero. This applies to j_n for every n.

The desired vanishing bounds may be checked after derived reduction modulo ℓ, so it is enough to treat $A/\ell A$-modules. But $A/\ell A$ is a finite-dimensional k-algebra, so the assumed ℓ-cohomological dimension bound also bounds the cohomology of sheaves of A-modules.

Therefore, it suffices to see that the derived inverse limit functor $\mathrm{R}\varprojlim_n$, along a countable index set, has finite cohomological dimension. Thanks to this countability, $\mathrm{R}\varprojlim_n A_n$ can be expressed as

$$\mathrm{R}\varprojlim_n A_n \cong \mathrm{cone}(\mathrm{R}\prod_n A_n \to \mathrm{R}\prod_n A_n)[-1] \tag{3.1.4}$$

where $\mathrm{R}\prod_n$ is the derived functor of the product, defined as the product of injective resolutions of the terms. By the standard generalities in homological algebra, $\mathrm{R}\prod_n$ can be calculated using a product of acyclic (for the global sections functor) resolutions, which can be taken to be of length bounded by the cohomological dimension of Y, which by assumption is $d < \infty$. Then from (3.1.4) we see that $\mathrm{R}\varprojlim_n$ has cohomological dimension at most $d+1$.

□

Corollary 3.1.13. *In the situation of Lemma 3.1.12, if $\mathcal{K} \in D^b_{\acute{e}t}(U;A)$ has finite tor-dimension then $\mathrm{R}j_*\mathcal{K} \in D_{\acute{e}t}(Y;A)$ has finite tor-dimension.*

Proof. This is a special case of [SGA73, Exposé XVII, Théorème 5.2.11], which applies because $\mathrm{R}j_*$ has finite cohomological dimension by Lemma 3.1.12 and the fact [Sch22, Proposition 14.3] that $Y_{\text{ét}}$ has enough points, thanks to Y being locally spatial.

□

3.2. **Fixed points and equivariant sheaves.** For an endomorphism σ of a v-stack Y, we write $\mathrm{Fix}(\sigma, Y) := Y^\sigma$ for the fibered product

$$\begin{array}{ccc} Y^\sigma & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}\times\sigma} \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

Observe that if Y is a separated locally spatial diamond, then Δ is a closed embedding, hence $Y^\sigma \to Y$ is a closed embedding. We will only be interested in this situation. From now on, we assume that σ is an automorphism of order ℓ and denote by $\Sigma = \langle\sigma\rangle$ the cyclic group that it generates. Recall the equivariant bounded derived category $D^b_{\text{ét}}(Y;\Lambda)^{B\Sigma}$, which is the homotopy category of $\mathcal{D}^b_{\text{ét}}(Y;\Lambda)^{B\Sigma}$, the limit of the action diagram $B\Sigma \to \mathsf{Cat}_\infty$ on the ∞-category $\mathcal{D}^b_{\text{ét}}(Y;\Lambda)$ of [Sch22, §17].

The Σ-action on Y induces the trivial Σ-action on Y^σ, for which we have an equivalence of derived categories

$$D^b_{\text{ét}}(Y^\sigma;\Lambda)^{B\Sigma} \cong D^b_{\text{ét}}(Y^\sigma;\Lambda[\Sigma]). \tag{3.2.1}$$

Indeed, this is a consequence of the following more general Lemma.

Lemma 3.2.1. *Let Y be a v-stack equipped with the trivial action of a finite discrete group Γ. Then we have a natural equivalence of derived categories*

$$D_{\acute{e}t}(Y;\Lambda)^{B\Gamma} \cong D_{\acute{e}t}(Y;\Lambda[\Gamma]). \tag{3.2.2}$$

where $\Lambda[\Gamma]$ is the group ring of Γ over Λ.

Proof. By definition (recall §2.6), both sides of (3.2.2) are defined by v-descent from strictly totally disconnected spaces, so we reduce to the case where Y is strictly totally disconnected. For discrete coefficients, $D_{\text{ét}}(Y;\Lambda)$ is (by definition) the derived category of the abelian category of Λ-modules on $Y_{\text{ét}}$, and similarly for $\Lambda[\Gamma]$.

Since Γ is discrete, $D_{\text{ét}}(Y;\Lambda)^{B\Gamma}$ is the derived category of the abelian category of Γ-equivariant Λ-module sheaves on $Y_{\text{ét}}$. It therefore suffices to identify this abelian category with the abelian category of $\Lambda[\Gamma]$-module sheaves on $Y_{\text{ét}}$. Since the Γ-action on Y is trivial, a Γ-equivariant structure on an étale sheaf $\mathcal{F}$ on Y is tautologically equivalent to an action of Γ on $\mathcal{F}$ over Y, or in other words, a $\Lambda[\Gamma]$-module

structure on $\mathcal{F}$, as desired. The adic case follows by passing to the inverse limit from the case of torsion coefficients, since formation of Γ-equivariant objects commutes with limits. □

Remark 3.2.2. If $f\colon Y \to Y'$ is a σ-equivariant map, then it induces a commutative diagram

$$\begin{array}{ccc} Y^\sigma & \longrightarrow & Y \\ \downarrow & & \downarrow \\ (Y')^\sigma & \longrightarrow & Y' \end{array}$$

In general, this is not Cartesian. However, if Y, Y' are separated locally spatial diamonds and f is a monomorphism, then it *is* Cartesian. (This special case will be used in the sequel.) Indeed, in this case we have $Y^\sigma(S) = Y(S)^\sigma$ for any affinoid perfectoid space S, and

$$Y(S) \times_{Y'(S)} (Y')^\sigma(S) = Y(S) \cap Y'(S)^\sigma = Y(S)^\sigma = Y^\sigma(S)$$

3.3. **Bounding tor-amplitude.** Here we establish technical results that show the "permanence" of finite tor-dimension under various operations.

Let Λ be a finite commutative $W(k)$-algebra; we will be most interested in the cases $\Lambda = k$ or $W(k)$. Recall that Σ is a cyclic group of order ℓ and $\Lambda[\Sigma]$ denotes its group ring.

Definition 3.3.1. Let Y be a small v-stack. We define $\mathrm{Flat}^b(Y;\Lambda[\Sigma]) \subset D^b_{\text{ét}}(Y;\Lambda[\Sigma])$ to be the full subcategory consisting of complexes with finite tor-dimension [Sta20, Tag 0651], i.e., which have tor-amplitude in $[a,b]$ for some $a, b \in \mathbf{Z}$.

For $\Lambda = k$, the following Lemma (pointed out by Jesper Grodal) gives an alternative characterization of the subcategory $\mathrm{Flat}^b(Y;k[\Sigma])$, showing that it is determined by conditions on geometric stalks.

Lemma 3.3.2. *Assume that Y is a locally spatial diamond. Then the subcategory $\mathrm{Flat}^b(Y;k[\Sigma]) \subset D^b_{\acute{e}t}(Y;k[\Sigma])$ coincides with the full subcategory spanned by objects whose stalks at all geometric points have finite tor-dimension over $k[\Sigma]$.*

Proof. We apply [Sta20, Tag 0DJJ(1)] which immediately gives one containment: if $\mathcal{K} \in D^b_{\text{ét}}(Y;k[\Sigma])$ has finite tor-dimension over $k[\Sigma]$, then all its stalks at geometric points have finite tor-dimension over $k[\Sigma]$.

Next we show the other containment. Since Y is a locally spatial diamond, [Sch22, Proposition 14.3] ensures that $Y_{\text{ét}}$ has enough points. Therefore we may apply [Sta20, Tag 0DJJ(2)]: it suffices to show that if all the geometric stalks of $\mathcal{K} \in D^b_{\text{ét}}(Y;k[\Sigma])$ have finite tor-amplitude, then there is a *uniform* bound on the tor-amplitude of the geometric stalks. To see this, since $\mathcal{K}$ is bounded we may pick a bounded complex representing it, say concentrated in degrees $[a,b]$. It then suffices to show: if $\mathcal{K}$ is a complex of $k[\Sigma]$-modules concentrated in degrees $[a,b]$ and has finite tor-dimension, then in fact its tor-amplitude lies in $[a,b]$.

Since $k[\Sigma]$ is Artinian, the properties of being flat and projective over $k[\Sigma]$ coincide by [Sta20, Tag 051E]. Therefore $\mathcal{K}$ also has finite projective dimension [Sta20, Tag 0A5M]. Furthermore, since $k[\Sigma]$ has finitistic dimension 0 (by [RG71], the finitistic dimension of a commutative ring is bounded by the Krull dimension), the projective amplitude of $\mathcal{K}$ lies in $[a,b]$. Hence $\mathcal{K}$ is represented by a complex of projective $k[\Sigma]$-modules supported in degrees $[a,b]$, and therefore has tor-amplitude in $[a,b]$. □

Lemma 3.3.3. *Let A be a finite commutative $\mathbb{O}$-algebra. Let Y and S be locally spatial diamonds, and let $f\colon Y \to S$ be shriekable with $\dim.\,\mathrm{trg}\, f < \infty$.*

(1) Then $\mathrm{R}f_!\colon D^b_{\acute{e}t}(Y;A) \to D^b_{\acute{e}t}(S;A)$ has cohomological dimension $\leq 3\dim.\,\mathrm{trg}\, f$.

(2) If furthermore S has ℓ-cohomological dimension $\leq d$ and f is ω_0-locally spatial, then $\mathrm{R}f_*\colon D^b_{\acute{e}t}(Y;A) \to D^b_{\acute{e}t}(S;A)$ has cohomological dimension $\leq 3\dim.\,\mathrm{trg}\, f + d + 1$.

Proof. As in the proof of Lemma 3.1.12, ℓ-adic devissage reduces the relevant cohomological dimension bounds for A-coefficients to the case of $A/\ell A$-coefficients.

(1) By the hypothesis that Y and S are locally spatial, we may write $Y \cong \varinjlim_i Y_i$ as a filtered colimit of spatial open subdiamonds $j_i\colon Y_i \hookrightarrow Y$. We have by definition [Sch22, p.134-135] that

$$\mathrm{R}f_! \cong \varinjlim_i \mathrm{R}(f|_{Y_i})_! j_i^*.$$

By [Sch22, Theorem 22.5], each $\mathrm{R}(f|_{Y_i})_!$ has cohomological dimension at most $3\dim.\,\mathrm{trg}\, f$. Since j_i^* is exact and filtered colimits are exact, we deduce that $\mathrm{R}f_!$ also has cohomological dimension at most $3\dim.\,\mathrm{trg}\, f$.

(2) Similarly to (1), we have

$$\mathrm{R}f_* \cong \mathrm{R}\varprojlim_i \mathrm{R}(f|_{Y_i})_* j_i^*$$

where now we may arrange the indexing set to be countable, thanks to the hypothesis that f is ω_0-locally spatial. Each $\mathrm{R}(f|_{Y_i})_* j_i^*$ has cohomological dimension at most $3\dim.\,\mathrm{trg}\, f$ by [Sch22, Theorem 22.5]. Then we bound the cohomological dimension of the derived inverse limit as in the proof of Lemma 3.1.12. □

Lemma 3.3.4. *Let A be a finite commutative $\mathbb{O}$-algebra. Let Y and S be locally spatial diamonds, and let $f\colon Y \to S$ be shriekable with $\dim.\,\mathrm{trg}\, f < \infty$.*

(1) Then $\mathrm{R}f_!\colon D^b_{\acute{e}t}(Y;A) \to D^b_{\acute{e}t}(S;A)$ carries $\mathrm{Flat}^b(Y;A)$ to $\mathrm{Flat}^b(S;A)$.

(2) If furthermore S is ℓ-bounded and f is ω_0-locally spatial, then $\mathrm{R}f_*$ carries $\mathrm{Flat}^b(Y;A)$ to $\mathrm{Flat}^b(S;A)$.

Proof. (1) Suppose $\mathcal{F} \in \mathrm{Flat}^b(Y;A)$. We need to verify that for any sheaf M (concentrated in degree 0) in $D^b_{\text{ét}}(S;A)$, we have $\mathrm{H}^{-i}(M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}f_!\mathcal{F}) = 0$ for $i \gg 0$. By the projection formula [Sch22, Proposition 22.23], we have

$$\mathrm{H}^{-i}(M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}f_!\mathcal{F}) \cong \mathrm{H}^{-i}(\mathrm{R}f_!(f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F})).$$

By the assumption that $\mathcal{F} \in \mathrm{Flat}^b(Y;A)$, $f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F}$ is bounded. Then by Lemma 3.3.3(1), $\mathrm{R}f_!(f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F})$ is bounded, so $\mathrm{H}^{-i}(\mathrm{R}f_!(f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F})) = 0$ for $i \gg 0$, as desired.

(2) This follows from [SGA73, Exposé XVII, Théorème 5.2.11], which applies because $\mathrm{R}f_*$ has finite cohomological dimension by Lemma 3.3.3 and the fact [Sch22, Proposition 14.3] that $S_{\text{ét}}$ has enough points, thanks to S being locally spatial. □

Lemma 3.3.5. *Let Y be a locally spatial diamond with a free action of Σ. Let Y/Σ be the quotient in the category of diamonds, and $q\colon Y \to Y/\Sigma$ denote the quotient map. Then for any $\mathcal{F} \in D^b_{\acute{e}t}(Y;k)^{B\Sigma}$, we have $q_*\mathcal{F} \in \mathrm{Flat}^b(Y/\Sigma;k[\Sigma]) \subset D^b_{\acute{e}t}(Y/\Sigma;k[\Sigma])$, where the $k[\Sigma]$-module structure on $q_*\mathcal{F}$ is induced by the equivariant structure on $\mathcal{F}$.*

Proof. By Lemma 3.3.2, it suffices to check finite tor-dimension on geometric stalks. Let $\overline{y} \to Y/\Sigma$ be a geometric point, and choose a geometric point $\overline{x} \to Y$ over $\overline{y}$. The fiber of q over $\overline{y}$ is a free Σ-orbit, and the equivariant structure on $\mathcal{F}$ identifies

$$(q_*\mathcal{F})_{\overline{y}} \cong \bigoplus_{\tau \in \Sigma} \mathcal{F}_{\tau\overline{x}} \cong k[\Sigma] \otimes_k \mathcal{F}_{\overline{x}}$$

as a complex of $k[\Sigma]$-modules. Since every k-vector space is free, this is a bounded complex of free $k[\Sigma]$-modules, hence has finite tor-dimension.

□

Corollary 3.3.6. *Let Y and S be locally spatial diamonds, and let $f\colon Y \to S$ be shriekable with $\dim.\,\mathrm{trg}\, f < \infty$. Suppose Σ acts trivially on S and freely on Y, and f is Σ-equivariant.*

(1) Let $\Lambda \in \{k, \mathbb{O}\}$. If $\mathcal{F} \in D^b_{\acute{e}t}(Y;\Lambda)^{B\Sigma}$ has finite tor-amplitude over Λ, then $\mathrm{R}f_!\mathcal{F}$ has finite tor-amplitude over $\Lambda[\Sigma]$. In particular, for $\Lambda = k$, the functor $\mathrm{R}f_!\colon D^b_{\acute{e}t}(Y;k)^{B\Sigma} \to D^b_{\acute{e}t}(S;k[\Sigma])$ lands in $\mathrm{Flat}^b(S;k[\Sigma])$.

(2) If furthermore S has ℓ-cohomological dimension $\le d$ and f is ω_0-locally spatial, then $\mathrm{R}f_*\colon D^b_{\acute{e}t}(Y;k)^{B\Sigma} \to D^b_{\acute{e}t}(S;k[\Sigma])$ lands in $\mathrm{Flat}^b(S;k[\Sigma])$.

Proof. By the hypotheses, we may factor f as the composition of Σ-equivariant maps

$$Y \xrightarrow{q} Y/\Sigma \xrightarrow{\overline{f}} S.$$

Since q is finite, $q_! = q_*$.

For (1), let $\mathcal{F}$ be as in the statement. Since Σ acts freely on Y, the map $q\colon Y \to Y/\Sigma$ is an étale Σ-torsor. Therefore, there exists an étale cover $V \to Y/\Sigma$ trivializing q, and choosing a section $s\colon V \to Y$ gives

$$(q_*\mathcal{F})|_V \cong \Lambda[\Sigma] \overset{\mathrm{L}}{\otimes}_\Lambda s^*\mathcal{F}.$$

Since finite tor-amplitude is étale-local, $q_*\mathcal{F}$ has finite tor-amplitude over $\Lambda[\Sigma]$. Lemma 3.3.4(1), applied to $\overline{f}$ with coefficients $\Lambda[\Sigma]$, shows that $\mathrm{R}\overline{f}_!q_*\mathcal{F} = \mathrm{R}f_!\mathcal{F}$ has finite tor-amplitude over $\Lambda[\Sigma]$. For (2), apply Lemma 3.3.5 to q_* and Lemma 3.3.4(2) to $\mathrm{R}\overline{f}_*$ with $A := k[\Sigma]$. □

Lemma 3.3.7. *Let Y be an ℓ-bounded, separated, ω_0-locally spatial diamond with an action of Σ. Assume that Y/Σ is ℓ-bounded and ω_0-locally spatial and that the induced map $\overline{i}\colon Y^\sigma \hookrightarrow Y/\Sigma$ is a relatively extra small closed immersion. Let $U := Y \setminus Y^\sigma$ and $j\colon U \hookrightarrow Y$ be its inclusion into Y. Then for any $\mathcal{F} \in D^b_{\acute{e}t}(U;k)^{B\Sigma}$, we have $i^*\mathrm{R}j_*\mathcal{F} \in \mathrm{Flat}^b(Y^\sigma;k[\Sigma])$.*

Proof. Let $q\colon Y \to Y/\Sigma$ be the quotient map, and let $i\colon Y^\sigma \hookrightarrow Y$ be the fixed-point closed immersion. Let $\overline{i}$ be the composition $Y^\sigma \xrightarrow{i} Y \xrightarrow{q} Y/\Sigma$.

We claim that the commutative square

$$\begin{array}{ccc} Y^\sigma & \xrightarrow{i} & Y \\ \| & & \downarrow q \\ Y^\sigma & \xrightarrow{\overline{i}} & Y/\Sigma \end{array} \tag{3.3.1}$$

is Cartesian. (Informally speaking, this expresses that q is "totally ramified over $Y^\sigma \subset Y$".) Temporarily denote the fibered product by Y'; then the commutative diagram induces a map $Y^\sigma \to Y'$, which we want to show is an isomorphism. We first observe that it is qcqs: in the sequence of maps $Y^\sigma \to Y' \to Y$, the second map is qcqs (being the base change of the qcqs map $Y^\sigma \to Y/\Sigma$) and the composite map is a closed embedding, hence $Y^\sigma \to Y'$ is qcqs, by the "cancellation property" for qcqs morphisms. Then by [Sch22, Lemma 11.11], we can check that $Y^\sigma \to Y'$ is an isomorphism by checking that it induces a bijection on $\mathrm{Spa}(C,C^+)$-points for all geometric points $\mathrm{Spa}(C,C^+)$. Let (y,z) be a geometric point of Y'. The equality $q(y) = q(z)$ in the v-sheaf quotient means that, after a v-cover of the test space, there is some $\tau \in \Sigma$ for which $y = \tau z$. Since z is Σ-fixed, this means that $y = z$. Thus $Y^\sigma \to Y'$ induces a bijection on geometric points, and proves the claim.

Next we claim that $\overline{i}$ is a closed immersion. This can be checked v-locally on the target [Sch22, Proposition 10.10(i)]. The Cartesian square (3.3.1) indicates that base changing $\overline{i}$ along the v-cover $Y \to Y/\Sigma$ gives the closed embedding $i\colon Y^\sigma \hookrightarrow Y$, establishing the claim.

Applying base change to (3.3.1), we have an isomorphism

$$i^*\mathrm{R}j_*\mathcal{F} \cong \overline{i}^*q_*\mathrm{R}j_*\mathcal{F} \in D^b_{\text{ét}}(Y^\sigma;k[\Sigma]). \tag{3.3.2}$$

From the commutativity of the diagram

$$\begin{array}{ccc} U & \xhookrightarrow{j} & Y \\ \downarrow{\scriptstyle q_U} & & \downarrow{\scriptstyle q} \\ U/\Sigma & \xhookrightarrow{\overline{j}} & Y/\Sigma \end{array}$$

$\overline{i}^*q_*\mathrm{R}j_*\mathcal{F} \cong \overline{i}^*\mathrm{R}\overline{j}_*q_{U*}\mathcal{F}$. Now Lemma 3.3.5 implies that $q_{U*}\mathcal{F}$ has finite tor-dimension. The relative extra smallness of $\overline{i}$ allows us to apply Corollary 3.1.13 to deduce that $\mathrm{R}\overline{j}_*q_{U*}\mathcal{F}$ also has finite tor-dimension, hence $\overline{i}^*\mathrm{R}\overline{j}_*q_{U*}\mathcal{F}$ also has finite tor-dimension. It is bounded because $q_{U*}\mathcal{F}$ is bounded (as $\mathcal{F}$ is bounded and q_U is a finite map) and $\mathrm{R}\overline{j}_*$ has finite cohomological dimension by Lemma 3.1.12. Then (3.3.2) implies that $i^*\mathrm{R}j_*\mathcal{F}$ has finite tor-dimension and is bounded, hence lies in $\mathrm{Flat}^b(Y^\sigma;k[\Sigma])$.

□

Lemma 3.3.8. *Retain the notation and assumptions from Lemma 3.3.7. For any $\mathcal{K} \in D^b_{\acute{e}t}(Y;k)^{B\Sigma}$, if we view $i^*\mathcal{K}, i^!\mathcal{K} \in D^b_{\acute{e}t}(Y^\sigma;k)^{B\Sigma} \overset{(3.2.1)}{\cong} D^b_{\acute{e}t}(Y^\sigma;k[\Sigma])$, then the cone of the natural map $i^!\mathcal{K} \to i^*\mathcal{K}$ belongs to $\mathrm{Flat}^b(Y^\sigma;k[\Sigma]) \subset D^b_{\acute{e}t}(Y^\sigma;k[\Sigma])$.*

Proof. We will first recall the construction of the natural map. Let $j\colon Y \setminus Y^\sigma \hookrightarrow Y$. Consider the exact triangle $i_*i^!\mathcal{K} \to \mathcal{K} \to \mathrm{R}j_*j^*\mathcal{K}$ in $D^b_{\text{ét}}(Y;k)^{B\Sigma}$. Applying i^* to it yields the exact triangle in $D^b_{\text{ét}}(Y^\sigma;k)^{B\Sigma}$,

$$i^!\mathcal{K} \to i^*\mathcal{K} \to i^*\mathrm{R}j_*j^*\mathcal{K},$$

in which the first map is the one from the statement of Lemma 3.3.8. Then Lemma 3.3.7 implies that $\mathrm{Cone}(i^!\mathcal{K} \to i^*\mathcal{K}) \cong i^*\mathrm{R}j_*j^*\mathcal{K}$ lies in $\mathrm{Flat}^b(Y^\sigma;k[\Sigma])$, as desired. □

3.4. **The Tate category.**

Definition 3.4.1. We define the *(large) Tate category of* Y (with respect to Λ) to be the Verdier quotient category

$$\mathrm{Shv}(Y;\mathcal{T}_\Lambda) := D^b_{\text{ét}}(Y;\Lambda[\Sigma])/\,\mathrm{Flat}^b(Y;\Lambda[\Sigma]).$$

We denote the tautological projection map from $D^b_{\text{ét}}(Y;\Lambda[\Sigma])$ to $\mathrm{Shv}(Y;\mathcal{T}_\Lambda)$ by

$$\mathbb{T}^*\colon D^b_{\text{ét}}(Y;\Lambda[\Sigma]) \to \mathrm{Shv}(Y;\mathcal{T}_\Lambda).$$

Remark 3.4.2. The applications will all use $\Lambda = k$, but the results in this subsection hold with the stated more general coefficients.

Definition 3.4.3. Let $\epsilon\colon \Lambda[\Sigma] \twoheadrightarrow \Lambda$ be the augmentation and $\epsilon^*\colon D^b_{\text{ét}}(Y;\Lambda) \to D^b_{\text{ét}}(Y;\Lambda[\Sigma])$ be pullback along ϵ. Define the functor

$$\mathbb{T} := \mathbb{T}^*\epsilon^*\colon D^b_{\text{ét}}(Y;\Lambda) \to \mathrm{Shv}(Y;\mathcal{T}_\Lambda).$$

Remark 3.4.4. The notation $\mathbb{T}, \mathbb{T}^*, \epsilon^*$ follows that of Leslie-Lonergan [LL21].

Let $f\colon Y \to S$ denote a Σ-equivariant map of separated locally spatial diamonds with Σ-action.

3.4.1. *Pullback.* Since the pullback functor $f^{\sigma*}\colon D^b_{\text{ét}}(S^\sigma;k)^{B\Sigma} \to D^b_{\text{ét}}(Y^\sigma;k)^{B\Sigma}$ preserves stalks, Lemma 3.3.2 implies that it carries $\mathrm{Flat}^b(S^\sigma;k[\Sigma])$ to $\mathrm{Flat}^b(Y^\sigma;k[\Sigma])$, hence induces a functor

$$f^{\sigma*}\colon \mathrm{Shv}(S^\sigma;\mathcal{T}_k) \to \mathrm{Shv}(Y^\sigma;\mathcal{T}_k).$$

3.4.2. *Pushforward.* Assume furthermore that $f\colon Y \to S$ is shriekable with $\dim.\,\mathrm{trg}\, f < \infty$. By Lemma 3.3.4(1), the functor $\mathrm{R}f^\sigma_!\colon D^b_{\text{ét}}(Y^\sigma;k[\Sigma]) \to D^b_{\text{ét}}(S^\sigma;k[\Sigma])$ carries $\mathrm{Flat}^b(Y^\sigma;k[\Sigma])$ to $\mathrm{Flat}^b(S^\sigma;k[\Sigma])$, hence induces a functor

$$\mathrm{R}f^\sigma_!\colon \mathrm{Shv}(Y^\sigma;\mathcal{T}_k) \to \mathrm{Shv}(S^\sigma;\mathcal{T}_k).$$

If in addition f^σ is ω_0-locally spatial and S^σ is ℓ-bounded, then Lemma 3.3.4(2) gives a functor

$$\mathrm{R}f^\sigma_*\colon \mathrm{Shv}(Y^\sigma;\mathcal{T}_k) \to \mathrm{Shv}(S^\sigma;\mathcal{T}_k).$$

3.5. **The Smith operation.** We define the analogue in p-adic geometry of the "Smith operation" Psm introduced by Treumann in [Tre19].

Definition 3.5.1. Let Y be a separated locally spatial diamond with an action of Σ and $i\colon Y^\sigma \hookrightarrow Y$ the inclusion of its Σ-fixed points. The *Smith operation* is the functor

$$\mathrm{Psm} := \mathbb{T}^* \circ i^*\colon D^b_{\text{ét}}(Y;k)^{B\Sigma} \to \mathrm{Shv}(Y^\sigma;\mathcal{T}_k) \tag{3.5.1}$$

defined as the composition of $i^*\colon D^b_{\text{ét}}(Y;k)^{B\Sigma} \to D^b_{\text{ét}}(Y^\sigma;k)^{B\Sigma} \overset{(3.2.1)}{\cong} D^b_{\text{ét}}(Y^\sigma;k[\Sigma])$ with the projection $\mathbb{T}^*$ to $\mathrm{Shv}(Y^\sigma;\mathcal{T}_k)$.

The following result may be viewed as a form of equivariant localization for spaces with Σ-action, comparing the (relative) cohomology of a space with that of its Σ-fixed points.

Proposition 3.5.2. *Let Y and S be separated locally spatial diamonds with actions of Σ, and let $f\colon Y \to S$ be Σ-equivariant and shriekable with $\dim.\,\mathrm{trg}\, f < \infty$. Denote by $f^\sigma\colon Y^\sigma \to S^\sigma$ the induced map on fixed points.*

(1) Then the following diagram commutes:

$$\begin{array}{ccc} D^b_{\acute{e}t}(Y;k)^{B\Sigma} & \xrightarrow{\mathrm{R}f_!} & D^b_{\acute{e}t}(S;k)^{B\Sigma} \\ \downarrow{\scriptstyle \mathrm{Psm}} & & \downarrow{\scriptstyle \mathrm{Psm}} \\ \mathrm{Shv}(Y^\sigma;\mathcal{T}_k) & \xrightarrow{\mathrm{R}f^\sigma_!} & \mathrm{Shv}(S^\sigma;\mathcal{T}_k) \end{array}$$

(2) Assume that Σ acts trivially on S, that f is ω_0-locally spatial, and that S, Y, and Y/Σ are ℓ-bounded and ω_0-locally spatial. Assume also that $\bar{i}\colon Y^\sigma \hookrightarrow Y/\Sigma$ is relatively extra small. Then the following diagram commutes:

$$\begin{array}{ccc} D^b_{\acute{e}t}(Y;k)^{B\Sigma} & \xrightarrow{\mathrm{R}f_*} & D^b_{\acute{e}t}(S;k)^{B\Sigma} \\ \downarrow{\scriptstyle \mathrm{Psm}} & & \downarrow{\scriptstyle \mathrm{Psm}} \\ \mathrm{Shv}(Y^\sigma;\mathcal{T}_k) & \xrightarrow{\mathrm{R}f^\sigma_*} & \mathrm{Shv}(S^\sigma;\mathcal{T}_k) \end{array}$$

Proof. For (1), base change for $\mathrm{R}f_!$ allows us to replace S by S^σ. In (2), the action on S is trivial by hypothesis.

(1) Let $\mathcal{F} \in D^b_{\acute{e}t}(Y;k)^{B\Sigma}$. Writing $j\colon (Y \setminus Y^\sigma) \hookrightarrow Y$ for the open complement of $i\colon Y^\sigma \hookrightarrow Y$, we have an exact triangle

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \in D^b_{\acute{e}t}(Y;k)^{B\Sigma}.$$

By definition Σ acts freely on $Y \setminus Y^\sigma$, hence Corollary 3.3.6(1), applied to $f \circ j\colon Y \setminus Y^\sigma \to S$, implies that $\mathrm{R}f_!(j_!j^*\mathcal{F}) \cong \mathrm{R}(f\circ j)_!(j^*\mathcal{F})$ lies in $\mathrm{Flat}^b(S;k[\Sigma])$. Then the cone of $\mathrm{R}f_!\mathcal{F} \to \mathrm{R}f^\sigma_!(i^*\mathcal{F})$ lies in $\mathrm{Flat}^b(S;k[\Sigma])$, and therefore becomes 0 in $\mathrm{Shv}(S;\mathcal{T}_k)$. Therefore

$$\mathbb{T}^*(\mathrm{R}f_!\mathcal{F}) \cong \mathbb{T}^*(\mathrm{R}f^\sigma_!(i^*\mathcal{F})) \cong \mathrm{R}f^\sigma_!\,\mathrm{Psm}(\mathcal{F}) \in \mathrm{Shv}(S;\mathcal{T}_k),$$

which exactly expresses the desired commutativity.

(2) The argument is similar to that for (1), but using instead the exact triangle

$$i_*i^!\mathcal{F} \to \mathcal{F} \to \mathrm{R}j_*j^*\mathcal{F}.$$

The Cartesian square (3.3.1) identifies $i\colon Y^\sigma \hookrightarrow Y$ with the base change of $\bar{i}$, so i is relatively extra small. Hence j and therefore $f\circ j$ are ω_0-locally spatial. Corollary 3.3.6(2), applied to $f\circ j\colon Y\setminus Y^\sigma \to S$, shows that the object $\mathrm{R}f_*\mathrm{R}j_*j^*\mathcal{F} \cong \mathrm{R}(f\circ j)_*(j^*\mathcal{F})$ lies in $\mathrm{Flat}^b(S;k[\Sigma])$, and therefore projects to 0 in $\mathrm{Shv}(S;\mathcal{T}_k)$. Lemma 3.3.8 shows that the first arrow in the zigzag

$$\mathrm{R}f_*i_*i^*\mathcal{F} \longleftarrow \mathrm{R}f_*i_*i^!\mathcal{F} \longrightarrow \mathrm{R}f_*\mathcal{F}$$

is obtained by applying $\mathrm{R}f^\sigma_*$ to a morphism whose cone lies in $\mathrm{Flat}^b(Y^\sigma;k[\Sigma])$. The map f^σ is ω_0-locally spatial, and since S is ℓ-bounded, Lemma 3.3.4(2) implies that the first arrow still has cone in $\mathrm{Flat}^b(S;k[\Sigma])$ after applying $\mathrm{R}f^\sigma_*$. Since the action of Σ on $Y\setminus Y^\sigma$ is free, the same holds for the second arrow. Both arrows therefore become isomorphisms in $\mathrm{Shv}(S;\mathcal{T}_k)$, which proves the claimed comparison. □

4. The Tate construction

In this section we define the Tate construction and relate it to Tate cohomology. We show that it factors through Smith–Treumann localization, which we use to establish equivariant localization results for the Tate construction.

4.1. **Tate cohomology.** For a $\Lambda[\Sigma]$-module M, its *Tate cohomology groups* were defined in (1.2.1). We extend the definition 2-periodically to define $\mathrm{T}^j(M)$ for all $j \in \mathbf{Z}$.

4.1.1. *Tate cohomology of complexes.* The Tate cohomology of a *complex* of $\Lambda[\Sigma]$-modules is defined in [Fen24, §3.4.1]. We summarize below. The exact sequence of $\Lambda[\Sigma]$-modules

$$0 \to \Lambda \to \Lambda[\Sigma] \xrightarrow{1-\sigma} \Lambda[\Sigma] \to \Lambda \to 0 \tag{4.1.1}$$

induces a morphism

$$\beta\colon \Lambda \to \Lambda[2] \in D^b(\Lambda[\Sigma]). \tag{4.1.2}$$

Note that it becomes an isomorphism in the Tate category $D^b(\Lambda[\Sigma])/\operatorname{Flat}^b(\Lambda[\Sigma])$, since the middle two terms project to 0. Given a bounded-below complex of $\Lambda[\Sigma]$-modules $C^\bullet$, we define its *Tate cohomology* to be

$$\mathrm{T}^n(C^\bullet) = \varinjlim_{j\to\infty} \operatorname{Hom}_{D^+(\Lambda[\Sigma])}(\Lambda, C^\bullet[n+2j]) \tag{4.1.3}$$

where the transition maps are the β from (4.1.2). Evidently $\mathrm{T}^n(C^\bullet)$ is 2-periodic in n, and we occasionally view the indexing as being $n \in \mathbf{Z}/2\mathbf{Z}$. It is clear that this construction descends to the derived category.

Now we state some assertions that may be specific[4] to $\Lambda = k$. By [Sta20, Tag 051E], a module over $k[\Sigma]$ is flat if and only if it is free. Since Tate cohomology of free $k[\Sigma]$-complexes vanishes (by inspection), this construction further factors through the Tate category, inducing

$$\mathrm{T}^n\colon \operatorname{Shv}(\mathrm{pt}; \mathcal{T}_k) \to \operatorname{Vect}_k,$$

where pt denotes a geometric point $\operatorname{Spa}(C, C^+)$, so that $D^b_{\text{ét}}(\mathrm{pt}; k[\Sigma]) \cong D^b(k[\Sigma])$. We write $\mathrm{T}^* := \bigoplus_n \mathrm{T}^n$ and $\mathrm{H}^* := \bigoplus_n \mathrm{H}^n$.

Remark 4.1.1. Since filtered colimits are exact, an exact triangle $C' \to C \to C''$ of bounded-below complexes induces a 2-periodic long exact sequence

$$\cdots \to \mathrm{T}^n(C') \to \mathrm{T}^n(C) \to \mathrm{T}^n(C'') \to \mathrm{T}^{n+1}(C') \to \cdots.$$

If C is bounded, then applying this to the truncation filtration of C gives a finite filtration of $\mathrm{T}^n(C)$, functorial in C, whose graded pieces are subquotients of the $\mathrm{T}^{n-i}(\mathrm{H}^i(C))$, hence in particular subquotients of the $\mathrm{H}^i(C)$.

4.1.2. *Tate cohomology sheaves.* Let S be a locally spatial diamond with trivial Σ-action, so that Σ-equivariant complexes on S may be regarded as complexes of $\Lambda[\Sigma]$-module sheaves by Lemma 3.2.1. Given $\mathcal{F} \in D^+_{\text{ét}}(S; \Lambda[\Sigma])$, we define *Tate cohomology sheaves*

$$\mathrm{T}^n(\mathcal{F}) := \varinjlim_{j\to\infty} \mathcal{H}om_{D^+_{\text{ét}}(S;\Lambda[\Sigma])}(\Lambda, \mathcal{F}[n+2j])$$

where the transition maps are induced by (4.1.2). The $\mathrm{T}^i(\mathcal{F})$ are étale sheaves of $\mathrm{T}^0(\Lambda)$-modules, where $\mathrm{T}^0(\Lambda)$ is the 0th Tate cohomology of the trivial Σ-module Λ.

Remark 4.1.2 (Tate cohomology as Hom in the Tate category)**.** If $C^\bullet \in D^b(k[\Sigma])$ is bounded, then the argument of [LL21, Proposition 4.5.1] gives a natural isomorphism

$$\mathrm{T}^i(C^\bullet) \cong \operatorname{Hom}_{\operatorname{Shv}(\mathrm{pt};\mathcal{T}_k)}(k, \mathbb{T}^* C^\bullet[i]).$$

Remark 4.1.3. For $\Lambda = k$ and $\mathcal{F} \in D^b_{\text{ét}}(S; k[\Sigma])$, as in Remark 4.1.2 we also have the description

$$\mathrm{T}^i(\mathcal{F}) \cong \mathcal{H}om_{\operatorname{Shv}(S;\mathcal{T}_k)}(k, \mathbb{T}^* \mathcal{F}[i]).$$

[4]Our argument uses $\Lambda = k$, but the assertions could be true in greater generality, as far as we know.

4.2. **The Tate construction.** Our upcoming definitions fit into the very general formalism of the *Tate construction*. In order to contextualize them properly, we recall this formalism, following the discussion of [NS18, §I.1]. Let X be a Σ-equivariant object of a stable ∞-category admitting limits and colimits indexed by $B\Sigma$. The *Tate construction* for X is [NS18, Definitions I.1.10 and I.1.13]

$$X^{t\Sigma} := \mathrm{cofib}(X_{h\Sigma} \xrightarrow{N} X^{h\Sigma}),$$

where N is the norm map.

We will apply the Tate construction in the following specific situation. Let Y be a small v-stack with trivial Σ-action, so that $D_{\text{ét}}(Y;k)^{B\Sigma} \cong D_{\text{ét}}(Y;k[\Sigma])$ by Lemma 3.2.1. For $\mathcal{F} \in D_{\text{ét}}(Y;k[\Sigma])$ we have $\mathcal{F}_{h\Sigma} = \mathcal{F} \overset{\mathrm{L}}{\otimes}_{k[\Sigma]} k$ and $\mathcal{F}^{h\Sigma} = \mathcal{RH}om_{k[\Sigma]}(k, \mathcal{F})$, so that

$$\mathcal{F}^{t\Sigma} = \mathrm{cofib}\left(\mathcal{F} \overset{\mathrm{L}}{\otimes}_{k[\Sigma]} k \xrightarrow{N} \mathcal{RH}om_{k[\Sigma]}(k, \mathcal{F})\right) \in D_{\text{ét}}(Y;k). \tag{4.2.1}$$

Example 4.2.1. If $\mathcal{F} = M$ is a sheaf in degree 0, then we have

$$\begin{array}{lccccccccc} \text{degree} & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ M_{h\Sigma} \cong & \cdots \xrightarrow{1-\sigma} M & \xrightarrow{N} M & \xrightarrow{1-\sigma} M & \longrightarrow 0 & \longrightarrow 0 & \longrightarrow \cdots \\ & & & & \downarrow N & & & \\ M^{h\Sigma} \cong & \cdots \longrightarrow 0 & \longrightarrow 0 & \longrightarrow M & \xrightarrow{1-\sigma} M & \xrightarrow{N} M & \xrightarrow{1-\sigma} \cdots \end{array} \tag{4.2.2}$$

and the norm map $M_{h\Sigma} \to M^{h\Sigma}$ is represented by $N = 1 + \sigma + \ldots + \sigma^{\ell-1}$ in degree 0, and 0 in the other degrees. Thus, we have

$$M^{t\Sigma} \cong [\ldots \xrightarrow{1-\sigma} M \xrightarrow{N} M \xrightarrow{1-\sigma} M \xrightarrow{N} M \xrightarrow{1-\sigma} \ldots] \tag{4.2.3}$$

In general, since $(-)_{h\Sigma}$ and $(-)^{h\Sigma}$ are exact functors, and the norm map is a natural transformation between them, $\mathcal{F}^{t\Sigma}$ can be computed explicitly by picking a complex representing $\mathcal{F}$, applying Example 4.2.1 to the terms to obtain a double complex, and then totalizing. We see from the construction that even for bounded $\mathcal{F}$, the Tate construction need not be bounded.

Lemma 4.2.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact, colimit-preserving functor between presentable stable ∞-categories equipped with t-structures. Assume that $\mathcal{D}$ is right separated, i.e. $\bigcap_n \mathcal{D}^{\geq n} = 0$, and that there exists $c \in \mathbf{Z}$ such that $F(\mathcal{C}^{\geq n}) \subset \mathcal{D}^{\geq n+c}$ for all n. Let Γ be a finite group and $A \in \mathcal{C}^{\geq a}$ an object with Γ-action. Then the natural maps*

$$F(A^{h\Gamma}) \to (FA)^{h\Gamma}, \qquad (FA)_{h\Gamma} \to F(A_{h\Gamma}) \tag{4.2.4}$$

are isomorphisms, inducing an isomorphism

$$F(A^{t\Gamma}) \cong (FA)^{t\Gamma} \tag{4.2.5}$$

Proof. Let Δ be the simplex category and $\Delta_{\leq s} \subset \Delta$ be the full subcategory on $[0], \ldots, [s]$. For a cosimplicial object $A^\bullet\colon \Delta \to \mathcal{C}$, we write

$$\mathrm{Tot}_s(A^\bullet) := \lim_{\Delta_{\leq s}} A^\bullet.$$

Restriction along $\Delta_{\leq s} \hookrightarrow \Delta_{\leq s+1}$ gives a tower

$$\ldots \to \mathrm{Tot}_2(A^\bullet) \to \mathrm{Tot}_1(A^\bullet) \to \mathrm{Tot}_0(A^\bullet), \quad \text{such that} \quad \mathrm{Tot}(A^\bullet) \cong \lim_s \mathrm{Tot}_s(A^\bullet).$$

For $A^\bullet$ the cosimplicial object $[r] \mapsto \prod_{\Gamma^r} A$, we have $A^{h\Gamma} = \varprojlim_s \mathrm{Tot}_s(A^\bullet)$. The fiber of the restriction map

$$\mathrm{Tot}_s(A^\bullet) \to \mathrm{Tot}_{s-1}(A^\bullet)$$

is the shift by $-s$ of a direct summand of a finite product of copies of A, so it lies in $\mathcal{C}^{\geq a+s}$. Since $\mathcal{C}^{\geq m}$ is closed under limits, $\mathrm{fib}(A^{h\Gamma} \to \mathrm{Tot}_s(A^\bullet))$ lies in $\mathcal{C}^{\geq a+s+1}$. Since F is exact, $F(\mathrm{Tot}_s(A^\bullet))$ is the sth partial totalization for FA, so the fiber of the map $F(A^{h\Gamma}) \to (FA)^{h\Gamma}$ lies in $\mathcal{D}^{\geq n}$ for every n, hence vanishes by the assumption that $\mathcal{D}$ is separated.

The homotopy orbits are the colimit over Δ^{op} of the simplicial bar construction $[r] \mapsto \bigoplus_{\Gamma^r} A$, so preservation of colimits by F gives the isomorphism $(FA)_{h\Gamma} \to F(A_{h\Gamma})$.

Finally, both comparisons are compatible with the norm maps, which gives the isomorphism $F(A^{t\Gamma}) \cong (FA)^{t\Gamma}$. □

Example 4.2.3. Lemma 4.2.2 applies, with $\Gamma = \Sigma$ and the standard t-structures, to the following functors:

(1) $g^*\colon D_{\text{ét}}(S;k) \to D_{\text{ét}}(Y;k)$ for any map $g\colon Y \to S$ of small v-stacks;
(2) $\mathrm{R}f_!\colon D_{\text{ét}}(Y;k) \to D_{\text{ét}}(S;k)$ for a shriekable map f of locally spatial diamonds with $\dim.\,\mathrm{trg}\, f < \infty$;
(3) $\mathcal{L} \overset{\mathrm{L}}{\otimes}_k (-)$ on $D_{\text{ét}}(Y;k)$, for $\mathcal{L} \in D^b_{\text{ét}}(Y;k)$.

In particular, for $\mathcal{F} \in D^+_{\text{ét}}(S;k[\Sigma])$ and $g\colon S' \to S$ we have $g^*(\mathcal{F}^{t\Sigma}) \cong (g^*\mathcal{F})^{t\Sigma}$ and $g^*\mathrm{T}^n(\mathcal{F}) \cong \mathrm{T}^n(g^*\mathcal{F})$. Thus the stalks of $\mathrm{T}^n(\mathcal{F})$ at geometric points are the Tate cohomology groups of the stalks of $\mathcal{F}$.

Proposition 4.2.4. *Let Y be a small v-stack with trivial Σ-action and $\mathcal{F} \in D^b_{\acute{e}t}(Y;k[\Sigma])$. Let $\beta\colon \mathcal{F}^{h\Sigma} \to \mathcal{F}^{h\Sigma}[2]$ be the map from* (4.1.2)*. The canonical map $\mathcal{F}^{h\Sigma} \to \mathcal{F}^{t\Sigma}$ extends to an isomorphism*

$$\varinjlim\big(\mathcal{F}^{h\Sigma} \xrightarrow{\beta} \mathcal{F}^{h\Sigma}[2] \xrightarrow{\beta} \mathcal{F}^{h\Sigma}[4] \to \cdots\big) \xrightarrow{\sim} \mathcal{F}^{t\Sigma}. \tag{4.2.6}$$

Consequently $\mathcal{F}^{t\Sigma}$ is 2-periodic, $\mathrm{H}^n(\mathcal{F}^{t\Sigma}) \cong \mathrm{T}^n(\mathcal{F})$ naturally in $\mathcal{F}$, and $\mathcal{F}^{t\Sigma} = 0$ if and only if all $\mathrm{T}^n(\mathcal{F})$ vanish.

Proof. By Example 4.2.3(1), all the constructions involved commute with pullback to a v-cover by strictly totally disconnected perfectoid spaces. The statements in question may be checked after such a cover, so we may and do assume that Y is strictly totally disconnected. In this case we have $D_{\text{ét}}(Y;k[\Sigma]) = D(Y_{\text{ét}};k[\Sigma])$ [Sch22, Propositions 14.11(iii) and 14.15], so we may represent $\mathcal{F}$ by a bounded complex of étale sheaves.

From (4.1.1) we see that the cone of the map $\beta\colon \mathcal{F} \to \mathcal{F}[2]$, obtained by tensoring $\mathcal{F}$ with (4.1.1) over k, with the diagonal Σ-action, is built from induced objects of the form $k[\Sigma] \otimes_k \mathcal{F}$, whose Tate construction vanishes. This implies that β acts invertibly on $\mathcal{F}^{t\Sigma}$, hence the natural transformation $\mathcal{F}^{h\Sigma} \to \mathcal{F}^{t\Sigma}$ factors through a natural transformation

$$\varinjlim_{\beta} \mathcal{F}^{h\Sigma} \to \mathcal{F}^{t\Sigma}. \tag{4.2.7}$$

To show that it is an isomorphism, we may by dévissage reduce (since $\mathcal{F}$ is bounded) to the case where $\mathcal{F}$ is an étale sheaf in degree 0. In this case, it follows from the description in Example 4.2.1 that (4.2.7) is an isomorphism.

The remaining assertions follow by passing to cohomology sheaves, which commute with filtered colimits and detect isomorphisms. □

Proposition 4.2.5. *The functor $\mathcal{F} \mapsto \mathcal{F}^{t\Sigma}$ of* (4.2.1) *factors through an exact "realization" functor*

$$\mathrm{Real}_Y\colon \mathrm{Shv}(Y;\mathcal{T}_k) \to D_{\acute{e}t}(Y;k), \qquad \mathrm{Real}_Y(\mathbb{T}^*\mathcal{F}) = \mathcal{F}^{t\Sigma}. \tag{4.2.8}$$

We have $\mathrm{H}^j \mathrm{Real}_Y(\mathbb{T}^*\mathcal{F}) \cong \mathrm{T}^j(\mathcal{F})$, and Real_Y commutes with pullback along maps $Y' \to Y$, naturally in $\mathcal{F}$.

Proof. Let $\mathcal{F} \in \mathrm{Flat}^b(Y;k[\Sigma])$. Each stalk of $\mathcal{F}$ at a geometric point is a bounded complex of finite tor-dimension over $k[\Sigma]$, hence is represented by a bounded complex of free $k[\Sigma]$-modules, and so has vanishing Tate cohomology. By Example 4.2.3 and Proposition 4.2.4, $\mathcal{F}^{t\Sigma} = 0$, since vanishing may be checked after pullback to a v-cover by a locally spatial diamond, where geometric stalks are conservative. Hence the exact functor $(-)^{t\Sigma}$ factors through the Verdier quotient $\mathrm{Shv}(Y;\mathcal{T}_k)$. The remaining assertions follow from the same two results. □

4.3. **Equivariant localization.** We translate the results of §3.5 into equivariant localization results for the Tate construction.

Proposition 4.3.1. *Let Y and S be separated locally spatial diamonds with Σ-action, and let $f\colon Y \to S$ be Σ-equivariant and shriekable with $\dim.\operatorname{trg} f < \infty$. Write $i_Y\colon Y^\sigma \hookrightarrow Y$ and $i_S\colon S^\sigma \hookrightarrow S$ for the inclusions of the fixed loci. For $\mathcal{F} \in D^b_{\acute{e}t}(Y;k)^{B\Sigma}$ there is an isomorphism, natural in $\mathcal{F}$ and compatible with base change in S,*

$$\big(i_S^* \mathrm{R}f_!\mathcal{F}\big)^{t\Sigma} \xrightarrow{\sim} \mathrm{R}f^\sigma_!\big((i_Y^*\mathcal{F})^{t\Sigma}\big). \tag{4.3.1}$$

In particular, if Σ acts trivially on S, then $(\mathrm{R}f_!\mathcal{F})^{t\Sigma} \cong \mathrm{R}f^\sigma_!\big((i_Y^*\mathcal{F})^{t\Sigma}\big)$.

Proof. Apply Real_{S^σ} to the isomorphism $\mathrm{Psm}(\mathrm{R}f_!\mathcal{F}) \cong \mathrm{R}f^\sigma_!\,\mathrm{Psm}(\mathcal{F})$ of Proposition 3.5.2(1). By (3.5.1) and (4.2.8), the left side becomes $(i_S^*\mathrm{R}f_!\mathcal{F})^{t\Sigma}$ and the right side becomes $(\mathrm{R}f^\sigma_! i_Y^*\mathcal{F})^{t\Sigma}$, which is naturally isomorphic to $\mathrm{R}f^\sigma_!((i_Y^*\mathcal{F})^{t\Sigma})$ by Example 4.2.3(2).

To justify the compatibility with base change in S: note that the isomorphism of Proposition 3.5.2(1) comes from base change and restriction to the fixed locus, so the desired compatibility follows from Proposition 4.2.5 and base change for $\mathrm{R}f_!$. □

Corollary 4.3.2 (Localization for Tate cohomology)**.** *Let $f\colon Y \to S$ be a Σ-equivariant shriekable map of separated locally spatial diamonds with $\dim.\operatorname{trg} f < \infty$, where Σ acts trivially on S. Let $i\colon Y^\sigma \hookrightarrow Y$ be the inclusion and $f^\sigma = f \circ i$. For $\mathcal{F} \in D^b_{\acute{e}t}(Y;k)^{B\Sigma}$, restriction to Y^σ induces isomorphisms*

$$\mathrm{T}^n(\mathrm{R}f_!\mathcal{F}) \xrightarrow{\sim} \mathrm{T}^n\big(\mathrm{R}f^\sigma_! i^*\mathcal{F}\big), \qquad n \in \mathbf{Z}. \tag{4.3.2}$$

In particular, if S is a geometric point then $\mathrm{T}^n(\mathrm{R}\Gamma_c(Y;\mathcal{F})) \cong \mathrm{T}^n(\mathrm{R}\Gamma_c(Y^\sigma; i^*\mathcal{F}))$.

Proof. Let j be the open embedding of the complement of Y^σ. Then we have an exact triangle

$$\mathrm{R}(f\circ j)_! j^*\mathcal{F} \to \mathrm{R}f_!\mathcal{F} \to \mathrm{R}f^\sigma_! i^*\mathcal{F},$$

and the leftmost term lies in $\mathrm{Flat}^b(S;k[\Sigma])$ by Corollary 3.3.6(1), because Σ acts freely on $Y \setminus Y^\sigma$. Its Tate cohomology vanishes by Proposition 4.2.5, and then the assertion follows from the long exact sequence of Remark 4.1.1. □

Remark 4.3.3. The isomorphisms (4.3.1) and (4.3.2) are equivariant for any action of a group Γ on Y, S and $\mathcal{F}$ that commutes with the Σ-actions and is compatible with f.

Remark 4.3.4. The same argument gives localization for compactly supported cohomology of separated schemes of finite type over an algebraically closed field of characteristic p, with coefficients in k; cf. [Fen24, §A.1.2]. This will be used in §10.3.

4.4. **Multiplicativity.** We establish compatibility of the Tate construction with tensor products and cup products.

Proposition 4.4.1. *Let Y and Z be small v-stacks with trivial Σ-actions, and let $\mathcal{F} \in D^b_{\acute{e}t}(Y;k[\Sigma])$ and $\mathcal{G} \in D^b_{\acute{e}t}(Z;k[\Sigma])$. Equip $\mathcal{F}\boxtimes\mathcal{G}$ with the diagonal Σ-action. Then there is a natural exterior cup product in $D_{\acute{e}t}(Y\times Z;k)$*

$$\mathcal{F}^{t\Sigma} \boxtimes \mathcal{G}^{t\Sigma} \to (\mathcal{F}\boxtimes\mathcal{G})^{t\Sigma}. \tag{4.4.1}$$

It is compatible with pullback, and for shriekable maps $f\colon Y \to S$ and $g\colon Z \to T$ of finite $\dim.\operatorname{trg}$, it is compatible with the Künneth isomorphism

$$\mathrm{R}f_!\mathcal{F} \boxtimes \mathrm{R}g_!\mathcal{G} \xrightarrow{\sim} \mathrm{R}(f\times g)_!(\mathcal{F}\boxtimes\mathcal{G}) \tag{4.4.2}$$

and with the projection formula for $\mathrm{R}f_!$. For Σ-equivariant maps as in Proposition 4.3.1, these products are compatible with the isomorphisms (4.3.1)*.*

Proof. The proof of [NS18, Theorem I.3.1], with spectra replaced by $D_{\text{ét}}(Y;k)$ (using [NS18, Theorem I.3.6 and Lemma I.3.8]), shows that the Tate construction (4.2.1) is lax symmetric monoidal, compatibly with the canonical map from homotopy invariants, and that this structure is unique. This constructs (4.4.1).

The map (4.4.1) is the composite of $\mathcal{F}^{t\Sigma}\boxtimes\mathcal{G}^{t\Sigma} \cong (p_1^*\mathcal{F})^{t\Sigma} \otimes (p_2^*\mathcal{G})^{t\Sigma}$, given by Example 4.2.3(1), with the above lax monoidal structure. The comparison maps of Lemma 4.2.2 are induced by the natural maps

for homotopy invariants and coinvariants, and the lax structure on $(-)^{t\Sigma}$ is determined by the one on $(-)^{h\Sigma}$ through the universal property of [NS18, Theorem I.3.6(ii)]. The Künneth and projection-formula compatibilities therefore reduce to those for $(-)^{h\Sigma}$.

Note that the restriction maps $\mathcal{F} \to i_{Y*}i_Y^*\mathcal{F}$ commute with exterior products and with (4.4.2). Applying $(-)^{t\Sigma}$ and Lemma 4.2.2 gives the compatibility with (4.3.1). □

4.5. **Some special cases.** We explicate Tate cohomology in some special cases.

Example 4.5.1 (Tate cohomology of trivial coefficients)**.** Equip k with the trivial Σ-action. (We emphasize our running notation from §2 that $|\Sigma| = \ell$ and ch $k = \ell$.)

$$\mathrm{T}^*(k) = \begin{cases} k[\beta, \beta^{-1}] \otimes_k \bigwedge_k(\eta), & \ell > 2, \\ k[\eta, \eta^{-1}], \quad \beta = \eta^2, & \ell = 2 \end{cases} \tag{4.5.1}$$

where $\beta \in \mathrm{T}^2(k)$ is induced by (4.1.2), and $\eta \in \mathrm{T}^1(k)$ is the class sending σ to 1.

Equip $\mathbb{O}$ with the trivial Σ-action. Then we have

$$\mathrm{T}^j(\mathbb{O}) \cong \begin{cases} k & j \equiv 0 \pmod 2, \\ 0 & j \equiv 1 \pmod 2. \end{cases}$$

Example 4.5.2 (Tate cohomology of trivial complexes)**.** If $C^\bullet \in D^b(\mathbb{O})$, then (cf. Definition 3.4.3 for notation) we have

$$\mathrm{T}^*(\epsilon^* C^\bullet) \cong \mathrm{H}^*(\mathrm{Tot}(\ldots \xrightarrow{\ell} C^\bullet \xrightarrow{0} C^\bullet \xrightarrow{\ell} C^\bullet \xrightarrow{0} \ldots))$$

so that

$$\mathrm{T}^n(\epsilon^* C^\bullet) \cong \bigoplus_{i \equiv n \pmod 2} \mathrm{H}^i(C^\bullet \overset{\mathrm{L}}{\otimes}_{\mathbb{O}} k).$$

If $C^\bullet \in D^b(k)$, then we have

$$\mathrm{T}^*(\epsilon^* C^\bullet) \cong \mathrm{H}^*(C^\bullet) \otimes_k \mathrm{T}^*(k) \tag{4.5.2}$$

where k is equipped with the trivial Σ-action. By Example 4.5.1, (4.5.2) simplifies to

$$\mathrm{T}^n(\epsilon^* C^\bullet) \cong \bigoplus_{i \in \mathbf{Z}} \mathrm{H}^i(C^\bullet).$$

For $\mathcal{L} \in D^b_{\text{ét}}(Y; k)$ with trivial Σ-action, Example 4.2.3(3) gives a natural isomorphism

$$(\epsilon^* \mathcal{L})^{t\Sigma} \cong \mathcal{L} \overset{\mathrm{L}}{\otimes}_k k_Y^{t\Sigma}, \qquad k_Y^{t\Sigma} := (\epsilon^* k_Y)^{t\Sigma}. \tag{4.5.3}$$

By Example 4.5.1, $k_Y^{t\Sigma}$ has cohomology sheaf k_Y in every degree, for every ℓ.

Example 4.5.3. Let V be a k-vector space. Then $V^{\otimes \ell}$ has an action of Σ, where σ acts by cyclic rotation of the factors, and a straightforward calculation shows that

$$\mathrm{T}^i(V^{\otimes \ell}) \cong V^{(\ell)} := V \otimes_{k, \mathrm{Frob}_\ell} k,$$

the *Frobenius twist* of V. Indeed, choosing a basis of V splits $V^{\otimes \ell}$ into a sum of free and trivial $k[\Sigma]$-modules. The Tate cohomology of the free modules vanish, and the Tate cohomology of trivial modules is calculated by Example 4.5.1.

5. Σ-fixed point calculations

Let G be a reductive group over E and σ an order $\ell \neq p$ automorphism of G such that $H := G^\sigma$ is (connected) reductive.

Example 5.0.1. If G is semisimple and simply connected, then H is automatically connected reductive. Indeed, after base change to $\overline{E}$, smoothness of G^σ and reductivity of $(G^\sigma)^\circ$ follow from [FS21, Proposition VIII.5.14] in positive characteristic and are classical in characteristic zero. Since σ has finite order invertible in $\overline{E}$, it is semisimple, and Steinberg's theorem [Ste68, Theorem 8.1] implies that G^σ is connected.

In this section we will analyze the fixed points of σ on various spaces affiliated with G, and relate them to the analogous spaces affiliated with H, such as:

- The $\mathrm{B}^+_{\mathrm{dR}}$-affine Grassmannians, as well as their variants such as Beilinson-Drinfeld affine Grassmannians, convolution affine Grassmannians, and "twisted" affine Grassmannians (which are the target of the generalized Grothendieck-Messing period maps).
- The moduli spaces of local shtukas.

These calculations are used later when applying Smith–Treumann localization to such spaces.

5.1. **Affine Grassmannians.** We begin by formulating the results for affine Grassmannians and their variants.

Proposition 5.1.1. *Let $\mathbf{C}/E$ be an algebraically closed field. Let $\mathrm{Gr}^{\mathrm{alg}}_{G,\mathbf{C}}, \mathrm{Gr}^{\mathrm{alg}}_{H,\mathbf{C}}$ be the algebraic affine Grassmannians over $\mathbf{C}$. Then the map $\mathrm{Gr}^{\mathrm{alg}}_{H,\mathbf{C}} \to \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,\mathbf{C}})$ is an isomorphism on underlying reduced ind-schemes.*

We first deduce the Beilinson–Drinfeld variants of Proposition 5.1.1, and then return to prove Proposition 5.1.1.

Let I be a non-empty finite set and let S be a separated locally spatial diamond over $(\mathrm{Div}^1_X)^I$. Then we define

$$\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I} := \mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I} \times_{(\mathrm{Div}^1_X)^I} S.$$

Remark 5.1.2. Note that these affine Grassmannians are partially proper, so that their $\mathrm{Spa}(C, C^+)$-points identify with their $\mathrm{Spa}(C, C^\circ)$-points, where $C^\circ \subset C$ is the ring of power-bounded elements (i.e., the maximal possible C^+). The fixed locus is closed in the partially proper diamond $\mathrm{Gr}_{G,C}$, and is therefore partially proper as well. For this reason, we do not encounter subtleties with higher rank points.

Proposition 5.1.3. *Let S be a separated locally spatial diamond over $(\mathrm{Div}^1_X)^I$. Then the natural map*

$$\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})$$

is an isomorphism.

Proof. Since $\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$ is separated, $\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})$ is a closed subdiamond of $\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$. The map $\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})$ is a closed embedding by the argument of [SW20, Lemma 19.1.5], hence qcqs, so by [Sch22, Lemma 11.11] it is an isomorphism if and only if it induces a bijection on $\mathrm{Spa}(C, C^+)$-points for all (C, C^+). Checking this reduces to the case where $S = \mathrm{Spa}(C, C^+)$, in which case we abbreviate $\mathrm{Gr}_{G,C} := \mathrm{Gr}_{G,\mathrm{Spa}(C,C^+)/(\mathrm{Div}^1_X)^I}$ and similarly for H.

A $\mathrm{Spa}(C, C^+)$-point over $(\mathrm{Div}^1_X)^I$ gives an I-tuple of untilts $\{C_i^\sharp/E\}_{i\in I}$. Let J be the set of distinct untilts occurring in this tuple, and write $C_\alpha^\sharp$ for the untilt corresponding to $\alpha \in J$. By the factorization property of the Beilinson–Drinfeld Grassmannian, a point of $\mathrm{Gr}_{G,C}$ is equivalently a collection, for $\alpha \in J$, of $(C_\alpha^\sharp, \mathcal{O}_{C_\alpha^\sharp})$-points of the one-leg $\mathrm{B}^+_{\mathrm{dR}}$-affine Grassmannian for G. Choose an identification $B^+_{\mathrm{dR}}(C_\alpha^\sharp) \cong C_\alpha^\sharp[[t]]$ as in the proof of [SW20, Proposition 19.2.1], this comparison is functorial in the group and compatible with the automorphism σ. Then, for each $\alpha \in J$, we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Gr}_H^{\mathrm{B}^+_{\mathrm{dR}}}(C_\alpha^\sharp, \mathcal{O}_{C_\alpha^\sharp}) & \hookrightarrow & \mathrm{Fix}(\sigma, \mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}})(C_\alpha^\sharp, \mathcal{O}_{C_\alpha^\sharp}) & \hookrightarrow & \mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}}(C_\alpha^\sharp, \mathcal{O}_{C_\alpha^\sharp}) \\
\| & & \| & & \| \\
\mathrm{Gr}^{\mathrm{alg}}_{H,C_\alpha^\sharp}(C_\alpha^\sharp) & \hookrightarrow & \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,C_\alpha^\sharp})(C_\alpha^\sharp) & \hookrightarrow & \mathrm{Gr}^{\mathrm{alg}}_{G,C_\alpha^\sharp}(C_\alpha^\sharp)
\end{array}$$

By Proposition 5.1.1, the map $\mathrm{Gr}^{\mathrm{alg}}_{H,C_\alpha^\sharp}(C_\alpha^\sharp) \to \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,C_\alpha^\sharp})(C_\alpha^\sharp)$ is a bijection, hence so is the parallel map in the top row. Taking the product over the finitely many distinct legs $\alpha \in J$ gives the desired bijection for the fiber over the collection of untilts $\{C_i^\sharp/E\}_{i\in I}$. Thus the map

$$\mathrm{Gr}_{H,C}(C, C^+) \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,C})(C, C^+)$$

is also a bijection, as desired. □

Remark 5.1.4. The same argument works for variants like $(\mathrm{Div}^1_{\mathcal{Y}})^I$ instead of $(\mathrm{Div}^1_X)^I$.

We also deduce variants for convolution Grassmannians and the "twisted" Grassmannians $\mathrm{Gr}^{\mathrm{tw}}_G$ from [SW20, §23.5].

Corollary 5.1.5. *Let $m \in \mathbf{Z}_{\geq 1}$.*

(1) Let $\mathbf{C}/E$ be an algebraically closed field. Let $\widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_{G,\mathbf{C}}$ be the m-step convolution Grassmannian for G, and similarly for H. Then the natural map

$$\widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_{H,\mathbf{C}} \to \mathrm{Fix}(\sigma, \widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_{G,\mathbf{C}})$$

is an isomorphism on underlying reduced ind-schemes.

(2) Let I be a non-empty finite set and let S be a separated locally spatial diamond with a map $S \to (\mathrm{Div}^1_X)^I$. Let $\widetilde{\mathrm{Gr}}^{(m)}_{G,S/(\mathrm{Div}^1_X)^I}$ be the m-step convolution Beilinson-Drinfeld Grassmannian for G, and similarly for H. Then the natural map

$$\widetilde{\mathrm{Gr}}^{(m)}_{H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \widetilde{\mathrm{Gr}}^{(m)}_{G,S/(\mathrm{Div}^1_X)^I})$$

is an isomorphism.

(3) Let I be a non-empty finite set. Let $\mathrm{Gr}^{\mathrm{tw}}_{G,I} = \varinjlim_{\mu_\bullet} \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,E_i,\leq\mu_\bullet}$ be the twisted affine Grassmannian (cf. [SW20, Definition 23.5.1]*). Then the natural map*

$$\mathrm{Gr}^{\mathrm{tw}}_{H,I} \xrightarrow{\sim} \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{tw}}_{G,I})$$

is an isomorphism.

Proof. (1) The map $\widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_G \to (\mathrm{Gr}^{\mathrm{alg}}_G)^m$ sending $(\mathcal{E}^0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \ldots \dashrightarrow \mathcal{E}_m)$ to $(\mathcal{E}^0 \dashrightarrow \mathcal{E}_1, \mathcal{E}^0 \dashrightarrow \mathcal{E}_2, \ldots, \mathcal{E}^0 \dashrightarrow \mathcal{E}_m)$ is a Σ-equivariant isomorphism. Under this isomorphism, the statement transports to (the mth power of) that of Proposition 5.1.1.

(2) As in the proof of Proposition 5.1.3, the statement reduces to the case where $S = \mathrm{Spa}(C, C^+)$. Indeed, over a geometric point the convolution Grassmannian factorizes over the finitely many distinct untilts. Then it follows from the algebraic case over each untilt, which was treated in (1).

(3) As in the proof of Proposition 5.1.3, the statement reduces to the case $S = \mathrm{Spa}(C, C^+)$. It is then a consequence of (2). □

5.2. **Borels.** Recall that a *Borus* of a reductive group G is a pair (B, T) of a Borel subgroup $B < G$ and a split maximal torus $T < B$.

Lemma 5.2.1. *Suppose Σ stabilizes a Borus (B, T) of G. Then (B^σ, T^σ) is a Borus of H.*

Proof. Suppose for the moment that we know B^σ is a Borel subgroup of H. We will argue that T^σ is a split maximal torus of H. Let U be the unipotent radical of B. Applying [CGP15, Proposition A.8.10(2)], which implies that the map on Σ-fixed points of a smooth Σ-equivariant morphism is again smooth, to $B \to B/U \cong T$ shows that $B^\sigma \to (B/U)^\sigma \cong T^\sigma$ is a smooth map whose non-empty fibers are U^σ-torsors. Since the section $T^\sigma \hookrightarrow B^\sigma$ shows that $B^\sigma \to T^\sigma$ is surjective, we conclude that the map $B^\sigma/U^\sigma \to (B/U)^\sigma \cong T^\sigma$ is an isomorphism. Note that U^σ is a normal unipotent subgroup of B^σ and is smooth by [CGP15, Proposition A.8.10(2)]. It is also connected: since σ preserves (B, T), U has a Σ-stable filtration by vector groups, whose Σ-fixed points are (connected) vector subgroups. Thus U^σ is contained in the unipotent radical of B^σ; on the other hand, the image of the unipotent radical in $B^\sigma/U^\sigma \cong T^\sigma$ is a unipotent subgroup of a torus, hence must necessarily be trivial. This shows that T^σ is a maximal torus of H (and in particular is connected).

Now we return to showing that B^σ is a Borel subgroup of H. The group B^σ is smooth by [CGP15, Proposition A.8.10(2)] and is solvable. By assumption G/B is projective, hence its closed subscheme $(G/B)^\sigma$ is projective. We will prove that H/B^σ is closed and open in $(G/B)^\sigma$, which will imply that H/B^σ is projective. Evidently H acts by translation on $(G/B)^\sigma$, and it suffices to show that each H-orbit in $(G/B)^\sigma$ is open, since then the orbits are finite in number and pairwise disjoint, hence open-closed. Indeed, for any geometric point g of G representing a point in $(G/B)^\sigma$, the orbit map $G^\sigma \to (G/B)^\sigma$ through gB is a smooth morphism by [CGP15, Proposition A.8.10(2)] again, hence has open image.

Indeed, the natural map

$$H/B^{\sigma,\circ} \longrightarrow H/B^{\sigma}$$

is finite. Hence projectivity of H/B^{σ} implies that $B^{\sigma,\circ}$ is parabolic. Since it is solvable, it is a Borel subgroup of H. It remains to show that $B^{\sigma} = B^{\sigma,\circ}$ is connected. The group B^{σ} normalizes its identity component, whereas a parabolic subgroup is self-normalizing by [Con, Theorem 1.3.1]. Thus we may conclude that $B^{\sigma} = B^{\sigma,\circ}$, completing the proof. □

Proposition 5.2.2. *Let S be a separated locally spatial diamond over $(\mathrm{Div}^1_X)^I$. Suppose that G has a Σ-stable Borel subgroup B. Let $B_H := B^{\sigma}$, a Borel subgroup of H by Lemma 5.2.1. Then the natural map*

$$\mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I})$$

is an isomorphism.

Proof. In the proof of [SW20, Lemma 19.1.5] there is a general argument for when a closed embedding of affine algebraic groups $G_1 \hookrightarrow G_2$ induces a closed embedding of affine Grassmannians. More precisely, the formulation of *loc. cit.* requires the groups to be reductive, but the proof requires only that G_2/G_1 be affine and that G_1-torsors on $\mathrm{Spec}\, B^+_{\mathrm{dR}}(R^\sharp)$ be étale locally trivial on $\mathrm{Spa}(R, R^+)$. We verify these conditions in the present case.

First we show that B/B_H is affine. This may be checked after base change to $\overline{E}$. We may write

$$B = T \ltimes U, \qquad B_H = T^{\sigma} \ltimes U^{\sigma},$$

for a maximal torus $T \subset B$ and the unipotent radical $U \subset B$. Then

$$B/B_H \simeq T \times^{T^{\sigma}} (U/U^{\sigma}).$$

The homogeneous space U/U^{σ} is clearly affine. Choosing a splitting of the sequence

$$1 \longrightarrow T^{\sigma} \longrightarrow T \longrightarrow T/T^{\sigma} \longrightarrow 1$$

gives an isomorphism $T \times^{T^{\sigma}} (U/U^{\sigma}) \cong (T/T^{\sigma}) \times (U/U^{\sigma})$, which is clearly affine.

Second, the local-triviality argument for B_H-torsors used in the proof of [SW20, Lemma 19.1.5] requires only that B_H be smooth: a B_H-torsor over $B^+_{\mathrm{dR}}(R^\sharp)$ acquires a section after an étale localization of $R^\sharp$, and this section lifts across the henselian pair $(B^+_{\mathrm{dR}}(R^\sharp), \ker\theta)$. Hence it is trivial étale-locally on $\mathrm{Spa}(R, R^+)$. Therefore, [SW20, Lemma 19.1.5] applies to show that the map

$$\mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I})$$

is a closed embedding, hence qcqs. Then by [Sch22, Lemma 11.11] it suffices to check that it induces a bijection on geometric points. A variation on [FS21, Proposition VI.3.1] shows that the maps $\mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I}$ and $\mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$ are bijections on geometric points. Hence in the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} & \longrightarrow & \mathrm{Fix}(\sigma, \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} & \longrightarrow & \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}) \end{array}$$

both vertical maps and the bottom horizontal map induce bijections on geometric points (the latter by Proposition 5.1.3). Therefore the upper horizontal map also induces a bijection on geometric points, concluding the proof.

□

5.3. **Proof of Proposition 5.1.1.** In this subsection, all loop groups are algebraic loop groups. We write LG^{alg} for the loop group of G with respect to the loop variable t, so $LG^{\mathrm{alg}}(R) = G(R((t)))$. We write $L^+G^{\mathrm{alg}} \subset LG^{\mathrm{alg}}$ for the arc group, with functor of points $L^+G^{\mathrm{alg}}(R) = G(R[[t]])$. The algebraic affine Grassmannian is the fppf quotient $LG^{\mathrm{alg}}/L^+G^{\mathrm{alg}}$, which has a natural ind-scheme structure.

For the purpose of proving Proposition 5.1.1, we may base change G to the algebraically closed field $\mathbf{C}$. Since σ is a semisimple automorphism of G over an algebraically closed field, a theorem of Steinberg [Ste68, Theorem 7.5] implies that σ stabilizes a Borus (B, T) of G. By Lemma 5.2.1, (B^σ, T^σ) is a Borus of H.

5.3.1. *Iwahori stratification.* Let Iw^G be the Iwahori subgroup of L^+G^{alg} corresponding to B. This induces a stratification by Iw^G-orbits (cf. [RW22, (4.6)])

$$(\mathrm{Gr}^{\mathrm{alg}}_{G,\mathbf{C}})_{\mathrm{red}} = \coprod_{\lambda \in X_*(T)} \mathrm{Gr}^{\mathrm{alg}}_{G,\lambda}\,.$$

Furthermore, Iw^G is stable under σ and $\mathrm{Iw}^H := (\mathrm{Iw}^G)^\sigma$ is the Iwahori subgroup of L^+H^{alg} corresponding to B^σ. Thus we have an analogous stratification

$$(\mathrm{Gr}^{\mathrm{alg}}_{H,\mathbf{C}})_{\mathrm{red}} = \coprod_{\lambda \in X_*(T^\sigma)} \mathrm{Gr}^{\mathrm{alg}}_{H,\lambda}\,.$$

The action of σ on G (stabilizing T) induces an action on $X_*(T)$. We will show that:

(1) If λ is not fixed by σ, then $\mathrm{Gr}^{\mathrm{alg}}_{G,\lambda} \cap (\sigma \cdot \mathrm{Gr}^{\mathrm{alg}}_{G,\lambda}) = \emptyset$.
(2) If λ is fixed by σ, then $\mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,\lambda}) \xleftarrow{\sim} \mathrm{Gr}^{\mathrm{alg}}_{H,\lambda}$.

The combination of (1) and (2) clearly suffices to prove Proposition 5.1.1

Item (1) is immediate: since we arranged σ to preserve Iw^G, we have $\sigma \cdot \mathrm{Gr}_{G,\lambda} = \mathrm{Gr}_{G,\sigma(\lambda)}$. Because distinct Iw^G-orbits are disjoint, the intersection of any two distinct Iw^G-orbits is empty.

Next we turn to (2). For this we analyze the structure of the strata.

5.3.2. *Fixed points of strata.* Suppose σ fixes λ. Let

$$\mathrm{Iw}^G_\lambda := \mathrm{Iw}^G \cap t^\lambda L^+G^{\mathrm{alg}} t^{-\lambda}$$

be the stabilizer of t^λ in Iw^G. Then

$$\mathrm{Gr}^{\mathrm{alg}}_{G,\lambda} \cong \mathrm{Iw}^G \,/\, \mathrm{Iw}^G_\lambda\,.$$

Similarly, since $\lambda \in X_*(T)^\sigma = X_*(T^\sigma)$, the stabilizer of t^λ in $\mathrm{Iw}^H = (\mathrm{Iw}^G)^\sigma$ is $\mathrm{Iw}^H_\lambda := (\mathrm{Iw}^G_\lambda)^\sigma$, and

$$\mathrm{Gr}^{\mathrm{alg}}_{H,\lambda} \cong \mathrm{Iw}^H \,/\, \mathrm{Iw}^H_\lambda\,.$$

We claim that the natural map

$$\mathrm{Iw}^H \,/\, \mathrm{Iw}^H_\lambda \longrightarrow \mathrm{Fix}(\sigma, \mathrm{Iw}^G \,/\, \mathrm{Iw}^G_\lambda)_{\mathrm{red}}$$

is an isomorphism on underlying reduced schemes. The map is a locally closed immersion of finite type reduced $\mathbf{C}$-schemes, being the restriction of the closed immersion $\mathrm{Gr}^{\mathrm{alg}}_{H,\mathbf{C}} \hookrightarrow \mathrm{Gr}^{\mathrm{alg}}_{G,\mathbf{C}}$ to the indicated strata. Hence it suffices to check that it induces a bijection on $\mathbf{C}$-points. Let $g\,\mathrm{Iw}^G_\lambda \in (\mathrm{Iw}^G \,/\, \mathrm{Iw}^G_\lambda)(\mathbf{C})$ be fixed by σ. By the long exact sequence in non-abelian cohomology, it suffices to show that the map

$$\mathrm{H}^1(\Sigma, \mathrm{Iw}^G_\lambda(\mathbf{C})) \longrightarrow \mathrm{H}^1(\Sigma, \mathrm{Iw}^G(\mathbf{C}))$$

is injective. Let

$$r\colon\ \mathrm{Iw}^G \longrightarrow B \longrightarrow B/U \simeq T$$

be evaluation at $t = 0$ followed by the quotient map, and put

$$N := \ker(r), \qquad N_\lambda := N \cap \mathrm{Iw}^G_\lambda\,.$$

Since the constant torus T is contained in both Iw^G and $t^\lambda L^+G^{\mathrm{alg}} t^{-\lambda}$, restriction of r to Iw^G_λ is surjective. We therefore have Σ-equivariantly split exact sequences

$$1 \to N_\lambda \to \mathrm{Iw}^G_\lambda \to T \to 1, \qquad 1 \to N \to \mathrm{Iw}^G \to T \to 1. \tag{5.3.1}$$

We claim that the nonabelian cohomology sequence for these two split extensions in (5.3.1) now shows that both maps

$$\mathrm{H}^1(\Sigma, \mathrm{Iw}_\lambda^G(\mathbf{C})) \longrightarrow \mathrm{H}^1(\Sigma, T(\mathbf{C})), \qquad \mathrm{H}^1(\Sigma, \mathrm{Iw}^G(\mathbf{C})) \longrightarrow \mathrm{H}^1(\Sigma, T(\mathbf{C}))$$

are bijective. Surjectivity is clear from the splitting. For injectivity, note that N_λ and N are filtered by Σ-stable vector groups, so $\mathrm{H}^1(\Sigma, N_\lambda)$ and $\mathrm{H}^1(\Sigma, N)$ are trivial because ℓ is invertible in the coefficient ring $\mathbf{C}$. □

5.4. **Moduli of local shtukas.** We now turn to examine the moduli spaces of local shtukas, for which a reference is Scholze's Berkeley lectures, especially [SW20, Lecture XXIII].

Let $(G, b, \{\mu_i\}_{i\in I})$ be a local shtuka datum (cf. [SW20, Definition 23.1.1], except we allow the equal characteristic case as well): G is a reductive group over E, $b \in B(G)$, and μ_i is a conjugacy class of cocharacters $\mathbf{G}_m \to G_{E^s}$. For any compact open subgroup $K \subset G(E)$ there is a *moduli space of local shtukas with K-level structure* $\mathrm{Sht}_{(G,b,\{\mu_i\}_{i\in I}),K}$, which is functorial in K. There is a moduli description of $\mathrm{Sht}_{(G,b,\{\mu_i\}_{i\in I}),K}$ given (in the mixed characteristic case) in [SW20, §23].

For $\{\mu_i\}_{i\in I} \leq \{\mu_i'\}_{i\in I}$ in the component-wise Bruhat order, there are closed embeddings $\mathrm{Sht}_{(G,b,\{\mu_i\}),K} \hookrightarrow \mathrm{Sht}_{(G,b,\{\mu_i'\}),K}$. It will also be convenient to consider the following variant without any cocharacter "bounds":

Definition 5.4.1. We define $\mathrm{Sht}_{(G,b,I),K} := \varinjlim_{\{\mu_i\}_{i\in I}} \mathrm{Sht}_{(G,b,\{\mu_i\}_{i\in I}),K}$.

5.4.1. *Grothendieck-Messing period map.* We recall some facts about the generalized Grothendieck-Messing period map

$$\pi_{\mathrm{GM}}^G \colon \mathrm{Sht}_{(G,b,I),K} \to \mathrm{Gr}_{G,I}^{\mathrm{tw}} \tag{5.4.1}$$

from [SW20, §23.5]. Note that we have not imposed any "bound" above; our (5.4.1) comes from the colimit over $\{\mu_i\}_{i\in I}$ of the map from [SW20, Corollary 23.5.3].

The image of π_{GM}^G is an open subset of $\mathrm{Gr}_{G,I}^{\mathrm{tw},a} \subset \mathrm{Gr}_{G,I}^{\mathrm{tw}}$ called the *admissible* locus. In terms of points, $\mathrm{Gr}_{G,I}^{\mathrm{tw}}$ parametrizes G-torsors $\mathcal{P}_\eta$ on $\mathcal{Y}_{(0,\infty)}$ plus additional data $\varphi_{\mathcal{P}_\eta}$ and ι_r from [SW20, Definition 23.5.1] that we do not need to reference right now. The admissible locus is cut out by the condition that the Newton point $\nu_{\mathcal{P}_\eta}$ and the Kottwitz point $\kappa_{\mathcal{P}_\eta}$ are both identically zero [SW20, Theorem 22.6.2].

Definition 5.4.2. For a basic $b \in B(G)$, denote by $\mathrm{Gr}_{G,I}^{\mathrm{tw},b}$ the subspace where $\nu_{\mathcal{P}_\eta} = \nu(b)$ and $\kappa_{\mathcal{P}_\eta} = \kappa(b)$ after pulling back to each geometric point. This is an open subspace, and when $b = 1_G$ this recovers the admissible locus, i.e., $\mathrm{Gr}_{G,I}^{\mathrm{tw},1_G} = \mathrm{Gr}_{G,I}^{\mathrm{tw},a} \subset \mathrm{Gr}_{G,I}^{\mathrm{tw}}$.

Example 5.4.3. If $b \in B(G)$ is basic, then there is an extended pure inner twist G_b (sometimes denoted J_b in the literature) and a canonical isomorphism

$$\mathrm{tr}_b \colon B(G) \xrightarrow{\sim} B(G_b) \tag{5.4.2}$$

sending $b \in B(G)$ to $1_{G_b} \in B(G_b)$. There is also a compatible isomorphism $\mathrm{Gr}_{G,I}^{\mathrm{tw}} \cong \mathrm{Gr}_{G_b,I}^{\mathrm{tw}}$ as in [FS21, §III.4.1], which takes $\mathrm{Gr}_{G,I}^{\mathrm{tw},b}$ to $\mathrm{Gr}_{G_b,I}^{\mathrm{tw},a}$. So we could have formulated the subspaces $\mathrm{Gr}_{G,I}^{\mathrm{tw},b}$ as admissible loci for extended pure inner twists G_b. However, we prefer to keep the distinction, for psychological reasons if nothing else.

Let $\iota \colon B(H) \to B(G)$ be the map induced by the inclusion $\iota \colon H \hookrightarrow G$. The induced map $\iota \colon \mathrm{Gr}_{H,I}^{\mathrm{tw}} \hookrightarrow \mathrm{Gr}_{G,I}^{\mathrm{tw}}$ is also a closed embedding (by the proof of [SW20, Lemma 19.1.5]).

Lemma 5.4.4. *We have*

$$\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,I}^{\mathrm{tw},a}) = \coprod_{b' \in \iota^{-1}(1_G)} \mathrm{Gr}_{H,I}^{\mathrm{tw},b'}.$$

as subdiamonds of $\mathrm{Gr}_{G,I}^{\mathrm{tw}}$.

Proof. In Corollary 5.1.5(3) we have seen that $\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,I}^{\mathrm{tw}}) = \mathrm{Gr}_{H,I}^{\mathrm{tw}}$ (with the obvious embedding). Therefore, we may interpret (say, by Remark 3.2.2)

$$\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,I}^{\mathrm{tw},a}) = \mathrm{Gr}_{G,I}^{\mathrm{tw},a} \cap \mathrm{Gr}_{H,I}^{\mathrm{tw}}.$$

The admissible locus $\mathrm{Gr}^{\mathrm{tw},a}_{G,I} \subset \mathrm{Gr}^{\mathrm{tw}}_{G,I}$ is cut out by the condition that $\nu(\mathcal{P}_\eta) = 0$ and $\kappa(\mathcal{P}_\eta) = 0$. Hence $\mathrm{Gr}^{\mathrm{tw},a}_{G,I} \cap \mathrm{Gr}^{\mathrm{tw}}_{H,I}$ is the pullback of the simultaneous vanishing loci of the Newton and Kottwitz maps on $\mathrm{Gr}^{\mathrm{tw}}_{G,I}$ along the closed embedding $\mathrm{Gr}^{\mathrm{tw}}_{H,I} \hookrightarrow \mathrm{Gr}^{\mathrm{tw}}_{G,I}$, which is precisely the expression on the RHS. □

5.4.2. *Fixed points of moduli of local shtukas.* We now recall more about the structure of $\mathrm{Sht}_{(G,b,I),K}$ in terms of the Grothendieck-Messing period map. According to [SW20, Corollary 23.5.3], $\mathrm{Gr}^{\mathrm{tw},a}_{G,I}$ carries a $\underline{G(E)}$-torsor $\mathbb{P}^G_\eta$, and $\pi^G_{\mathrm{GM}} \colon \mathrm{Sht}_{(G,b,I),K} \to \mathrm{Gr}^{\mathrm{tw},a}_{G,I}$ is the étale cover parametrizing K-lattices $\mathbb{P}^G \subset \mathbb{P}^G_\eta$.

We write $\iota^* K = K \cap H(E) = K^\sigma$.

Lemma 5.4.5. *Let K be a profinite group of prime-to-ℓ pro-order with a continuous Σ-action. If a v-sheaf X has a compatible Σ-action and a free right $\underline{K}$-action, and $X \to X/\underline{K}$ is a v-torsor, then*

$$X^\Sigma/\underline{K}^\Sigma \xrightarrow{\sim} (X/\underline{K})^\Sigma. \tag{5.4.3}$$

Proof. We want to show that the map $X^\Sigma/K^\Sigma \to (X/\underline{K})^\Sigma$ is an isomorphism of v-sheaves. It is clearly an injection of v-sheaves, so the point is to show surjectivity. By definition, any section y of $(X/\underline{K})^\Sigma$ lifts v-locally to $x \in X$; we want to adjust the lift to lie in X^Σ after possible further v-localization. By the assumed freeness of the action of $\underline{K}$, there is a unique cocycle $(\sigma \mapsto a_\sigma)$ with values in $\underline{K}$ such that $\sigma(x) = x a_\sigma$. Since K has prime-to-ℓ pro-order, we have $H^1(\Sigma, K) = 0$, hence the coboundary map $K \to Z^1(\Sigma, K)$ is a continuous surjection of profinite sets, so after a further v-cover we may write $a_\sigma = k\sigma(k)^{-1}$. Then xk is a section of X^Σ lying over y.

□

Proposition 5.4.6. *Let I be nonempty and finite, and let $K \subset G(E)$ be a Σ-stable compact open subgroup of prime-to-ℓ pro-order. The natural embedding identifies*

$$\mathrm{Sht}_{(H,1_H,I),\iota^* K} \xrightarrow{\sim} \mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,1_G,I),K}). \tag{5.4.4}$$

This identification is compatible with base change in $\prod_{i\in I} \operatorname{Spd} \breve{E}$ and with level covers.

Proof. It suffices to show the statement at infinite level:

$$\mathrm{Sht}_{(H,1_H,I),\infty} \xrightarrow{\sim} \mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,1_G,I),\infty}). \tag{5.4.5}$$

Indeed, once we know this, (5.4.4) follows from it by Lemma 5.4.5.

We will analyze the Σ-fixed points by examining the infinite-level Grothendieck–Messing period map π^G_{GM}, first identifying the Σ-fixed points of the period space, and then the Σ-fixed points of the fibers. By Lemma 5.4.4, the former is

$$\mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{tw},a}_{G,I}) = \coprod_{b' \in \iota^{-1}(1_G)} \mathrm{Gr}^{\mathrm{tw},b'}_{H,I}.$$

For an S-valued point $x \in \mathrm{Gr}^{\mathrm{tw},b'}_{H,I}$, let $\mathcal{E}_{b'}$ denote its associated H-bundle on X_S, of geometric class b'. The fiber of π^G_{GM} over $\iota(x)$ is the $\underline{G(E)}$-torsor $\mathrm{Isom}_G(\mathcal{E}_{1_G}, \mathcal{E}_{b'} \times^H G)$, given by

$$(T \to S) \mapsto \mathrm{Isom}_{G\text{-bundles on } X_T}(\mathcal{E}_{1_G}|_{X_T}, (\mathcal{E}_{b'} \times^H G)|_{X_T}).$$

Its Σ-fixed points form the $H(E)$-pseudo-torsor $\mathrm{Isom}_H(\mathcal{E}_{1_H}, \mathcal{E}_{b'})$, as can be checked by a local calculation using $H = G^\sigma$. This shows that:

- For $b' \neq 1_H$, $\mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,1_G,I),\infty})$ has empty fiber over $\iota(x)$.
- For $b' = 1_H$, the fibral maps from $(\pi^H_{\mathrm{GM}})^{-1}(x) = \mathrm{Isom}_H(\mathcal{E}_{1_H}, \mathcal{E}_{b'})$ to $\mathrm{Fix}(\sigma, (\pi^G_{\mathrm{GM}})^{-1}(\iota(x)))$ are isomorphisms.

This completes the proof. □

6. The Brauer functor

Let G be a reductive group over E with an action of Σ, and let $H = G^\sigma \xrightarrow{\iota} G$. We assume that H is reductive. Treumann–Venkatesh defined in [TV16] the *(normalized) Brauer homomorphism* br from the spherical Hecke algebra of G to that of H. In this section we will categorify it to a *Brauer functor* from the Satake category of G to that of H.

6.1. **Geometric Satake.** We first record some generalities on the Geometric Satake equivalence.

6.1.1. *The Satake category.* Let S be any small v-stack over Div^1_X. We denote the *Satake category*

$$\mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda) \subset D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$$

to be the full subcategory spanned by flat perverse sheaves over S, in the sense of Fargues–Scholze [FS21, §VI.7]. We recall some details of this definition for the convenience of the reader.

Definition 6.1.1. The perverse t-structure on $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is determined by the condition that $A \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ lies in ${}^pD^{\leq 0}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ if and only if, after pullback to every geometric point of S, its restriction to the open Schubert cell indexed by μ lies in degrees $\leq -\langle 2\rho,\mu\rangle$.

By [FS21, Definition–Proposition VI.7.1], these conditions define a unique t-structure, which we call the *perverse t-structure*. Finally, a perverse sheaf A is "flat" if $A \otimes^L_\Lambda M$ is perverse for every Λ-module M; see [FS21, Proposition VI.7.7 and Definition VI.7.8].

By the same argument as for [FS21, Proposition VI.7.2], the pullback functor

$$\mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda) \to D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$$

is fully faithful, so we may regard $\mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda) \subset D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$.

A similar story holds for multiple legs. For a finite non-empty set I, we abbreviate $\mathrm{Sat}^I_G(k) := \mathrm{Sat}(\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I};k)$. There is an analogous relative perverse t-structure on $D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/(\mathrm{Div}^1_X)^I};k)^{\mathrm{bd}}$ for any $S \to (\mathrm{Div}^1_X)^I$. For this t-structure, pullback along any $S' \to S$ is t-exact by [FS21, Proposition VI.7.4].

6.1.2. *The Geometric Satake equivalence.* For a v-stack S, we let $\mathrm{Loc}(S;\Lambda)$ be the category of Λ-free étale local systems on S. Let $\pi_{G,S}\colon \mathrm{Gr}_{G,S/\mathrm{Div}^1_X} \to S$ be the natural projection. Fargues–Scholze establish in [FS21, §VI.9–VI.11] the Geometric Satake equivalence,

$$\mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};\Lambda) \cong \mathrm{Rep}_{\mathrm{Loc}(\mathrm{Div}^1_X;\Lambda)}(\check{G}), \tag{6.1.1}$$

where the right side is the category of representations of a certain reductive group object in the category $\mathrm{Loc}(\mathrm{Div}^1_X;\Lambda)$. The equivalence is symmetric monoidal, with the underlying monoidal structure given by convolution on the left and tensor product on the right, and carries the fiber functor $\bigoplus_i \mathrm{R}^i\pi_{G,\mathrm{Div}^1_X *}$ on the left to the forgetful functor on the right.

The category $\mathrm{Loc}(\mathrm{Div}^1_X;\Lambda)$ is symmetric monoidal under tensor product. Note that there is an action of $\mathrm{Loc}(\mathrm{Div}^1_X;\Lambda)$ on $\mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};\Lambda)$ via pullback and tensoring. Pullback and tensoring likewise define an action of $\mathrm{Loc}(S;\Lambda)$ on $\mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$.

Let $2\rho_G$ be the sum of the positive roots with respect to a Borus of G_{E^s}. Since $\langle 2\rho_G,-\rangle$ is even on coroots, its reduction modulo 2 factors through a Galois-invariant homomorphism

$$\begin{aligned} \dagger_G\colon \pi_1(G) &\to \mathbf{Z}/2\mathbf{Z}, \\ \dagger_G(\lambda) &= \langle 2\rho_G,\lambda\rangle \pmod 2. \end{aligned} \tag{6.1.2}$$

As in [FS21, §VI.9, before Definition/Proposition VI.9.4], the commutativity constraint of the Satake category is modified by a sign whenever two objects supported on odd components (for $\dagger_G$) are interchanged. With this convention, the fiber functors $\mathsf{F}^{G,I} := \bigoplus_n \mathrm{R}^n\pi_{G*}$ are symmetric monoidal (see the discussion after [FS21, Definition/Proposition VI.9.4]), and so are the normalized constant term functors $\mathrm{CT}_B[\deg_G]$ of §6.1.4, by [FS21, Proposition VI.9.6]. We omit I from the notation when $|I| = 1$.

6.1.3. *Fusion.* For a partition $I = I_1 \sqcup I_2$, let j be the inclusion of the open subset of $(\mathrm{Div}^1_X)^I$ where legs in different blocks are disjoint. Restriction along j is fully faithful on Satake categories, and the exterior product over this open subset extends to the fusion product [FS21, Proposition VI.9.3 and Definition/Proposition VI.9.4]. Fusion commutes with pullback along maps of finite sets, and its restriction to the diagonal is convolution. Normalized constant terms commute with fusion [FS21, Proposition VI.9.6].

Remark 6.1.2. There is the following geometric description of fusion, from [FS21, VI.9]. Any map $\zeta\colon I \to J$ of non-empty finite sets induces a closed embedding $\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I} \times_{(\mathrm{Div}^1_X)^I} (\mathrm{Div}^1_X)^J \hookrightarrow \mathrm{Gr}_{G,(\mathrm{Div}^1_X)^J}$, and the fusion product $\zeta\colon \mathrm{Sat}^I_G(k) \to \mathrm{Sat}^J_G(k)$ can be interpreted as pull-push along the diagram

$$\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I} \xleftarrow{p_\zeta} \mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I} \times_{(\mathrm{Div}^1_X)^I} (\mathrm{Div}^1_X)^J \xhookrightarrow{q_\zeta} \mathrm{Gr}_{G,(\mathrm{Div}^1_X)^J} \tag{6.1.3}$$

Writing the two arrows in (6.1.3) as p_ζ and q_ζ, this functor is $q_{\zeta,*}p_\zeta^*$, where $q_{\zeta,*} = q_{\zeta,!}$ since q_ζ is a closed embedding. When ζ is not surjective, the map q_ζ simply inserts the unit object at the legs in $J \setminus \mathrm{im}(\zeta)$.

6.1.4. *Constant terms.* Suppose that G is split, and choose a Borus (B,T). Consider the natural maps

$$\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I} \xleftarrow{p_G} \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I} \xrightarrow{q_G} \mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I}$$

and define the functor

$$\mathrm{CT}_B = \mathrm{R}q_{G!}p_G^*\colon D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/(\mathrm{Div}^1_X)^I};k)^{\mathrm{bd}} \to D_{\text{ét}}(\mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I};k). \tag{6.1.4}$$

Let $\deg_G\colon \mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I} \to \mathbf{Z}$ be the locally constant function of [FS21, Proposition VI.7.4], obtained by pairing $2\rho_G$, the sum of the roots of T in B, with the sum of the relative positions of all modifications. According to [FS21, Propositions VI.4.2 and VI.7.4], the functor $\mathrm{CT}_B[\deg_G]$ is conservative and perverse t-exact, hence exact and faithful on perverse sheaves. By [FS21, Proposition VI.6.4] a complex M with bounded support is ULA if and only if $\mathrm{R}\pi_{T*}\,\mathrm{CT}_B(M)$ is locally constant with perfect fibers.

Example 6.1.3. We consider the case of a split torus T. For $\boldsymbol{\lambda} = (\lambda_i)_i \in X_*(T)^I$ we have the section

$$\begin{aligned} s_{\boldsymbol{\lambda}}\colon (\mathrm{Div}^1_X)^I &\to \mathrm{Gr}_{T,(\mathrm{Div}^1_X)^I} \\ (D_i)_i &\mapsto \sum_i \lambda_i D_i. \end{aligned}$$

Every $M \in \mathrm{Sat}(\mathrm{Gr}_{T,(\mathrm{Div}^1_X)^I};k)$ has a canonical (finite) decomposition

$$M \cong \bigoplus_{\boldsymbol{\lambda}\in X_*(T)^I} s_{\boldsymbol{\lambda}*}V_{\boldsymbol{\lambda}}, \qquad V_{\boldsymbol{\lambda}} \in \mathrm{Loc}((\mathrm{Div}^1_X)^I;k). \tag{6.1.5}$$

Indeed, the functor $(V_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda}} \mapsto \bigoplus_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}*}V_{\boldsymbol{\lambda}}$ from $X_*(T)^I$-graded local systems is fully faithful: the sections $s_{\boldsymbol{\lambda}}$ are disjoint over the open subset of $(\mathrm{Div}^1_X)^I$ where the legs are pairwise distinct, and restriction to this open subset is fully faithful on Satake categories and on local systems [FS21, Proposition VI.9.3]. It is essentially surjective because under the Satake equivalence of [FS21, Proposition VI.10.3 and Theorem VI.11.1], $s_{\boldsymbol{\lambda}*}V$ corresponds to V placed in weight $\boldsymbol{\lambda}$.

6.2. **Brauer homomorphisms.** Let $K \subset G(E)$ be an open compact subgroup. The *Hecke algebra of G with respect to K with coefficients in Λ* is

$$\mathscr{H}(G,K;\Lambda) := \mathrm{Fun}_c(K\backslash G(E)/K,\Lambda), \tag{6.2.1}$$

the compactly supported functions on $K\backslash G(E)/K$ valued in Λ. This forms an algebra under convolution, normalized so that the indicator function $\mathbb{1}_K$ is the unit.

For a Σ-stable subgroup $K \subset G(E)$, we write $\iota^*K := \iota^{-1}(K) \subset H(E)$. We say that a compact open subgroup $K \subset G(E)$ is a *plain subgroup* if the natural map $H(E)/\iota^*K \to [G(E)/K]^\sigma$ is a bijection. We may view $\mathscr{H}(G,K;\Lambda)$ as the ring $\mathrm{Fun}_c^{G(E)}([G(E)/K]\times[G(E)/K],\Lambda)$ of $G(E)$-invariant (for the diagonal action) functions (valued in Λ) on $[G(E)/K]\times[G(E)/K]$, with compact support modulo $G(E)$, under convolution.

6.2.1. *The un-normalized Brauer homomorphism.* Now suppose that $\Lambda = k$ has characteristic ℓ. In this special situation, Treumann-Venkatesh observed that if $K \subset G(E)$ is a plain subgroup, then the *restriction map*

$$\begin{aligned} \mathscr{H}(G,K;k)^\sigma = \mathrm{Fun}_c^{G(E)}([G(E)/K]\times[G(E)/K],k)^\sigma \\ \xrightarrow{\text{restrict}} \mathrm{Fun}_c^{H(E)}([H(E)/\iota^*K]\times[H(E)/\iota^*K],k) = \mathscr{H}(H(E),\iota^*K;k) \end{aligned} \tag{6.2.2}$$

is an algebra homomorphism (cf. [Fen24, Lemma 6.5] for a proof). This map was introduced in [TV16, §4] and called the *(un-normalized) Brauer homomorphism*. We denote it

$$\mathrm{Br}\colon \mathscr{H}(G,K;k)^{\sigma} \to \mathscr{H}(H,\iota^*K;k). \tag{6.2.3}$$

6.2.2. *Frobenius twist of algebras.* Let Frob be the absolute (ℓ-power) Frobenius of k. Given a commutative k-algebra A, we denote by $A^{(\ell)} := A \otimes_{k,\mathrm{Frob}} k$ its Frobenius twist. The map $\phi\colon A \to A^{(\ell)}$ sending $a \mapsto a \otimes 1$ is a Frob-semilinear (i.e., $\phi(\lambda a) = \mathrm{Frob}(\lambda)\phi(a)$) isomorphism.

If A is equipped with an $\mathbf{F}_\ell$-structure $\varphi_0\colon A \cong A_0 \otimes_{\mathbf{F}_\ell} k$, then there is a k-linear isomorphism $f_{\varphi_0}\colon A \xrightarrow{\sim} A^{(\ell)}$, characterized by the property that it sends $A_0 \subset A$ to $A_0 \otimes 1 \subset A^{(\ell)}$ via $\mathrm{Id} \otimes 1$. We denote by $\mathrm{Frob}_{\varphi_0} = f_{\varphi_0}^{-1} \circ \phi\colon A \xrightarrow{\sim} A$; it is characterized as the unique Frob-semilinear automorphism of A which restricts to the identity on A_0.

Suppose $f\colon A \to B$ is any Frob-semilinear homomorphism (i.e., $f(\lambda a) = \lambda^\ell f(a)$ for $\lambda \in k$). Then f factors uniquely through a k-linear homomorphism

$$\begin{array}{ccc} A & & \\ {\scriptstyle \mathrm{Frob}_{\varphi_0}}\downarrow & \searrow^{f} & \\ A & \dashrightarrow & B \end{array}$$

The k-linear homomorphism $A \to B$ obtained by precomposing f with the inverse of $\mathrm{Frob}_{\varphi_0}$ (i.e., the dashed arrow in the diagram above) will be called the *linearization* of f (with respect to φ_0).

Definition 6.2.1 (The Tate diagonal)**.** Let A be a commutative k-algebra equipped with an action of Σ by k-algebra automorphisms, and with an $\mathbf{F}_\ell$-structure φ_0. Let $\mathrm{Nm}\colon A \to \mathrm{T}^0(A)$ be the *Tate diagonal* map sending a to the class of $a \cdot ({}^{\sigma}a) \cdot \ldots \cdot ({}^{\sigma^{\ell-1}}a)$. One checks that this is a ring homomorphism, which is evidently Frob-semilinear. Let $\mathrm{Nm}^{(\ell^{-1})}\colon A \to \mathrm{T}^0(A)$ be the linearization of Nm with respect to φ_0.

6.2.3. *The normalized Brauer homomorphism.* Next suppose that $\mathscr{H}(G(E),K;k)$ and $\mathscr{H}(H(E),\iota^*K;k)$ are commutative. The Tate diagonal provides a map (cf. §4.1 for notation on Tate cohomology)

$$\mathrm{Nm}\colon \mathscr{H}(G,K;k) \to \mathrm{T}^0(\mathscr{H}(G,K;k)) = \frac{\mathscr{H}(G,K;k)^{\sigma}}{N \cdot \mathscr{H}(G,K;k)}.$$

One observes that the Brauer homomorphism Br factors over $\mathrm{T}^0(\mathscr{H}(G,K;k))$, hence we obtain a map

$$\mathrm{Br} \circ \mathrm{Nm}\colon \mathscr{H}(G,K;k) \to \mathscr{H}(H,\iota^*K;k).$$

This map is a Frob-semilinear ring homomorphism, and the *normalized Brauer homomorphism* is its linearization with respect to the canonical $\mathbf{F}_\ell$-structure

$$\mathscr{H}(G,K;k) \cong \mathscr{H}(G,K;\mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} k.$$

Equivalently, we may write

$$\mathrm{br} = \mathrm{Br} \circ \mathrm{Nm}^{(\ell^{-1})}\colon \mathscr{H}(G,K;k) \to \mathscr{H}(H,\iota^*K;k). \tag{6.2.4}$$

We will categorify br to a Tannakian functor; see [DM82, §2] for a reference on Tannakian categories.

6.3. **Construction of the Brauer functor.** We now construct the functors $\widetilde{br}^{I}$. We remark that in this construction, it is permitted for Σ to act trivially on G.

For a sheaf of k-modules M, let

$$M^{(\ell^{-1})} = M \otimes_{k,\mathrm{Frob}^{-1}} k \tag{6.3.1}$$

be its inverse Frobenius twist. This is an exact tensor autoequivalence, Frob^{-1}-semilinear on morphisms, which commutes with the sheaf operations and preserves Satake objects and their supports. For $X \in \mathrm{Sat}_G^I(k)$, define

$$\mathrm{Nm}(X) := X \star \sigma^* X \star \cdots \star (\sigma^{\ell-1})^* X \tag{6.3.2}$$

and

$$\mathrm{Nm}^{(\ell^{-1})}(X) := \mathrm{Nm}(X^{(\ell^{-1})}). \tag{6.3.3}$$

Using the commutativity constraint, cyclic rotation of the factors gives isomorphisms $\sigma^* \operatorname{Nm}(X) \cong \operatorname{Nm}(X)$. Since $\operatorname{Nm}(X)$ lies in the abelian category $\operatorname{Sat}_G^I(k)$, this promotes $\operatorname{Nm}(X)$ into a Σ-equivariant object. The functor $\operatorname{Nm}^{(\ell^{-1})}$ is compatible with composition of morphisms, but it is not additive; additivity will be restored after passing to Tate constructions (Proposition 6.4.2).

Definition 6.3.1 (The Brauer functor)**.** Let

$$\delta := \iota^* \dagger_G - \dagger_H \colon \pi_1(H) \to \mathbf{Z}/2\mathbf{Z}, \tag{6.3.4}$$

where $\iota^*\dagger_G$ is the restriction of $\dagger_G$ to $\pi_1(H)$, and write $[\delta]$ for the shift by the representative of δ in $\{0,1\}$ on each connected component. Let $\iota_I \colon \mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}^1_X)^I} \to \mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}^1_X)^I}$ be the natural map. It is Σ-equivariant for the trivial action on its source, so $\iota_I^* \operatorname{Nm}^{(\ell^{-1})}(X)$ is a $k[\Sigma]$-module object, and we set

$$A_I(X) = [\delta]\big(\iota_I^* \operatorname{Nm}^{(\ell^{-1})} X\big)^{t\Sigma} \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}^1_X)^I}; k)^{\mathrm{bd}}. \tag{6.3.5}$$

Define the *Brauer functor*

$$\widetilde{br}^I(X) = {}^p\mathrm{H}^0(A_I(X)) \in \operatorname{Sat}_H^I(k), \tag{6.3.6}$$

and abbreviate $\widetilde{br} = \widetilde{br}^{\{1\}}$. We show in Proposition 6.4.2 that $\widetilde{br}^I(X)$ is a Satake object.

Remark 6.3.2. We indicate why this can be thought of as a categorification of the Brauer homomorphism. By Proposition 5.1.3 and Example 4.2.3(1), the pullback of $A_I(X)$ to $\mathrm{Gr}_{H,(\mathrm{Div}^1_X)^I}$ is $[\delta]\operatorname{Real}(\operatorname{Psm}(\operatorname{Nm}^{(\ell^{-1})} X))$, so we can think of $A_I(X)$ as the "periodic realization" (up to shift) of $\operatorname{Psm}(\operatorname{Nm}^{(\ell^{-1})} X)$, the Smith–Treumann localization of $\operatorname{Nm}^{(\ell^{-1})} X$. The latter object corresponds to the normalized Brauer homomorphism, under the function-sheaf dictionary.

6.4. **Constant terms and perversity.** If G has a Σ-stable Borus (B,T), then Lemma 5.2.1 implies that $(B_H, T_H) = (B^\sigma, T^\sigma)$ is a Borus of H. For $\lambda \in X_*(T)$, we define

$$N(\lambda) = \sum_{r=0}^{\ell-1} \sigma^r(\lambda) \in X_*(T)^\sigma = X_*(T_H).$$

When N is applied to tuples of coweights, it means apply N entrywise.

Lemma 6.4.1. *Suppose G has a Σ-stable Borus (B,T). Let M be a bounded Σ-equivariant complex on $\mathcal{H}\mathrm{ck}_{G,S/(\mathrm{Div}^1_X)^I}$ with bounded support, and let $\iota_{T,I} \colon \mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I}$ be the natural map. There is a natural isomorphism*

$$\mathrm{CT}_{B_H}\big((\iota_I^* M)^{t\Sigma}\big) \cong \big(\iota_{T,I}^* \mathrm{CT}_B(M)\big)^{t\Sigma}, \tag{6.4.1}$$

compatible with pullback in S and with the cup products (4.4.1).

Proof. Define Y to be the fibered product as in the diagram

$$\begin{array}{ccccc} & & Y & \xrightarrow{q} & \mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I} \\ & \swarrow^{p} & \downarrow & \lrcorner & \downarrow{\scriptstyle \iota_{T,I}} \\ \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I} & \xleftarrow[p_G]{} & \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I} & \xrightarrow[q_G]{} & \mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I} \end{array}$$

Base change gives

$$\iota_{T,I}^* \mathrm{CT}_B(M) \cong \mathrm{R}q_! p^* M. \tag{6.4.2}$$

The group Σ acts on Y over $\mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I}$, and $Y^\sigma = \mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I}$ by Propositions 5.1.3 and 5.2.2. The inclusion $i \colon \mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} = Y^\sigma \hookrightarrow Y$ of the fixed locus fits into the commutative diagram

$$\begin{array}{ccccc} \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} & \xleftarrow{p_H} & \mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} & \xrightarrow{q_H} & \mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I} \\ {\scriptstyle \iota_I}\downarrow & & \downarrow{\scriptstyle i} & & \| \\ \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I} & \xleftarrow[p]{} & Y & \xrightarrow[q]{} & \mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I} \end{array}$$

so that $i^*p^*M \cong p_H^*\iota_I^*M$ and $q \circ i = q_H$. Since M has bounded support and p_G is bijective on geometric points and a locally closed immersion on each connected component of $\mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I}$ [FS21, Proposition VI.3.1], the complex p^*M is supported on a Σ-stable closed subspace of Y that is spatial and of finite dimension over S, and meets only finitely many connected components. On this subspace Proposition 4.3.1 applies to q; together with Example 4.2.3(1), it gives

$$(\mathrm{R}q_!p^*M)^{t\Sigma} \cong \mathrm{R}q_{H!}\big((p_H^*\iota_I^*M)^{t\Sigma}\big) \cong \mathrm{R}q_{H!}p_H^*\big((\iota_I^*M)^{t\Sigma}\big) = \mathrm{CT}_{B_H}\big((\iota_I^*M)^{t\Sigma}\big).$$

The compatibilities with pullback and with cup products follow from those of base change, of (4.3.1), and of Proposition 4.4.1. □

Proposition 6.4.2. *The construction* (6.3.6) *defines an exact, faithful, k-linear functor $\widetilde{br}^I\colon \mathrm{Sat}_G^I(k) \to \mathrm{Sat}_H^I(k)$, whose formation commutes with pullback along any $S \to (\mathrm{Div}^1_X)^I$.*

Moreover, assume G has a Σ-stable Borus (B,T). If $X \in \mathrm{Sat}_G^I(k)$ and

$$\mathrm{CT}_B[\deg_G](X) = \bigoplus_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}*}(V_{\boldsymbol{\lambda}}) \tag{6.4.3}$$

in the notation of (6.1.5)*, then there is a natural isomorphism*

$$\mathrm{CT}_{B_H}[\deg_H](\widetilde{br}^I(X)) \cong \bigoplus_{\boldsymbol{\lambda}} s_{N\boldsymbol{\lambda}*}(V_{\boldsymbol{\lambda}}). \tag{6.4.4}$$

Proof. We first observe that the formation of (6.3.6) commutes with pullback along any $S \to (\mathrm{Div}^1_X)^I$: the norm, ι_I^* and $[\delta]$ commute with such pullbacks, as do the Tate construction (Example 4.2.3(1)) and ${}^p\mathrm{H}^0$, since pullback is perverse t-exact [FS21, Proposition VI.7.4].

The remaining assertions of the first paragraph reduce to the case where G has a Σ-stable Borus (B,T) with T split. Indeed, since σ has finite order, such a Borus exists after replacing E by a finite extension E' [Ste68, Theorem 7.5], which amounts to pulling back along the finite étale cover $\pi\colon (\mathrm{Div}^1_{X'})^I \to (\mathrm{Div}^1_X)^I$. By the previous paragraph, $\pi^* \circ \widetilde{br}^I$ is the Brauer functor for $G_{E'}$ composed with π^*. The functor π^* is exact and faithful on complexes with bounded support. This pullback preserves and reflects perversity, since π^* is perverse t-exact [FS21, Proposition VI.7.4] and conservative, and it preserves and reflects the ULA property, since local acyclicity descends along v-covers of the base [FS21, Proposition IV.2.5]. Hence it suffices to show that $\widetilde{br}^I(X)$ is a Satake object and that $\widetilde{br}^I$ is additive, k-linear, exact and faithful after applying π^*, that is, for $G_{E'}$.

Now suppose that G has a Σ-stable Borus (B,T). We deduce the asserted properties of (6.3.6) from (6.4.4). Indeed, the explicit formula (6.4.4) implies that $\mathrm{CT}_{B_H}[\deg_H] \circ \widetilde{br}^I$ is additive, k-linear and exact. As $\mathrm{CT}_{B_H}[\deg_H]$ is exact, faithful and conservative on perverse sheaves, we may conclude that $\widetilde{br}^I$ is additive, k-linear, exact and faithful. After undoing the shifts by $\deg_H$, the pushforward of (6.4.4) to $(\mathrm{Div}^1_X)^I$ is a finite direct sum of shifted local systems of finite rank, so $\widetilde{br}^I(X)$ is ULA by [FS21, Proposition VI.6.4], and hence lies in $\mathrm{Sat}_H^I(k)$.

By definition of $\widetilde{br}^I$ and the perverse t-exactness of $\mathrm{CT}_{B_H}[\deg_H]$ [FS21, Proposition VI.7.4], we have

$$\mathrm{CT}_{B_H}[\deg_H](\widetilde{br}^I(X)) = \mathrm{H}^0\big(\mathrm{CT}_{B_H}[\deg_H]A_I(X)\big). \tag{6.4.5}$$

Consider the locally constant function

$$d := \delta + \deg_H - \deg_G \circ \iota_{T,I}\colon \mathrm{Gr}_{T_H,(\mathrm{Div}^1_X)^I} \to \mathbf{Z},$$

where δ is the function from Definition 6.3.1 restricted to T_H. Lemma 6.4.1 identifies $\mathrm{CT}_{B_H}[\deg_H]A_I(X)$ with

$$\big(\iota_{T,I}^*\,\mathrm{CT}_B[\deg_G](\mathrm{Nm}^{(\ell^{-1})}(X))\big)^{t\Sigma}[d], \tag{6.4.6}$$

which we now compute.

Since B is Σ-stable, $\mathrm{CT}_B[\deg_G]$ commutes with σ^*, and since it is symmetric monoidal it carries $\mathrm{Nm}^{(\ell^{-1})}(X)$, with its Σ-equivariant structure, to

$$\bigotimes_{r=0}^{\ell-1}(\sigma^r)^*\Big(\bigoplus_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}*}V_{\boldsymbol{\lambda}}^{(\ell^{-1})}\Big) = \bigoplus_{(\boldsymbol{\lambda}_0,\dots,\boldsymbol{\lambda}_{\ell-1})} s_{\boldsymbol{\mu}*}\big(V_{\boldsymbol{\lambda}_0}^{(\ell^{-1})}\otimes\cdots\otimes V_{\boldsymbol{\lambda}_{\ell-1}}^{(\ell^{-1})}\big), \qquad \boldsymbol{\mu}=\sum_r \sigma^{-r}\boldsymbol{\lambda}_r,$$

with Σ permuting the words $(\boldsymbol{\lambda}_0,\dots,\boldsymbol{\lambda}_{\ell-1})$ cyclically.

The orbit of a nonconstant word is free, so the corresponding summand is induced from the trivial subgroup, hence has vanishing Tate construction and does not contribute to (6.4.6). The constant word $\boldsymbol{\lambda}$ contributes $s_{N\boldsymbol{\lambda}*}$ of $(V_{\boldsymbol{\lambda}}^{(\ell^{-1})})^{\otimes\ell}$, on which σ acts by cyclic permutation.[5] The section $s_{N\boldsymbol{\lambda}}$ lands in the connected component of $\mathrm{Gr}_{T_H,(\mathrm{Div}^1_X)^I}$ indexed by $\sum_i N\lambda_i \in X_*(T_H)$, where d takes the value $m_{\boldsymbol{\lambda}}$ below. So the contribution of $\boldsymbol{\lambda}$ to (6.4.6) is

$$s_{N\boldsymbol{\lambda}*}\Big(\big((V_{\boldsymbol{\lambda}}^{(\ell^{-1})})^{\otimes\ell}\big)^{t\Sigma}[m_{\boldsymbol{\lambda}}]\Big), \tag{6.4.7}$$

where

- $\delta(-)$ is as in Definition 6.3.1 composed with the obvious set map $\mathbf{Z}/2\mathbf{Z}\to\{0,1\}$.
- $m_{\boldsymbol{\lambda}} = \delta\Big(\sum_i N\lambda_i\Big) + \sum_i\langle 2\rho_H, N\lambda_i\rangle - \sum_i\langle 2\rho_G, N\lambda_i\rangle$. By (6.3.4), $m_{\boldsymbol{\lambda}}$ is even.

Let $\beta\in \mathrm{T}^2(k)$ be the periodicity class of Example 4.5.1. By Proposition 4.2.4 and Example 4.5.3, the Tate diagonal followed by multiplication by $\beta^{m_{\boldsymbol{\lambda}}/2}$ gives a natural isomorphism of local systems

$$V_{\boldsymbol{\lambda}} \xrightarrow{\sim} \mathrm{H}^0\Big(\big((V_{\boldsymbol{\lambda}}^{(\ell^{-1})})^{\otimes\ell}\big)^{t\Sigma}[m_{\boldsymbol{\lambda}}]\Big). \tag{6.4.8}$$

The Frobenius twists are arranged to make this identification k-linear. Since $s_{N\boldsymbol{\lambda}*}$ is exact, taking H^0 of the sum of the contributions (6.4.7) and using (6.4.5) gives the isomorphism (6.4.4). From the definition, we see that it is natural in X.

□

Proposition 6.4.3. *For every $n\in\mathbf{Z}$ there is an isomorphism, natural in X and compatible with pullback in the base,*

$${}^p\mathrm{H}^n(A_I(X)) \cong \widetilde{br}^{\,I}(X). \tag{6.4.9}$$

For even n it is induced by multiplication by $\beta^{n/2}$, and for odd n by multiplication by $\eta\beta^{(n-1)/2}$, where $\eta\in\mathrm{T}^1(k)$ and $\beta\in\mathrm{T}^2(k)$ are as in Example 4.5.1.

Proof. By the lax monoidality of the Tate construction (§4.4), $A_I(X)$ is a module over $k^{t\Sigma}$, so β and η act on it by maps $A_I(X)\to A_I(X)[2]$ and $A_I(X)\to A_I(X)[1]$. Multiplying by $\beta^{n/2}$ or $\eta\beta^{(n-1)/2}$, depending on whether n is even or odd, gives a natural transformation

$$\widetilde{br}^{\,I} = {}^p\mathrm{H}^0\circ A_I \to {}^p\mathrm{H}^n\circ A_I.$$

To check that this is an isomorphism, we may pull back along a finite étale cover of $(\mathrm{Div}^1_X)^I$, with which the formation of A_I and of ${}^p\mathrm{H}^n$ commutes (see the proof of Proposition 6.4.2). Hence we may assume that G has a Σ-stable Borus (B,T) with T split. To see that this is an isomorphism, we may apply $\mathrm{CT}_{B_H}[\deg_H]$, which is t-exact, conservative, and by Lemma 6.4.1 compatible with these module structures. By (6.4.7), this reduces to showing that multiplication by $\beta^{n/2}$ and $\eta\beta^{(n-1)/2}$ are both isomorphisms on the Tate cohomology of $(V^{(\ell^{-1})})^{\otimes\ell}$ with its cyclic rotation action, which is Example 4.5.3. □

[5] In principle there could be a Koszul sign character, but it equals $(-1)^{(\ell-1)\sum_i\langle 2\rho_G,\lambda_i\rangle}=1$.

6.5. **Tensor structure.** Recall that $\mathrm{Sat}_G^I(k)$ has a symmetric monoidal structure coming from fusion [FS21, §VI.9]. Next we will promote $\widetilde{br}^I$ to a tensor functor, with respect to this structure.

Theorem 6.5.1. *For each non-empty finite set I, the functors*

$$\widetilde{br}^I \colon \mathrm{Sat}_G^I(k) \to \mathrm{Sat}_H^I(k)$$

are exact, faithful and symmetric monoidal, and linear over $\mathrm{Loc}((\mathrm{Div}_X^1)^I; k)$. *They commute with fusion and with the "coalescence maps" induced by pullback along maps of finite sets $I \to J$, compatibly with base change along any $S \to (\mathrm{Div}_X^1)^I$.*

Proof. The assertions can be checked after pullback along a finite étale cover of $(\mathrm{Div}_X^1)^I$, with which the formation of $\widetilde{br}^I$ commutes by Proposition 6.4.2. As in the proof of that proposition, we may therefore reduce to the case that G has a Σ-stable Borus (B, T).

Fix a partition $I = I_1 \sqcup I_2$, and let

$$j \colon \big((\mathrm{Div}_X^1)^{I_1} \times (\mathrm{Div}_X^1)^{I_2}\big)_{\mathrm{disj}} \hookrightarrow (\mathrm{Div}_X^1)^I$$

be the inclusion of the open subset where every leg indexed by I_1 is disjoint from every leg indexed by I_2. Over this subset the Hecke stack factors [FS21, §VI.9, before Definition/Proposition VI.9.4]:

$$j^* \mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}_X^1)^I} \cong j^* \big(\mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}_X^1)^{I_1}} \times \mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}_X^1)^{I_2}}\big),$$

where j^* denotes base change along j, and similarly for H. We write $P \boxtimes Q$ for the exterior product of complexes P and Q on the two factors, restricted to this open subset. By [FS21, Proposition VI.9.3 and Definition/Proposition VI.9.4], restriction along j is fully faithful on Satake categories, and for $X \in \mathrm{Sat}_G^{I_1}(k)$ and $Y \in \mathrm{Sat}_G^{I_2}(k)$ the fusion product $X * Y \in \mathrm{Sat}_G^I(k)$ is characterized as the unique object of $\mathrm{Sat}_G^I(k)$ satisfying $j^*(X * Y) \cong X \boxtimes Y$. We first construct an isomorphism

$$\widetilde{br}^{I_1}(X) \boxtimes \widetilde{br}^{I_2}(Y) \cong j^* \widetilde{br}^I(X * Y). \tag{6.5.1}$$

Step 1: perverse truncation and exterior products. Let $P \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}_X^1)^{I_1}}; k)^{\mathrm{bd}}$ and $Q \in D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}_X^1)^{I_2}}; k)^{\mathrm{bd}}$ be such that ${}^p\mathrm{H}^0(P)$ and ${}^p\mathrm{H}^0(Q)$ are Satake. It is clear from the definition that the exterior product preserves ${}^pD^{\leq 0}$. Moreover, exterior products of Satake objects are Satake [FS21, §VI.9]. Applying ${}^p\mathrm{H}^0$ to the truncation triangles and to ${}^p\tau^{\leq 0}P \boxtimes {}^p\tau^{\leq 0}Q \to P \boxtimes Q$ therefore gives a natural map

$${}^p\mathrm{H}^0(P) \boxtimes {}^p\mathrm{H}^0(Q) \cong {}^p\mathrm{H}^0\big({}^p\tau^{\leq 0}P \boxtimes {}^p\tau^{\leq 0}Q\big) \to {}^p\mathrm{H}^0(P \boxtimes Q). \tag{6.5.2}$$

Step 2: the comparison map. After restriction along j, reordering the factors by the commutativity constraint gives $j^* \mathrm{Nm}^{(\ell^{-1})}(X * Y) \cong \mathrm{Nm}^{(\ell^{-1})}(X) \boxtimes \mathrm{Nm}^{(\ell^{-1})}(Y)$, compatibly with the Σ-equivariant structures, and ι_I restricts to $\iota_{I_1} \times \iota_{I_2}$. Hence the cup product (4.4.1) gives a map

$$\big(\iota_{I_1}^* \mathrm{Nm}^{(\ell^{-1})} X\big)^{t\Sigma} \boxtimes \big(\iota_{I_2}^* \mathrm{Nm}^{(\ell^{-1})} Y\big)^{t\Sigma} \xrightarrow{\cup} j^* \big(\iota_I^* \mathrm{Nm}^{(\ell^{-1})}(X * Y)\big)^{t\Sigma}.$$

Now we need to take care of sign issues. We use the standard convention that $C[a] \otimes D[b] \to (C \otimes D)[a+b]$ sends $x \otimes y$ to $(-1)^{b|x|} x \otimes y$, where $|x|$ is the degree of x in C. On connected components c and c' of the two factors, put

$$a(c, c') = \tfrac{1}{2}\big(\delta(c) + \delta(c') - \delta(c + c')\big) \in \{0, 1\}, \qquad \epsilon(c, c') = (-1)^{\delta(c)\dagger_H(c')}. \tag{6.5.3}$$

We define a map

$$A_{I_1}(X) \boxtimes A_{I_2}(Y) \to j^* A_I(X * Y) \tag{6.5.4}$$

as the composite

$$\begin{array}{ccc} A_{I_1}(X) \boxtimes A_{I_2}(Y) & \xrightarrow{\sim} & \Big(\big(\iota_{I_1}^* \mathrm{Nm}^{(\ell^{-1})} X\big)^{t\Sigma} \boxtimes \big(\iota_{I_2}^* \mathrm{Nm}^{(\ell^{-1})} Y\big)^{t\Sigma}\Big)[\delta(c) + \delta(c')] \\ & & \big\downarrow \cup \\ j^* A_I(X * Y) & \xleftarrow{\epsilon\beta^{-a}} & j^* \big(\iota_I^* \mathrm{Nm}^{(\ell^{-1})}(X * Y)\big)^{t\Sigma}[\delta(c) + \delta(c')] \end{array} \tag{6.5.5}$$

where the horizontal map $\epsilon\beta^{-a}$ in the bottom row denotes multiplication by $\epsilon(c,c')\beta^{-a(c,c')}$. Applying ${}^p\mathrm{H}^0$ and combining with (6.5.2), we obtain a natural transformation

$$\widetilde{br}^{I_1}(X) \boxtimes \widetilde{br}^{I_2}(Y) \xrightarrow{(6.5.2)} {}^p\mathrm{H}^0\big(A_{I_1}(X) \boxtimes A_{I_2}(Y)\big) \xrightarrow{(6.5.5)} j^*\widetilde{br}^{I}(X * Y). \tag{6.5.6}$$

Step 3: analyzing constant terms. Recall that we have reduced to the case where G has a Σ-stable Borus (B,T), with Σ-fixed points (B_H, T_H). In order to check that (6.5.6) is an isomorphism, we apply $\mathrm{CT}_{B_H}[\deg_H] \boxtimes \mathrm{CT}_{B_H}[\deg_H]$ to it. Write

$$\mathrm{CT}_B[\deg_G](X) = \bigoplus_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}*} V_{\boldsymbol{\lambda}}, \qquad \mathrm{CT}_B[\deg_G](Y) = \bigoplus_{\boldsymbol{\mu}} s_{\boldsymbol{\mu}*} W_{\boldsymbol{\mu}}$$

as in (6.1.5), with $\boldsymbol{\lambda} \in X_*(T)^{I_1}$ and $\boldsymbol{\mu} \in X_*(T)^{I_2}$. Since normalized constant terms commute with fusion [FS21, Proposition VI.9.6], the corresponding decomposition of $\mathrm{CT}_B[\deg_G](X * Y)$ has the summands $s_{(\boldsymbol{\lambda},\boldsymbol{\mu})*}(V_{\boldsymbol{\lambda}} \boxtimes W_{\boldsymbol{\mu}})$. Applying (6.4.4) to X, Y and $X * Y$, the source and the target of (6.5.6) become, after $\mathrm{CT}_{B_H}[\deg_H]$,

$$\bigoplus_{\boldsymbol{\lambda},\boldsymbol{\mu}} s_{N\boldsymbol{\lambda}*} V_{\boldsymbol{\lambda}} \boxtimes s_{N\boldsymbol{\mu}*} W_{\boldsymbol{\mu}} \qquad \text{and} \qquad j^* \bigoplus_{\boldsymbol{\lambda},\boldsymbol{\mu}} s_{N(\boldsymbol{\lambda},\boldsymbol{\mu})*}\big(V_{\boldsymbol{\lambda}} \boxtimes W_{\boldsymbol{\mu}}\big),$$

which agree summand by summand. It therefore suffices to compute the map on the summand indexed by a pair $(\boldsymbol{\lambda},\boldsymbol{\mu})$; write $V = V_{\boldsymbol{\lambda}}$ and $W = W_{\boldsymbol{\mu}}$. Lemma 6.4.1 is compatible with cup products and with the Künneth isomorphism (4.4.2), so the map itself is computed by cup products of the classes (6.4.8). This explicit presentation shows that (6.5.6) is an isomorphism.

Step 4: coherence. Normalized constant terms carry the modified commutativity constraint of the Satake categories to the ordinary commutativity constraint of local systems, and the classes (6.4.8) have even degree, so (4.4.1) introduces no sign under symmetry. Hence, after the faithful functor $\mathrm{CT}_{B_H}[\deg_H]$, the unit, associativity and symmetry diagrams for (6.5.6) become the corresponding diagrams of tensor isomorphisms of local systems, and they commute.

Step 5: extension across collisions. Both $\widetilde{br}^{I}(X*Y)$ and $\widetilde{br}^{I_1}(X)*\widetilde{br}^{I_2}(Y)$ lie in $\mathrm{Sat}_H^I(k)$, so by the full faithfulness in [FS21, Proposition VI.9.3] the isomorphism (6.5.6) and the attendant coherent diagrams extend uniquely across $(\mathrm{Div}^1_X)^I$. Restricting to diagonals gives the symmetric monoidal structure on $\widetilde{br}^{I}$ for each I [FS21, Definition/Proposition VI.9.4], and the compatibility with base change in Proposition 6.4.2 identifies these structures under all coalescence maps.

The final assertion to establish is linearity over tensoring with $L \in \mathrm{Loc}((\mathrm{Div}^1_X)^I; k)$. This follows from the argument of Example 4.5.3, which supplies a natural isomorphism

$$\mathrm{H}^0\Big(\big((L^{(\ell^{-1})})^{\otimes \ell}\big)^{t\Sigma}\Big) \cong L.$$

□

Lemma 6.5.2. *Suppose that G has a Σ-stable Borus (B,T) with T split, and let N_* be the tensor functor on torus Satake objects which sends a summand $s_{\boldsymbol{\lambda}*}V$ to $s_{N\boldsymbol{\lambda}*}V$. There is an isomorphism of tensor functors*

$$\mathrm{CT}_{B_H}[\deg_H] \circ \widetilde{br}^{I} \cong N_* \circ \mathrm{CT}_B[\deg_G], \tag{6.5.7}$$

compatible with fusion and coalescence.

Proof. The underlying isomorphism of additive functors is (6.4.4). It is compatible with the tensor structures away from the collision locus by the proof of Theorem 6.5.1, and hence everywhere by [FS21, Proposition VI.9.3]. □

Proposition 6.5.3. *After passing to an extension of E over which there is a split Σ-stable Borus, there is an isomorphism of fiber functors*

$$\mathsf{F}^{H,I} \circ \widetilde{br}^{I} \cong \mathsf{F}^{G,I}, \tag{6.5.8}$$

and in particular on geometric fibers. There are also natural isomorphisms $\widetilde{br}^{I} \circ \sigma^* \cong \widetilde{br}^{I}$, compatible with fusion.

Proof. As remarked in the proof of [FS21, Theorem VI.11.1], the fiber functor is identified (as tensor functors) with

$$\bigoplus_i H^i \circ \mathrm{R}\pi_{G*} \cong H^0 \circ \mathrm{R}\pi_{T*}\, \mathrm{CT}_B[\deg_G]$$

In view of this, the identification (6.5.8) follows from Lemma 6.5.2. The last assertion follows from observing that cyclic rotation of the factors identifies $\mathrm{Nm}(\sigma^* X)$ with $\mathrm{Nm}(X)$ as Σ-equivariant objects, compatibly with the commutativity constraints. □

7. The σ-dual homomorphism

In this section, we apply Tannaka duality to the Brauer functor to construct the σ-dual homomorphism

$${}^L\psi\colon {}^L H \to {}^L G$$

which lifts the Brauer homomorphism on spherical Hecke algebras. Such a map was constructed in [TV16], when G is simply connected and H is semisimple, unless G has type E_6. Our construction has the following advantages: it works in full generality (without any technical assumptions on G or H), and it is canonical (while the map in [TV16] is not uniquely characterized).

7.1. **The c-group.** Actually, the most canonical object uses a different incarnation of the L-group. There are (at least) two normalizations of the dual group of G/E.

- We write ${}^L G \cong \widehat{G} \rtimes W_E$ for the *L-group* in the sense of Langlands.
- There is also the *c-group* ${}^c G$ as in [TV14, §7.8].

Over a p-adic local field, a choice of square root of the cyclotomic character (which always exists, for example by choosing a square root of the residue cardinality) induces an isomorphism ${}^L G \cong {}^c G$ by [Zhu17b, Lemma 5.5.7]. However, because of the non-canonicity of such choices, it is necessary to distinguish these groups for certain purposes. While the L-group is the incarnation more typically treated in the literature, the c-group is the incarnation that naturally shows up in the Geometric Satake equivalence (see [FS21, VI.11] and [Zhu17b, Remark 5.5.11]):

$$\mathrm{Rep}_{\mathrm{Loc}(\mathrm{Div}^1_X;\Lambda)}(\check{G}) \cong \mathrm{Rep}_{\mathrm{Rep}_\Lambda(W_E)}(\check{G}) \cong \mathrm{Rep}_\Lambda({}^c G).$$

Definition 7.1.1. For homomorphisms $\phi_1, \phi_2\colon {}^L H \to {}^L G$ over W_E, we write $\phi_1 \sim \phi_2$ if they are conjugate by an element of $\widehat{G}(k)$, and similarly for homomorphisms ${}^c H \to {}^c G$ over W_E, with $\check{G}(k)$ in place of $\widehat{G}(k)$. A diagram of such homomorphisms *commutes up to* $\sim$ if its two composites are related by $\sim$.

7.2. **Construction of the σ-dual homomorphism.**

Theorem 7.2.1. *The functor*

$$\widetilde{br}\colon \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};k) \to \mathrm{Sat}(\mathrm{Gr}_{H,\mathrm{Div}^1_X};k)$$

corresponds under the Geometric Satake equivalence to restriction along a continuous homomorphism ${}^c\psi\colon {}^c H \to {}^c G$ over W_E, which is algebraic on $\check{H}$ and well defined up to $\check{G}(k)$-conjugacy.

Proof. Choose a tensor identification of the geometric fiber functors supplied by Proposition 6.5.3. The assertion follows from Theorem 6.5.1 by relative Tannakian duality over $\mathrm{Loc}(\mathrm{Div}^1_X;k)$, as in [FS21, §VI.10]; changing the identification conjugates the resulting homomorphism by $\check{G}(k)$. □

We fix a representative ${}^c\psi$, and write $\check{\psi}\colon \check{H} \to \check{G}$ for its restriction to $\check{H}$.

Theorem 7.2.2. *The Tannakian functors*

$$\widetilde{br}^I\colon \mathrm{Sat}^I_G(k) \to \mathrm{Sat}^I_H(k)$$

from (6.3.6) *satisfy the following properties.*

(1) Under the Geometric Satake equivalence (6.1.1) *of Fargues-Scholze, $\widetilde{br}^I$ corresponds to the restriction ${}^c\psi^*\colon \mathrm{Rep}_k({}^c G)^{\otimes I} \to \mathrm{Rep}_k({}^c H)^{\otimes I}$ induced by the σ-dual homomorphism ${}^c\psi\colon {}^c H \to {}^c G$ from Theorem 7.2.1. (In particular, $\widetilde{br}^{\{1\}}$ agrees with the $\widetilde{br}$ from* (6.3.6)*.)*

(2) (Naturality in I) For any map of non-empty finite sets $\zeta\colon I \to J$, inducing the fusion product $\zeta\colon \mathrm{Sat}_G^I(k) \to \mathrm{Sat}_G^J(k)$ and similarly for H, there is a commutative square

$$\begin{array}{ccc} \mathrm{Sat}_G^I(k) & \xrightarrow{\widetilde{br}^I} & \mathrm{Sat}_H^I(k) \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta} \\ \mathrm{Sat}_G^J(k) & \xrightarrow{\widetilde{br}^J} & \mathrm{Sat}_H^J(k) \end{array}$$

such that the implicit natural isomorphisms are compatible with compositions $I \to J \to K$.

Proof. Write $I = \bigsqcup_{i\in I}\{i\}$. We bootstrap from the case $|I| = 1$ using the convolution Hecke stack

$$\widetilde{\mathcal{H}\mathrm{ck}}_G^I := \mathcal{H}\mathrm{ck}_G^{I;\{i\}_{i\in I}} \to (\mathrm{Div}_X^1)^I$$

defined in [FS21, p. 226]. We have projection maps $p_i\colon \widetilde{\mathcal{H}\mathrm{ck}}_G^I \to \widetilde{\mathcal{H}\mathrm{ck}}_{G,\mathrm{Div}_X^1}^{\{i\}}$ for each $i \in I$, as well as a convolution map $m\colon \widetilde{\mathcal{H}\mathrm{ck}}_G^I \to \mathcal{H}\mathrm{ck}_{G,(\mathrm{Div}_X^1)^I}$. The map

$$(\mathcal{K}_i)_{i\in I} \mapsto Rm_!(\otimes_{i\in I}\, p_i^*\mathcal{K}_i) \in \mathrm{Sat}_G^I(k)$$

induces an equivalence

$$\mathrm{conv}\colon \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}_X^1})^{\otimes I} \xrightarrow{\sim} \mathrm{Sat}_G^I(k). \tag{7.2.1}$$

Here $(-)^{\otimes I}$ denotes Deligne's tensor product over k, and conv is an equivalence by [FS21, Proposition VI.10.3].

The same considerations apply to H. By the compatibility with fusion in Theorem 6.5.1, $\widetilde{br}^I$ makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Sat}_G(k)^{\otimes I} & \xrightarrow{\widetilde{br}^{\otimes I}} & \mathrm{Sat}_H(k)^{\otimes I} \\ {\scriptstyle\sim}\downarrow{\scriptstyle \mathrm{conv}} & & {\scriptstyle\sim}\downarrow{\scriptstyle \mathrm{conv}} \\ \mathrm{Sat}_G^I(k) & \dashrightarrow^{\widetilde{br}^I} & \mathrm{Sat}_H^I(k) \end{array} \tag{7.2.2}$$

Then property (1) follows from the $|I| = 1$ case, which was arranged in Theorem 7.2.1, and Tannakian reconstruction [FS21, §VI.10].

Property (2), including the compatibility with compositions, is the compatibility with fusion and coalescence in Theorem 6.5.1.

□

On maximal tori, Lemma 6.5.2 gives the following.

Corollary 7.2.3. *Let $\check{T}_H, \check{T}$ be the canonical maximal tori in $\check{H}, \check{G}$, respectively. Some $\check{G}(k)$-conjugate of $\check{\psi}$ carries $\check{T}_H$ into $\check{T}$, and its restriction $\check{T}_H \to \check{T}$ corresponds to the map $X^*(\check{T}) \to X^*(\check{T}_H) = X^*(\check{T})^\sigma$ given by applying $N = (1 + \sigma + \ldots + \sigma^{\ell-1})$.*

Given choices of isomorphisms ${}^LG \xrightarrow{\sim} {}^cG$ and ${}^LH \xrightarrow{\sim} {}^cH$, we can transport ${}^c\psi$ to a map ${}^L\psi\colon {}^LH \to {}^LG$ characterized by a commutative diagram

$$\begin{array}{ccc} {}^cH & \xrightarrow{{}^c\psi} & {}^cG \\ \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\ {}^LH & \xrightarrow{{}^L\psi} & {}^LG \end{array}$$

However, the choices are not canonical, and this can create genuine incompatibilities when comparing σ-dual homomorphisms with the same domain and codomain that arise in different ways, as will be seen in the example of base change. Also, in the context of global number fields Treumann-Venkatesh pointed out [TV14, §7.8] that ${}^L\psi$ may not exist. Without the simply connected hypotheses, a σ-dual homomorphism need not exist with the "usual" L-group defined by Langlands, and they predicted that using instead the "c-group" might fix this issue.

Remark 7.2.4. The second main theorem of Treumann-Venkatesh (see [TV16, §1.3]) is the construction of a σ-dual homomorphism when G is simply connected and H is semisimple, with three possible exceptions when G has type E_6. Their proof is based on classification of all possible examples, and then case-by-case analysis of each. By contrast, our construction is completely uniform.

7.3. **Compatibilities.** We record some compatibilities of ${}^c\psi$.

7.3.1. *Base change.* Let E'/E be a finite extension. Then $H_{E'} = G^{\sigma}_{E'}$, and Theorem 7.2.1 gives a map

$$ {}^c(\psi_{E'})\colon {}^c(H_{E'}) \to {}^c(G_{E'}). \tag{7.3.1}$$

On the other hand, let $(\bullet)|_{E'}$ denote restriction along the inclusion $W_{E'} \hookrightarrow W_E$. Applying this to ${}^c\psi\colon {}^cH \to {}^cG$, we obtain a map

$$ ({}^c\psi)|_{E'}\colon ({}^cH)|_{E'} \to ({}^cG)|_{E'} \tag{7.3.2}$$

Note that we have natural isomorphisms $({}^cH)|_{E'} \cong {}^c(H_{E'})$ and $({}^cG)|_{E'} \cong {}^c(G_{E'})$. The following Lemma expresses compatibility of the formation of ${}^c\psi$ with respect to the base change along $E \to E'$.

Lemma 7.3.1. *Under the identifications $({}^cH)|_{E'} \cong {}^c(H_{E'})$ and $({}^cG)|_{E'} \cong {}^c(G_{E'})$, we have $({}^c\psi)|_{E'} \sim {}^c(\psi_{E'})$; that is, the diagram*

$$\begin{array}{ccc} ({}^cH)|_{E'} & \xrightarrow{({}^c\psi)|_{E'}} & ({}^cG)|_{E'} \\ \downarrow\wr & & \downarrow\wr \\ {}^c(H_{E'}) & \xrightarrow{{}^c(\psi_{E'})} & {}^c(G_{E'}) \end{array}$$

commutes (up to $\check{G}(k)$-conjugacy).

Proof. Let $\mathrm{Div}^1_{X'} \to \mathrm{Div}^1_X$ be the finite étale cover corresponding to E'. The construction (6.3.5) commutes with pullback along this map by Example 4.2.3(1), and so does perverse truncation. Hence the pullback of $\widetilde{br}$ to $\mathrm{Div}^1_{X'}$ is isomorphic, as a tensor functor, to the Brauer functor for $G_{E'}$ (Theorem 6.5.1), and the assertion follows by Tannaka duality. □

Remark 7.3.2. If the square root of the cyclotomic character used over E' is the restriction of the one used over E, then the same compatibility holds for ${}^L\psi\colon {}^LH \to {}^LG$ on all of $W_{E'}$. For independently chosen unramified square roots, it holds at least after restriction to the subgroup of $W_{E'}$ mapping to $2\mathbf{Z}$ under the valuation.

7.3.2. *Products.* Suppose we have factorizations $G \cong G_1 \times G_2$, $\sigma \cong \sigma_1 \times \sigma_2$, hence $H = H_1 \times H_2$ where $H_i = G_i^{\sigma_i}$. Theorem 7.2.1, applied to each factor, gives

$$ {}^c\psi_1\colon {}^cH_1 \to {}^cG_1, \quad \text{and} \quad {}^c\psi_2\colon {}^cH_2 \to {}^cG_2.$$

Lemma 7.3.3. *Under the isomorphisms ${}^cH \cong {}^cH_1 \times_{W_E} {}^cH_2$ and ${}^cG \cong {}^cG_1 \times_{W_E} {}^cG_2$, we have*

$$ {}^c\psi \sim {}^c\psi_1 \times_{W_E} {}^c\psi_2.$$

Proof. We have $\mathrm{Gr}_{G,\mathrm{Div}^1_X} = \mathrm{Gr}_{G_1,\mathrm{Div}^1_X} \times_{\mathrm{Div}^1_X} \mathrm{Gr}_{G_2,\mathrm{Div}^1_X}$, and similarly for H. For Satake objects X_i for G_i, the norm of $X_1 \boxtimes X_2$ is $\mathrm{Nm}(X_1) \boxtimes \mathrm{Nm}(X_2)$, so the cup product (4.4.1), normalized as in (6.5.5), gives a natural isomorphism

$$ \widetilde{br}(X_1 \boxtimes X_2) \cong \widetilde{br}_1(X_1) \boxtimes \widetilde{br}_2(X_2). \tag{7.3.3}$$

As in the proof of Theorem 6.5.1, after normalized constant terms this is the identity on weight spaces; hence it is compatible with the tensor structures. Applying Tannaka duality gives the claim. □

7.4. **Maps inducing isomorphisms on adjoint groups.** Let $G \to G'$ be a homomorphism of reductive groups over E which induces an isomorphism $G_{\mathrm{ad}} \xrightarrow{\sim} G'_{\mathrm{ad}}$ (e.g., a central isogeny). Then its kernel is central, it identifies the roots of G with those of G', and it restricts to an isomorphism $G_{\mathrm{sc}} \xrightarrow{\sim} G'_{\mathrm{sc}}$ of the simply connected covers of the derived groups. The induced map $g\colon \mathrm{Gr}_{G,\mathrm{Div}^1_X} \to \mathrm{Gr}_{G',\mathrm{Div}^1_X}$ maps each geometric connected component isomorphically onto a connected component. Under Geometric Satake, the restriction along the dual homomorphism ${}^cG' \to {}^cG$ corresponds to the pushforward

$$\mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X}; k) \xrightarrow{\mathrm{R}g_*} \mathrm{Sat}(\mathrm{Gr}_{G',\mathrm{Div}^1_X}; k). \tag{7.4.1}$$

Now suppose that σ acts compatibly on G, G' with fixed subgroups H, H', and let $f\colon \mathrm{Gr}_{H,\mathrm{Div}^1_X} \to \mathrm{Gr}_{H',\mathrm{Div}^1_X}$ be the induced map. The map $H \to H'$ again induces an isomorphism on adjoint groups (recall that $H' = G'^{\sigma}$ is assumed connected).

Lemma 7.4.1. *Let ${}^c\psi\colon {}^cH \to {}^cG$ and ${}^c\psi'\colon {}^cH' \to {}^cG'$ be the σ-dual homomorphisms. Then the diagram*

$$\begin{array}{ccc} {}^cH' & \longrightarrow & {}^cH \\ \downarrow{\scriptstyle {}^c\psi'} & & \downarrow{\scriptstyle {}^c\psi} \\ {}^cG' & \longrightarrow & {}^cG \end{array} \tag{7.4.2}$$

commutes (up to $\check{G}(k)$-conjugacy).

Proof. By Tannaka duality it suffices to give a tensor isomorphism $\mathrm{R}f_* \circ \widetilde{br} \cong \widetilde{br}' \circ \mathrm{R}g_*$, compatible with fusion. Only finitely many connected components meet any bounded support, so on bounded supports f and g are proper, and $\mathrm{R}f_*$, $\mathrm{R}g_*$ are perverse t-exact because f and g map components isomorphically onto components. Since the maps $G \to G'$ and $H \to H'$ identify roots, $\dagger_G$ and $\dagger_H$ are pulled back from $\dagger_{G'}$ and $\dagger_{H'}$, so δ is the pullback of δ'. The tensor functor $\mathrm{R}g_*$ commutes with the norms, and f is the map induced by g on fixed points (Proposition 5.1.3), so Proposition 4.3.1 gives

$$\mathrm{R}f_* A_1(X) \cong A'_1(\mathrm{R}g_* X). \tag{7.4.3}$$

Taking ${}^p\mathrm{H}^0$ implies that $\mathrm{R}f_* \circ \widetilde{br} \cong \widetilde{br}' \circ \mathrm{R}g_*$. This isomorphism is compatible with exterior products and fusion, which proves the lemma. □

7.5. **Examples of the σ-dual homomorphism.**

Proposition 7.5.1. *Suppose that $G = T$ is a torus and that $H = T^{\sigma}$ is connected, and let $N\colon X_*(T) \to X_*(H)$ be the norm. Then the σ-dual homomorphism may be taken to be $\check{\psi} \rtimes \mathrm{Id}_{W_E}$, where $\check{\psi}\colon \check{H} \to \check{T}$ is dual to N. In particular, if Σ acts trivially on T, then $\check{\psi}$ is the ℓ-power map.*

Proof. This follows from Proposition 6.4.2. □

Proposition 7.5.2 (Cyclic base change)**.** *Let E'/E be cyclic of degree ℓ, let $G = \mathrm{Res}_{E'/E}(H_{E'})$, and let $\Sigma = \mathrm{Gal}(E'/E)$ act in the natural way on G, so that $G^{\sigma} = H$. Setting $\Omega := \mathrm{Hom}_E(E', E^s)$, identify $\check{G} = \prod_{\tau\in\Omega} \check{H}$, with W_E-action*

$$w \cdot ((g_\tau)_\tau) = (w \cdot (g_{w^{-1}\tau}))_\tau. \tag{7.5.1}$$

Let $\Delta\colon \check{H} \to \check{G}$ be the diagonal. Then ${}^c\psi \sim {}^c\psi_{\mathrm{BC}}$, where

$${}^c\psi_{\mathrm{BC}}(h, w) = (\Delta(h), w). \tag{7.5.2}$$

Proof. Choose a Borus (B, T) of H_{E^s} and use the product Borus of $G_{E^s} = \prod_\Omega H_{E^s}$, which is Σ-stable. Then Proposition 6.5.3 gives a comparison

$$\xi_{B,T}\colon \mathsf{F}^H \circ \widetilde{br} \xrightarrow{\sim} \mathsf{F}^G. \tag{7.5.3}$$

We claim that $\xi_{B,T}$ does not depend on (B, T). As $H(E^s)$ acts transitively on its Borus pairs, it suffices to see that $\xi_{B,T}$ is unchanged by simultaneous conjugation by $u \in H(E^s)$, acting through $\Delta(u)$ on Gr_G. Every map entering the construction of $\xi_{B,T}$ commutes with this conjugation, so it suffices that u and $\Delta(u)$ act trivially on the fiber functors. This holds because the action of the connected group L^+H on

the cohomology of an equivariant complex with bounded support is constant, and hence trivial [FS21, Propositions VI.1.10, VI.2.3 and VI.2.8]. By transport of structure,

$$w \cdot \xi_{B,T} = \xi_{wB,wT} \quad \text{for all } w \in W_E,$$

and combined with the claim, this implies that $\xi := \xi_{B,T}$ is W_E-equivariant. This equivariance implies that with the choice of fiber functors in (7.5.3), ${}^c\psi$ has the form $(h, w) \mapsto (\phi(h), w)$ for some W_E-equivariant map $\phi\colon \check{H} \to \check{G}$. With respect to the factorization $\check{G} \cong \prod_{\tau\in\Omega} \check{H}$, write

$$\phi = (\phi_\tau)_{\tau\in\Omega}.$$

On the cocharacter lattice $X_*(T)^\Omega$ of the product torus, the norm map $N\colon X_*(T)^\Omega \to X_*(T)$ is the sum of the coordinates, so by Lemma 6.5.2 the restriction of ϕ to $\check{T}_H$ is the diagonal. Thus each coordinate homomorphism $\phi_\tau\colon \check{H} \to \check{H}$ restricts to the identity on $\check{T}_H$, and is therefore conjugation by an element of $\check{T}_H(k)$.

We have established that $\phi = \mathrm{Ad}(t) \circ \Delta$ for some $t \in \check{T}_G(k)$. Since ϕ and Δ are W_E-equivariant and the centralizer of $\Delta(\check{H})$ is $Z(\check{G})$, the image $\bar{t}$ of t in $\check{T}_G/Z(\check{G})$ is fixed by W_E. Let $Q \subset X = X^*(\check{T}_G)$ be the root lattice and Γ the finite image of W_E in $\mathrm{Aut}(X)$. As Γ permutes the simple roots, the coinvariant group Q_Γ is free, so $Q_\Gamma \to X_\Gamma$ is injective (being injective rationally). Viewing $\bar{t}$ as a character of Q_Γ and using divisibility of $k^\times$, we extend it to X_Γ and obtain a W_E-fixed $g \in \check{T}_G(k)$ with image $\bar{t}$. Conjugating by g^{-1} gives (7.5.2).

□

8. Tate cohomology of moduli of local shtukas

8.1. **Moduli spaces of local shtukas.** The definitions of local shtukas in p-adic geometry are developed in Scholze's Berkeley Lectures on p-adic geometry, especially [SW20, Lecture XXIII]. Properties of the cohomology of their moduli spaces are established in [FS21, §IX.3]. We recall some relevant aspects here.

8.1.1. *Setup.* Recall that G is a reductive group over E. For each finite set I, $b \in B(G)$, and compact open subgroup $K < G(E)$, we have a moduli space of local shtukas $\mathrm{Sht}_{(G,b,I),K}$. By [SW20, Theorem 23.1.4], it is an inductive limit of locally spatial diamonds with finite cohomological dimension along a countable index set. Furthermore, there is an action of $G_b(E)$ on $\mathrm{Sht}_{(G,b,I),K}$.

The space $\mathrm{Sht}_{(G,b,I),K}$ comes equipped with "leg" maps

$$f_K\colon \mathrm{Sht}_{(G,b,I),K} \to \prod_{i\in I} \mathrm{Spd}\, \breve{E}$$

which are partially proper and ind-shriekable (cf. [FS21, §IX.3]), and a Grothendieck-Messing period map

$$\pi_K\colon \mathrm{Sht}_{(G,b,I),K} \to \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}},$$

which is étale [SW20, Proposition 23.3.3, Corollary 23.4.4].

8.1.2. *Satake sheaves.* As in [FS21, §IX], fix an unramified square root $\chi_{\mathrm{cyc}}^{1/2}\colon W_E \to k^\times$ of the cyclotomic character. It gives an isomorphism $\alpha_G\colon {}^LG \xrightarrow{\sim} {}^cG$ over W_E, identifying $\widehat{G}$ with $\check{G}$ and changing the Weil splitting to account for the cyclotomic twist in Geometric Satake. We use the same square root for H.

Let Q be a finite quotient of W_E through which both the action on $\widehat{G}$ and $\chi_{\mathrm{cyc}}^{1/2}$ factor. For any finite set I and $W \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, the Geometric Satake equivalence gives a relative perverse sheaf $\mathcal{S}_W$ on $\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}}$ (cf. [FS21, §I.9, §IX.3]) which we pull back via π_K^* to $\mathrm{Sht}_{(G,b,I),K}$, and we also denote the resulting complex by $\mathcal{S}_W \in D^b_{\text{ét}}(\mathrm{Sht}_{(G,b,I),K}; k)$.

8.1.3. *Cohomology of moduli spaces of local shtukas.* By [FS21, Corollary I.7.3, Proposition IX.3.2], we may regard

$$\mathrm{R}f_{K!}\mathcal{S}_W \in D(\mathrm{Rep}_k^{\mathrm{sm}}\, G_b(E))^{B\prod_{i\in I} W_E} \tag{8.1.1}$$

as a (derived) representation of $G_b(E)$ with a commuting action of $\prod_{i\in I} W_E$. To demystify this a bit: the $G_b(E)$-action on cohomology is induced by the $G_b(E)$-action on the space $\mathrm{Sht}_{(G,b,I),K}$, and the $\prod_{i\in I} W_E$-action comes from a natural descent (using an interpretation via Hecke operators) of $\mathrm{R}f_{K!}\mathcal{S}_W$ from $\prod_{i\in I}\mathrm{Spd}\,\breve{E}$ to $\prod_{i\in I}[\mathrm{Spd}\,\breve{E}/\varphi] \cong \prod_{i\in I}[\mathrm{Spd}\,C/W_E]$ where C is a completed algebraic closure of E. The isomorphism $[\mathrm{Spd}\,\breve{E}]/\varphi \cong [\mathrm{Spd}\,C/W_E]$ is on [FS21, p.164].

Example 8.1.1. For $W = \mathbb{1}$, the sheaf $\mathcal{S}_{\mathbb{1}}$ is supported on the trivial-modification locus, which is non-empty only if $b = 1_G$. In that case, $\mathrm{R}f_{K!}\mathcal{S}_{\mathbb{1}}$ is the constant local system on $\prod_{i\in I}\mathrm{Spd}\,\breve{E}$ with fiber $\text{c-Ind}_K^{G(E)}(k)$, equipped with the obvious $G(E)$-action and the trivial W_E-action.

The functor

$$\mathrm{Rep}_k((\widehat{G}\rtimes Q)^I) \to D(\mathrm{Rep}_k^{\mathrm{sm}}\, G_b(E))^{B\prod_{i\in I} W_E}$$

sending $W \mapsto \mathrm{R}f_{K!}\mathcal{S}_W$ satisfies the following *fusion compatibility*. Any map of finite non-empty sets $\zeta\colon I \to J$ induces a map $({}^LG)^J \to ({}^LG)^I$. Let $\mathrm{Res}_\zeta\colon \mathrm{Rep}_k(({}^LG)^I) \to \mathrm{Rep}_k(({}^LG)^J)$ be the restriction functor along this map. By construction, Geometric Satake is arranged so that the corresponding functor $\mathrm{Sat}_G^I(k) \to \mathrm{Sat}_G^J(k)$ is the fusion product. For surjective ζ, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sht}_{(G,b,J),K} & \longrightarrow & \mathrm{Sht}_{(G,b,I),K} \\
\downarrow{\scriptstyle \pi_K} & & \downarrow{\scriptstyle \pi_K} \\
\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{j\in J}\mathrm{Spd}\,\breve{E}} & \longrightarrow & \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}} \\
\downarrow & & \downarrow \\
\prod_{j\in J}\mathrm{Spd}\,\breve{E} & \longrightarrow & \prod_{i\in I}\mathrm{Spd}\,\breve{E}
\end{array}$$

The Geometric Satake equivalence is constructed in [FS21, §VI] so that one has $\mathcal{S}_W|_{\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{j\in J}\mathrm{Spd}\,\breve{E}}} \cong \mathcal{S}_{\mathrm{Res}_\zeta W}$, naturally in $W \in \mathrm{Rep}_k((\widehat{G}\rtimes Q)^I)$ and compatibly under compositions of finite sets. This induces a natural isomorphism

$$(\mathrm{R}f_{K!}\mathcal{S}_W)\,\big|_{\prod_{j\in J}\mathrm{Spd}\,\breve{E}} \cong \mathrm{R}f_{K!}\mathcal{S}_{\mathrm{Res}_\zeta(W)} \tag{8.1.2}$$

of functors $\mathrm{Rep}_k((\widehat{G}\rtimes Q)^I) \to D(\mathrm{Rep}_k^{\mathrm{sm}}\, G_b(E))^{B\prod_{j\in J} W_E}$. For arbitrary ζ, factor it as $I \twoheadrightarrow \zeta(I) \hookrightarrow J$ and use the pull-push construction of Remark 6.1.2, inserting trivial modifications at the unused legs. This gives (8.1.2) for all ζ. Moreover, the natural isomorphism (8.1.2) is compatible under compositions of maps of finite sets.

8.2. **Excursion algebra.** An *excursion datum* for $\widehat{G}$ (over k) [FS21, Definition VIII.4.2] is a tuple $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ where:

- I is a finite set and $\gamma_i \in W_E$ for each $i \in I$,
- $V \in \mathrm{Rep}_k((\widehat{G}\rtimes Q)^I)$ for varying Q, and $\mathbb{1} \xrightarrow{\alpha} V|_{\widehat{G}}$ and $V|_{\widehat{G}} \xrightarrow{\beta} \mathbb{1}$ are maps of $\widehat{G}$-representations. (Here $\mathbb{1}$ is the trivial representation.)

The *excursion algebra* $\mathrm{Exc}_k(W_E, \widehat{G})$ is the k-algebra on generators $S_{\mathcal{D}}$ indexed by excursion data $\mathcal{D}$, with relations as in [Fen24, §2.4]. Another definition appears in [FS21, Definition VIII.3.4].

For an excursion datum $\mathcal{D}$, let

$$f_{\mathcal{D}}((g_i)_i) = \beta \circ \rho_V((g_i)_i) \circ \alpha(1).$$

This function is invariant under left and right translation by the diagonal copy of $\widehat{G}$. In the presentation of [FS21, Definition VIII.3.4 and §VIII.4] by free groups, $S_{\mathcal{D}}$ is determined by the restriction of $f_{\mathcal{D}}$ to the component $\widehat{G}^I \cdot (\bar{\gamma}_i)_i$, where $\bar{\gamma}_i$ is the image of γ_i in Q.

An *L-parameter* (with coefficients in k) is a class in $\mathrm{H}^1(W_E; \widehat{G}(k))$, or equivalently a section $W_E \to \widehat{G}(k) \rtimes W_E$ up to $\widehat{G}(k)$-conjugation. An L-parameter is *semisimple* if whenever it factors through a parabolic ${}^L P(k) \subset {}^L G(k)$, it also factors through a Levi ${}^L M(k) \subset {}^L P(k)$ [FS21, Definition VIII.3.1].

Combining [FS21, Proposition VIII.3.8] and the statement above [FS21, Definition VIII.3.4] that the natural map from $\mathrm{Exc}(W_E, \widehat{G})$ to the *spectral Bernstein center* $\mathcal{O}(Z^1(W_E, \widehat{G}))^{\widehat{G}}$ is a universal homeomorphism, we obtain a canonical bijection between

$$\{\text{characters } \mathrm{Exc}_k(W_E, \widehat{G}) \to k\} \longleftrightarrow \{\text{semisimple } L\text{-parameters } \rho \in \mathrm{H}^1(W_E; \widehat{G}(k))\}.$$

Lemma 8.2.1. *Let ${}^L\psi\colon {}^L H \to {}^L G$ be an L-homomorphism, and write ${}^L\psi(h, w) = (\phi(h)z(w), w)$. Composition with ${}^L\psi$ induces a k-algebra homomorphism*

$$ {}^L\psi^*\colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Exc}_k(W_E, \widehat{H}), \tag{8.2.1}$$

which depends only on the $\widehat{G}(k)$-conjugacy class of ${}^L\psi$. For an excursion datum $\mathcal{D}$, the element ${}^L\psi^* S_{\mathcal{D}}$ is represented on the component selected by $(\gamma_i)_i$ by the function

$$(h_i)_i \mapsto f_{\mathcal{D}}\big((\phi(h_i)z(\gamma_i), \bar{\gamma}_i)_i\big),$$

and if ${}^L\psi^* V$ has finite Weil image then ${}^L\psi^* S_{\mathcal{D}} = S_{{}^L\psi^*\mathcal{D}}$, where ${}^L\psi^*\mathcal{D} = (I, {}^L\psi^* V, {}^L\psi^*\alpha, {}^L\psi^*\beta, (\gamma_i)_i)$. On k-points, ${}^L\psi^*$ sends (the point corresponding to) a semisimple L-parameter $\rho \in \mathrm{H}^1(W_E; \widehat{H}(k))$ to (the point corresponding to) the semisimple L-parameter $({}^L\psi \circ \rho)^{\mathrm{ss}} \in \mathrm{H}^1(W_E; \widehat{G}(k))$.

Proof. For each map $F_n \to W_E$ from a free group, composition with ${}^L\psi$ defines a ϕ-equivariant morphism $Z^1(F_n, \widehat{H}) \to Z^1(F_n, \widehat{G})$, compatible with maps of free groups and with passage to the quotients by open subgroups of wild inertia as in [FS21, §VIII.3.2]. Pulling back invariant functions gives (8.2.1) by [FS21, Definition VIII.3.4]. Conjugating ${}^L\psi$ by $\widehat{G}(k)$ does not change the pullback of invariant functions. The remaining assertions follow by evaluating on cocycles. □

Remark 8.2.2. Strictly speaking, the excursion operators of [FS21] are only defined for data where the W_E-action on V factors through a finite quotient, while ${}^L\psi^* V$ need not have this property, since the Weil component z of ${}^L\psi$ may have infinite image. We explain that the same formula in Lemma 8.2.1 computes the excursion operators attached to ${}^L\psi^*\mathcal{D}$ when the W_E^I-action on ${}^L\psi^* V$ is only continuous. Let U be a finite-dimensional continuous representation of $({}^L H)^I$. Inertia acts on U through a finite quotient, so there are powers of Frobenius in each factor which centralize the image of inertia and act trivially on $\widehat{H}^I$. Their simultaneous generalized eigenspaces are subrepresentations, and on each of them an unramified twist makes these powers unipotent, hence of finite order since k has characteristic ℓ. Thus U is a finite direct sum of unramified twists of representations with finite Weil image. Twisting by unramified characters $(\chi_i)_i$ multiplies both the excursion operator and the function $f_{\mathcal{D}}$ by $\prod_i \chi_i(\gamma_i)$, so by additivity the formula reduces to the case where W_E has finite image. In particular, the excursion operator attached to ${}^L\psi^*\mathcal{D}$ is $\mathrm{FS}_H({}^L\psi^* S_{\mathcal{D}})$.

8.3. **The Bernstein center.** Recall that in (6.2.1) we defined the Hecke algebra $\mathscr{H}(G, K; \Lambda)$ for a compact open subgroup $K \subset G(E)$. We let $\mathfrak{Z}(G, K; \Lambda) := Z(\mathscr{H}(G, K; \Lambda))$ be the center of $\mathscr{H}(G, K; \Lambda)$.

If $K \subset K'$ have prime-to-ℓ pro-order (e.g., this will be true as long as they are sufficiently small), then convolution with the idempotent $[K' : K]^{-1} \mathbb{1}_{K'}$ gives a homomorphism $\mathfrak{Z}(G, K; \Lambda) \to \mathfrak{Z}(G, K'; \Lambda)$. The *Bernstein center of G (with coefficients in Λ)* is

$$\mathfrak{Z}(G; \Lambda) := \varprojlim_K \mathfrak{Z}(G, K; \Lambda),$$

where the transition maps are as above, and the inverse limit is taken over K with prime-to-ℓ pro-order.

If $\Lambda = k$, we abbreviate $\mathscr{H}(G, K) := \mathscr{H}(G, K; k)$, $\mathfrak{Z}(G, K) := \mathfrak{Z}(G, K; k)$, and $\mathfrak{Z}(G) := \mathfrak{Z}(G; k)$.

The Bernstein center $\mathfrak{Z}(G)$ may also be identified with the ring of endomorphisms of the identity functor of the category $\mathrm{Rep}_k^{\mathrm{sm}}(G(E))$ (cf. [FS21, Definition I.9.2]). In particular, any irreducible smooth representation Π of $G(E)$ over k induces a character of $\mathfrak{Z}(G)$.

8.4. **Review of Fargues-Scholze correspondence.** Fargues-Scholze construct in [FS21] a k-algebra homomorphism

$$\mathrm{FS}_G\colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathfrak{Z}(G;k). \tag{8.4.1}$$

Via (8.4.1), $\mathrm{Exc}_k(W_E,\widehat{G})$ acts through a character on any irreducible admissible representation $\Pi \in \mathrm{Rep}_k^{\mathrm{sm}}(G(E))$, and the corresponding semisimple L-parameter is denoted $\rho_\Pi \in \mathrm{H}^1(W_E;\widehat{G}(k))$.

We present the construction of (8.4.1) on [FS21, p.36]. Fix $b := 1_G \in B(G)$. Let

$$\overline{x}\colon \mathrm{Spd}\,\widehat{\overline{E}} \to \prod_{i\in I} \mathrm{Spd}\,\breve{E}$$

be the geometric diagonal, and $f_K^\Delta\colon \mathrm{Sht}^\Delta_{(G,1_G,I),K} \to \mathrm{Spd}\,\widehat{\overline{E}}$ be the base change of f_K along $\overline{x}$.

Let $K \subset G(E)$ be a compact open subgroup. Recall from Example 8.1.1 that the underlying $G(E)$-representation of $\mathrm{R}f^\Delta_{K!}\mathcal{S}_{\mathbb{1}}$ is $\text{c-Ind}_K^{G(E)}\, k$. Hence for each excursion datum $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ we have a composition

$$\begin{array}{ccccccccccc}
\text{c-Ind}_K^{G(E)}\, k & & \xrightarrow{\qquad\qquad\qquad\qquad \mathrm{FS}_G(S_\mathcal{D}) \qquad\qquad\qquad\qquad} & & & & & & & & \text{c-Ind}_K^{G(E)}\, k \\
\| & & & & & & & & & & \| \\
\mathrm{R}f^\Delta_{K!}\mathcal{S}_{\mathbb{1}} & \xrightarrow{\alpha} & \mathrm{R}f^\Delta_{K!}\mathcal{S}_{V|_{\widehat{G}}} & = & (\mathrm{R}f_{K!}\mathcal{S}_V)_{\overline{x}} & \xrightarrow{(\gamma_i)_{i\in I}} & (\mathrm{R}f_{K!}\mathcal{S}_V)_{\overline{x}} & = & \mathrm{R}f^\Delta_{K!}\mathcal{S}_{V|_{\widehat{G}}} & \xrightarrow{\beta} & \mathrm{R}f^\Delta_{K!}\mathcal{S}_{\mathbb{1}}
\end{array} \tag{8.4.2}$$

(in the middle step we used (8.1.1) to obtain an action of $\prod_{i\in I} W_E$ on $(\mathrm{R}f_{K!}\mathcal{S}_V)_{\overline{x}}$) which defines an element

$$\mathrm{FS}_G(S_\mathcal{D}) \in \mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)}\, k) \cong \mathscr{H}(G,K).$$

By [FS21, Proposition I.9.3], excursion operators commute with Hecke operators, so this element lies in $\mathfrak{Z}(G,K)$. As K varies, these elements are compatible under convolution, hence assemble to an element of $\mathfrak{Z}(G)$. The map (8.4.1) sends $S_\mathcal{D}$ to this element of $\mathfrak{Z}(G)$.

8.5. **Excursion operators on Tate cohomology.** Recall the notion of *Tate cohomology* from §4.1. We will now study excursion operators on the Tate cohomology of $\mathrm{Sht}_{(G,1_G,I),K}$. We will distinguish two actions of excursion operators on Tate cohomology. The first is the native action, obtained by applying Tate cohomology to the usual excursion action. The second is the normed action, obtained by first passing from the excursion algebra to its Tate cohomology via the Tate diagonal. The comparison between these two actions is controlled by Frobenius linearization, and this will be the source of the Frobenius twists appearing later.

8.5.1. *Tate cohomology of smooth representations.* Let Σ act on $G(E)$ through its action on G, so that $G(E)^\sigma = H(E)$. The formula (4.1.3) applies verbatim to $\mathcal{K} \in D^b(\mathrm{Rep}^{\mathrm{sm}}_\Lambda G(E))^{B\Sigma} \cong D^b(\mathrm{Rep}^{\mathrm{sm}}_\Lambda(G(E) \rtimes \Sigma))$:

$$\mathrm{T}^n(\mathcal{K}) = \varinjlim_{j\to\infty} \mathrm{Hom}_{D^+(\Lambda[\Sigma])}\big(\Lambda, \mathcal{K}[n+2j]\big), \tag{8.5.1}$$

where on the right $\mathcal{K}$ denotes the underlying complex of $\Lambda[\Sigma]$-modules. Computing with the explicit resolutions of Example 4.2.1, each term of the resulting complex is a finite direct sum of terms of $\mathcal{K}$, and the differentials and transition maps are built from $1-\sigma$ and N. These commute with $H(E)$, so (8.5.1) is naturally a smooth representation of $H(E)$, and we obtain functors

$$\mathrm{T}^n\colon D^b(\mathrm{Rep}^{\mathrm{sm}}_\Lambda G(E))^{B\Sigma} \to \mathrm{Rep}^{\mathrm{sm}}_\Lambda H(E).$$

The same description shows that T^n commutes with direct sums, and with filtered colimits, of complexes concentrated in a fixed finite range of degrees.

8.5.2. *Tate cohomology of moduli of shtukas.* In this subsection we take $b = 1_G$ and assume that $K \subset G(E)$ is Σ-invariant. The canonical Σ-equivariant structure on the trivial G-bundle induces a Σ-action on $\mathrm{Sht}_{(G,1_G,I),K}$.

The given action of Σ on G induces an action $V \mapsto {}^{\sigma}V$ of Σ on $\mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$. Suppose we have a Σ-equivariant representation $W \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$. Then $\mathcal{S}_W$ has the structure of a Σ-equivariant sheaf on $\mathrm{Sht}_{(G,1_G,I),K}$. This equips its cohomology with a Σ-equivariant structure, so that we may regard, using (8.1.1),

$$\mathrm{R}f_{K!}\mathcal{S}_W \in D(\mathrm{Rep}_k^{\mathrm{sm}} G(E))^{B(\prod_{i\in I} W_E \times \Sigma)}. \tag{8.5.2}$$

Remark 8.5.1. This comes from the following more general principle: let Γ and Γ' be groups, and let $\mathcal{C}$ be an ∞-category with a Γ-action, classified by a functor

$$\mathcal{F}\colon B\Gamma \to \mathsf{Cat}_\infty.$$

Suppose $\mathcal{F}$ has a Γ'-equivariant structure; more informally, Γ' acts on Γ and on $\mathcal{C}$ in a compatible way. Equivalently, the Γ-action and the Γ'-action combine to an action of the semidirect product $\Gamma \rtimes \Gamma'$ on $\mathcal{C}$, i.e., a functor

$$\widetilde{\mathcal{F}}\colon B(\Gamma \rtimes \Gamma') \to \mathsf{Cat}_\infty.$$

Then there is a canonical equivalence

$$(\mathcal{C}^{B\Gamma})^{B\Gamma'} \cong \varprojlim_{B\Gamma'}(\varprojlim_{B\Gamma} \mathcal{F}) \cong \varprojlim_{B(\Gamma\rtimes\Gamma')} \widetilde{\mathcal{F}}.$$

Thanks to (8.5.2), we can form the jth Tate cohomology of $\mathrm{R}f_{K!}\mathcal{S}_W$ in the sense of (8.5.1) (for $j \in \mathbf{Z}/2\mathbf{Z}$), which we denote

$$\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; W) := \mathrm{T}^j(\mathrm{R}f_{K!}\mathcal{S}_W) \in (\mathrm{Rep}_k^{\mathrm{sm}} H(E))^{B\prod_{i\in I} W_E}, \tag{8.5.3}$$

as we will want to record the dependence on G, I.

Example 8.5.2. For the trivial representation $W = \mathbb{1}$, Example 8.1.1 gives

$$\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; \mathbb{1}) \cong \mathrm{T}^j(\text{c-Ind}_K^{G(E)} k) \in \mathrm{Rep}_k^{\mathrm{sm}}(H(E))$$

with the trivial $\prod_{i\in I} W_E$-action.

A similar story applies to H, with $b = 1_H$ and $K \subset H(E)$. Note that Σ acts trivially on $\mathrm{Sht}_{(H,1_H,I),K}$, so if Σ also acts trivially on $W \in \mathrm{Rep}_k((\widehat{H} \rtimes Q)^I)$, then by Example 4.5.2 and Example 4.5.1, we have for each $j \in \mathbf{Z}/2\mathbf{Z}$ a natural isomorphism

$$\mathrm{T}^j(\mathrm{Sht}_{(H,1_H,I),K}; W) \cong \bigoplus_{n\in\mathbf{Z}} \mathrm{H}^n(\mathrm{R}f_{K!}\mathcal{S}_W) = \mathrm{H}^*(\mathrm{R}f_{K!}\mathcal{S}_W). \tag{8.5.4}$$

8.5.3. *Excursion action.* The Σ action on G induces a Σ-action on $\mathrm{Exc}_k(W_E, \widehat{G})$ by transport of structure. Concretely, Σ acts on excursion data by

$$\sigma \cdot (I, V, \alpha, \beta, (\gamma_i)_{i\in I}) = (I, {}^{\sigma}V, \sigma(\alpha), \sigma(\beta), (\gamma_i)_{i\in I})$$

and then $\sigma \cdot S_{\mathcal{D}} = S_{\sigma\cdot\mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})$.

In general, given a $k[\Sigma]$-algebra A and a Σ-equivariant A-module M with compatible actions, there is a natural $\mathrm{T}^0(A) = A^\sigma/(N\cdot A)$-module structure on $\mathrm{T}^*(M)$. In particular, this equips $\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; W)$ with a natural $\mathrm{T}^0(\mathrm{Exc}_k(W_E, \widehat{G}))$-action. We are most interested in the special case where $W = \mathbb{1}$, in which case we have $\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; \mathbb{1}) = \mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$ at the level of underlying $H(E)$-representations (cf. Example 8.5.2). If an excursion datum $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ is Σ-invariant, then $S_{\mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})^\sigma$

and its action on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)}\, k)$ can be described more concretely using (8.4.2): it is given by composition

$$\begin{array}{ccccccc}
\mathrm{T}^j(\text{c-Ind}_K^{G(E)}\, k) & & \xrightarrow{\qquad\qquad S_{\mathcal{D}}\qquad\qquad} & & & & \mathrm{T}^j(\text{c-Ind}_K^{G(E)}\, k) \\
\| \text{Ex. 8.5.2} & & & & & & \| \\
\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K};\mathbb{1}) & \xrightarrow{\alpha} & \mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K};V) & \xrightarrow{(\gamma_i)_{i\in I}} & \mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K};V) & \xrightarrow{\beta} & \mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K};\mathbb{1})
\end{array} \tag{8.5.5}$$

8.5.4. *Interaction with Frobenius.* Equip $\mathrm{Exc}_k(W_E,\widehat{G})$ with the $\mathbf{F}_\ell$-structure transported from the c-group via the isomorphism α_G of §8.1.2. Let

$$\Phi_G := \alpha_G^{-1}\circ \mathrm{Fr}_\ell \circ \alpha_G \colon {}^L G \longrightarrow {}^L G, \qquad F_G := \Phi_G^* \colon \mathrm{Exc}_k(W_E,\widehat{G}) \longrightarrow \mathrm{Exc}_k(W_E,\widehat{G}), \tag{8.5.6}$$

where $\mathrm{Fr}_\ell \colon {}^c G \to {}^c G$ is Frobenius on $\check{G}$ and the identity on W_E. Thus F_G is the linearized Frobenius for this transported $\mathbf{F}_\ell$-structure (§6.2.2).

Lemma 8.5.3. *Let Π be an irreducible smooth representation of $G(E)$, and let*

$$\chi_\Pi \colon \mathrm{Exc}_k(W_E,\widehat{G}) \to k$$

be the corresponding character. Then we have $\chi_{\Pi^{(\ell)}} = \chi_\Pi \circ F_G$.

Proof. More generally, let A_0 be an $\mathbf{F}_\ell$-algebra and $A := A_0 \otimes_{\mathbf{F}_\ell} k$. Then the Frobenius on A, given by $\mathrm{Frob}_A(a) = a^\ell$, factors as the composition of the linearized Frobenius $F_A = (\mathrm{Frob}_\ell)_{A_0} \otimes \mathrm{Id}$ and $\varphi_A = \mathrm{Id}_{A_0} \otimes (\mathrm{Frob}_\ell)_k$.

For an A-module M, its Frobenius twist $M^{(\ell)} := M \otimes_{k,\mathrm{Frob}} k$ has the natural A-module structure $M^{(\ell)} \cong (\varphi_A^{-1})_* M$. Since Frob_A does not change supports, we have

$$\mathrm{supp}_A(M^{(\ell)}) = \mathrm{supp}_A((\varphi_A^{-1})_* M) = \mathrm{supp}_A((\mathrm{Frob}_A \circ \varphi_A^{-1})_* M) = \mathrm{supp}_A(F_{A*} M)$$

In particular, if $M \cong k$ corresponds to the character $\chi \colon A \to k$, then $M^{(\ell)}$ corresponds to the character

$$a \mapsto \mathrm{Frob}_\ell\big(\chi_M(\varphi_A^{-1}(a))\big) = \chi(F_A(a)). \tag{8.5.7}$$

The proof is completed by applying this to $A = \mathrm{Exc}_k(W_E,\widehat{G})$ and $\chi = \chi_\Pi$. □

Remark 8.5.4. In the usual pinned L-group coordinates, (8.5.6) is

$$\Phi_G(g,w) = \big(z_G(w)\,\mathrm{Fr}_\ell(g), w\big), \qquad z_G(w) = (2\rho_{\widehat{G}}^\vee)\left(\frac{\chi_{\mathrm{cyc}}^{1/2}(w)^\ell}{\chi_{\mathrm{cyc}}^{1/2}(w)}\right). \tag{8.5.8}$$

Here Fr_ℓ on $\widehat{G}$ uses its pinned $\mathbf{F}_\ell$-structure, $\rho_{\widehat{G}}^\vee$ is the half-sum of positive coroots, and z_G is an unramified cocycle valued in $Z(\widehat{G}_{\mathrm{der}})(k)^{W_E}$. Thus Lemma 8.5.3 says $\rho_{\Pi^{(\ell)}} \sim z_G \cdot \mathrm{Fr}_\ell \circ \rho_\Pi$ in these coordinates.

By construction, the Brauer functor $\widetilde{br}^I$ is defined with $\mathbf{F}_\ell$-coefficients. Consequently ${}^L\psi^*$ respects the $\mathbf{F}_\ell$-structures transported from the c-groups. Defining F_H as in (8.5.6), we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Exc}_k(W_E,\widehat{G}) & \xrightarrow{F_G} & \mathrm{Exc}_k(W_E,\widehat{G}) \\
\big\downarrow{\scriptstyle {}^L\psi^*} & & \big\downarrow{\scriptstyle {}^L\psi^*} \\
\mathrm{Exc}_k(W_E,\widehat{H}) & \xrightarrow{F_H} & \mathrm{Exc}_k(W_E,\widehat{H})
\end{array} \tag{8.5.9}$$

8.5.5. *Normed excursion action.* Recall from Definition 6.2.1 that for any commutative $k[\Sigma]$-algebra A, there is the *Tate diagonal* morphism

$$A \xrightarrow{\mathrm{Nm}} \mathrm{T}^0(A)$$

sending a to the class of $\mathrm{Nm}(a) = a \cdot \sigma(a) \cdots \sigma^{\ell-1}(a)$. This is Frob-semilinear, but an $\mathbf{F}_\ell[\Sigma]$-structure on A induces a linearization $\mathrm{Nm}^{(\ell^{-1})} \colon A \to \mathrm{T}^0(A)$, which is a k-algebra homomorphism.

We apply this to $A := \mathrm{Exc}_k(W_E, \widehat{G})$, with the transported $\mathbf{F}_\ell$-structure of §8.5.4. In §8.5.3 we saw an action of $\mathrm{T}^0(\mathrm{Exc}_k(W_E, \widehat{G}))$ on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$. Inflating this action through $\mathrm{Nm}^{(\ell^{-1})}$ gives an action of $\mathrm{Exc}_k(W_E, \widehat{G})$ on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$, which we call the *normed excursion action.*

8.5.6. *Native excursion action.* The Σ-invariant subalgebra $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ acts naturally on $\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; V)$ via the quotient $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma \twoheadrightarrow \mathrm{T}^0(\mathrm{Exc}_k(W_E, \widehat{G}))$ and then the mechanism of §8.5.3. We refer to this as the *native excursion action* of $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ on $\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; V)$. This action is *not* the same as the restriction of the normed excursion action to the subalgebra $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma \subset \mathrm{Exc}_k(W_E, \widehat{G})$; this discrepancy will be responsible for Frobenius twists which show up later.

In general, for a commutative k-algebra A the absolute Frobenius $A \to A$ is Frob-semilinear. If A has an $\mathbf{F}_\ell$-structure $A \cong A_0 \otimes_{\mathbf{F}_\ell} k$, then the absolute Frobenius may be linearized to a map $F \colon A \to A$, as explained in §6.2.2. The map F is characterized as the unique k-linear homomorphism that sends $a_0 \mapsto a_0^\ell$ for all $a_0 \in A_0 \subset A$.

The preceding discussion applies to $A := \mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ (with the $\mathbf{F}_\ell$-structure of §8.5.4). In these terms, the normed excursion action of $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ is the native excursion action composed with the map $F \colon \mathrm{Exc}_k(W_E, \widehat{G})^\sigma \to \mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ which is the linearization of the absolute Frobenius.

8.5.7. *Norm of excursion data.* For a finite quotient Q of W_E through which both the action on $\widehat{G}$ and $\chi_{\mathrm{cyc}}^{1/2}$ factor, we define a functor

$$\mathrm{Nm} \colon \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I) \to \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$

as follows:

- For a representation $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we set
$$\mathrm{Nm}(V) := V \otimes_k ({}^\sigma V) \otimes_k \ldots \otimes_k \left({}^{\sigma^{\ell-1}} V\right) \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$
with the obvious Σ-equivariant structure. Under Geometric Satake this corresponds to the norm (6.3.2).
- Given $h \colon V \to V' \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we set
$$\mathrm{Nm}(h) := h \otimes {}^\sigma h \otimes \ldots \otimes {}^{\sigma^{\ell-1}} h \colon \mathrm{Nm}(V) \to \mathrm{Nm}(V').$$

Note that Nm is *not* an additive functor, nor is it even k-linear.

Definition 8.5.5. The *linearized norm* is $\mathrm{Nm}^{(\ell^{-1})}(V) := \mathrm{Nm}(V^{(\ell^{-1})})$, where $(-)^{(\ell^{-1})}$ is the inverse coefficient twist on c-group representations, transported to L-group representations via α_G. Thus $\mathcal{S}_{V^{(\ell^{-1})}} \cong (\mathcal{S}_V)^{(\ell^{-1})}$, with the right-hand side defined by (6.3.1). It satisfies $\mathrm{Nm}^{(\ell^{-1})}(ah) = a\,\mathrm{Nm}^{(\ell^{-1})}(h)$ for $a \in k$ and morphisms h, but it is still not additive; compare Proposition 6.4.2.

Definition 8.5.6. For $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we denote by

$$N \cdot V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$

the σ-equivariant representation $V \oplus {}^\sigma V \oplus \ldots \oplus {}^{\sigma^{\ell-1}} V$, with the obvious Σ-equivariant structure.

For $h \colon V \to V' \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we denote by

$$N \cdot h \colon N \cdot V \to N \cdot V'$$

the σ-equivariant map $h \oplus {}^\sigma h \oplus \ldots \oplus {}^{\sigma^{\ell-1}} h$. Let $\Delta_\ell \colon \mathbb{1} \to \mathbb{1}^{\oplus \ell}$ denote the diagonal map and $\nabla_\ell \colon \mathbb{1}^{\oplus \ell} \to \mathbb{1}$ denote the sum map.

Definition 8.5.7. Let $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ be an excursion datum for $\widehat{G}$. Define the excursion data

$$\mathrm{Nm}^{(\ell^{-1})}\,\mathcal{D} := (I, \mathrm{Nm}^{(\ell^{-1})}\,V, \mathrm{Nm}^{(\ell^{-1})}\,\alpha, \mathrm{Nm}^{(\ell^{-1})}\,\beta, (\gamma_i)_{i\in I})$$

and

$$N\cdot\mathcal{D} := (I, N\cdot V, (N\cdot\alpha)\circ\Delta_\ell, \nabla_\ell\circ(N\cdot\beta), (\gamma_i)_{i\in I}),$$

which are excursion data for $\widehat{G}$.

Straightforward calculations yield:

Lemma 8.5.8 ([Fen24, Lemma 5.16])**.** *For all excursion data $\mathcal{D}$, we have*

$$\mathrm{Nm}^{(\ell^{-1})}(S_{\mathcal{D}}) = [S_{\mathrm{Nm}^{(\ell^{-1})}(\mathcal{D})}] \in \mathrm{T}^0\big(\mathrm{Exc}_k(W_E, \widehat{G})\big)$$

and

$$N\cdot S_{\mathcal{D}} = S_{N\cdot\mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})^\sigma.$$

Proof. The second identity appears in [Fen24, Lemma 5.16], which also proves that

$$\mathrm{Nm}(S_{\mathcal{D}}) = [S_{\mathrm{Nm}(\mathcal{D})}] \in \mathrm{T}^0\big(\mathrm{Exc}_k(W_E, \widehat{G})\big)$$

from which the first identity follows. □

8.6. **Equivariant localization for local shtukas.** We use the square root of the cyclotomic character fixed in §8.1, for both G and H, to transport ${}^c\psi$ to ${}^L\psi$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_{(H,1_H,I),\iota^*K} & \xrightarrow{\iota} & \mathrm{Sht}_{(G,1_G,I),K} \\ {\scriptstyle f^H_{\iota^*K}}\downarrow & & \downarrow{\scriptstyle f^G_K} \\ \prod_{i\in I}\mathrm{Spd}\,\breve{E} & = & \prod_{i\in I}\mathrm{Spd}\,\breve{E} \end{array} \tag{8.6.1}$$

of shtukas, where ι is the embedding of Proposition 5.4.6. The period maps fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_{(H,1_H,I),\iota^*K} & \xrightarrow{\iota} & \mathrm{Sht}_{(G,1_G,I),K} \\ {\scriptstyle \pi^H_{\iota^*K}}\downarrow & & \downarrow{\scriptstyle \pi^G_K} \\ \mathrm{Gr}^{\mathrm{tw}}_{H,\prod_{i\in I}\mathrm{Spd}\,\breve{E}} & \xrightarrow{\iota} & \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}} \end{array} \tag{8.6.2}$$

All sheaves have coefficients in k.

Proposition 8.6.1. *Let I be a nonempty finite set, $K\subset G(E)$ a Σ-stable compact open subgroup of prime-to-ℓ pro-order, and $W\in\mathrm{Rep}_k((\widehat{G}\rtimes Q)^I)^{B\Sigma}$. There is a natural $H(E)$-equivariant isomorphism*

$$\big((\mathrm{R}f^G_{K!}\mathcal{S}_W)|_{H(E)}\big)^{t\Sigma} \xrightarrow{\sim} \mathrm{R}f^H_{\iota^*K!}(\pi^H_{\iota^*K})^*\big((\iota^*\mathcal{S}_W)^{t\Sigma}\big), \tag{8.6.3}$$

compatible with base change in $\prod_{i\in I}\mathrm{Spd}\,\breve{E}$, with change of level, and with the fusion isomorphisms (8.1.2)*.*

Proof. Choose a Σ-stable closed subspace $Z\subset \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}}$, a finite union of Schubert varieties, containing the support of $\mathcal{S}_W$, and put

$$\mathrm{Sht}^G_Z := (\pi^G_K)^{-1}(Z)\subset \mathrm{Sht}_{(G,1_G,I),K}, \qquad \mathrm{Sht}^H_Z := (\pi^H_{\iota^*K})^{-1}(\iota^{-1}Z)\subset \mathrm{Sht}_{(H,1_H,I),\iota^*K}\,.$$

The restriction of the leg map f^G_K to Sht^G_Z is separated and shriekable with finite dim. trg , and $\mathrm{R}f^G_{K!}\mathcal{S}_W$ is computed on Sht^G_Z. By Proposition 5.4.6, ι identifies $\mathrm{Sht}_{(H,1_H,I),\iota^*K}$ with the Σ-fixed locus of $\mathrm{Sht}_{(G,1_G,I),K}$, so by the commutative square (8.6.2) it identifies Sht^H_Z with the Σ-fixed locus of Sht^G_Z. By the same commutative square again, the restriction of $\mathcal{S}_W = (\pi^G_K)^*\mathcal{S}_W$ to this fixed locus is $(\pi^H_{\iota^*K})^*\iota^*\mathcal{S}_W$. Now the desired comparison (8.6.3) follows from Proposition 4.3.1 (equivariant localization for Tate cohomology), together with Example 4.2.3(1) for $(\pi^H_{\iota^*K})^*$ and Remark 4.3.3 for the $H(E)$-action. The compatibilities follow from base change for $\mathrm{R}f_!$. □

The isomorphism (8.6.3) is only asserted to be $H(E)$-equivariant. We also want to compare the W_E^I-actions. The LHS has a natural W_E^I-action, thanks to (8.1.1). By transport, this induces a W_E^I-action on the RHS which we call the *transported action*.

The RHS of (8.6.3) is the cohomology of $\mathrm{Sht}_{(H,1_H,I),\iota^*K}$ with coefficients $(\iota^*\mathcal{S}_W)^{t\Sigma}$. This is not a Satake sheaf, so (8.1.1) does not give it an a priori action of W_E^I. However, if $W = \mathrm{Nm}^{(\ell^{-1})} V$ for some $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, then we will equip said RHS with a "perverse filtration" whose associated graded has a natural W_E^I-action, which we will call the *native action*. On $\mathrm{Sht}_{(H,1_H,I),\iota^*K}$ the coefficient complex $(\iota^*\mathcal{S}_W)^{t\Sigma}$ agrees with $A_I(\mathcal{S}_V)$.[6] By Proposition 6.4.3 and Theorem 7.2.2(1),

$$ {}^p\mathrm{H}^b(A_I(\mathcal{S}_V)) \cong \widetilde{br}^I(\mathcal{S}_V) \cong \mathcal{S}_{\mathcal{L}_{\psi^*V}} \quad \text{for all } b \in \mathbf{Z}. $$

So the perverse truncations of $A_I(\mathcal{S}_V)$ induce a filtration of the right side of (8.6.3) whose graded pieces $\mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{\mathcal{L}_{\psi^*V}}[-b]$ carry an action of W_E^I from (8.1.1) for H.

The following Proposition establishes the compatibility of these two W_E^I-actions.

Proposition 8.6.2. *Let K be as in Proposition 8.6.1, let $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, and let $W := \mathrm{Nm}^{(\ell^{-1})} V$. The filtration of the RHS of* (8.6.3) *induced by the perverse truncations ${}^p\tau^{\le b}A_I(\mathcal{S}_V)$ is stable under the transported action of W_E^I, and on its graded pieces the transported action agrees with the native action on $\mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{\mathcal{L}_{\psi^*V}}[-b]$. The filtration is functorial in V and compatible with base change and with coalescence of legs.*

The rest of this subsection proves Proposition 8.6.2. We use the following abstract observation.

Lemma 8.6.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $J\colon \mathcal{E} \to \mathcal{D}$ be exact, colimit-preserving functors of stable ∞-categories, with J fully faithful and $\mathcal{E}$ admitting sequential colimits. Suppose $\mathcal{C}$ carries a t-structure such that $A = \varinjlim_n \tau^{\le n}A$ for every bounded-below $A \in \mathcal{C}$.[7] If $F(A)$ and $F(\mathrm{H}^n A)$ lie in the essential image of J for all $n \in \mathbf{Z}$, then so do the $F(\tau^{\le b}A)$ for all $b \in \mathbf{Z}$, and the maps*

$$ F(\tau^{\le b}A) \to F(\tau^{\le b+1}A), \quad F(\tau^{\le b}A) \to F(A), \quad F(\tau^{\le b}A) \to F(\mathrm{H}^b A[-b]) $$

descend uniquely through J.

Proof. Each canonical truncation $\tau^{\le n}\tau^{\ge b}A$ is a finite extension of shifts of the $\mathrm{H}^r(A)$, so $F(\tau^{\le n}\tau^{\ge b}A)$ lies in the essential image of J, which is a stable subcategory closed under colimits. Hence $F(\tau^{\ge b}A) = \varinjlim_n F(\tau^{\le n}\tau^{\ge b}A)$ lies in the essential image of J for every b, and therefore so does $F(\tau^{\le b}A) = \mathrm{fib}(F(A) \to F(\tau^{\ge b+1}A))$. The assertions about the maps follow from full faithfulness of J. □

To apply Lemma 8.6.3, we use the description of cohomology of local shtukas in terms of Hecke operators on Bun_G, from the proof of [FS21, Proposition IX.3.2]:

$$ \mathrm{R}f^G_{K!}\mathcal{S}_W \cong j_G^* T_W\big(j_{G!}\,\mathrm{c\text{-}Ind}_K^{G(E)}\, k\big)\Big|_{\prod_{i\in I}\mathrm{Spd}\,\breve{E}}, \tag{8.6.4} $$

where $j_G\colon \mathrm{Bun}_G^1 \cong [*/G(E)] \hookrightarrow \mathrm{Bun}_G$ is the inclusion of the stratum of the trivial bundle, so that $j_{G!}\,\mathrm{c\text{-}Ind}_K^{G(E)}\, k \in D_{\text{ét}}(\mathrm{Bun}_G; k)$ via the equivalence $D(\mathrm{Rep}_k^{\mathrm{sm}}\, G(E)) \cong D_{\text{ét}}([*/G(E)]; k)$ [FS21, Theorem V.1.1], and T_W is the Hecke operator of [FS21, §IX.2]. In terms of the Hecke correspondence

$$ \mathrm{Bun}_G \xleftarrow{h_1} \mathrm{Hck}_G^I \xrightarrow{h_2} \mathrm{Bun}_G \times (\mathrm{Div}_X^1)^I, \tag{8.6.5} $$

it is the functor

$$ \begin{aligned} T_W\colon D_{\text{ét}}(\mathrm{Bun}_G; k) &\to D_{\text{ét}}(\mathrm{Bun}_G \times (\mathrm{Div}_X^1)^I; k), \\ A &\mapsto \mathrm{R}h_{2!}\big(h_1^* A \overset{\mathrm{L}}{\otimes}_k \mathcal{S}_W|_{\mathrm{Hck}_G^I}\big), \end{aligned} $$

[6] The shift $[\delta]$ vanishes (see the proof of Proposition 8.6.2).

[7] This holds, for example, if the t-structure on $\mathcal{C}$ is right-complete.

where $\mathcal{S}_W|_{\mathrm{Hck}^I_G}$ is the pullback of $\mathcal{S}_W$ from the local Hecke stack; h_2 is proper on its support. By [FS21, Corollary IX.2.3], $T_W(A)$ is the pullback of an object of $D_{\text{ét}}(\mathrm{Bun}_G;k)^{BW_E^I}$. For $A := j_{G!}(\text{c-Ind}_K^{G(E)}\, k)$, restriction to Bun_G^1 gives the W_E^I-action of (8.1.1). The analogous statements hold for H.

Let

$$F\colon D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}^1_X)^I};k)^{\mathrm{bd}} \to D_{\text{ét}}([*/H(E)]\times(\mathrm{Div}^1_X)^I;k),$$
$$B \mapsto j_H^*\mathrm{R}h_{2!}\big(h_1^*j_{H!}(\text{c-Ind}_{\iota^*K}^{H(E)}\, k)\overset{\mathrm{L}}{\otimes}_k B|_{\mathrm{Hck}^I_H}\big),$$

where h_1 and h_2 are analogous maps to (8.6.5) but for H. Here j_H^* is restriction to $\mathrm{Bun}_H^1\times(\mathrm{Div}^1_X)^I \cong [*/H(E)]\times(\mathrm{Div}^1_X)^I$. Thus $F(\mathcal{S}_U) = j_H^*T_U(j_{H!}\,\text{c-Ind}_{\iota^*K}^{H(E)}\, k)$, and by (8.6.4) its pullback to $[*/H(E)]\times\prod_{i\in I}\mathrm{Spd}\,\breve{E}$ is $\mathrm{R}f^H_{\iota^*K!}\mathcal{S}_U$, with the W_E^I-action of (8.1.1). We give the domain of F the relative perverse t-structure; the colimit condition of Lemma 8.6.3 holds, as one checks after applying normalized constant terms, which on bounded support are conservative, t-exact and colimit-preserving. Let J be the pullback functor

$$J\colon D(\mathrm{Rep}_k^{\mathrm{sm}}\,H(E))^{BW_E^I} \to D_{\text{ét}}([*/H(E)]\times(\mathrm{Div}^1_X)^I;k), \tag{8.6.6}$$

where the source is identified with $D_{\text{ét}}([*/H(E)]\times[*/W_E^I];k)$ by [FS21, Theorem V.1.1]. It is exact, colimit-preserving, and fully faithful by [FS21, Corollary IV.7.2].

Let $V\in\mathrm{Rep}_k((\widehat{G}\rtimes Q)^I)$. We apply Proposition 4.3.1 to the Hecke correspondence defining (8.6.4) for $W = \mathrm{Nm}^{(\ell^{-1})}V$, with both bundles trivial, keeping the framing of the target bundle and dividing the framing of the source by K. The Σ-fixed points of the Hecke correspondence for G at infinite level agrees with the Hecke correspondence for H, since a Σ-fixed modification of trivial G-bundles is the same as a modification of trivial H-bundles (as in the proof of Proposition 5.4.6). Then Lemma 5.4.5 implies that the analogous statement holds even at a finite level K of prime-to-ℓ pro-order. Hence we obtain

$$F(A_I(\mathcal{S}_V)) \cong \Big(\big(j_G^*T_{\mathrm{Nm}^{(\ell^{-1})}V}(j_{G!}\,\text{c-Ind}_K^{G(E)}\,k)\big)\big|_{[*/H(E)]\times(\mathrm{Div}^1_X)^I}\Big)^{t\Sigma} \tag{8.6.7}$$
$$\cong J\Big(\big((\mathrm{R}f^G_{K!}\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}V})|_{H(E)}\big)^{t\Sigma}\Big), \tag{8.6.8}$$

where (8.6.7) is Proposition 4.3.1 together with Lemma 4.2.2 for $\mathrm{R}h_{2!}$, and (8.6.8) is Lemma 4.2.2 for J, with the transported W_E^I-action on the right. In particular, $F(A_I(\mathcal{S}_V))$ lies in the essential image of J.

On the other hand, by Proposition 6.4.3 every ${}^p\mathrm{H}^nA_I(\mathcal{S}_V)$ is isomorphic to the Satake object $\widetilde{br}^I(\mathcal{S}_V)$, so $F({}^p\mathrm{H}^nA_I(\mathcal{S}_V))$ also lies in the image of J. Lemma 8.6.3 therefore descends the objects $F({}^p\tau^{\le b}A_I(\mathcal{S}_V))$, with their transition maps, to objects $F_b(V)$ of $D(\mathrm{Rep}_k^{\mathrm{sm}}\,H(E))^{BW_E^I}$. They form a W_E-equivariant filtration

$$F_b(V) \to \big((\mathrm{R}f^G_{K!}\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}V})|_{H(E)}\big)^{t\Sigma},\qquad b\in\mathbf{Z}, \tag{8.6.9}$$

and hence, through (8.6.3), a filtration of $\mathrm{R}f^H_{\iota^*K!}(\pi^H_{\iota^*K})^*A_I(\mathcal{S}_V)$. By Theorem 7.2.2(1), its graded pieces are

$$\mathrm{gr}_b\,F_\bullet(V) \cong \mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{L_{\psi^*V}}[-b], \tag{8.6.10}$$

with their native W_E^I-actions. By naturality of localization and t-exactness of pullback in the base, the filtration is functorial in V and compatible with base change and with coalescence of legs. This completes the proof of Proposition 8.6.2. □

8.7. **Functoriality for excursion operators.** Let K be Σ-stable of prime-to-ℓ pro-order. Taking $I=\{0\}$ and $W=\mathbb{1}$ in (8.6.3) and passing to degree-zero cohomology gives the first isomorphism in

$$\mathrm{T}^0(\mathrm{Sht}_{(G,1_G,\{0\}),K};\mathbb{1}) \cong \mathrm{T}^0(\mathrm{Sht}_{(H,1_H,\{0\}),\iota^*K};\mathbb{1}) \cong \text{c-Ind}_{\iota^*K}^{H(E)}\,k, \tag{8.7.1}$$

and the second isomorphism is (8.5.4) combined with Example 8.1.1.

Theorem 8.7.1. *Let $K<G(E)$ be a compact open subgroup with prime-to-ℓ pro-order, stable under Σ. Then the isomorphism* (8.7.1) *is equivariant for the action of $\mathrm{Exc}_k(W_E,\widehat{G})$, acting on the LHS via the normed excursion action (§8.5.5) and on the RHS through the homomorphism ${}^L\psi^*\colon \mathrm{Exc}_k(W_E,\widehat{G})\to\mathrm{Exc}_k(W_E,\widehat{H})$ from* (8.2.1) *followed by the native excursion action (§8.5.6).*

Proof. It suffices to treat an excursion operator of the form $S_{\mathcal{D}}$, with $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_i)$, since these generate the excursion algebra. Let $\Delta\colon \operatorname{Spd} \breve{E} \to \prod_{i \in I} \operatorname{Spd} \breve{E}$ be the diagonal. By Lemma 8.5.8, the normed action of $S_{\mathcal{D}}$ is T^0 of the composite of Σ-equivariant maps

$$\mathrm{R}f^G_{K!}\mathcal{S}_{\mathbb{1}} \xrightarrow{\mathrm{Nm}^{(\ell^{-1})}\alpha} \Delta^*\mathrm{R}f^G_{K!}\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}V} \xrightarrow{(\gamma_i)_i} \Delta^*\mathrm{R}f^G_{K!}\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}V} \xrightarrow{\mathrm{Nm}^{(\ell^{-1})}\beta} \mathrm{R}f^G_{K!}\mathcal{S}_{\mathbb{1}}, \tag{8.7.2}$$

where the outer maps are the creation and annihilation maps, induced by $\mathrm{Nm}^{(\ell^{-1})}\alpha$ and $\mathrm{Nm}^{(\ell^{-1})}\beta$ through the fusion isomorphism (8.1.2), and the middle map is the W_E^I-action.

Apply the Tate construction to (8.7.2), restricting the $G(E)$-action to $H(E)$. The equivariant localization isomorphism (8.6.3) gives a commutative diagram

$$\begin{array}{ccc}
\big((\mathrm{R}f^G_{K!}\mathcal{S}_{\mathbb{1}})|_{H(E)}\big)^{t\Sigma} & \xrightarrow{\sim} & \mathrm{R}f^H_{\iota^*K!}(\pi^H_{\iota^*K})^*\big((\iota^*\mathcal{S}_{\mathbb{1}})^{t\Sigma}\big) \\
{\scriptstyle \mathrm{Nm}^{(\ell^{-1})}\alpha}\downarrow & & \downarrow{\scriptstyle (\iota^*\mathrm{Nm}^{(\ell^{-1})}\alpha)^{t\Sigma}} \\
\Delta^*\big((\mathrm{R}f^G_{K!}\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}V})|_{H(E)}\big)^{t\Sigma} & \xrightarrow{\sim} & \Delta^*\mathrm{R}f^H_{\iota^*K!}(\pi^H_{\iota^*K})^*A_I(\mathcal{S}_V) \\
{\scriptstyle (\gamma_i)_i}\downarrow & & \downarrow{\scriptstyle (\gamma_i)_i} \\
\Delta^*\big((\mathrm{R}f^G_{K!}\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}V})|_{H(E)}\big)^{t\Sigma} & \xrightarrow{\sim} & \Delta^*\mathrm{R}f^H_{\iota^*K!}(\pi^H_{\iota^*K})^*A_I(\mathcal{S}_V) \\
{\scriptstyle \mathrm{Nm}^{(\ell^{-1})}\beta}\downarrow & & \downarrow{\scriptstyle (\iota^*\mathrm{Nm}^{(\ell^{-1})}\beta)^{t\Sigma}} \\
\big((\mathrm{R}f^G_{K!}\mathcal{S}_{\mathbb{1}})|_{H(E)}\big)^{t\Sigma} & \xrightarrow{\sim} & \mathrm{R}f^H_{\iota^*K!}(\pi^H_{\iota^*K})^*\big((\iota^*\mathcal{S}_{\mathbb{1}})^{t\Sigma}\big)
\end{array} \tag{8.7.3}$$

The left column is the Tate construction of (8.7.2), so its composite induces the normed action of $S_{\mathcal{D}}$ on H^0.

Next we filter the right column by the perverse filtration (8.6.9). By Proposition 8.6.2, the middle map preserves the filtration. The outer maps also preserve it: they are induced by α and β after pullback along Δ, and the norm, the Tate construction and perverse truncation commute with this pullback (the last by [FS21, Proposition VI.7.4]). Passing to the associated graded, we obtain for each b a commutative diagram

$$\begin{array}{ccc}
F_b(\mathbb{1}) & \longrightarrow & \mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{\mathbb{1}}[-b] \\
\downarrow & & \downarrow{\scriptstyle {}^L\psi^*\alpha} \\
\Delta^*F_b(V) & \longrightarrow & \Delta^*\mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{{}^L\psi^*V}[-b] \\
{\scriptstyle (\gamma_i)_i}\downarrow & & \downarrow{\scriptstyle (\gamma_i)_i} \\
\Delta^*F_b(V) & \longrightarrow & \Delta^*\mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{{}^L\psi^*V}[-b] \\
\downarrow & & \downarrow{\scriptstyle {}^L\psi^*\beta} \\
F_b(\mathbb{1}) & \longrightarrow & \mathrm{R}f^H_{\iota^*K!}\mathcal{S}_{\mathbb{1}}[-b]
\end{array} \tag{8.7.4}$$

The left column is the filtered piece of the right column of (8.7.3), and the horizontal maps are the projections to the graded pieces, identified with the cohomology of Satake sheaves for H by (8.6.10). The right column is computed as follows: the outer maps are ${}^L\psi^*\alpha$ and ${}^L\psi^*\beta$ by the naturality of (6.4.9) and the compatibility of $\widetilde{br}^I$ with fusion (Theorem 6.5.1), and the middle map is the native action of W_E^I by Proposition 8.6.2. Thus the right column is the native excursion operator of ${}^L\psi^*\mathcal{D}$, shifted by $[-b]$.

It remains to pass from the graded pieces to T^0. By Proposition 8.6.1 with $W = \mathbb{1}$, Example 8.1.1, (4.5.3) and the projection formula, we have an isomorphism of (derived) smooth $H(E)$-representations

$$\big((\text{c-Ind}_K^{G(E)} k)|_{H(E)}\big)^{t\Sigma} \cong \text{c-Ind}_{\iota^*K}^{H(E)} k \overset{\mathrm{L}}{\otimes}_k k^{t\Sigma}. \tag{8.7.5}$$

Since $A_{\{0\}}(\mathcal{S}_{\mathbb{1}})$ is supported on the unit section, which has relative dimension 0, perverse truncation there is ordinary truncation [FS21, Definition/Proposition VI.7.1], which gives the isomorphism of (derived)

smooth $H(E)$-representations

$$F_0(\mathbb{1}) \cong \text{c-Ind}_{\iota^* K}^{H(E)} k \overset{\mathrm{L}}{\otimes}_k \tau^{\leq 0} k^{t\Sigma}. \tag{8.7.6}$$

The natural maps

$$\mathrm{H}^0(\text{c-Ind}_{\iota^* K}^{H(E)} k \overset{\mathrm{L}}{\otimes}_k k^{t\Sigma}) \xleftarrow{\sim} \mathrm{H}^0 F_0(\mathbb{1}) \xrightarrow{\sim} \mathrm{H}^0 \operatorname{gr}_0 F_\bullet(\mathbb{1}) = \text{c-Ind}_{\iota^* K}^{H(E)} k \tag{8.7.7}$$

are isomorphisms: the first because $\tau^{\leq 0} k^{t\Sigma} \to k^{t\Sigma}$ is an isomorphism on H^0, and the second because H^0 and H^1 of $F_{-1}(\mathbb{1}) = \text{c-Ind}_{\iota^* K}^{H(E)} k \overset{\mathrm{L}}{\otimes}_k \tau^{\leq -1} k^{t\Sigma}$ vanish. Both maps intertwine the endomorphisms induced by the composites in (8.7.3) and (8.7.4) for $b = 0$.

- On the left, this endomorphism is the normed action of $S_{\mathcal{D}}$ on $\mathrm{T}^0(\mathrm{Sht}_{(G,1_G,\{0\}),K}; \mathbb{1}) = \mathrm{H}^0(\text{c-Ind}_{\iota^* K}^{H(E)} k \overset{\mathrm{L}}{\otimes}_k k^{t\Sigma})$.
- On the right, it is H^0 of (8.7.4) for $b = 0$, which is the native action of ${}^L\psi^* S_{\mathcal{D}}$ by Lemma 8.2.1 and Remark 8.2.2.

The resulting identification of $\mathrm{H}^0(\text{c-Ind}_{\iota^* K}^{H(E)} k \overset{\mathrm{L}}{\otimes}_k k^{t\Sigma})$ with $\text{c-Ind}_{\iota^* K}^{H(E)} k$ is (8.7.1). This completes the proof. □

9. Derived Treumann–Venkatesh Conjecture

In this section we will reap the applications of the preceding material, especially the functoriality relations for excursion operators from Theorem 8.7.1. In §9.1 we formulate the "derived" version of Treumann-Venkatesh's Conjecture 1.2.3(2), and prove it in §9.4. In §9.2 we will construct the Treumann-Venkatesh homomorphism alluded to in (1.4.2), and in §9.3 we establish the commutative square (1.4.3). Finally, in §9.5 we prove the existence of functorial lifts along σ-dual homomorphisms, including Theorem 1.4.1.

9.1. **Formulation.** By the realization of the Bernstein center $\mathfrak{Z}(G)$ as the ring of endomorphisms of the identity functor on $\mathrm{Rep}_k^{\mathrm{sm}} G(E)$, there is a tautological action of $\mathfrak{Z}(G)$ on any $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$. Composing this with the Fargues-Scholze map $\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathfrak{Z}(G; k)$ discussed in §8.4, we obtain an action of $\mathrm{Exc}_k(W_E, \widehat{G})$ on any such Π. In particular, to each Π we may attach a subset

$$\operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\Pi)) \subset \operatorname{Spec} \mathrm{Exc}_k(W_E, \widehat{G}),$$

which can be interpreted as the "set of semisimple L-parameters attached to Π". For example, if $\Pi \in \mathrm{Rep}_k^{\mathrm{sm}} G(E)$ is irreducible admissible, then $\operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\Pi)$ is necessarily a single closed point, corresponding to the *Fargues-Scholze parameter* $\rho_\Pi \in H^1(W_E; \widehat{G}(k))$. This defines the map (1.3.1).

Let $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}}(G(E) \rtimes \Sigma)) = D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))^{B\Sigma}$, the equivalence being as in Lemma 3.2.1. Then we may form the Tate cohomology $\mathrm{T}^j(\Pi)$, which is a smooth representation of $H(E)$ by §8.5.1.

Using the square root fixed in §8.1, we transport the σ-dual homomorphism to ${}^L\psi := \alpha_G^{-1} \circ {}^c\psi \circ \alpha_H \colon {}^L H \to {}^L G$.

Theorem 9.1.1. *Let $F = F_G \colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Exc}_k(W_E, \widehat{G})$ be the linearized Frobenius of* (8.5.6). *This induces a functor F_* on $\mathrm{Exc}_k(W_E, \widehat{G})$-modules.*

For any $j \in \mathbf{Z}/2\mathbf{Z}$, $\operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{H})}(\mathrm{T}^j(\Pi))$ lies over $F_* \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\Pi))$ with respect to the map $\operatorname{Spec} \mathrm{Exc}_k(W_E, \widehat{H}) \xrightarrow{{}^L\psi_*} \operatorname{Spec} \mathrm{Exc}_k(W_E, \widehat{G})$ induced by the σ-dual homomorphism ${}^L\psi \colon {}^L H \to {}^L G$. In other words, there is a commutative diagram

$$\begin{array}{ccc} \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{H})}(\mathrm{T}^j(\Pi)) & \hookrightarrow & \operatorname{Spec} \mathrm{Exc}_k(W_E, \widehat{H}) \\ \downarrow & & \downarrow{\scriptstyle {}^L\psi_*} \\ F_* \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\Pi)) & \hookrightarrow & \operatorname{Spec} \mathrm{Exc}_k(W_E, \widehat{G}) \end{array}$$

Example 9.1.2 (Treumann-Venkatesh functoriality conjecture)**.** If Π is an irreducible smooth representation of $G(E)$ (hence admissible, by [DHKM24]), then $\mathrm{Exc}_k(W_E, \widehat{G})$ acts on Π through a character χ_Π. By (8.5.7), the composition $\chi_\Pi \circ F$ is the character of $\mathrm{Exc}_k(W_E, \widehat{G})$ associated to the Frobenius twist $\Pi^{(\ell)} := \Pi \otimes_{k,\mathrm{Frob}_\ell} k$, so we have

$$F_* \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\Pi) = \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(F_*\Pi) = \{\chi_{\Pi^{(\ell)}}\}.$$

Suppose furthermore that $\Pi \cong {}^\sigma\Pi \in \mathrm{Rep}_k^{\mathrm{sm}} G(E)$. Then by [TV16, Proposition 6.1], the $G(E)$-action on Π extends uniquely to a $G(E) \rtimes \Sigma$-action. Then Theorem 9.1.1 and Lemma 8.2.1 imply that for each $j \in \mathbf{Z}/2\mathbf{Z}$ and every irreducible $H(E)$-subquotient π of $\mathrm{T}^j(\Pi)$, we have

$$\rho_{\Pi^{(\ell)}} \sim ({}^L\psi \circ \rho_\pi)^{\mathrm{ss}}.$$

This proves Theorem 1.3.1.

Example 9.1.3. Suppose σ is inner, induced by conjugation by an element $s \in G(E)$ of order ℓ. Then there is an isomorphism

$$G(E) \rtimes \Sigma \xrightarrow{\sim} G(E) \times \Sigma, \qquad (g, \sigma^i) \longmapsto (gs^i, \sigma^i).$$

Hence we get a functor

$$D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E)) \to D^b(\mathrm{Rep}_k^{\mathrm{sm}}(G(E) \times \Sigma)) \xrightarrow{\sim} D^b(\mathrm{Rep}_k^{\mathrm{sm}}(G(E) \rtimes \Sigma)).$$

This equips every representation $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$ with a canonical Σ-equivariant structure, where σ acts by $\Pi(s)$.

9.2. **The Treumann-Venkatesh homomorphism.** We next establish some technical lemmas which aid in studying the properties of the Brauer homomorphism.

Proposition 9.2.1. *Let $K \subset G(E)$ be a Σ-stable compact open subgroup with prime-to-ℓ pro-order. Then the natural map*

$$\iota^* K \backslash H(E) / \iota^* K \to K \backslash G(E) / K$$

is injective.

Proof. Let $a, b \in \iota^* K \backslash H(E)$ be two elements whose images in $K\backslash G(E)$ lie in the same orbit for the right translation of K. In other words, $a = b\kappa$ for some $\kappa \in K$. Applying σ to this equation, and using that a, b are fixed by σ, we obtain $a = b\sigma(\kappa)$, so $\sigma(\kappa)\kappa^{-1} \in \mathrm{Stab}_K(b)$. Since Σ is of order ℓ while K is prime-to-ℓ, we have $\mathrm{H}^1(\Sigma; \mathrm{Stab}_K(b)) = 0$. Hence there exists $y \in \mathrm{Stab}_K(b)$ such that $\sigma(\kappa)\kappa^{-1} = \sigma(y)y^{-1}$. Then $y^{-1}\kappa$ is fixed by σ, so $y^{-1}\kappa \in H(E) \cap K = \iota^* K$. But then

$$a = b\kappa = (by^{-1})\kappa = b(y^{-1}\kappa),$$

which shows that a and b lie in the same orbit for the right translation of $\iota^* K$ on $\iota^* K \backslash H(E)$. □

Recall that we defined a compact open subgroup $K \subset G(E)$ to be *plain* if the natural map $H(E)/\iota^* K \to [G(E)/K]^\sigma$ is a bijection. By a similar argument involving the vanishing of non-abelian cohomology (cf. [Fen24, Lemma 6.6]), any compact open subgroup $K \subset G(E)$ with prime-to-ℓ pro-order (henceforth abbreviated *prime-to-ℓ subgroup*) is plain, so the (un-normalized) Brauer homomorphism (cf. §6.2.1) $\mathrm{Br}\colon \mathscr{H}(G, K)^\sigma \to \mathscr{H}(H, \iota^* K)$ is defined for any such K.

Corollary 9.2.2. *Let $K \subset G(E)$ be a Σ-stable prime-to-ℓ compact open subgroup. Then* Br *induces a map of centers,*

$$\mathrm{Br}\colon Z(\mathscr{H}(G, K)^\sigma) \to Z(\mathscr{H}(H, \iota^* K)). \tag{9.2.1}$$

Proof. Proposition 9.2.1 implies that $\mathrm{Br}\colon \mathscr{H}(G, K)^\sigma \to \mathscr{H}(H, \iota^* K)$ is a surjective k-algebra homomorphism, so it maps the center to the center. □

It is evident from the definition that the map (9.2.1) factors over the quotient $Z(\mathscr{H}(G, K)^\sigma)/N \cdot \mathfrak{Z}(G, K)$, which induces a map

$$\mathrm{T}^0 \mathfrak{Z}(G, K) = \frac{\mathfrak{Z}(G, K)^\sigma}{N \cdot \mathfrak{Z}(G, K)} \hookrightarrow \frac{Z(\mathscr{H}(G, K)^\sigma)}{N \cdot \mathfrak{Z}(G, K)} \to \mathfrak{Z}(H, \iota^* K). \tag{9.2.2}$$

Note that we have a natural $\mathbf{F}_\ell$-structure on $\mathfrak{Z}(G,K;k)$ given by

$$\mathfrak{Z}(G,K;k) = \mathfrak{Z}(G,K;\mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} k. \tag{9.2.3}$$

The following is a "normalized" (cf. §6.2.3) version of (9.2.1), in the sense of the normalized Brauer homomorphism (6.2.4).

Definition 9.2.3. Let $K \subset G(E)$ be a Σ-stable prime-to-ℓ compact open subgroup. We define the *Treumann-Venkatesh homomorphism*

$$\mathfrak{Z}_{\mathrm{TV},K} \colon \mathfrak{Z}(G,K) \to \mathfrak{Z}(H,\iota^*K)$$

to be the composition of (9.2.2) with the map $\mathrm{Nm}^{(\ell^{-1})} \colon \mathfrak{Z}(G,K) \to \mathrm{T}^0(\mathfrak{Z}(G,K))$ from Definition 6.2.1, with respect to the $\mathbf{F}_\ell$-structure (9.2.3).

9.3. **Modular functoriality for Bernstein centers.** For an inclusion $K \subset K' \subset G(E)$ of Σ-stable prime-to-ℓ compact open subgroups, we have the map $e_G^{K\to K'} \colon \mathfrak{Z}(G,K) \to \mathfrak{Z}(G,K')$ given by convolution with the idempotent $[K':K]^{-1}\mathbb{1}_{K'}$ (the index is prime to ℓ). Similarly, we have $e_H^{K\to K'} \colon \mathfrak{Z}(H,\iota^*K) \to \mathfrak{Z}(H,\iota^*K')$ given by convolution with $[\iota^*K':\iota^*K]^{-1}\mathbb{1}_{\iota^*K'}$. The diagram

$$\begin{array}{ccc} \mathfrak{Z}(G,K) & \xrightarrow{\mathfrak{Z}_{\mathrm{TV},K}} & \mathfrak{Z}(H,\iota^*K) \\ \downarrow{\scriptstyle e_G^{K\to K'}} & & \downarrow{\scriptstyle e_H^{K\to K'}} \\ \mathfrak{Z}(G,K') & \xrightarrow{\mathfrak{Z}_{\mathrm{TV},K'}} & \mathfrak{Z}(H,\iota^*K') \end{array}$$

commutes.

Definition 9.3.1 (Base change homomorphism for Bernstein centers)**.** We define the *Treumann-Venkatesh homomorphism* (for Bernstein centers) $\mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G) \to \mathfrak{Z}(H)$ as

$$\varprojlim_K \mathfrak{Z}_{\mathrm{TV},K} \colon \varprojlim_K \mathfrak{Z}(G,K) \to \varprojlim_K \mathfrak{Z}(H,\iota^*K)$$

where the limit is taken over Σ-stable prime-to-ℓ compact open subgroups $K \subset G(E)$.

Recall that in Theorem 7.2.1 we have constructed the σ-dual homomorphism ${}^c\psi \colon {}^cH \to {}^cG$ over k. Using the fixed choice of $\sqrt{q} \in k$, we may transport it to ${}^L\psi \colon {}^LH \to {}^LG$.

Theorem 9.3.2. *The diagram* (9.3.1) *commutes.*

$$\begin{array}{ccc} \mathrm{Exc}_k(W_E,\widehat{G}) & \xrightarrow{{}^L\psi^*} & \mathrm{Exc}_k(W_E,\widehat{H}) \\ {\scriptstyle \mathrm{FS}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{FS}_H} \\ \mathfrak{Z}(G) & \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} & \mathfrak{Z}(H) \end{array} \tag{9.3.1}$$

Note that if K is a plain subgroup of $G(E)$, then we have

$$\mathrm{T}^0(\text{c-Ind}_K^{G(E)} k) \cong \text{c-Ind}_{\iota^*K}^{H(E)} k \in \mathrm{Rep}_k^{\mathrm{sm}} H(E). \tag{9.3.2}$$

(This follows, for example, from Lemma 10.4.1.)

In preparation for the proof of Theorem 9.3.2, we record the following interpretation of the Brauer homomorphism.

Lemma 9.3.3. *Under the identifications* $\mathscr{H}(G,K) \cong \mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)} k)$ *and* $\mathscr{H}(H,\iota^*K) \cong \mathrm{End}_{H(E)}(\text{c-Ind}_{\iota^*K}^{H(E)} k)$*, the map* $\mathrm{Br} \colon \mathrm{T}^0\mathscr{H}(G,K) \to \mathscr{H}(H,\iota^*K)$ *induced by the Brauer homomorphism is identified with the map*

$$\mathrm{T}^0 \mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)} k) \to \mathrm{End}_{H(E)}(\mathrm{T}^0(\text{c-Ind}_K^{G(E)} k)) \overset{(9.3.2)}{\cong} \mathrm{End}_{H(E)}(\text{c-Ind}_{\iota^*K}^{H(E)} k)$$

sending a Σ*-equivariant endomorphism of* $\text{c-Ind}_K^{G(E)} k$ *to the induced endomorphism on its Tate cohomology.*

Proof. Apply [Fen24, Lemma 6.7] (which comes from [TV16, §6.2]) with $\Pi := \text{c-Ind}_K^{G(E)} k$. □

Proof of Theorem 9.3.2. For any prime-to-ℓ, Σ-stable compact open subgroup $K \subset G(E)$, we also denote by

$$\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathfrak{Z}(G, K)$$

the composition of FS_G with the projection $\mathfrak{Z}(G) \to \mathfrak{Z}(G, K)$. Applying Tate cohomology, this induces

$$\mathrm{T}^0 \mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathrm{T}^0 \mathfrak{Z}(G, K). \tag{9.3.3}$$

We have identifications

$$\begin{array}{ccccc} \mathrm{T}^0(\text{c-Ind}_K^{G(E)} k) & \xrightarrow[\sim]{\text{Ex. 8.5.2}} & \mathrm{T}^0(\mathrm{Sht}_{(G,1_G,\{0\}),K}; \mathbb{1}) & \xrightarrow[\sim]{(8.7.1)} & \mathrm{T}^0(\mathrm{Sht}_{(H,1_H,\{0\}),\iota^* K}; \mathbb{1}) \\ & & & & \Big\downarrow{\scriptstyle \text{Ex. 8.5.2}}{\scriptstyle \sim} \\ & & \text{c-Ind}_{\iota^* K}^{H(E)} k & \xleftarrow[\text{Ex. 4.5.2}]{\sim} & \mathrm{T}^0(\text{c-Ind}_{\iota^* K}^{H(E)} k) \end{array} \tag{9.3.4}$$

Here the vertical identification is the analogue of Example 8.5.2 for H. Now, for any excursion datum $\mathcal{D}$ for $\widehat{G}$, Theorem 8.7.1 implies that the identifications in (9.3.4) intertwine

$$\begin{pmatrix} \text{the action of } S_{\mathrm{Nm}^{(\ell^{-1})}\mathcal{D}} \\ \text{on } \mathrm{T}^0(\text{c-Ind}_K^{G(E)} k) \end{pmatrix} = \begin{pmatrix} \text{the action of } {}^L\psi^*(S_{\mathcal{D}}) \\ \text{on c-Ind}_{\iota^* K}^{H(E)} k \end{pmatrix}. \tag{9.3.5}$$

The diagram

$$\begin{array}{ccc} \mathrm{Exc}_k(W_E, \widehat{G}) & \xrightarrow{\mathrm{Nm}^{(\ell^{-1})}} & \mathrm{T}^0 \mathrm{Exc}_k(W_E, \widehat{G}) \\ \Big\downarrow{\scriptstyle \mathrm{FS}_G} & & \Big\downarrow{\scriptstyle (9.3.3)} \\ \mathfrak{Z}(G, K) & \xrightarrow{\mathrm{Nm}^{(\ell^{-1})}} & \mathrm{T}^0 \mathfrak{Z}(G, K) \end{array} \tag{9.3.6}$$

commutes by the definition of Nm, and the fact that FS_G respects the transported $\mathbf{F}_\ell$-structure of §8.5.4 on the source and (9.2.3) on the target. Therefore, unraveling the definition of $\mathfrak{Z}_{\mathrm{TV}}$ and using Lemma 8.5.8, it suffices to show that

$$\mathrm{Br}(\mathrm{FS}_G(S_{\mathrm{Nm}^{(\ell^{-1})}\mathcal{D}})) = \mathrm{FS}_H({}^L\psi^*(S_{\mathcal{D}})) \tag{9.3.7}$$

for all excursion data $\mathcal{D}$ for $\widehat{G}$. By Lemma 9.3.3, $\mathrm{Br}(\mathrm{FS}_G(S_{\mathrm{Nm}^{(\ell^{-1})}\mathcal{D}}))$ is the endomorphism of $\mathrm{T}^0(\text{c-Ind}_K^{G(E)} k)$, identified via (9.3.2) with an endomorphism of $\text{c-Ind}_{\iota^* K}^{H(E)} k$, obtained by taking T^0 of the action of $S_{\mathrm{Nm}^{(\ell^{-1})}\mathcal{D}}$ on $\text{c-Ind}_K^{G(E)} k$, which according to (9.3.5) is the endomorphism given by $\mathrm{FS}_H({}^L\psi^*(S_{\mathcal{D}}))$. □

Corollary 9.3.4. *For any irreducible $H(E)$-representation π, the character*

$$\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G) \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} \mathfrak{Z}(H) \xrightarrow{\chi_\pi} k$$

has the property that for any irreducible $G(E)$-representation Π on which $\mathfrak{Z}(G)$ acts through $\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}}$, we have $\rho_\Pi \sim ({}^L\psi \circ \rho_\pi)^{\mathrm{ss}}$.

9.4. **Proof of Theorem 9.1.1.**

Lemma 9.4.1. *Let $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))^{B\Sigma}$ and $j \in \mathbf{Z}/2\mathbf{Z}$. Then the normed excursion action of $\mathrm{Exc}_k(W_E, \widehat{G})$ on $\mathrm{T}^j(\Pi)$ agrees with the action obtained from*

$$\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{{}^L\psi^*} \mathrm{Exc}_k(W_E, \widehat{H}) \xrightarrow{\mathrm{FS}_H} \mathfrak{Z}(H).$$

Moreover, ${}^L\psi^* \circ \sigma = {}^L\psi^*$ on $\mathrm{Exc}_k(W_E, \widehat{G})$, and hence ${}^L\psi^*(N \cdot \mathrm{Exc}_k(W_E, \widehat{G})) = 0$.

Proof. Proposition 6.5.3 gives a tensor isomorphism $\widetilde{br} \circ {}^\sigma(-) \cong \widetilde{br}$, so Tannakian reconstruction implies that $\sigma \circ {}^L\psi \sim {}^L\psi$. This implies that ${}^L\psi^*(\sigma S) = {}^L\psi^*(S)$ for all excursion operators $S \in \mathrm{Exc}_k(W_E, \widehat{G})$, hence

$${}^L\psi^*(N \cdot S) = \sum_{i=0}^{\ell-1} {}^L\psi^*(\sigma^i S) = \ell \, {}^L\psi^*(S) = 0.$$

It remains to compare the two actions. It suffices to do so on a generator $S_{\mathcal{D}}$. Put

$$\xi_G(\mathcal{D}) := \mathrm{FS}_G(S_{\mathrm{Nm}^{(\ell^{-1})}\mathcal{D}}) \in \mathfrak{Z}(G)^\sigma, \qquad \xi_H(\mathcal{D}) := \mathrm{FS}_H({}^L\psi^*(S_{\mathcal{D}})) \in \mathfrak{Z}(H).$$

By Lemma 8.5.8, the normed action of $S_{\mathcal{D}}$ is via $\xi_G(\mathcal{D})$.

Let K range over a cofinal system of Σ-stable compact open subgroups of prime-to-ℓ pro-order. By smoothness, $\Pi = \varinjlim_K \Pi^K$, and Tate cohomology commutes with this filtered colimit, so every vector in $\mathrm{T}^j(\Pi)$ is the image of some $u \in \mathrm{T}^j(\Pi^K)$. The normed action of $S_{\mathcal{D}}$ on u is through $\mathrm{Br}(\mathrm{FS}_G(S_{\mathrm{Nm}^{(\ell-1)}\mathcal{D}}))$, which by (9.3.7) identifies with $\mathrm{FS}_H({}^L\psi^*(S_{\mathcal{D}}))$, as desired. □

Proof of Theorem 9.1.1. By Lemma 9.4.1, restriction along ${}^L\psi^*$ of the native $\mathrm{Exc}_k(W_E, \widehat{H})$-action on $\mathrm{T}^j(\Pi)$ is the normed $\mathrm{Exc}_k(W_E, \widehat{G})$-action. Hence ${}^L\psi_*$ carries

$$\mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi) \xrightarrow{{}^L\psi_*} \mathrm{supp}^{\mathrm{normed}}_{\mathrm{Exc}_k(W_E,\widehat{G})} \mathrm{T}^j(\Pi), \tag{9.4.1}$$

where the native excursion action is defined in §8.5.6.

We will compare the support of the normed and native actions of $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ on $\mathrm{T}^j(\Pi)$. By the discussion in §8.5.5, we have for any $j \in \mathbf{Z}/2\mathbf{Z}$ an equality

$$\mathrm{supp}^{\mathrm{normed}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{T}^j(\Pi)) = F_* \mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{T}^j(\Pi)) \subset \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G})^\sigma. \tag{9.4.2}$$

By the functorial filtration of Remark 4.1.1, each $\mathrm{T}^j(\Pi)$ has a finite filtration each of whose graded pieces is a subquotient of $\mathrm{H}^*(\Pi)$, hence we have

$$F_* \mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{T}^j(\Pi)) \subset F_* \mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{H}^*(\Pi)) = \mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(F_*\mathrm{H}^*(\Pi)). \tag{9.4.3}$$

Let $q\colon \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ be the map of spectra induced by the obvious inclusion. Combining (9.4.2) and (9.4.3) with (9.4.1) yields

$$q \circ {}^L\psi_* \left(\mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi)\right) \subset F_* \mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{H}^*(\Pi)). \tag{9.4.4}$$

The surjection $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma \twoheadrightarrow \mathrm{T}^0\,\mathrm{Exc}_k(W_E, \widehat{G})$ induces a closed embedding on spectra, as in the diagram

$$\begin{array}{ccc} & & \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G}) \\ & & \big\downarrow q \\ \mathrm{Spec}\ \mathrm{T}^0\,\mathrm{Exc}_k(W_E, \widehat{G}) & \hookrightarrow & \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G})^\sigma \end{array}$$

Now, [Fen24, Lemma 5.15] says that any character from $\mathrm{T}^0\,\mathrm{Exc}_k(W_E, \widehat{G})$ to any perfect field has a unique extension to $\mathrm{Exc}_k(W_E, \widehat{G})$, which implies that the map $\mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ is one-to-one (and surjective) over the closed subscheme $\mathrm{Spec}\ \mathrm{T}^0\,\mathrm{Exc}_k(W_E, \widehat{G}) \subset \mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G})^\sigma$. Since ${}^L\psi^*$ annihilates $N \cdot \mathrm{Exc}_k(W_E, \widehat{G})$ by Lemma 9.4.1, every prime in the image of ${}^L\psi_*$ contains this ideal. Therefore $q \circ {}^L\psi_* \left(\mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi)\right)$ lands inside this subspace, so we may lift (9.4.4) to the containment

$${}^L\psi_* \left(\mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi)\right) \subset F_* \mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\Pi)), \tag{9.4.5}$$

as desired. □

9.5. **Existence of functorial lifts.** We generalize the proof of [Fen24, Theorem 6.26] to prove the existence of functorial liftings along any σ-dual homomorphism.

Theorem 9.5.1. *Let π be an irreducible representation of $H(E)$ over k, with Fargues-Scholze parameter $\rho_\pi \in \mathrm{H}^1(W_E; \widehat{H}(k))$. Then there is an irreducible representation Π of $G(E)$ over k such that $\rho_\Pi \sim ({}^L\psi \circ \rho_\pi)^{\mathrm{ss}} \in \mathrm{H}^1(W_E; \widehat{G}(k))$.*

Proof. Choose a Σ-stable pro-p compact open subgroup $K < G(E)$ such that $\pi^{\iota^* K} \neq 0$. From Theorem 9.3.2 we have a commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathfrak{Z}(H, \iota^* K) & \xrightarrow{\text{Spec } \mathfrak{Z}_{\mathrm{TV}}} & \text{Spec } \mathfrak{Z}(G, K) \\ \downarrow{\scriptstyle \text{Spec FS}_H} & & \downarrow{\scriptstyle \text{Spec FS}_G} \\ \text{Spec Exc}_k(W_E, \widehat{H}) & \xrightarrow{{}^L\psi_*} & \text{Spec Exc}_k(W_E, \widehat{G}) \end{array} \tag{9.5.1}$$

The representation π gives a k-point of Spec $\mathfrak{Z}(H, \iota^* K)$ lying over ρ_π viewed as a k-point of Spec $\mathrm{Exc}_k(W_E, \widehat{H})$. The commutativity of the diagram then tells us that there is a k-point of Spec $\mathfrak{Z}(G, K)$, say with maximal ideal $\mathfrak{m}$, lying over the k-point of Spec $\mathrm{Exc}_k(W_E, \widehat{G})$ corresponding to $({}^L\psi \circ \rho_\pi)^{\mathrm{ss}}$.

Recall that the functor $\Pi \mapsto \Pi^K$ induces a bijection between irreducible admissible $G(E)$-representations with non-zero K-invariants and irreducible $\mathscr{H}(G, K)$-modules (cf. [Vig96, §I.4]). It therefore suffices to construct an irreducible representation of $\mathscr{H}(G, K)$ on which the $\mathfrak{Z}(G, K)$-action factors over $\mathfrak{m}$.

By a result of Dat-Helm-Kurinczuk-Moss [DHKM24, Theorem 1.1] $\mathscr{H}(G, K)$ is a finite $\mathfrak{Z}(G, K)$-module. The localization of $\mathscr{H}(G, K)$ at $\mathfrak{m}$ is non-zero, since $\mathfrak{Z}(G, K) \hookrightarrow \mathscr{H}(G, K)$, so Nakayama's Lemma implies that the left $\mathscr{H}(G, K)$-module quotient $\mathscr{H}(G, K)/\mathscr{H}(G, K)\mathfrak{m}$ is finite-dimensional and non-zero. By design, it is supported over $\mathfrak{Z}(G, K)/\mathfrak{m}$, so it has an irreducible $\mathscr{H}(G, K)$-subquotient on which the $\mathfrak{Z}(G, K)$-action factors over $\mathfrak{m}$, as was to be shown.

□

10. Fargues-Scholze parameters of toral supercuspidals

In this section we will demonstrate how the preceding theory may be used to calculate the Fargues-Scholze parameters attached to explicitly constructed representations $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$. We will apply Theorem 9.1.1 with the automorphism σ being conjugation by a strongly regular ℓ-torsion element $s \in G(E)$. Then H is a *torus* $T < G$, so the Fargues-Scholze correspondence is completely understood for H. Using Theorem 9.1.1, we can describe $\rho_\Pi \in \mathrm{H}^1(W_E; \widehat{G}(k))$ in terms of $\mathrm{T}^j(\Pi)$ and the σ-dual homomorphism ${}^L\psi$.

We will focus our attention on $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$ related to the "(Howe-unramified) toral supercuspidals" studied by Chan-Oi [CO]. (The method has broader scope but we regard this as a sufficiently interesting proof-of-concept for now.) Then T must be taken to be an unramified elliptic torus, and we compute the σ-dual embedding in §10.2: it turns out to be the canonical L-embedding corresponding to an unramified maximal torus, composed with Frobenius.

The relevant Π arise as compact inductions of representations cut out from the cohomology of so-called "deep level Deligne-Lusztig varieties" studied by Chan-Ivanov [CI21]. In §10.3 we use equivariant localization to calculate the Tate cohomology of these deep level Deligne-Lusztig varieties. We feed the answer into §10.4 in order to compute the Tate cohomology of the compact inductions, and then identify an explicit point in their Fargues–Scholze parameter support in Theorem 10.4.2.

10.1. Review of tori.

Let G be a connected reductive group over E.

10.1.1. *The abstract Cartan.* To G we can associate a canonical E-torus $\mathbf{T}$, which we call the "abstract Cartan" of G. (We make no claim that $\mathbf{T}$ admits an E-rational embedding into G.) If G is quasi-split, then $\mathbf{T}$ is defined as the colimit over E-rational Borel subgroups $B < G$ of B/U, where U is the unipotent radical of B. In general, we pass to an extension of E where it becomes quasi-split, and then descend this construction.

Given an extension E'/E and a Borus (B, T) in $G_{E'}$, the composition

$$T \hookrightarrow B \twoheadrightarrow B/U \to \mathbf{T}_{E'}$$

defines an isomorphism $i_B \colon T \xrightarrow{\sim} \mathbf{T}_{E'}$.

We denote by $\mathbf{W}$ the Weyl group of $\mathbf{T}$, defined as the colimit of $N_{G_{E^s}}(T)/T$ over the category of Boruses (B, T) in G_{E^s}.

10.1.2. *The cocycle of a torus.* Let $T \subset G$ be a maximal torus over E. Over E^s, we can find a Borus (B, T_{E^s}) inside G_{E^s}, which gives an identification $i_B \colon T_{E^s} \xrightarrow{\sim} \mathbf{T}_{E^s}$.

For $\gamma \in \mathrm{Gal}(E^s/E)$, γB is another Borel subgroup of G containing T_{E^s}, so we have another identification $i_{\gamma B} \colon T_{E^s} \xrightarrow{\sim} \mathbf{T}_{E^s}$.

For $\gamma \in \mathrm{Gal}(E^s/E)$, we denote by $\gamma_{\mathbf{T}}$ (resp. γ_T) the endomorphism of $\mathbf{T}_{E^s}$ (resp. T_{E^s}) induced by γ. From the definition of the abstract Cartan, we see that $\gamma_{\mathbf{T}}$ is the composition

$$\mathbf{T}_{E^s} \xrightarrow{i_B^{-1}} T_{E^s} \xrightarrow{\gamma_T} T_{E^s} \xrightarrow{i_{\gamma B}} \mathbf{T}_{E^s}.$$

For each $\gamma \in \mathrm{Gal}(E^s/E)$, the automorphism $i_B i_{\gamma B}^{-1}$ of $\mathbf{T}_{E^s}$ is given by an element of $\mathbf{W}$. Hence the function $z_{T,B} \colon \gamma \mapsto (i_B i_{\gamma B}^{-1})$ defines a cocycle in $Z^1(\mathrm{Gal}(E^s/E); \mathbf{W})$. Choosing a different B alters $z_{T,B}$ by a coboundary, hence the cohomology class $h_T := [z_{T,B}] \in \mathrm{H}^1(\mathrm{Gal}(E^s/E); \mathbf{W})$ is independent of B. Then we have

$$i_B \gamma_T i_B^{-1} = i_B (i_{\gamma B}^{-1} \gamma_{\mathbf{T}} i_B) i_B^{-1} = (i_B i_{\gamma B}^{-1}) \gamma_{\mathbf{T}}.$$

In other words, the cocycle $z_{T,B}$ expresses the "difference" between the $\mathrm{Gal}(E^s/E)$-action on T and on $\mathbf{T}$.

10.1.3. *The canonical L-embedding.* Recall from the work of Langlands-Shelstad [LS87] that given a maximal torus $T \subset G$ together with a choice of "χ-data", there is a $\widehat{G}$-conjugacy class of admissible dual embeddings ${}^L j \colon {}^L T \hookrightarrow {}^L G$. In general there is no distinguished choice of ${}^L j$, but if T is *unramified* then it has a distinguished choice of χ-data, which gives a *canonical* conjugacy class of embeddings ${}^L j \colon {}^L T \to {}^L G$. We will describe it more explicitly.

Using $\widehat{G}$-conjugation, we can arrange that $\widehat{j}$ sends $\widehat{T}$ isomorphically to $\widehat{\mathbf{T}}$. Then ${}^L j$ is specified by a cocycle with values in $N_{\widehat{G}}(\widehat{\mathbf{T}})(k)$ lifting $h_T \in \mathrm{H}^1(W_E; \mathbf{W})$. Since T is unramified, the actions factor through $\mathrm{val} \colon W_E \twoheadrightarrow \langle \varphi \rangle \cong \mathbf{Z}$, and we choose a lift inflated from $\langle \varphi \rangle$.

Fix the representative $z_{T,B}$ of h_T from §10.1.2, and choose $n \in N_{\widehat{G}}(\widehat{\mathbf{T}})(k)$ lifting $z_{T,B}(\varphi)$. Since $\langle \varphi \rangle \cong \mathbf{Z}$, the assignment $\varphi \mapsto n$ defines a cocycle lifting $z_{T,B}$. The lifts of this representative, modulo $\widehat{\mathbf{T}}(k)$-conjugacy, are parametrized by $\mathrm{H}^1(\langle \varphi \rangle; \widehat{T}(k))$. We will explicate the correct lift only in the case that T is *elliptic*, i.e., anisotropic mod center.

First suppose G is semisimple: then T is anisotropic, so $t \mapsto t\,\varphi(t)^{-1}$ is an isogeny of $\widehat{T}$, which is surjective on k-points because k is algebraically closed. Hence

$$\mathrm{H}^1(\langle \varphi \rangle; \widehat{T}(k)) = \widehat{T}(k)/\varphi\text{−conj} = \{1\},$$

hence ${}^L j$ is uniquely determined in this case by the condition of lifting h_T.

In general, consider the adjoint quotient $G \twoheadrightarrow G_{\mathrm{ad}}$. Let $\overline{T} := T/Z(G)$, so we have a pullback square

$$\begin{array}{ccc} T & \longrightarrow & \overline{T} \\ \downarrow & & \downarrow \\ G & \longrightarrow & G_{\mathrm{ad}} \end{array}$$

On the dual side we have $\widehat{G_{\mathrm{ad}}} \to \widehat{G}$ and $\widehat{\overline{T}} \to \widehat{T}$, forming a pushout square. Let ${}^L j_{\mathrm{ad}} \colon {}^L \overline{T} \to {}^L G_{\mathrm{ad}}$ be the L-embedding specified in the preceding paragraph. Then we have a commutative square

$$\begin{array}{ccc} {}^L \overline{T} & \longrightarrow & {}^L T \\ \downarrow{\scriptstyle {}^L j_{\mathrm{ad}}} & & \downarrow{\scriptstyle {}^L j} \\ {}^L G_{\mathrm{ad}} & \longrightarrow & {}^L G \end{array}$$

This discussion is phrased for the usual L-group. In §7.2, the σ-dual homomorphism is constructed first for the c-group and is then transported to the usual L-group after choosing a square root of the residue cardinality; the comparison below uses that same choice.

10.2. **Calculation of the σ-dual homomorphism.** Let $T \subset G$ be an unramified elliptic maximal torus defined over E. We use the square root fixed in §8.1 to identify ${}^L G \cong {}^c G$.

Proposition 10.2.1. *Suppose $T(E)$ contains an element s of order ℓ which maps to a strongly regular element of $G_{\mathrm{ad}}(E)$. Let $\sigma = \mathrm{conj}_s$ be the conjugation action of s on G, so $G^\sigma = T$. Then the σ-dual homomorphism ${}^L\psi \colon {}^L T \to {}^L G$ satisfies ${}^L\psi \sim {}^L j \circ \mathrm{Fr}_\ell$, where Fr_ℓ is induced by the Frobenius endomorphism $\widehat{T} \to \widehat{T}$ (corresponding to multiplication by ℓ on character groups).*

We begin with some preliminaries before turning to the proof. Any choice of $B \supset T_{E^s}$ gives a commutative diagram

$$\begin{array}{ccccc} T_{E^s} & = & T_{E^s} & = & T_{E^s} \\ \downarrow{\scriptstyle i_B} & & \downarrow & & \downarrow \\ \mathbf{T}_{E^s} & \longleftarrow & B & \longrightarrow & G_{E^s} \end{array} \tag{10.2.1}$$

By Lemma 6.5.2 and Corollary 7.2.3, diagram (10.2.1) induces an isomorphism

$$\widetilde{br} \cong (\mathrm{Fr}_\ell)^* i_B^* \, \mathrm{CT}_B[\deg_G] \colon \mathrm{Rep}_k(\widehat{G}) \longrightarrow \mathrm{Rep}_k(\widehat{T}); \tag{10.2.2}$$

here we make no assertion about W_E-actions yet. Thus the choice of B gives an identification $\widehat{i}_B^{-1} \colon \widehat{T} \xrightarrow{\sim} \widehat{\mathbf{T}}$ in such a way that the map $\check{\psi}$, the restriction of ${}^L\psi$ to the identity component, factors as in the diagram below

$$\widehat{T} \xrightarrow[\check{\phi}_B]{} \widehat{\mathbf{T}} \hookrightarrow \check{G}, \qquad \widehat{T} \xrightarrow{\check{\psi}} \check{G} \tag{10.2.3}$$

where $\check{\phi}_B$ is $\widehat{i}_B^{-1}$ composed with the Frobenius endomorphism Fr_ℓ of $\widehat{T}$. As discussed in §10.1.2, the map $\widehat{i}_B$ carries the W_E-action on $\widehat{T}$ to the W_E-action on $\widehat{\mathbf{T}}$ twisted by the cocycle $z_{T,B}$, so for all $\gamma \in W_E$ we have

$$\check{\psi} \circ \gamma_{\widehat{T}} = z_{T,B}(\gamma) \cdot (\gamma_{\widehat{\mathbf{T}}} \circ \check{\phi}_B). \tag{10.2.4}$$

This has the following consequence. By (10.2.4), the homomorphism ${}^L\psi$ carries $W_E \subset {}^L T$ into $N_{\widehat{G}}(\widehat{\mathbf{T}}) \rtimes W_E$, so its projection to the first component defines a class $\mathrm{pr}_1({}^L\psi) \in \mathrm{H}^1(W_E; N_{\widehat{G}}(\widehat{\mathbf{T}}))$. Denote by $[\mathrm{pr}_1({}^L\psi)] \in \mathrm{H}^1(W_E; \mathbf{W})$ the projection of $\mathrm{pr}_1({}^L\psi)$ along $N_{\widehat{G}}(\widehat{\mathbf{T}}) \twoheadrightarrow \mathbf{W}$. From (10.2.4) we conclude:

Lemma 10.2.2. *We have $h_T = [\mathrm{pr}_1({}^L\psi)] \in \mathrm{H}^1(W_E; \mathbf{W})$.*

Lemma 10.2.3. *The cohomology class $\mathrm{pr}_1({}^L\psi) \in \mathrm{H}^1(W_E; N_{\widehat{G}}(\widehat{\mathbf{T}}))$ is inflated from $\mathrm{H}^1(\mathbf{Z}; N_{\widehat{G}}(\widehat{\mathbf{T}}))$.*

Proof. We need to see that the cocycle can be trivialized on $I_E \subset W_E$. Since T is unramified, it can be split after passing to a finite unramified extension. Choose a finite unramified extension E'/E splitting T. Compatibility with unramified base change (Lemma 7.3.1 and Remark 7.3.2) identifies the restrictions of the classes $\mathrm{pr}_1({}^L\psi)$ and $\mathrm{pr}_1({}^L\psi_{E'})$ to $I_E = I_{E'}$. Since $T_{E'}$ is a split maximal torus, $G_{E'}$ is split; choosing a Borel subgroup over E' makes (10.2.2) $W_{E'}$-equivariant. Thus $\mathrm{pr}_1({}^L\psi_{E'})$ is cohomologically trivial on $I_{E'}$, and so is $\mathrm{pr}_1({}^L\psi)$ on I_E. Note that the warning in Remark 7.3.2 causes no difficulty here: inertia maps to 0 under the valuation $W_E \to \mathbf{Z}$, hence lies in the subgroup on which the chosen square-root identifications of L-groups and c-groups are compatible. □

Proof of Proposition 10.2.1. First consider the semisimple case. Then T is anisotropic in this case, so $\mathrm{H}^1(\langle\varphi\rangle; \widehat{T}(k)) = \{1\}$ as explained in §10.1.3, and the canonical unramified L-embedding ${}^L j$ is uniquely characterized by the requirement that its $\mathbf{W}$-valued cocycle be h_T. Hence Lemmas 10.2.2 and 10.2.3 already imply that Proposition 10.2.1 holds in this case.

If G is a torus, the assertion follows from Proposition 7.5.1. Otherwise, put $\overline{T} = T/Z(G)$. The image of s in G_{ad} has order ℓ and centralizer $\overline{T}$, which is unramified elliptic. The discussion of the semisimple case applies to $(G_{\mathrm{ad}}, \overline{T})$. Compatibility with the central quotient (Lemma 7.4.1) gives a commutative

diagram

$$\begin{array}{ccc} {}^L\overline{T} & \longrightarrow & {}^LT \\ \downarrow{\scriptstyle {}^L\psi_{\mathrm{ad}}} & & \downarrow{\scriptstyle {}^L\psi} \\ {}^LG_{\mathrm{ad}} & \longrightarrow & {}^LG \end{array}$$

Corollary 7.2.3 implies that it factors as

$$\begin{array}{ccc} {}^L\overline{T} & \longrightarrow & {}^LT \\ \downarrow{\scriptstyle \mathrm{Fr}_\ell} & & \mathrm{Fr}_\ell\downarrow \\ {}^L\overline{T} & \longrightarrow & {}^LT \\ \downarrow{\scriptstyle \zeta_{\mathrm{ad}}} & & \zeta\downarrow \\ {}^LG_{\mathrm{ad}} & \longrightarrow & {}^LG \end{array}$$

with the outer arrows ${}^L\psi_{\mathrm{ad}}$ and ${}^L\psi$.

Choose compatible representatives of the L-homomorphisms with $\zeta|_{\widehat{T}} = \widehat{j}$ and $\zeta_{\mathrm{ad}}|_{\widehat{\overline{T}}} = \widehat{j}_{\mathrm{ad}}$, using (10.2.3). Since $\zeta_{\mathrm{ad}} \sim {}^L j_{\mathrm{ad}}$, choose $g \in \widehat{G_{\mathrm{ad}}}(k)$ conjugating ζ_{ad} to ${}^L j_{\mathrm{ad}}$. Their restrictions to $\widehat{\overline{T}}$ agree, so g centralizes the maximal torus $\widehat{j}_{\mathrm{ad}}(\widehat{\overline{T}})$ and hence lies in it. Its image in $\widehat{G}(k)$ therefore centralizes $\widehat{j}(\widehat{T})$. Conjugating ζ by this image preserves its restriction to $\widehat{T}$ and makes it agree with ${}^L j$ on the image of ${}^L\overline{T}$. These two subgroups generate ${}^L T$, since ${}^L\overline{T} \to {}^L T$ induces the identity on W_E. Thus $\zeta \sim {}^L j$. □

10.3. **Tate cohomology of deep level Deligne-Lusztig varieties.** We briefly recall the generalized Deligne-Lusztig representations appearing in [CI21].

10.3.1. *Group-theoretic setup.* Let $T \hookrightarrow G$ be an unramified maximal torus and $x \in \mathcal{B}(G/E)$ be a point of the Bruhat-Tits building of G that lies in the apartment of T. Corresponding to x we have by Bruhat-Tits theory a parahoric group scheme $\mathcal{G}/\mathcal{O}_E$, whose generic fiber is G/E.

Recall that $\mathbf{F}_q$ is the residue field of E. By assumption, T splits over $\breve{E}$. Choose a $\breve{E}$-rational Borel subgroup of $G_{\breve{E}}$ containing $T_{\breve{E}}$, and let U be its unipotent radical.

For $r \in \mathbf{Z}_{\geq 0}$, we have group schemes $\mathbb{G}_r, \mathbb{T}_r$ over $\mathbf{F}_q$ as in [CO, §6.1][8] corresponding to subquotients of the Moy-Prasad filtration at x, such that

$$\mathbb{G}_r(\mathbf{F}_q) = G_{x,0:r+} := G_{x,0}/G_{x,r+} \quad \text{and} \quad \mathbb{T}_r(\mathbf{F}_q) = T_{x,0:r+} := T_{x,0}/T_{x,r+}.$$

We also have a group scheme $\mathbb{U}_r \subset (\mathbb{G}_r)_{\overline{\mathbf{F}}_q}$ corresponding to U.

10.3.2. *Deep level Deligne-Lusztig varieties.* Let Fr_q be the q-power Frobenius for schemes over $\mathbf{F}_q$. We recall certain schemes constructed in [CI21, §4]: the "deep level Deligne-Lusztig varieties"

$$S_{\mathbb{T}_r,\mathbb{U}_r} := \{g \in (\mathbb{G}_r)_{\overline{\mathbf{F}}_q} \colon g^{-1}\,\mathrm{Fr}_q(g) \in \mathbb{U}_r\}.$$

(The variety $S_{\mathbb{T}_r,\mathbb{U}_r}$ is called X_r in [CO].) It is a separated, smooth, finite type scheme over $\overline{\mathbf{F}}_q$, with an action of $\mathbb{G}_r(\mathbf{F}_q) \times \mathbb{T}_r(\mathbf{F}_q)$ by multiplication on the left and right.The action of $\mathbb{T}_r(\mathbf{F}_q)$ is free, and we define

$$X_{\mathbb{T}_r,\mathbb{U}_r} := S_{\mathbb{T}_r,\mathbb{U}_r}/\mathbb{T}_r(\mathbf{F}_q).$$

Example 10.3.1. When $r = 0$, quotienting $S_{\mathbb{T}_0,\mathbb{U}_0}$ by $\mathbb{U}_0 \cap \mathrm{Fr}_q^{-1}(\mathbb{U}_0)$ and $\mathbb{T}_0(\mathbf{F}_q)$ gives the classical Deligne–Lusztig variety in the sense of [DL76] (see [CI21, Remark 3.2]).

Definition 10.3.2 (Deep level Deligne-Lusztig induction)**.** Let $\theta\colon \mathbb{T}_r(\mathbf{F}_q) \to k^\times$ be a character. Let $\mathcal{L}_\theta$ be the corresponding local system on $X_{\mathbb{T}_r,\mathbb{U}_r}$. We define

$$R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) := \mathrm{R}\Gamma_c(X_{\mathbb{T}_r,\mathbb{U}_r}; \mathcal{L}_\theta) \in D^b(\mathrm{Rep}_k\,\mathbb{G}_r(\mathbf{F}_q)).$$

[8]The indexing seems to differ from that of [CI21, §2.5, 2.6] by 1.

10.3.3. *Calculation of Tate cohomology.* Following [CI21, §2.8], we define $W_x(T)$ to be the subgroup of $W(T,G)$ generated by the reflections corresponding to the vector parts of affine roots ψ of G for which $\psi(x)=0$.

Following [CI21, Definition 5.1], we say that $s \in G_x(\mathcal{O}_{\breve{E}})$ is *unramified very regular* (with respect to x) if s is regular semisimple in $G_{\breve{E}}$, its connected centralizer $Z_G^\circ(s)$ is an $\breve{E}$-split maximal torus of $G_{\breve{E}}$, and $\alpha(s) \not\equiv 1 \pmod{\varpi_E}$ for all roots α of $G_{\breve{E}}$ relative to $Z_G^\circ(s)$. We say that $s \in \mathbb{G}_r(\overline{\mathbf{F}}_q)$ is unramified very regular if it is the image of an unramified very regular element of $G_x(\mathcal{O}_{\breve{E}})$.

Proposition 10.3.3. *Let $s \in \mathbb{T}_r(\mathbf{F}_q)$ be unramified very regular in $\mathbb{G}_r(\overline{\mathbf{F}}_q)$ and of order ℓ. Let Σ act on $\mathbb{G}_r$ by conjugation by s. Then for each $j \in \mathbf{Z}/2\mathbf{Z}$,*

$$\mathrm{T}^j(R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)) \cong \bigoplus_{w \in W_x(T)^{\mathrm{Fr}_q}} \theta^w \in \mathrm{Rep}_k\, \mathbb{T}_r(\mathbf{F}_q). \tag{10.3.1}$$

where $\theta^w = \theta \circ w^{-1}$ is the translate of θ by the action of w.

Proof. In the canonical σ-equivariant structure on $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$, the action of σ is induced by left multiplication by s on $X_{\mathbb{T}_r,\mathbb{U}_r}$.

By [CI21, Proposition 5.6], we have

$$\mathrm{Fix}(\sigma, X_{\mathbb{T}_r,\mathbb{U}_r}) = \bigcup_{w \in W_x(T)^{\mathrm{Fr}_q}} [\dot{w}] \tag{10.3.2}$$

where $[\dot{w}]$ is the coset $\dot{w}\mathbb{T}_r(\mathbf{F}_q)$ of any Frobenius-fixed lift $\dot{w} \in \mathbb{G}_r(\mathbf{F}_q)$ of w (which exists by Lang's theorem for the connected group $\mathbb{T}_r$).

By the equivariant localization theory for schemes in Remark 4.3.4, we have

$$\mathrm{T}^j(R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)) = \mathrm{T}^j(\mathrm{R}\Gamma_c(X_{\mathbb{T}_r,\mathbb{U}_r}; \mathcal{L}_\theta)) \cong \mathrm{T}^j(\bigcup_{w \in W_x(T)^{\mathrm{Fr}_q}} [\dot{w}]; \mathcal{L}_\theta) \cong \bigoplus_{w \in W_x(T)^{\mathrm{Fr}_q}} \mathcal{L}_\theta|_{[\dot{w}]}.$$

Here $\mathcal{L}|_{[\dot{w}]}$ is a $\mathbb{T}_r(\mathbf{F}_q)$-equivariant sheaf on a point, i.e., a representation of $\mathbb{T}_r(\mathbf{F}_q)$. Note that the Σ-action on the rank-one stalk of $\mathcal{L}_\theta$ has ℓ-power order and is therefore trivial in characteristic ℓ. Writing $t\dot{w} = \dot{w}(\dot{w}^{-1}t\dot{w})$, we see that $\mathcal{L}_\theta|_{[\dot{w}]} \cong \theta^w \in \mathrm{Rep}_k\, \mathbb{T}_r(\mathbf{F}_q)$.

□

10.4. **Toral compact inductions.** We may regard $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ as a (derived) smooth representation of $G_{x,0}$ by inflation. Choose some extension $\widetilde{\theta}$ of θ to a character of $T(E)$, which we use to regard $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ as a (derived) smooth representation of $T(E)G_{x,0}$. Then we define

$$\pi_{T,U,\widetilde{\theta}} := \text{c-Ind}^{G(E)}_{T(E)G_{x,0}} R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) \in D^b(\mathrm{Rep}^{\mathrm{sm}}_k G(E)).$$

Note that since σ is inner, Example 9.1.3 applies to equip any $\Pi \in D^b(\mathrm{Rep}^{\mathrm{sm}}_k G(E))$ with a canonical Σ-equivariant structure. In particular, we use this to view $\pi_{T,U,\widetilde{\theta}} \in D^b(\mathrm{Rep}^{\mathrm{sm}}_k G(E))^{B\Sigma}$.

Let $K \subset G(E)$ be a Σ-stable open subgroup and M a bounded complex of smooth k-representations of $K \rtimes \Sigma$. We view $\text{c-Ind}^{G(E)}_K M$ as the complex of locally constant functions $f\colon G(E) \to M$ with $f(gk) = k^{-1}f(g)$ and compact support modulo K. There is a natural action of Σ on $\text{c-Ind}^{G(E)}_K M$ by $(\sigma f)(g) = \sigma(f(\sigma^{-1}(g)))$.

Lemma 10.4.1 (Tate cohomology of compact inductions)**.** *Let $K \subset G(E)$ be a Σ-stable open subgroup and M a bounded complex of smooth k-representations of $K \rtimes \Sigma$. Define*

$$\ker^1(\Sigma; K, G) := \ker\big(\mathrm{H}^1(\Sigma; K) \to \mathrm{H}^1(\Sigma; G(E))\big).$$

For each $\xi \in \ker^1(\Sigma; K, G)$ choose $g_\xi \in G(E)$ such that $c_\xi = g_\xi^{-1}\sigma(g_\xi)$ lies in K and represents ξ. Let $H_\xi = H(E) \cap g_\xi K g_\xi^{-1}$, and let M_ξ be M with H_ξ acting through $h \mapsto g_\xi^{-1}hg_\xi$ and with σ acting by $\sigma_\xi(m) = c_\xi \cdot \sigma(m)$. Then σ_ξ commutes with H_ξ, and there is a natural isomorphism

$$\mathrm{T}^n\big(\text{c-Ind}^{G(E)}_K M\big) \cong \bigoplus_{\xi \in \ker^1(\Sigma;K,G)} \text{c-Ind}^{H(E)}_{H_\xi} \mathrm{T}^n(M_\xi). \tag{10.4.1}$$

The summand of the trivial class is $\text{c-Ind}_{K^\sigma}^{H(E)} \mathrm{T}^n(M)$. *In particular, if* $\mathrm{H}^1(\Sigma; K) = 1$ *then*

$$\mathrm{T}^n\big(\text{c-Ind}_K^{G(E)} M\big) \cong \text{c-Ind}_{K^\sigma}^{H(E)} \mathrm{T}^n(M). \tag{10.4.2}$$

Proof. Note that $\text{c-Ind}_K^{G(E)} M$ is the direct sum of the functions supported on the union of the free Σ-orbits in $G(E)/K$ and those supported on $(G(E)/K)^\sigma$, and both summands are stable under $H(E)$ and Σ. The first summand is a direct sum of induced $k[\Sigma]$-complexes, one for each free orbit, all concentrated in the same degrees as M, so its Tate cohomology vanishes by §8.5.1.

A coset gK is σ-fixed if and only if $g^{-1}\sigma(g) \in K$. Replacing g by gk changes this cocycle to $k^{-1}g^{-1}\sigma(g)\sigma(k)$, and two elements with the same cocycle differ by left multiplication by $G(E)^\sigma = H(E)$. Hence the map $g \mapsto [g^{-1}\sigma(g)]$ identifies $H(E)\backslash(G(E)/K)^\sigma$ with $\ker^1(\Sigma; K, G)$, and the orbit of ξ is $H(E)g_\xi K/K \cong H(E)/H_\xi$. Functions supported on this orbit are identified as $H(E)$-representations with $\text{c-Ind}_{H_\xi}^{H(E)} M_\xi$ via the map sending such a function f to $\varphi(h) = f(hg_\xi)$. Since $\sigma^{-1}(g_\xi) = g_\xi \sigma^{-1}(c_\xi)^{-1}$ we have

$$(\sigma\varphi)(h) = \sigma\big(f(hg_\xi\sigma^{-1}(c_\xi)^{-1})\big) = c_\xi \cdot \sigma(\varphi(h)).$$

The cocycle condition gives $\sigma_\xi^\ell = 1$, and σ_ξ commutes with H_ξ because $c_\xi\sigma(g_\xi^{-1}hg_\xi)c_\xi^{-1} = g_\xi^{-1}hg_\xi$ for $h \in H_\xi$. Now Σ acts on $\text{c-Ind}_{H_\xi}^{H(E)} M_\xi$ pointwise, and compact induction from an open subgroup is exact and commutes with direct sums, so computing with the periodic resolution gives $\mathrm{T}^n(\text{c-Ind}_{H_\xi}^{H(E)} M_\xi) \cong \text{c-Ind}_{H_\xi}^{H(E)} \mathrm{T}^n(M_\xi)$. This proves (10.4.1).

□

Theorem 10.4.2. *Let $T < G$ be an elliptic unramified maximal torus. Assume that $T(E)$ contains an element s of order ℓ, which is unramified very regular with respect to $x \in \mathcal{B}(G/E)$ and strongly regular in $G_{\mathrm{ad}}(E)$. Then for any $r \in \mathbf{Z}_{\geq 0}$, any $\theta\colon \mathbb{T}_r(\mathbf{F}_q) \to k^\times$, and any extension $\widetilde{\theta}\colon T(E) \to k^\times$ of θ, $\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\pi_{T,U,\widetilde{\theta}}))$ contains the point of* $\mathrm{Spec}\ \mathrm{Exc}_k(W_E, \widehat{G})$ *corresponding to the semisimple L-parameter*

$$W_E \xrightarrow{{}^L\widetilde{\theta}} {}^L T(k) \xrightarrow{{}^L j} {}^L G(k) \tag{10.4.3}$$

where ${}^L\widetilde{\theta}$ is the L-parameter given by class field theory, and ${}^L j$ is the canonical L-embedding of T (§10.1.3).

Proof. Let $\sigma \in \mathrm{Aut}(G)$ be the conjugation by s, so $G^\sigma = T$. Note that the hypotheses on s imply that Proposition 10.2.1 holds in this situation.

Let $K = T(E)G_{x,0}$. By the unramifiedness and ellipticity of T, we have $T(E)G_{x,0} = Z_G(E)G_{x,0}$, where Z_G denotes the center of G; this is recorded in [CO, p. 1508], and follows from the standard equality $T(E) = Z_G(E)T_{x,0}$ for elliptic tori T. From the long exact sequence in non-abelian cohomology

$$1 \to K^\sigma \to G(E)^\sigma \to (G(E)/K)^\sigma \to \mathrm{H}^1(\Sigma; K) \to \mathrm{H}^1(\Sigma; G(E))$$

and the torsor-shifting discussion in [Ser94, §I.5.3–I.5.4], we see that

$$(G(E)/K)^\sigma = \bigcup_{\xi \in \ker[\mathrm{H}^1(\Sigma;K) \to \mathrm{H}^1(\Sigma;G(E))]} G(E)^\sigma/({}_\xi K)^\sigma$$

where ${}_\xi K$ is the twist of K corresponding to the torsor ξ. Henceforth we abbreviate

$$\ker^1(\Sigma; K, G) := \ker[\mathrm{H}^1(\Sigma; K) \to \mathrm{H}^1(\Sigma; G(E))].$$

We claim that each quotient $G(E)^\sigma/({}_\xi K)^\sigma$ is a point, which we will denote by $[\xi]$. Indeed, a coset $gK \in G(E)/K$ is fixed by σ if and only if $s \in gKg^{-1}$, hence s fixes $g \cdot x \in \mathcal{B}(G/E)$. But by tame descent [KP23, Ch. 12], the fixed-point set of s on the reduced building $\mathcal{B}(G/E)$ is the image of $\mathcal{B}(Z_G(s)^\circ/E) = \mathcal{B}(T/E)$, which is the singleton x because T is elliptic. We deduce that $g \cdot x = x$, so g normalizes $G_{x,0}$, and then

$$gKg^{-1} = Z_G(E)\, gG_{x,0}g^{-1} = Z_G(E)G_{x,0} = K \supset T(E).$$

Thus the stabilizer of gK in $G(E)^\sigma = T(E)$ is all of $T(E)$, proving the claim. Lemma 10.4.1, applied to $M = R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ extended to K by $\widetilde{\theta}$ and with σ acting through s, gives an isomorphism of $T(E)$-representations

$$\mathrm{T}^j(\pi_{T,U,\widetilde{\theta}}) \cong \bigoplus_{\xi \in \ker^1(\Sigma;K,G)} \mathrm{T}^j\big(R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)_\xi\big),$$

since $H_\xi = T(E)$ for every ξ. For the trivial class, σ acts through s as before, so Proposition 10.3.3 identifies this summand with a direct sum of Weyl conjugates of $\widetilde{\theta}$.

Taking $w = 1$ in (10.3.1) shows that $\mathrm{T}^j(\pi_{T,U,\widetilde{\theta}})$ contains $\widetilde{\theta}$. Since Tate cohomology commutes with coefficient twists, Theorem 9.1.1, applied to $\pi^{(\ell^{-1})}_{T,U,\widetilde{\theta}}$, shows that $F_{G*} \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})} \mathrm{H}^*(\pi^{(\ell^{-1})}_{T,U,\widetilde{\theta}})$ contains the image under ${}^L\psi$ of the parameter of $\widetilde{\theta}^{(\ell^{-1})}$. For tori, this is the parameter given by class field theory [FS21, Proposition IX.6.5],

$$W_E \xrightarrow{{}^L(\widetilde{\theta}^{(\ell^{-1})})} {}^L T(k). \tag{10.4.4}$$

By Proposition 10.2.1, this image is

$${}^L\psi \circ {}^L(\widetilde{\theta}^{(\ell^{-1})}) \sim {}^L j \circ \mathrm{Fr}_\ell \circ {}^L(\widetilde{\theta}^{(\ell^{-1})}) = {}^L j \circ {}^L\widetilde{\theta}.$$

The support identity in the proof of Lemma 8.5.3 gives

$$F_{G*} \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})} \mathrm{H}^*(\pi^{(\ell^{-1})}_{T,U,\widetilde{\theta}}) = \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})} \mathrm{H}^*(\pi_{T,U,\widetilde{\theta}}),$$

which proves the assertion. □

Corollary 10.4.3. *There is a non-zero irreducible $G(E)$-subquotient Π of $\mathrm{H}^i(\pi_{T,U,\widetilde{\theta}})$, for some i, such that ρ_Π is given by* (10.4.3)*. In particular, if $\pi_{T,U,\widetilde{\theta}}$ is concentrated in a single cohomological degree and is irreducible, then its Fargues-Scholze parameter is given by* (10.4.3)*.*

Proof. Let $\mathfrak{m} \subset \mathrm{Exc}_k(W_E, \widehat{G})$ be the maximal ideal corresponding to (10.4.3). By Theorem 10.4.2, $\mathfrak{m}$ lies in the support of the $\mathrm{Exc}_k(W_E, \widehat{G})$-module $M := \mathrm{H}^i(\pi_{T,U,\widetilde{\theta}})$ for some i. It then suffices to show that there exists an actual irreducible subquotient of M on which $\mathrm{Exc}_k(W_E, \widehat{G})$ acts with kernel $\mathfrak{m}$. The representation M is compactly induced from a finite-dimensional representation, hence is finitely generated. Then [DHKM24, Corollary 3.5(1)] says that M is admissible over $\mathrm{Exc}_k(W_E, \widehat{G})$, which implies that it has a simple subquotient with support $\mathfrak{m}$, as desired. □

Remark 10.4.4. Note that Theorem 10.4.2 imposes no regularity conditions on θ. We expect that if θ is sufficiently regular, and ℓ is not too small, then $\pi_{T,U,\widetilde{\theta}}$ should be concentrated in a single cohomological degree and irreducible, and cuspidal. There are interesting classes of examples where this is known; a notable one pertains to the *depth-zero supercuspidals* studied in [KV06, DR09], which correspond to the case where:

- $x \in \mathcal{B}(G/E)$ is a vertex (so G_x is a maximal parahoric),
- $r = 0$, so $X_{\mathbb{T}_0,\mathbb{U}_0}$ is a usual Deligne-Lusztig variety up to an affine fibration,
- $\mathbb{T}_0$ is not contained in any proper $\mathbf{F}_q$-rational parabolic subgroup of $\mathbb{G}_0$, so the Deligne-Lusztig induction is cuspidal [DL76, §8], and θ is non-singular (meaning that it is not orthogonal to any coroot, cf. [DL76, Definition 5.15(1)]).

Letting $\mathcal{L}_\theta$ be the character sheaf on $X_{\mathbb{T}_0,\mathbb{U}_0}$ associated with the non-singular character $\theta\colon \mathbb{T}_0(\mathbf{F}_q) \to k^\times$, it follows from [DL76, Lemma 9.14] and [Bro90, Lemma 3.5] that in this situation $\mathrm{H}^*_c(X_{\mathbb{T}_0,\mathbb{U}_0}; \mathcal{L}_\theta)$ concentrates in a single degree; and if the prime-to-ℓ component of θ is in general position, then $\mathrm{H}^*_c(X_{\mathbb{T}_0,\mathbb{U}_0}; \mathcal{L}_\theta)$ is irreducible by [Bro90, Corollary 3.6].

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