MODULAR FUNCTORIALITY IN THE LOCAL LANGLANDS CORRESPONDENCE

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Abstract. We develop a theory of Smith-Treumann localization and relative parity sheaves in the context of Fargues-Scholze's Geometrization of the Local Langlands Correspondence. We then apply this theory to prove some conjectures of Treumann-Venkatesh concerning mod-ℓ Local Langlands functoriality between a reductive group G and its fixed subgroup under an order ℓ automorphism. As another application, we explicitly calculate the Fargues-Scholze parameters of certain mod-ℓ toral representations.

Contents

1. Introduction

1.1. **Modular Langlands functoriality.** Let E be a local field of residue characteristic p and G be a reductive group over E. For simplicity, we assume for now that G is split; beyond the Introduction, the text always treats general G. The Local Langlands Correspondence predicts, vaguely speaking, that the smooth representations of $G(E)$ over a field k are controlled by the *Langlands dual group* $\check{G}$ over k.

For example, letting $W_E \subset \mathrm{Gal}(E^s/E)$ be the Weil group of E, the Local Langlands Correspondence predicts that there should be a natural parametrization of irreducible smooth representations of $G(E)$ over k by "L-parameters", which are homomorphisms $W_E \to \check{G}(k)$ up to conjugacy. The existence of such a parametrization has surprising implications for representation theory: for example, a homomorphism $\check{\psi} \colon \check{H} \to \check{G}$ of dual groups induces an obvious map

$$\{L\text{-parameters for } \check{H}\} \xrightarrow{\check{\psi}_*} \{L\text{-parameters for } \check{G}\}$$

and therefore suggests some *Langlands functoriality* operation from (packets of) irreducible representations of $H(E)$ to (packets of) irreducible representations of $G(E)$. In practice, it is usually difficult to construct such operations directly, or to describe them explicitly in representation-theoretic terms.

In this paper we investigate Langlands functoriality for a specific class of dual homomorphisms $\check{\psi}$ that was identified by Treumann-Venkatesh in [TV16]. It considers the situation where the reductive group H arises as the fixed points of an order ℓ automorphism σ of another reductive group G, where ℓ is a prime different from p. Furthermore, we take k to be a field of characteristic ℓ. In this situation (and under some hypotheses), Treumann-Venkatesh constructed a dual homomorphism $\check{\psi} \colon \check{H} \to \check{G}$ over k, with a specific property in terms of the Satake isomorphism that we will discuss later. We refer to this situation as *modular functoriality*, because its construction depends on special features of modular arithmetic (and this paper depends in turn on special features of modular representation theory). By contrast, Langlands' original conjectures were

Type	A_n	B_n, D_n	C_n	G_2, F_4, E_6	E_7	E_8
$b(\Phi)$	1	2	n	3	19	31

FIGURE 1. Excluded primes for each root system.

focused on the case where k has characteristic zero. We note that in many examples the map $\check{\psi}$ does not lift to characteristic zero, and the resulting functoriality is truly specific to modular arithmetic.

The story extends to non-split groups. In that case, the dual group $\check{G}$ should be augmented to the L-group ${}^L G \cong \widehat{G} \rtimes W_E$, where the action of W_E on $\widehat{G}$ reflects the twisting of G over E relative to its split form. This generalization is necessary to treat some of the most interesting examples.

Example 1.1.1. A familiar example is cyclic base change, where $G = \mathrm{Res}_{E'/E} H$ for a cyclic ℓ-extension E'/E and σ is a generator of $\mathrm{Gal}(E'/E)$ acting on G in the natural way, so that $G^\sigma = H$. This particular example lifts to characteristic 0, but many do not; several examples are tabulated in the ArXiv version of Treumann-Venkatesh's paper [TV14]. We also consider here some interesting examples that are ruled out by the hypotheses of [TV14, TV16]: a useful one is where σ is conjugation by a strongly regular element of G, in which case H is a (not necessarily split!) maximal torus. Note that the work of Treumann-Venkatesh only treats examples where G is simply connected and H is semi-simple.

1.2. **Conjectures of Treumann-Venkatesh.** We describe some conjectures of Treumann-Venkatesh that will be proved in this paper, up to technical hypotheses that exclude small ℓ. We again restrict our attention to the case where G and H are split, for simplicity. We let $\mathscr{H}(G, K)$ be the spherical Hecke algebra of G (with respect to some maximal compact subgroup K stable under σ) with coefficients in k. Then the Satake isomorphism supplies a k-algebra isomorphism

$$\mathscr{H}(G, K) \cong \mathcal{O}(\check{G} /\!\!/ \check{G})$$

Under some technical assumptions, Treumann-Venkatesh construct a k-algebra homomorphism $\mathrm{br} \colon \mathscr{H}(G, K) \to \mathscr{H}(H, K^\sigma)$ which they call the *(normalized) Brauer homomorphism.* (The construction uses in an essential way the assumption that char $k = \ell = \mathrm{ord}(\sigma)$.) This implies the existence of a commutative diagram

$$\begin{tikzcd} \mathscr{H}(G, K) \arrow[r, "\sim", "\text{Satake}"'] \arrow[d, "\mathrm{br}"] & \mathcal{O}(\check{G} /\!\!/ \check{G}) \arrow[d, dashed] \\ \mathscr{H}(H, K^\sigma) \arrow[r, "\sim", "\text{Satake}"'] & \mathcal{O}(\check{H} /\!\!/ \check{H}) \end{tikzcd}$$

It is then natural to ask if the dashed map is induced by restriction along a homomorphism $\check{\psi} \colon \check{H} \to \check{G}$. If so, then $\check{\psi}$ will be called a "σ-dual homomorphism". One of the main theorems of Treumann-Venkatesh [TV16, §1.3] is that if G is simply connected and H is semisimple, then a σ-dual homomorphism exists, with three possible exceptions if G has type E_6. In [TV14, §1.4(iii)], Treumann-Venkatesh pose the open problem of constructing a σ-dual homomorphism in full generality. Our first theorem addresses this question.

Theorem 1.2.1. *Suppose $\ell > \max\{b(\check{G}), b(\check{H})\}$ where $b(\check{G})$ is the maximum over the bad primes $b(\check{\Phi})$ over all root systems $\check{\Phi}$ of the simple factors of G, tabulated in Figure 1, and $b(\check{H})$ is defined similarly. Then a σ-dual homomorphism $\check{\psi} \colon \check{H} \to \check{G}$ exists.*

Remark 1.2.2. Theorem 1.2.1 is not a strict improvement on the work of Treumann-Venkatesh, as our characteristic assumption actually rules out many interesting examples. On the other hand, relaxing the condition that H be semisimple is also interesting in examples; even the case where H is a torus is very useful, as we shall discuss below.

The construction of the σ-dual homomorphism in [TV16] is a tour de force: it invokes classification theorems to tabulate all examples of modular functoriality, and analyzes them case-by-case. By contrast, the proof of Theorem 1.2.1 is completely uniform across all cases, and makes no use of classification theorems. The idea is quite natural: instead of contemplating the dual groups, we try to *categorify* the Brauer homomorphism br to a *Brauer functor* $\widetilde{br}$ from the Satake category of G to the Satake category of H. After equipping this Brauer functor with a Tannakian structure, we obtain the σ-dual homomorphism for free from the Geometric Satake equivalence.[1]

$$\begin{tikzcd} \mathrm{Sat}(\mathrm{Gr}_G; k) \arrow[r, "\sim", "\text{Geom. Satake}"'] \arrow[d, "\widetilde{br}"] & \mathrm{Rep}_k(\check{G}) \arrow[d, dashed] \\ \mathrm{Sat}(\mathrm{Gr}_H; k) \arrow[r, "\sim", "\text{Geom. Satake}"'] & \mathrm{Rep}_k(\check{H}) \end{tikzcd}$$

Although this idea is simple, its implementation is quite involved and will be elaborated upon later; for now we just mention that for the sole purpose of producing the σ-dual homomorphism, it should be easy to improve the assumption $\ell > \max\{b(\check{G}), b(\check{H})\}$ to "ℓ is a good prime for $\check{G}$ and $\check{H}$", which for example holds as long as $\ell > 5$. The extra inefficiency in our Theorem 1.2.1 comes from our desire not to work with the usual Geometric Satake equivalence, but with the version on the $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannians occurring in the work of Fargues-Scholze [FS21], whose geometric representation theory is not as developed.

We next turn to describe conjectures of Treumann-Venkatesh pertaining to the Local Langlands Correspondence. We write $\mathbf{Z}[\sigma]$ for the group ring of $\langle\sigma\rangle \cong \mathbf{Z}/\ell\mathbf{Z}$ and let $N := 1 + \sigma + \ldots + \sigma^{\ell-1} \in \mathbf{Z}[\sigma]$. If Π is a representation of $G(E) \rtimes \sigma$, then its *Tate cohomology* groups $\mathrm{T}^j(\Pi)$, for $j \in \mathbf{Z}/2\mathbf{Z}$, are defined as

$$\mathrm{T}^0(\Pi) := \frac{\ker(1-\sigma \mid \Pi)}{\mathrm{Im}\,(N \mid \Pi)} \qquad \mathrm{T}^1(\Pi) := \frac{\ker(N \mid \Pi)}{\mathrm{Im}\,(1-\sigma \mid \Pi)} \tag{1.1}$$

and the $G(E)$-action on Π induces an $H(E)$-action on each $\mathrm{T}^j(\Pi)$.

Conjecture 1.2.3 ([TV16, Conjecture 6.3])**.** *Let Π be an irreducible smooth representation of $G(E)$ whose isomorphism class is fixed by σ, so that the $G(E)$-action on Π uniquely extends to a $G(E) \rtimes \sigma$-action by* [TV16, Proposition 6.1]. *Then for each $j \in \mathbf{Z}/2\mathbf{Z}$:*

(1) (Admissibility Conjecture) $\mathrm{T}^j(\Pi)$ is admissible as a representation of $H(E)$.

(2) (Functoriality Conjecture) The L-parameter of every irreducible $H(E)$-subquotient of $\mathrm{T}^j(\Pi)$ is sent by the σ-dual homomorphism $\check{\psi}$ to the L-parameter for $\Pi^{(\ell)} := \Pi \otimes_{k,\mathrm{Frob}} \ell$, the Frobenius twist of Π.

We do not consider Part (1) of Conjecture 1.2.3 in this paper; it is the subject of current work-in-progress. We will focus on Part (2). At the time of its formulation, the Local Langlands Correspondence was constructed only for certain families of groups, so the precise meaning of Conjecture 1.2.3(2) was left vague for general groups. We may now formulate a precise version for all G using the work of Fargues-Scholze [FS21]; this is what we will describe next.

Example 1.2.4. For $G = \mathrm{GL}_n$, Vignéras had constructed in [Vig01] the Local Langlands Correspondence over k (using the characteristic zero version due to Harris-Taylor [HT01]), many years before the work of Fargues-Scholze. This gives a precise meaning to Conjecture 1.2.3(2) when G and H are both general linear groups, and in the context of cyclic base change, special cases have been proven by other authors, as will be discussed more in §1.6. Even for this case, where the statements of our results can be formulated in classical terms, the proofs will utilize the work of Fargues-Scholze.

1.3. **The Fargues-Scholze correspondence.**

An output of the work [FS21] of Fargues-Scholze is a map

$$\left\{ \begin{matrix} \text{irreducible admissible representations} \\ \Pi \text{ of } G(E) \text{ over } k \end{matrix} \right\} / \sim \longrightarrow \left\{ \begin{matrix} \text{semi-simple } L\text{-parameters} \\ \rho_\Pi \colon W_E \to {}^L G(k) \end{matrix} \right\} / \sim \tag{1.2}$$

that we call the *Fargues-Scholze correspondence*. We refer to ρ_Π, which is most naturally regarded as an element of $\mathrm{H}^1(W_E; \widehat{G}(k))$, as the *Fargues-Scholze parameter* of Π. The map $\Pi \mapsto \rho_\Pi$ is expected to be the semi-simplification of "the" hypothetical Local Langlands Correspondence. For specific groups including tori and GL_n, the Local Langlands Correspondence has been constructed previously (by class field theory and by Harris-Taylor, respectively), and in these cases it is known that the Fargues-Scholze correspondence is compatible with the previous construction.

[1] A subtle but important point found by Treumann-Venkatesh is that, when taking into account the full L-group, a σ-dual homomorphism may not exist with the "usual" notion of L-group due to Langlands. Treumann-Venkatesh suggest in [TV14, §7.8] that this problem might be repaired by instead using the "c-group", which is the variant of the L-group that naturally comes out of the geometric Satake equivalence [Zhu17, Remark 5.5.11]. For non-archimedean local fields there is actually an isomorphism between the L-group and the c-group (possibly depending on a choice of square root of p in k), so this distinction is not essential for our purposes, but it seems to support the morality of our approach.

For most groups, the Fargues-Scholze correspondence is quite mysterious. For example, it is expected that (1.2) is surjective and has finite fibers, but proving this is wide open. For *regular supercuspidal representations* of quite general groups, Kaletha [Kal19] has prescribed explicit constructions for the L-parameters, which are strongly supported by the expected endoscopic character relations; while for general groups almost nothing is known about the Fargues-Scholze parameters of such representations. We remark that in the cases where the Fargues-Scholze correspondence has been explicated, there have been important geometric implications; an example is the work of Koshikawa [Kos21] towards torsion vanishing in the cohomology of Shimura varieties, where the relevant compatibility with the classical LLC is proven in [HKW22].

1.3.1. *The functoriality conjecture.* We prove the following result concerning modular functoriality in the Fargues-Scholze correspondence, which we take as fulfilling Conjecture 1.2.3(2), away from small ℓ.

Theorem 1.3.1. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. Let Π be an irreducible smooth representation of $G(E)$ whose isomorphism class is fixed by σ. Then for each $j \in \mathbf{Z}/2\mathbf{Z}$ and every $H(E)$-irreducible subquotient π of $\mathrm{T}^j(\Pi)$, the σ-dual homomorphism $\check{\psi}_* \colon \mathrm{H}^1(W_E; \widehat{H}(k)) \to \mathrm{H}^1(W_E; \widehat{G}(k))$ sends $\rho_\pi \mapsto \rho_{\Pi^{(\ell)}}$.*

This result appears as Corollary 9.3.4 in the main text. It is a consequence of a more powerful statement, Theorem 9.1.1, which treats "derived smooth representations" $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$. The derived version is more useful for calculations but less elementary to formulate, so we do not state it here.

Theorem 1.3.1 does not seem to be made any easier by assuming the full categorical conjectures of [FS21], or the conjectural compatibility with Kaletha's explicit Local Langlands Correspondence, or any other standard conjectures about the Local Langlands Correspondence that we know. On the other hand, it may not be unreasonable to speculate that Theorem 1.3.1 could be useful for proving some of these other conjectures. For example, as an application of (the derived version of) Theorem 1.3.1, we calculate explicitly the Fargues-Scholze parameters of certain classes of representations.

1.3.2. *Explicit calculation of Fargues-Scholze parameters.* The basic idea is that if the L-parameter $\rho_\Pi \colon W_E \to {}^L G(k)$ admits some factorization

$$\begin{tikzcd} W_E \arrow[rr, "\rho_\Pi"] \arrow[dr, dashed] & & {}^L G(k) \\ & {}^L H(k) \arrow[ur, "{}^L\psi"'] & \end{tikzcd} \tag{1.3}$$

through an L-parameter into ${}^L H(k)$ where H is a *torus* arising as the fixed points of some appropriate σ, then Theorem 1.3.1 identifies ρ_Π explicitly in terms of the L-parameter of $\mathrm{T}^j(\Pi)$, which is computable since the Fargues-Scholze correspondence is completely understood for tori. Note that "most" supercuspidal parameters factor through the L-group of *some* (not necessarily split) torus; for example, all of them factor in this way if G is tamely ramified and p does not divide the order of the Weyl group of G.

We apply this idea to the toral representations considered in work of Chan-Oi [CO]. They are generalizations of depth-zero supercuspidals, with an analogous construction but instead using "deeper-level Deligne-Lusztig representations" studied in work of Chan-Ivanov [CI21]; consequently, they include supercuspidal representations with arbitrarily high depth. The input for the (modular version of the) Chan-Oi construction is an elliptic unramified torus $T \subset G$ and a character $\theta \colon T(E) \to k^\times$.

Under the assumption that $\ell > \max\{b(\check{G}), b(\check{H})\}$, and that T contains a strongly regular element of order ℓ, we prove (Corollary 10.4.2) that some irreducible constituent in the Chan-Oi construction has Fargues-Scholze parameter of the form

$$W_E \xrightarrow{{}^L\theta} {}^L T(k) \xrightarrow{{}^L j} {}^L G(k), \tag{1.4}$$

where ${}^L j$ is the canonical embedding of the L-group of an unramified maximal torus. This is morally in accordance with the prediction with Kaletha's explicit Local Langlands Correspondence for regular supercuspidal representations in [Kal19]; we say "morally" because Kaletha does not consider modular coefficients. Because the precise definition of a toral representation is complicated, we defer the precise formulation of our result to Theorem 10.4.1.

We emphasize that the simple appearance of (1.4) belies the intricacy of Kaletha's recipe, which for example involves twisting the most natural (from a representation-theoretic perspective) guess for the L-parameter by a subtle "twisting character". In the relatively simple example of epipelagic representations of unitary groups, these characters were explicated in [FRT20] and found to be already very complicated there. The geometric construction of Chan-Oi somehow bakes this twisting character into the geometry (see [CO, §8] for more discussion of this point) which is then reflected in our computation.

1.4. **Further results.** We mention some further results.

1.4.1. *Existence of functorial lifts.* The following Theorem guarantees the existence of functorial lifts along any σ-dual homomorphism.

Theorem 1.4.1. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. Let π be an irreducible smooth representation of $H(E)$ over k, with Fargues-Scholze parameter $\rho_\pi \in \mathrm{H}^1(W_E; \widehat{H}(k))$. Then there exists an irreducible smooth representation Π of $G(E)$ over k with Fargues-Scholze parameter $\rho_\Pi \cong \check{\psi} \circ \rho_\pi \in \mathrm{H}^1(W_E; \widehat{G}(k))$.*

The more general version (allowing non-split groups) is Theorem 9.5.1. The more general version allows to treat further examples such as the following.

Example 1.4.2 (Base change)**.** Taking $G = \mathrm{Res}_{E'/E} H$ for a cyclic ℓ-extension E'/E and σ a generator of $\mathrm{Gal}(E'/E)$ acting in the natural way, Theorem 1.4.1 asserts the existence of base change along E'/E. In equal characteristic and for the Genestier-Lafforgue correspondence, this was established in [Fen24, Theorem 1.1].

Example 1.4.3. Take σ to be conjugation by a strongly regular ℓ-torsion element of G. Then H is a torus, so every $\rho \in \mathrm{H}^1(W_E; \widehat{T}(k))$ is realized as a Fargues-Scholze parameter. Then Theorem 1.4.1 implies that every $\rho \in \mathrm{H}^1(W_E; \widehat{G}(k))$ that factors through ${}^L\psi$ is realized as a Fargues-Scholze parameter. It would be interesting to investigate which ${}^L\psi$ can arise as a σ-dual homomorphism, as this might help to show surjectivity of the Fargues-Scholze correspondence over k: recall that if p is not too small relative to G, then every supercuspidal L-parameter factors through the L-group of some torus, and we know that the Fargues-Scholze correspondence is surjective for tori.

1.4.2. *Functoriality for the Bernstein center.* Let $\mathfrak{Z}(G; k)$ be the Bernstein center of G with coefficients in k, and similarly for H. The Fargues-Scholze correspondence (1.2) is deduced from the construction of a k-algebra homomorphism

$$\mathrm{FS}_G \colon \mathrm{Exc}_k(W_E; \widehat{G}) \to \mathfrak{Z}(G; k) \tag{1.5}$$

where $\mathrm{Exc}_k(W_E; \widehat{G})$ is the *excursion algebra* (over k).

Building on ideas of Treumann-Venkatesh, we construct a map of Bernstein centers

$$\mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G; k) \to \mathfrak{Z}(H; k) \tag{1.6}$$

which we call the *Treumann-Venkatesh homomorphism.* We show in Theorem 9.3.2 that if $\ell > \max\{b(\check{G}), b(\check{H})\}$, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Exc}_k(W_E, \widehat{G}) \arrow[r, "\check{\psi}^*"] \arrow[d, "\mathrm{FS}_G"'] & \mathrm{Exc}_k(W_E, \widehat{H}) \arrow[d, "\mathrm{FS}_H"] \\ \mathfrak{Z}(G) \arrow[r, "\mathfrak{Z}_{\mathrm{TV}}"] & \mathfrak{Z}(H) \end{tikzcd} \tag{1.7}$$

This shows that $\mathfrak{Z}_{\mathrm{TV}}$ realizes functoriality for the Bernstein center. In fact, it is the main ingredient in the proof of Theorem 1.4.1.

1.5. **Methods.** The proofs of the aforementioned results synthesize recent breakthroughs in several different areas of mathematics, including:

(1) The work [TV16, Tre19] of Treumann-Venkatesh on *Smith theory* and Langlands functoriality.
(2) The work [JMW14, JMW16] of Juteau-Mautner-Williamson on *parity sheaves* in modular representation theory.
(3) The work [FS21] of Fargues-Scholze on the *geometrization of the Local Langlands Correspondence.*

Let us sketch vaguely how these ingredients fit together. In the sketch below, we use some abbreviated and simplified notation which does not match that of the main body of the text.

We begin by commenting on the proof of Theorem 1.2.1. As explained earlier, the proof is based on categorifying the Brauer homomorphism of Treumann-Venkatesh to a *Brauer functor*, from perverse sheaves on the affine Grassmannian Gr_G for G to perverse sheaves on the affine Grassmannian Gr_H for H, which should be Tannakian (i.e., additive, symmetric monoidal, and compatible with the fiber functor).

The construction of the Brauer homomorphism is based on the observation that the *restriction of σ-equivariant functions* from $G(E)/K$ to $H(E)/K^\sigma$ has the miraculous property of being compatible with convolution specifically in characteristic ℓ (the order of σ). The naive categorification of this operation would be restriction of sheaves from Gr_G to Gr_H, but the miracle does not (naively) persist to the level of sheaves. Instead we apply an operation that we call *Smith-Treumann localization*, which is restriction followed by a certain Verdier quotient. The Smith-Treumann localization functor is then monoidal to some extent, but does not interact well with perversity. However it turns out that it can be made to interact well with *parity sheaves* in the sense of Juteau-Mautner-Williamson. This allows to lift Smith-Treumann localization to a functor from perverse parity sheaves on Gr_G to perverse parity sheaves on Gr_H. Then studying the interaction of parity and perversity on affine Grassmannians allows to extend the functor to all perverse sheaves (under conditions on ℓ).

The preceding construction could have been executed on the "classical" affine Grassmannian as soon as ℓ is good for $\check{G}$ and $\check{H}$.[2] However, at the next step we want to combine it with the constructions of Fargues-Scholze, so we actually need carry everything out on the $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannians and their Beilinson-Drinfeld variants, which are built out of the period rings of p-adic Hodge theory. As those objects live in the world of p-adic geometry, which behaves very differently from algebraic geometry in several key aspects (for example, there is no good theory of constructible sheaves), there are substantial technical difficulties to overcome in order to develop the appropriate technology in this new setting. We defer discussion of these technical difficulties to the individual sections in which they appear. The formalism that we develop here lays the groundwork for further applications of geometric representation theory to p-adic geometry, which we hope to pursue in future work.

$$
\begin{tikzcd}
\mathrm{Rep}_k(\check{G}) \arrow[r, "\check{\psi}^*"] \arrow[d, "\text{Geom. Satake}"] & \mathrm{Rep}_k(\check{H}) \arrow[d, "\text{Geom. Satake}"'] \\
\mathrm{Sat}(\mathrm{Gr}_G; k) \arrow[r, "\widetilde{br}"] \arrow[d, "\mathrm{Sat}_G"] & \mathrm{Sat}(\mathrm{Gr}_H; k) \arrow[d, "\mathrm{Sat}_H"'] \\
\boxed{\mathrm{Tate}(\mathrm{R}\Gamma_c(\mathrm{Sht}_G, -))} \arrow[r, leftrightarrow, dashed, "\text{equiv. localization}"] & \boxed{\mathrm{Tate}(\mathrm{R}\Gamma_c(\mathrm{Sht}_H, -))}
\end{tikzcd}
$$

FIGURE 2. This cartoon (produced with the aid of ChatGPT-3.5 after much coaxing) depicts the Brauer functor $\widetilde{br}$ interfacing with the Tate cohomology of moduli of local shtukas and with the σ-dual homomorphism $\check{\psi}$.

Next we turn towards the proof of Theorem 1.3.1. For this we need to integrate the Brauer functor with the construction of the Fargues-Scholze correspondence for G and for H. The Fargues-Scholze correspondence for G can be obtained by constructing excursion operators on the cohomology of *moduli spaces* Sht_G *of local G-shtukas* with coefficients in Satake sheaves coming from Gr_G. We need to compare such cohomology groups to the ones obtained from applying the Brauer functor to get Satake sheaves on Gr_H, transporting them to the moduli spaces Sht_H of local H-shtukas, and then taking cohomology. This comparison is mediated by *equivariant localization* for Tate cohomology. Figure 2 depicts a cartoon of the strategy. From this

[2]Explicitly, this means that $\ell > 2$ if $\check{G}$ has simple factors of type B, C or D; $\ell > 3$ if $\widehat{G}$ has simple factors of type G_2, F_4, E_6, E_7; and $\ell > 5$ if $\widehat{G}$ has simple factors of type E_8.

comparison, we extract relations among certain excursion operators for $\check{G}$ and for $\check{H}$, which are ultimately used to establish Theorem 1.3.1.

1.6. **Related work.** We note some related work on the problems studied here.

1.6.1. *The functoriality conjecture.* We focus first on the "functoriality conjecture", Conjecture 1.2.3(2). Ronchetti [Ron16] studied Conjecture 1.2.3(2) in the special case of cyclic base change for GL_n, proving it for depth-zero cuspidal representations "of level zero and minimal-maximal type".

Dhar-Nadimpalli [DN23] studied Conjecture 1.2.3(2) in the special case of cyclic base change for GL_n, proving it for generic representations (i.e., those possessing mod-ℓ Whittaker models).

The earlier work of the author [Fen24] proved the analogous statement to Theorem 1.3.1 for the Genestier-Lafforgue correspondence (in equal characteristic) in the special case of cyclic base change, and in doing so introduced a primordial form of some ideas developed here. We point out several differences.

- The methods of [Fen24] were based on local-global compatibility, and were thus fundamentally restricted to the function field setting. Here our methods are purely local, and we encompass both mixed characteristic and equal characteristic local fields. To do this, we work in the context of p-adic geometry, which presents substantial new difficulties.
- The argument of [Fen24] was restricted to the case of cyclic base change, for both local and global reasons. On the *local* side, the σ-dual homomorphism is obvious for cyclic base change, and the construction of the Brauer functor in that case exploited some simplifying special features of cyclic base change; the present paper debuts more general and conceptual arguments to treat the local aspects in arbitrary generality (as long as ℓ is not too small). We still do not know how to generalize the *global* arguments even in the function field case, but fortunately they are irrelevant for the new approach.

The author's work [BFH$^+$] with Böckle-Harris-Khare-Thorne explicates Conjecture 1.2.3(2) for the Genestier-Lafforgue correspondence in the special case of cyclic base change for toral supercuspidals, assuming several conjectures about torsion in the cohomology of deep-level Deligne-Lusztig varieties. The analogous results for the Fargues-Scholze correspondence are subsumed by the L-parameter calculations in §10.

1.6.2. *The σ-dual homomorphism.* The work [RW22] of Riche-Williamson, which gives a geometric proof of the linkage principle, is not logically related to problems we study here, but has some philosophical similarities. We apply Smith-Treumann localization from Gr_G to Gr_H coming from an automorphism of G, while Riche-Williamson apply (a slightly different form of) Smith-Treumann localization to a self-embedding of Gr_G coming from the action of $\mu_\ell \subset \mathbf{G}_m$ via loop rotation.

The paper [LL21] of Leslie-Lonergan also applies Smith-Treumann localization along this self-embedding, in attempt to give a geometric construction of the Frobenius contraction functor on $\mathrm{Rep}_k(\check{G})$. The formalism of localization that they develop is more similar to the one that we use in this paper.

1.7. **Organization of the paper.** We now indicate the structure of the paper.

In §2 we collect some notation and abbreviations used commonly throughout the paper.

In §3 we develop a general formalism of Smith-Treumann localization for diamonds, which refers to a certain type of sheaf restriction operation from a diamond Y to its σ-fixed points. Then in §4 we calculate the σ-fixed points of various diamonds associated to G, such as the $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannian and its Beilinson-Drinfeld or twisted variants, as well as moduli spaces of local G-shtukas. These calculations are used when applying Smith-Treumann localization to such spaces.

In §5, we develop a notion of "relative parity complexes" on the Beilinson-Drinfeld affine Grassmannians arising in p-adic geometry. Then in §6 we prove that away from small characteristics, (normalized) indecomposable relative parity complexes are relative perverse and correspond under the Geometric Satake equivalence to tilting modules; this is analogous to a theorem of Juteau-Mautner-Williamson from [JMW16]. In fact, each of §3, §5, and §6 is parallel to some existing theory in algebraic geometry, but there are non-trivial new issues encountered in the setting of p-adic geometry, which we try to illuminate at the beginnings of the respective sections.

In §7, we construct the Brauer functor and establish its properties, proving Theorem 1.2.1. Then in §8 we study the (Tate) cohomology of moduli spaces of local shtukas, and integrate it with the Brauer functor in order to prove certain identities of excursion operators. The applications ripen for picking in §9, where we combine the preceding sections to prove Theorem 1.3.1, Theorem 1.4.1, construct the Treumann-Venkatesh homomorphism and establish the commutative diagram (1.7).

Finally, in §10 we study the example where σ is conjugation by a strongly regular order-ℓ element of an unramified elliptic maximal torus of G. We calculate the σ-dual embedding, the Tate cohomology of deep level Deligne-Lusztig varieties and their compact inductions, and deduce explicit computations of Fargues-Scholze parameters.

The beginnings of individual sections summarize their contents in more detail.

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2. Notation

We fix a prime p.

2.1. **Local fields.** We let E be a local field of residue characteristic p, $\mathcal{O}_E$ its ring of integers, ϖ_E a uniformizer, and $\mathbf{F}_q = \mathcal{O}_E/\varpi_E$ its residue field.

Let $\breve{E}$ be the completion of the maximal unramified extension of E. For a reductive group G/E, $B(G)$ denotes the Kottwitz set of G, i.e., elements of $G(\breve{E})$ modulo Frobenius-conjugacy.

Let E^s be a separable closure of E and $W_E \subset \mathrm{Gal}(E^s/E)$ be the Weil group of E.

2.2. **The group Σ.** Throughout, ℓ denotes a prime number different from p and Σ denotes a finite cyclic group of order ℓ. The notation σ always denotes a generator of Σ; conversely, if an automorphism σ is constructed first then Σ always denotes the group generated by σ. Some constructions (e.g., Tate cohomology) are phrased in terms of σ, but ultimately all constructions are independent of the choice of σ.

We denote by N the element $1 + \sigma + \ldots + \sigma^{\ell-1} \in \mathbf{Z}[\Sigma]$.

For an object Y with an action of Σ, we denote by Y^σ or $\mathrm{Fix}(\sigma, Y)$ the σ-fixed points of Y.

2.3. **Reductive groups.** Throughout, G denotes a reductive group over E. When the notation H appears, it refers to a reductive group arising as the subgroup of G fixed by an action of Σ. We denote by $\iota \colon H \to G$ the tautological embedding, and we use the same notation for induced maps such as $H(E) \to G(E)$, $B(H) \to B(G)$, $\mathrm{Gr}_H \to \mathrm{Gr}_G$, etc.

For a torus T, we denote by $X_*(T)$ and $X^*(T)$ the cocharacter and character groups of T, respectively, and by $X_*(T)^+$ and $X^*(T)^+$ the subsets of dominant (co)characters.

We denote by $\mathrm{Rep}_k^{\mathrm{sm}} G(E)$ the category of smooth representations of $G(E)$ over k.

2.4. **Perfectoid spaces.** The notation (C, C^+) will always mean that C is an algebraically closed perfectoid field and $C^+ \subset C$ is an open bounded valuation subring.

Following [FS21], we denote by $\mathrm{Perf} = \mathrm{Perf}_{\overline{\mathbf{F}}_p}$ the category of perfectoid spaces over $\overline{\mathbf{F}}_p$.

2.5. **Coefficients.** Throughout, k denotes an algebraically closed field of characteristic ℓ. We denote by $\mathbb{O} := W(k)$ the Witt vectors of k.

We write $\mathrm{Frob} \colon k \to k$ for the absolute Frobenius automorphism $x \mapsto x^\ell$.

2.6. **Sheaves.** We will use Λ to denote a coefficient ring which is finite over $\mathbb{O}$. In particular, Λ is ℓ-adically complete. The six-functor formalism for such sheaves is developed in [Sch22] (the adic case is in §27 of *loc. cit.*).

Definition 2.6.1. We say that a map $f\colon Y \to Y'$ of small v-stacks on Perf is *shriekable* if it is compactifiable, representable in locally spatial diamonds, and has locally finite dim. trg . This is exactly the hypothesis in [Sch22, §1] for $Rf_!\colon D_{\text{ét}}(Y;\Lambda) \to D_{\text{ét}}(Y';\Lambda)$ to exist, and under the same assumptions its right adjoint $Rf^!\colon D_{\text{ét}}(Y';\Lambda) \to D_{\text{ét}}(Y;\Lambda)$ exists.

For a shriekable map $f\colon Y \to Y'$, we denote by $\mathbb{D}_{Y/Y'}$ the relative Verdier duality functor,

$$\mathbb{D}_{Y/Y'}(-) := \mathcal{RH}om_{D_{\text{ét}}(Y;\Lambda)}(-, f^!\Lambda).$$

We abuse notation and write $\mathbb{D}_{Y/Y'} := \mathbb{D}_{Y/Y'}(\Lambda) = f^!\Lambda$ for the relative dualizing sheaf. We omit Y' from the notation in the case $Y' = \operatorname{Spd}\overline{\mathbf{F}}_p$.

2.7. **Categories.** For an abelian category C with an action of a group Γ, we let $\mathsf{C}^{B\Gamma}$ denote the category of Γ-equivariant objects in C. This comes equipped with a forgetful functor to C.

For a triangulated category C with a natural stable ∞-categorical enhancement, then by $\mathsf{C}^{B\Gamma}$ we mean the homotopy category of the Γ-equivariant objects in its stable ∞-categorical enhancement.

For an A-linear abelian category C and a commutative ring homomorphism $A \to B$, we abbreviate

$$\mathsf{C} \otimes_A B := \mathsf{C} \otimes_{A-\text{Mod}} (B - \text{Mod}).$$

When $A = \mathbb{O}$ and $B = k$, we write

$$\mathbb{F}\colon \mathsf{C} \to \mathsf{C} \otimes_{\mathbb{O}} k$$

for the tautological base change.

3. Smith-Treumann localization for diamonds

In this section we develop for diamonds a form of *Smith-Treumann localization*, which refers to a type of sheaf-theoretic equivariant localization from a Σ-equivariant space to its Σ-fixed points. There is a loose analogy between Smith-Treumann localization and the perhaps more familiar hyperbolic localization for spaces with a $\mathbf{G}_m$-action.

The output of the theory looks similar to that for complex algebraic varieties or schemes (developed in [Tre19] and [RW22]). However, the details of the proofs are quite different, owing to the different behavior of étale sheaves on adic spaces. For example, the theory relies crucially on finiteness results, which in the case of schemes comes from quasi-compactness (among other things). On the other hand, non-quasicompactness is ubiquitous in p-adic geometry. For example, the complement of a closed subspace is usually not quasi-compact, hence the extension-by-zero from such a complement is not constructible; this example already illustrates that we will frequently need to contend with non-constructible sheaves.

The contents of this section are as follows. In §3.1 we codify the notion of "extra small" v-stacks. The adjective "extra small" is a slight strengthening of "small", and guarantees that we can approximate the v-stack by quasicompact subspaces. In §3.2, we define the (large) "Tate category" of a locally spatial diamond, following ideas of Treumann [Tre19] but with modifications as in [Fen24, §3] to allow sheaves with infinite-dimensional stalks. In §3.4 we prove technical bounds for the tor-dimension of various sheaf operations, in order to show in that these operations pass to the Tate category, which we do in §3.5. In §3.6 we define the "Smith operation", a kind of localization functor to the Σ-fixed points, and establish compatibility properties that may be thought of as forms of equivariant localization. Finally, in §3.7 we define Tate cohomology and various generalizations of it.

3.1. **Extra small v-stacks.** Let ω_1 be the first uncountable cardinal. Recall that an ω_1-*cofiltered* inverse system I is one for which any functor $J \to I$, with J a countable category, extends to $J^{\triangleleft} \to I$, where the cone $J^{\triangleleft}$ consists of J plus an extra object with a unique map to each object of J.

Definition 3.1.1 (Extra small v-stacks)**.** We say that a v-stack Y is *extra small* if for any ω_1-cofiltered inverse system $\{S_i = \operatorname{Spa}(R_i, R_i^+)\}_{i\in I}$ of affinoid perfectoid spaces with inverse limit $S = \operatorname{Spa}(R, R^+)$, the natural map

$$Y(S) \to \varinjlim_{i\in I} Y(S_i)$$

is an isomorphism.

The following Lemma justifies the terminology.

Lemma 3.1.2. *If a v-stack Y is extra small, then Y is small.*

Proof. The proof is contained in the first paragraph of [FS21, Proof of Proposition III.1.3]. □

Remark 3.1.3. Informally speaking, all the small v-stacks that have come up in our experience are also extra small. The remark below [FS21, Proposition III.1.3] suggests that all "reasonable" v-stacks are extra small.

Example 3.1.4. The last paragraph of the proof of [FS21, Proposition III.1.3] shows that Bun_G is extra small. By minor variations on this argument, the following v-stacks are also extra small.

(1) The local Hecke stack $\mathcal{H}\mathrm{ck}_{G/(\mathrm{Div}^1_X)^I}$ [FS21, Definition VI.1.6], and the global Hecke stack Hck^I_G [FS21, §IX.2].
(2) The Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I}$ [FS21, Definition VI.I.8].
(3) The moduli spaces of local shtukas $\mathrm{Sht}_{(G,b,\mu_\bullet),K}$ [SW20, Lecture 23].

The importance of extra smallness (for us) is to control the cohomological dimension of direct image along open embeddings. For quasi-compact open embeddings, the direct image is actually exact, but we will typically be contending with non-quasicompact open embeddings. Under hypotheses that spaces are extra small, we will be able to approximate open embeddings by quasicompact ones, as articulated in the following Lemma (pointed out by Peter Scholze).

Lemma 3.1.5. *Let Y be a locally spatial diamond admitting a countable open cover by spatial diamonds Y_n. Let $Z \hookrightarrow Y$ be a closed embedding such that $Z \cap Y_n$ is extra small for each n. Then $U := Y \setminus Z$ is a countable union of quasicompact open subspaces.*

Proof. Let $Z_n := Z \cap Y_n$. If $Y_n \setminus Z_n$ is a countable union of quasicompact open subsets for each n, then

$$Y \setminus Z = \bigcup_n (Y_n \setminus Z_n)$$

is also a countable union of quasicompact open subsets. Therefore, renaming Y_n to Y and Z_n to Z, we may assume that Y is spatial and Z is extra small, with the goal of showing that $U := Y \setminus Z$ is a countable union of quasicompact open subsets.

Consider the collection $\mathcal{Z}$ of closed sub-diamonds $Z_i \subset Y$ such that

- $Z_i \supset Z$, and
- $Y \setminus Z_i$ is a countable union of quasi-compact open subsets.

Viewing $\mathcal{Z}$ as a partially ordered set under inclusion, we claim that $\mathcal{Z}$ is ω_1-cofiltered. Indeed, if I is countable then we have that

$$Y \setminus \bigcap_{i \in I} Z_i = \bigcup_{i \in I} (Y \setminus Z_i)$$

is a countable union of countable unions of quasi-compact open subsets, so $\bigcap_{i \in I} Z_i \in \mathcal{Z}$.

There is an obvious map

$$Z \to \varprojlim_{Z_i \in \mathcal{Z}} Z_i, \tag{3.1}$$

which we claim is an isomorphism. Recall that closed sub-diamonds of Y are determined by their underlying topological spaces, as follows from [Sch22, Proposition 11.20] (the generalization of this statement to v-stacks is [AGLR22, Proposition 1.3]), so it suffices to check (3.1) at the level of topological spaces. Any point of the open subspace $|Y \setminus Z|$ lies in a quasicompact open subset thereof, since Y is spatial. This shows the second equality in the sequence of identifications

$$|\varprojlim_{Z_i \in \mathcal{Z}} Z_i| = \varprojlim_{Z_i \in \mathcal{Z}} |Z_i| = |Z|,$$

verifying the claim.

Since Y is spatial, there exists [Sch22, Proposition 11.24] a quasi-pro-étale cover of Y by a strictly totally disconnected space, which is of the form

$$\mathrm{Spa}(B, B^+) \twoheadrightarrow Y \tag{3.2}$$

by [Sch22, Proposition 1.15]. Recall that a closed subspace of a strictly totally disconnected space is again strictly totally disconnected by [Sch22, Proposition 7.16], and then affinoid by [Sch22, Proposition 1.15]. Hence for each $Z_i \in \mathcal{Z}$, the cover (3.2) pulls back to a cover $\mathrm{Spa}(A_i, A_i^+) \twoheadrightarrow Z_i$ where $\mathrm{Spa}(A_i, A_i^+)$ is strictly totally disconnected. We showed that the system $\{\mathrm{Spa}(A_i, A_i^+)\}_{Z_i \in \mathcal{Z}}$ is ω_1-cofiltered. Then, letting $\mathrm{Spa}(A, A^+) := \varprojlim_{Z_i \in \mathcal{Z}} \mathrm{Spa}(A_i, A_i^+)$,[3] the composite map

$$\mathrm{Spa}(A, A^+) = \varprojlim_{Z_i \in \mathcal{Z}} \mathrm{Spa}(A_i, A_i^+) \to \varprojlim_{Z_i \in \mathcal{Z}} Z_i = Z$$

factors through some $\mathrm{Spa}(A_i, A_i^+) \to Z$ since Z is extra small. This gives a retract $Z_i \to Z$ over Y, which must then be an isomorphism since these are closed subdiamonds of Y. Hence $Z \in \mathcal{Z}$, which means by definition that $Y \setminus Z$ is a countable union of quasi-compact open subsets. □

Motivated by Lemma 3.1.5, we make the following definition.

Definition 3.1.6. We say that a locally spatial diamond Y is *(ω_0-)locally spatial* if Y can be written as a (countable) union $\bigcup_n Y_n$ of open spatial sub-diamonds. (Recall that $\omega_0 = \aleph_0$ is the cardinality of the natural numbers.) We say that a closed subdiamond $Z \hookrightarrow Y$ is *ω_0-locally extra small* if such a union can be arranged so that $Y_n \cap Z$ is extra small for each n.

We say that a morphism of v-stacks $f : Y' \to Y$ is (ω_0-)*locally spatial* if for any spatial diamond X and any map $X \to Y$, the fibered product $X \times_Y Y'$ is (ω_0-)locally spatial.

Example 3.1.7. Let $i : Z \hookrightarrow Y$ be a closed embedding of ω_0-locally spatial diamonds. If $Z \subset Y$ is ω_0-locally extra small, then Lemma 3.1.5 says that the complementary open embedding $j : U \hookrightarrow Y$ is ω_0-locally spatial.

Definition 3.1.8. For $d \geq 0$, we say that a locally spatial diamond Y has *ℓ-cohomological dimension $\leq d$* if its cohomology on étale $\mathbf{F}_\ell$-sheaves has amplitude in $[0, d]$. We say that Y is *ℓ-bounded* if has ℓ-cohomological dimension $\leq d$ for some $d < \infty$.

Example 3.1.9. The following statements are used to bound the ℓ-cohomological dimension in practice. Recall from [Sch22, Definition 21.1] that the *dimension* of a locally spectral space is the supremum of the lengths of chains of specializations. By [Sch22, Proposition 21.11], if Y is a spatial diamond of finite dimension, then

$$Y \text{ has } \ell\text{-cohomological dimension} \leq \dim Y + \sup_y \mathrm{cd}_\ell\, y$$

where y runs through maximal points of Y. According to [Sch22, Proposition 21.16], if $f : Y \to \mathrm{Spa}(C, C^+)$ is a map of locally spatial diamonds, then $\mathrm{cd}_\ell\, y \leq \dim.\mathrm{trg}\, f$ for all maximal points $y \in Y$.

Lemma 3.1.10. *Let A be a noetherian ring. Let Y be an ω_0-locally spatial diamond with ℓ-cohomological dimension $\leq d$. Let $Z \hookrightarrow Y$ be an ω_0-locally extra small closed subdiamond with (open) complement $j : U \hookrightarrow Y$. Then $\mathrm{R}j_* : D_{\acute{e}t}(U; A) \to D_{\acute{e}t}(Y; A)$ has cohomological dimension $\leq d + 1$.*

Proof. The statement is local, so we may immediately reduce to the case that Y is a spatial diamond and $Z \hookrightarrow Y$ is extra small.

By Lemma 3.1.5, there is a countable sequence of quasicompact open embeddings $j_n : U_n \hookrightarrow Y$ such that $U_n \subset U$ and $\varinjlim_n U_n = U$. Hence we have

$$\mathrm{R}j_* \cong \mathrm{R}\varprojlim_n \mathrm{R}j_{n*} j_n^* : D_{\text{ét}}(U; A) \to D_{\text{ét}}(Y; A).$$

By [Sch22, Lemma 21.13], the (derived) direct image along a *quasi-compact* open embedding is exact, hence has cohomological dimension zero. This applies to j_n for every n. Therefore, it suffices to see that the derived inverse limit functor $\mathrm{R}\varprojlim_n$, along a countable index set, has finite cohomological dimension. Thanks to this countability, $\mathrm{R}\varprojlim_n A_n$ can be expressed as

$$\mathrm{R}\varprojlim_n A_n \cong \mathrm{cone}(\mathrm{R}\prod_n A_n \to \mathrm{R}\prod_n A_n)[-1] \tag{3.3}$$

[3] So $A = \varinjlim A_i$, which we note is already complete since any Cauchy sequence already lies in some A_i by the ω_1-cofiltered condition.

where $\mathrm{R}\prod_n$ is the derived functor of the product, defined as the product of injective resolutions of the terms. By the standard generalities in homological algebra, $\mathrm{R}\prod_n$ can be calculated using a product of acyclic (for the global sections functor) resolutions, which can be taken to be of length bounded by the cohomological dimension of Y, which by assumption is $d < \infty$. Then from (3.3) we see that $\mathrm{R}\varprojlim_n$ has cohomological dimension at most $d+1$.

∎

Corollary 3.1.11. *In the situation of Lemma 3.1.10, if $\mathcal{K} \in D^b_{ét}(U; A)$ has finite tor-dimension then $\mathrm{R}j_*\mathcal{K} \in D_{ét}(Y; A)$ has finite tor-dimension.*

Proof. This is a special case of [SGA73, Exposé XVII, Théorème 5.2.11], which applies because $\mathrm{R}j_*$ has finite cohomological dimension by Lemma 3.1.10 and the fact [Sch22, Proposition 14.3] that $Y_{\text{ét}}$ has enough points, thanks to Y being locally spatial.

∎

3.2. **The Tate category.** Let Λ be a finite commutative $W(k)$-algebra; we will be most interested in the cases $\Lambda = k$ or $W(k)$. Recall that Σ is a cyclic group of order ℓ and $\Lambda[\Sigma]$ denotes its group ring.

Definition 3.2.1. Let Y be a small v-stack. We define $\mathrm{Flat}^b(Y; \Lambda[\Sigma]) \subset D^b_{\text{ét}}(Y; \Lambda[\Sigma])$ to be the full subcategory consisting of complexes with finite tor-dimension [Sta20, Tag 0651], i.e., which have tor-amplitude in $[a, b]$ for some $a, b \in \mathbf{Z}$.

Definition 3.2.2. We define the *(large) Tate category of* Y (with respect to Λ) to be the Verdier quotient category

$$\mathrm{Shv}(Y; \mathcal{T}_\Lambda) := D^b_{\text{ét}}(Y; \Lambda[\Sigma]) / \mathrm{Flat}^b(Y; \Lambda[\Sigma]).$$

(A "small" variant of the Tate category will appear later in §5.2.) We denote the tautological projection map from $D^b_{\text{ét}}(Y; \Lambda[\Sigma])$ to $\mathrm{Shv}(Y; \mathcal{T}_\Lambda)$ by

$$\mathbb{T}^* \colon D^b_{\text{ét}}(Y; \Lambda[\Sigma]) \to \mathrm{Shv}(Y; \mathcal{T}_\Lambda).$$

Remark 3.2.3. We will only be using the large Tate category for $\Lambda = k$, but some arguments are repeated in the small version which also gets used for $\Lambda = W(k)$, so we formulate it with more general coefficients.

For $\Lambda = k$, the following Lemma (pointed out by Jesper Grodal) gives an alternative characterization of the subcategory $\mathrm{Flat}^b(Y; k[\Sigma])$, showing that it is determined by conditions on geometric stalks.

Lemma 3.2.4. *The subcategory $\mathrm{Flat}^b(Y; k[\Sigma]) \subset D^b_{ét}(Y; k[\Sigma])$ coincides with the full subcategory spanned by objects whose stalks at all geometric points have finite tor-dimension over $k[\Sigma]$.*

Proof. We apply [Sta20, Tag 0DJJ] which immediately gives one containment: if $\mathcal{K} \in D^b_{\text{ét}}(Y; k[\Sigma])$ has finite tor-dimension over $k[\Sigma]$, then all its stalks at geometric points have finite tor-dimension over $k[\Sigma]$.

For the other containment, again by [Sta20, Tag 0DJJ] it suffices to show that if all the geometric stalks of $\mathcal{K} \in D^b_{\text{ét}}(Y; k[\Sigma])$ have finite tor-amplitude, then there is a *uniform* bound on the tor-amplitude of the geometric stalks. To see this, since $\mathcal{K}$ is bounded we may pick a bounded complex representing it, say concentrated in degrees $[a, b]$. It then suffices to show: if $\mathcal{K}$ is a complex of $k[\Sigma]$-modules concentrated in degrees $[a, b]$ and has finite tor-dimension, then in fact its tor-amplitude lies in $[a, b]$.

Since $k[\Sigma]$ is Artinian, the properties of being flat and projective over $k[\Sigma]$ coincide by [Sta20, Tag 051E]. Therefore $\mathcal{K}$ also has finite projective dimension [Sta20, Tag 0A5M]. Furthermore, since $k[\Sigma]$ has finitistic dimension 0 (by [RG71], the finitistic dimension of a commutative ring is bounded by the Krull dimension), the projective amplitude of $\mathcal{K}$ lies in $[a, b]$. Hence $\mathcal{K}$ is represented by a complex of projective $k[\Sigma]$-modules supported in degrees $[a, b]$, and therefore has tor-amplitude in $[a, b]$. ∎

Definition 3.2.5. Let $\epsilon \colon \Lambda[\Sigma] \twoheadrightarrow \Lambda$ be the augmentation and $\epsilon^* \colon D^b_{\text{ét}}(Y; \Lambda) \to D^b_{\text{ét}}(Y; \Lambda[\Sigma])$ be pullback along ϵ. Define the functor

$$\mathbb{T} := \mathbb{T}^* \epsilon^* \colon D^b_{\text{ét}}(Y; \Lambda) \to \mathrm{Shv}(Y; \mathcal{T}_\Lambda).$$

Remark 3.2.6. The notation $\mathbb{T}, \mathbb{T}^*, \epsilon^*$ follows that of Leslie-Lonergan [LL21].

3.3. **Fixed points.** For an endomorphism σ of a v-stack Y, we write $\mathrm{Fix}(\sigma, Y) := Y^\sigma$ for the fibered product

$$\begin{tikzcd} Y^\sigma \arrow[r] \arrow[d] & Y \arrow[d, "\mathrm{Id} \times \sigma"] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}$$

Observe that if Y is a separated locally spatial diamond, then Δ is a closed embedding, hence $Y^\sigma \to Y$ is a closed embedding. We will only be interested in this situation. We denote by $\Sigma = \langle \sigma \rangle$ the cyclic group of order ℓ. Recall the equivariant bounded derived category $D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma}$, which is the homotopy category of $\mathcal{D}^b_{\text{ét}}(Y; \Lambda)^{B\Sigma}$, the ∞-category of functors from $B\Sigma$ to the ∞-category $\mathcal{D}^b_{\text{ét}}(Y; \Lambda)$ from [Sch22, §17].

The Σ-action on Y induces the trivial Σ-action on Y^σ, for which we have an equivalence of derived categories

$$D^b_{\text{ét}}(Y^\sigma; \Lambda)^{B\Sigma} \cong D^b_{\text{ét}}(Y^\sigma; \Lambda[\Sigma]). \tag{3.4}$$

3.4. **Bounding tor-dimension.** Here we establish technical results that show the "permanence" of finite tor-dimension under various operations.

Lemma 3.4.1. *Let Y be a locally spatial diamond with a free action of Σ. Let $q\colon Y \to Y/\Sigma$ denote the quotient. Then for any $\mathcal{F} \in D^b_{\acute{e}t}(Y; \Lambda[\Sigma])$, we have $q_*\mathcal{F} \in \mathrm{Flat}^b(Y/\Sigma; \Lambda[\Sigma]) \subset D^b_{\acute{e}t}(Y/\Sigma; \Lambda[\Sigma])$.*

Proof. It suffices to show that for any open subset $U \subset Y/\Sigma$, the $\Lambda[\Sigma]$-module $q_*\mathcal{F}(U)$ is free. Indeed, since Σ acts freely on Y we have

$$q^{-1}(U) = \coprod_{i \in \mathbf{Z}/\ell\mathbf{Z}}^{\ell} U_i$$

where q maps each U_i isomorphically to U and σ cyclically permutes $U_i \mapsto U_{i+1}$. Then

$$q_*\mathcal{F}(U) \cong \mathcal{F}(q^{-1}(U)) \cong \prod_{i=1}^{\ell} \mathcal{F}(U_i)$$

where σ acts by sending $(f_i \in \mathcal{F}(U_i))_{i \in \mathbf{Z}/\ell\mathbf{Z}} \mapsto (\sigma^*(f_i) \in \mathcal{F}(U_{i-1}))_{i \in \mathbf{Z}/\ell\mathbf{Z}}$, which is visibly free. □

Lemma 3.4.2. *Let Y be an ℓ-bounded, separated, ω_0-locally spatial diamond with an action of Σ. Assume that the embedding of the Σ-fixed points $i\colon Y^\sigma \hookrightarrow Y$ is ω_0-locally extra small. Let $U := Y \setminus Y^\sigma$ and $j\colon U \hookrightarrow Y$ be its inclusion into Y. Then for any $\mathcal{F} \in D^b_{\acute{e}t}(U; \Lambda[\Sigma])$, we have $i^*\mathrm{R}j_*\mathcal{F} \in \mathrm{Flat}^b(Y^\sigma; \Lambda[\Sigma])$.*

Proof. Let Y/Σ be the quotient diamond; then Y/Σ is also ℓ-bounded, ω_0-locally spatial, and $Y^\sigma \hookrightarrow Y/\Sigma$ is ω_0-locally extra small. Since the map $q\colon Y \to Y/\Sigma$ is totally ramified over Y^σ, the composition

$$Y^\sigma \xrightarrow{i} Y \xrightarrow{q} Y/\Sigma$$

is a closed embedding, which we denote $\bar{i}$, and the square

$$\begin{tikzcd} Y^\sigma \arrow[r, "i"] \arrow[d, equal] & Y \arrow[d, "q"] \\ Y^\sigma \arrow[r, "\bar{i}"] & Y/\Sigma \end{tikzcd} \tag{3.5}$$

is Cartesian. Applying base change to it, we have an isomorphism

$$i^*\mathrm{R}j_*\mathcal{F} \cong \bar{i}^* q_* \mathrm{R}j_*\mathcal{F} \in D^b_{\text{ét}}(Y^\sigma; \Lambda[\Sigma]). \tag{3.6}$$

From the commutativity of the diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[d, "q_U"] & Y \arrow[d, "q"] \\ U/\Sigma \arrow[r, hook, "\bar{j}"] & Y/\Sigma \end{tikzcd}$$

we have $\bar{i}^* q_* \mathrm{R}j_*\mathcal{F} \cong \bar{i}^* \mathrm{R}\bar{j}_* q_{U*}\mathcal{F}$. Now Lemma 3.4.1 implies that $q_{U*}\mathcal{F}$ has finite tor-dimension. The extra smallness hypothesis allows us to apply Corollary 3.1.11 to deduce that $\mathrm{R}\bar{j}_* q_{U*}\mathcal{F}$ also has finite tor-dimension, hence $\bar{i}^* \mathrm{R}\bar{j}_* q_{U*}\mathcal{F}$ also has finite tor-dimension, and then we conclude using (3.6).

□

Lemma 3.4.3. *Retain the notation and assumptions from Lemma 3.4.2. Then for any $\mathcal{K} \in D^b_{ét}(Y;\Lambda)^{B\Sigma}$, the cone of the natural map $i^!\mathcal{K} \to i^*\mathcal{K}$ belongs to* $\mathrm{Flat}^b(Y^\sigma;\Lambda[\Sigma])$.

Proof. We will first recall the construction of the natural map. Let $j\colon Y \setminus Y^\sigma \hookrightarrow Y$. Consider the exact triangle $i_*i^!\mathcal{K} \to \mathcal{K} \to j_*j^*\mathcal{K}$ in $D^b_{\text{ét}}(Y;\Lambda)^{B\Sigma}$. Applying i^* to it yields the exact triangle in $D^b_{\text{ét}}(Y^\sigma;\Lambda)^{B\Sigma}$,

$$i^!\mathcal{K} \to i^*\mathcal{K} \to i^*\mathrm{R}j_*j^*\mathcal{K},$$

in which the first map is the one from the statement of Lemma 3.4.3. Then Lemma 3.4.2 implies that $\mathrm{Cone}(i^!\mathcal{K} \to i^*\mathcal{K}) \cong i^*\mathrm{R}j_*j^*\mathcal{K}$ lies in $\mathrm{Flat}^b(Y^\sigma;\Lambda[\Sigma])$, as desired. □

Lemma 3.4.4. *Let A be a finite Λ-algebra (not necessarily commutative[4]). Let Y and S be locally spatial diamonds, and let $f\colon Y \to S$ be shriekable.*

(1) Then $\mathrm{R}f_!\colon D^b_{ét}(Y;A) \to D^b_{ét}(S;A)$ has cohomological dimension $\leq 3\dim.\mathrm{trg}\, f$.

(2) If furthermore S has ℓ-cohomological dimension $\leq d$ and f is ω_0-locally spatial, then $\mathrm{R}f_*\colon D^b_{ét}(Y;A) \to D^b_{ét}(S;A)$ has cohomological dimension $\leq 3\dim.\mathrm{trg}\, f + d + 1$.

Proof. (1) By the hypothesis that Y and f are locally spatial, we may write $Y \cong \varinjlim_i Y_i$ as a filtered colimit of spatial open subdiamonds $j_i\colon Y_i \hookrightarrow Y$. We have by definition [Sch22, p.134-135] that

$$\mathrm{R}f_! \cong \varinjlim_i \mathrm{R}(f|_{Y_i})_! j_i^*.$$

By [Sch22, Theorem 22.5], each $\mathrm{R}(f|_{Y_i})_!$ has cohomological dimension at most $3\dim.\mathrm{trg}\, f$. Since j_i^* is exact and filtered colimits are exact, we deduce that $Rf_!$ also has cohomological dimension at most $3\dim.\mathrm{trg}\, f$.

(2) Similarly to (1), we have

$$\mathrm{R}f_* \cong \mathrm{R}\varprojlim_i \mathrm{R}(f|_{Y_i})_* j_i^*$$

where now we may arrange the indexing set to be countable, thanks to the hypothesis that f is ω_0-locally spatial. Each $\mathrm{R}(f|_{Y_i})_*j_i^*$ has cohomological dimension at most $3\dim.\mathrm{trg}\, f$ by [Sch22, Theorem 22.5]. Then we bound the cohomological dimension of the derived inverse limit as in the proof of Lemma 3.1.10.

□

Lemma 3.4.5. *Let A be a finite Λ-algebra (not necessarily commutative). Let Y and S be locally spatial diamonds, and let $f\colon Y \to S$ be shriekable.*

(1) Then $\mathrm{R}f_!\colon D^b_{ét}(Y;A) \to D^b_{ét}(S;A)$ carries $\mathrm{Flat}^b(Y;A)$ to $\mathrm{Flat}^b(S;A)$.

(2) If furthermore S is ℓ-bounded and f is ω_0-locally spatial, then $\mathrm{R}f_*$ carries $\mathrm{Flat}^b(Y;A)$ to $\mathrm{Flat}^b(S;A)$.

Proof. (1) Suppose $\mathcal{F} \in \mathrm{Flat}^b(Y;A)$. We need to verify that for any sheaf M (concentrated in degree 0) in $D^b_{\text{ét}}(S;A)$, we have $\mathrm{H}^{-i}(M \overset{\mathrm{L}}{\otimes} \mathrm{R}f_!\mathcal{F}) = 0$ for $i \gg 0$. By the projection formula, we have

$$\mathrm{H}^{-i}(M \overset{\mathrm{L}}{\otimes}_A \mathrm{R}f_!\mathcal{F}) \cong \mathrm{H}^{-i}(\mathrm{R}f_!(f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F})).$$

By the assumption that $\mathcal{F} \in \mathrm{Flat}^b(Y;A)$, $f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F}$ is bounded. Then by Lemma 3.4.4(1), $\mathrm{R}f_!(f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F})$ is bounded, so $\mathrm{H}^{-i}(\mathrm{R}f_!(f^*M \overset{\mathrm{L}}{\otimes}_A \mathcal{F})) = 0$ for $i \gg 0$, as desired.

(2) This follows from [SGA73, Exposé XVII, Théorème 5.2.11], which applies because $\mathrm{R}f_*$ has finite cohomological dimension by Lemma 3.4.4 and the fact [Sch22, Proposition 14.3] that $S_{\text{ét}}$ has enough points, thanks to S being locally spatial. □

Corollary 3.4.6. *Let Y and S be locally spatial diamonds, and let $f\colon Y \to S$ be shriekable. Suppose Σ acts trivially on S and freely on Y, and f is Σ-equivariant.*

(1) Then $\mathrm{R}f_!\colon D^b_{ét}(Y;\Lambda[\Sigma]) \to D^b_{ét}(S;\Lambda[\Sigma])$ lands in $\mathrm{Flat}^b(Y;\Lambda[\Sigma])$.

(2) If furthermore S has ℓ-cohomological dimension $\leq d$ and f is ω_0-locally spatial, then $\mathrm{R}f_*$ carries $\mathrm{Flat}^b(Y;\Lambda[\Sigma])$ to $\mathrm{Flat}^b(S;\Lambda[\Sigma])$.

[4] Although we will only apply this Lemma in the case where A is commutative.

Proof. By the hypotheses, we may factor f as the composition of Σ-equivariant maps

$$Y \xrightarrow{q} Y/\Sigma \xrightarrow{\overline{f}} S.$$

For (1), apply Lemma 3.4.1 to $q_!$ and Lemma 3.4.5(1) to $\mathrm{R}\overline{f}_!$ with $A := \Lambda[\Sigma]$. For (2), apply Lemma 3.4.1 to $q_* = q_!$ and Lemma 3.4.5(2) to $\mathrm{R}\overline{f}_*$ with $A := \Lambda[\Sigma]$. □

3.5. **Functors on Tate categories.** Let $f: Y \to S$ denote a Σ-equivariant map of separated locally spatial diamonds with Σ-action.

3.5.1. *Pullback.* Since the pullback functor $f^*: D^b_{\text{ét}}(S^\sigma; k)^{B\Sigma} \to D^b_{\text{ét}}(Y^\sigma; k)^{B\Sigma}$ preserves stalks, Lemma 3.2.4 implies that it carries $\mathrm{Flat}^b(S^\sigma; k[\Sigma])$ to $\mathrm{Flat}^b(Y^\sigma; k[\Sigma])$, hence induces a functor

$$f^*: \mathrm{Shv}(S^\sigma; \mathcal{T}_k) \to \mathrm{Shv}(Y^\sigma; \mathcal{T}_k).$$

3.5.2. *Pushforward.* Assume furthermore that $f: Y \to S$ is shriekable. By Lemma 3.4.5(1), the functor $\mathrm{R}f_!: D^b_{\text{ét}}(Y^\sigma; \Lambda[\Sigma]) \to D^b_{\text{ét}}(S^\sigma; \Lambda[\Sigma])$ carries $\mathrm{Flat}^b(Y^\sigma; \Lambda[\Sigma])$ to $\mathrm{Flat}^b(S^\sigma; \Lambda[\Sigma])$, hence induces a functor

$$\mathrm{R}f_!: \mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Shv}(S^\sigma; \mathcal{T}_\Lambda).$$

If f is furthermore ω_0-locally spatial and $i: Y^\sigma \hookrightarrow Y$ is ω_0-locally extra small, then similarly using Lemma 3.4.5(2) gives a functor

$$\mathrm{R}f_*: \mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Shv}(S^\sigma; \mathcal{T}_\Lambda).$$

3.6. **The Smith operation.** We define the analogue in p-adic geometry of the "Smith operation" Psm introduced by Treumann in [Tre19].

Definition 3.6.1. Let Y be a separated locally spatial diamond with an action of Σ and $i: Y^\sigma \hookrightarrow Y$ the inclusion of its Σ-fixed points. The *Smith operation* is the functor

$$\mathrm{Psm} := \mathbb{T}^* \circ i^*: D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma} \to \mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda) \tag{3.7}$$

defined as the composition of $i^*: D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma} \to D^b_{\text{ét}}(Y^\sigma; \Lambda)^{B\Sigma} \overset{(3.4)}{\cong} D^b_{\text{ét}}(Y^\sigma; \Lambda[\Sigma])$ with the projection $\mathbb{T}^*$ to $\mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda)$.

The following result may be viewed as a form of equivariant localization for spaces with Σ-action, comparing the (relative) cohomology of a space with that of its Σ-fixed points.

Proposition 3.6.2. *Let Y and S be separated locally spatial diamonds, and let $f: Y \to S$ be shriekable. Denote be $f^\sigma: Y^\sigma \to S^\sigma$ the induced map on fixed points.*

(1) Then the following diagram commutes:

$$\begin{tikzcd} D^b_{\acute{e}t}(Y;\Lambda)^{B\Sigma} \arrow[r, "\mathrm{R}f_!"] \arrow[d, "\mathrm{Psm}"] & D^b_{\acute{e}t}(S;\Lambda)^{B\Sigma} \arrow[d, "\mathrm{Psm}"] \\ \mathrm{Shv}(Y^\sigma;\mathcal{T}_\Lambda) \arrow[r, "\mathrm{R}f^\sigma_!"] & \mathrm{Shv}(S^\sigma;\mathcal{T}_\Lambda) \end{tikzcd}$$

(2) If f is ω_0-locally spatial and $i: Y^\sigma \hookrightarrow Y$ is ω_0-locally extra small, then the following diagram commutes:

$$\begin{tikzcd} D^b_{\acute{e}t}(Y;\Lambda)^{B\Sigma} \arrow[r, "\mathrm{R}f_*"] \arrow[d, "\mathrm{Psm}"] & D^b_{\acute{e}t}(S;\Lambda)^{B\Sigma} \arrow[d, "\mathrm{Psm}"] \\ \mathrm{Shv}(Y^\sigma;\mathcal{T}_\Lambda) \arrow[r, "\mathrm{R}f^\sigma_*"] & \mathrm{Shv}(S^\sigma;\mathcal{T}_\Lambda) \end{tikzcd}$$

Proof. For both statements, we may replace S by S^σ and thereby reduce to the case where the Σ-action on S is trivial.

(1) Let $\mathcal{F} \in D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma}$. Writing $j: (Y \setminus Y^\sigma) \hookrightarrow Y$ for the open complement of $i: Y^\sigma \hookrightarrow Y$, we have an exact triangle

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \in D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma}.$$

By definition Σ acts freely on $Y \setminus Y^\sigma$, hence Lemma 3.4.5(1) implies that $\mathrm{R}f_! \circ (j_! j^* \mathcal{F}) \in \mathrm{Flat}^b(S; \Lambda[\Sigma])$. Then the cone of $\mathrm{R}f_! \mathcal{F} \to \mathrm{R}f^\sigma_!(i^* \mathcal{F})$ lies in $\mathrm{Flat}^b(S; \Lambda[\Sigma])$, and therefore becomes 0 in $\mathrm{Shv}(S; \mathcal{T}_\Lambda)$. Therefore

$$\mathbb{T}^*(\mathrm{R}f_! \mathcal{F}) \cong \mathbb{T}^*(\mathrm{R}f^\sigma_!(i^* \mathcal{F})) \cong \mathrm{R}f^\sigma_! \,\mathrm{Psm}(\mathcal{F}) \in \mathrm{Shv}(S; \mathcal{T}_\Lambda),$$

which exactly expresses the desired commutativity.

(2) The argument is similar to that for (1), but using instead the exact triangle

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}.$$

By Lemma 3.4.5(2), the cone of $\mathrm{R}f_* i_* i^! \mathcal{F} \to \mathrm{R}f_* \mathcal{F}$ lies in $\mathrm{Flat}^b(S; \Lambda[\Sigma])$, and therefore projects to 0 in $\mathrm{Shv}(S; \mathcal{T}_\Lambda)$. Then using Lemma 3.4.3, the maps $\mathrm{R}f_* i_* i^! \mathcal{F} \to \mathrm{R}f_* i_* i^* \mathcal{F} \to \mathrm{R}f_* \mathcal{F}$ project to isomorphisms in $\mathrm{Shv}(S; \mathcal{T}_\Lambda)$.

□

3.7. **Tate cohomology.** For a $\Lambda[\Sigma]$-module M, its *Tate cohomology groups* were defined in (1.1). We extend the definition 2-periodically to define $\mathrm{T}^j(M)$ for all $j \in \mathbf{Z}$.

Example 3.7.1 (Tate cohomology of trivial coefficients)**.** Equip k with the trivial Σ-action. Then we have

$$\mathrm{T}^j(k) \cong k \text{ for all } j \in \mathbf{Z}.$$

Equip $\mathbb{O}$ with the trivial Σ-action. Then we have

$$\mathrm{T}^j(\mathbb{O}) \cong \begin{cases} k & j \equiv 0 \pmod 2, \\ 0 & j \equiv 1 \pmod 2. \end{cases}$$

Example 3.7.2. Let V be a k-vector space. Then $V^{\otimes \ell}$ has an action of Σ, where σ acts by cyclic rotation of the factors, and a straightforward calculation shows that

$$\mathrm{T}^0(V^{\otimes \ell}) \cong V^{(\ell)} := V \otimes_{k, \mathrm{Frob}_\ell} k,$$

the *Frobenius twist* of V.

Next we consider generalizations of Tate cohomology to complexes and sheaves.

3.7.1. *Tate cohomology of complexes.* The Tate cohomology of a complex of $\Lambda[\Sigma]$-modules is defined in [Fen24, §3.4.1]. We summarize below. The exact sequence of $\Lambda[\Sigma]$-modules

$$0 \to \Lambda \to \Lambda[\Sigma] \xrightarrow{1-\sigma} \Lambda[\Sigma] \to \Lambda \to 0$$

induces a morphism

$$\Lambda \to \Lambda[2] \in D^b(\Lambda[\Sigma]). \tag{3.8}$$

Note that it becomes an isomorphism in the Tate category $D^b(\Lambda[\Sigma]) / \mathrm{Flat}^b(\Lambda[\Sigma])$, since the middle two terms project to 0. Given a bounded-below complex of $\Lambda[\Sigma]$-modules $C^\bullet$, we define its *Tate cohomology* to be

$$\mathrm{T}^n(C^\bullet) = \varinjlim_{j \to \infty} \mathrm{Hom}_{D^+(\Lambda[\Sigma])}(\Lambda, C^\bullet[n+2j])$$

where the transition maps are those induced by (3.8). Evidently $\mathrm{T}^n(C^\bullet)$ is 2-periodic in n, and we occasionally view the indexing as being $n \in \mathbf{Z}/2\mathbf{Z}$. It is clear that this construction descends to the derived category.

Now we state some assertions that may be specific[5] to $\Lambda = k$. By [Sta20, Tag 051E], a module over $k[\Sigma]$ is flat if and only if it is free. Since Tate cohomology of free $k[\Sigma]$-complexes vanishes (by inspection), this construction further factors through the Tate category, inducing

$$\mathrm{T}^n \colon \mathrm{Shv}(\mathrm{pt}; \mathcal{T}_k) \to \mathrm{Vect}_{/k}.$$

We write $\mathrm{T}^* := \bigoplus_n \mathrm{T}^n$ and $\mathrm{H}^* := \bigoplus_n \mathrm{H}^n$.

Remark 3.7.3 (Tate cohomology as Hom in the Tate category)**.** If $C^\bullet \in D^b(k[\Sigma])$ is bounded, then the argument of [LL21, Proposition 4.5.1] gives a natural isomorphism

$$\mathrm{T}^i(C^\bullet) \cong \mathrm{Hom}_{\mathrm{Shv}(\mathrm{pt};\mathcal{T}_k)}(k, \mathbb{T}^* C^\bullet[i]).$$

[5]Our argument uses $\Lambda = k$, but the assertions could be true in greater generality, as far as we know.

Example 3.7.4 (Tate cohomology of trivial complexes)**.** If $C^\bullet \in D^b(\mathbb{O})$, then (cf. Definition 3.2.5 for notation) we have

$$\mathrm{T}^*(\epsilon^* C^\bullet) \cong \mathrm{H}^*(\mathrm{Tot}(\ldots \xrightarrow{\ell} C^\bullet \xrightarrow{0} C^\bullet \xrightarrow{\ell} C^\bullet \xrightarrow{0} \ldots))$$

so that

$$\mathrm{T}^n(\epsilon^* C^\bullet) \cong \bigoplus_{i \equiv n \pmod 2} \mathrm{H}^i(C^\bullet \overset{\mathrm{L}}{\otimes}_{\mathbb{O}} k).$$

If $C^\bullet \in D^b(k)$, then we have

$$\mathrm{T}^*(\epsilon^* C^\bullet) \cong \mathrm{H}^*(C^\bullet) \otimes_k \mathrm{T}^*(k) \tag{3.9}$$

where k is equipped with the trivial Σ-action. By Example 3.7.1, (3.9) simplifies to

$$\mathrm{T}^n(\epsilon^* C^\bullet) \cong \bigoplus_{i \in \mathbf{Z}} \mathrm{H}^i(C^\bullet).$$

3.7.2. *Tate cohomology sheaves.* Let S be a locally spatial diamond with trivial Σ-action, so that $\mathrm{Shv}(S; \mathcal{T}_\Lambda)$ is defined. Given $\mathcal{F} \in D^+_{\text{ét}}(S; \Lambda[\Sigma])$, we define *Tate cohomology sheaves*

$$\mathrm{T}^n(\mathcal{F}) := \varinjlim_{j \to \infty} \mathcal{H}om_{D^+_{\text{ét}}(S;\Lambda[\Sigma])}(\Lambda, \mathcal{F}[n+2j])$$

where the transition maps are induced by (3.8). The $\mathrm{T}^i(\mathcal{F})$ are étale sheaves of $T^0(\Lambda)$-modules, where $T^0(\Lambda)$ is the 0th Tate cohomology of the trivial Σ-module Λ.

Remark 3.7.5. For $\Lambda = k$, as in Remark 3.7.3 we also have the description

$$\mathrm{T}^i(\mathcal{F}) \cong \mathcal{H}om_{\mathrm{Shv}(S;\mathcal{T}_k)}(k, \mathbb{T}^* \mathcal{F}[i]).$$

3.7.3. *Tate cohomology for a morphism.* Assume that Y and S are separated locally spatial diamonds with an action of Σ, and $f \colon Y \to S$ is a Σ-equivariant shriekable morphism.

For $\mathcal{F} \in D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma}$, we have $\mathrm{R}f_!(\mathcal{F}) \in D^b_{\text{ét}}(S; \Lambda)^{B\Sigma}$. If S has the trivial Σ-action, then we can form the "relative Tate cohomology" sheaves $\mathrm{T}^n(\mathrm{R}f_! \mathcal{F})$ on S.

Remark 3.7.6. Note that if Σ acts trivially on Y and on S, then the construction $\mathcal{F} \mapsto \mathrm{R}f_!(\mathcal{F})$ factors over $\mathrm{Shv}(Y; \mathcal{T}_k)$ by Lemma 3.4.5. In this situation we will also regard $\mathrm{T}^i(\mathrm{R}f_!)$ as a functor on $\mathrm{Shv}(Y; \mathcal{T}_k)$.

3.7.4. *The long exact sequence for Tate cohomology.* Let assumptions be as in §3.7.3, and suppose furthermore that the Σ-action on S is trivial. Given a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \in D^b_{\text{ét}}(Y; \Lambda)^{B\Sigma}$, we have a long exact sequence of étale sheaves on S,

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad \ldots \longrightarrow \mathrm{T}^{-1}(\mathrm{R}f_!(\mathcal{F}'')) \\ \longrightarrow \mathrm{T}^0(\mathrm{R}f_!(\mathcal{F}')) \longrightarrow \mathrm{T}^0(\mathrm{R}f_!(\mathcal{F})) \longrightarrow \mathrm{T}^0(\mathrm{R}f_!(\mathcal{F}'')) \\ \longrightarrow \mathrm{T}^1(\mathrm{R}f_!(\mathcal{F}')) \longrightarrow \ldots \end{array}$$

4. Σ-fixed point calculations

Let G be a reductive group over E and σ an order $\ell \neq p$ automorphism of G such that $H := G^\sigma$ is (connected) reductive.

Example 4.0.1. By [FS21, Theorem VIII.5.14] and [Ste68, Theorem 8.1], H is automatically (connected) reductive if G is semisimple simply connected.

In this section we will analyze the fixed points of σ on various spaces affiliated with G, and relate them to the analogous spaces affiliated with H, such as:

- The $\mathrm{B}^+_{\mathrm{dR}}$-affine Grassmannians, as well as their variants such as Beilinson-Drinfeld affine Grassmannians, convolution affine Grassmannians, and "twisted" affine Grassmannians (which are the target of the generalized Grothendieck-Messing period maps).
- The moduli spaces of local shtukas.

These calculations are used later when applying Smith-Treumann localization to such spaces.

4.1. **Affine Grassmannians.** We begin by formulating the results for affine Grassmannians and their variants.

Proposition 4.1.1. *Let* $\mathbf{C}$ *be an algebraically closed field. Let* $\mathrm{Gr}^{\mathrm{alg}}_{G,\mathbf{C}}, \mathrm{Gr}^{\mathrm{alg}}_{H,\mathbf{C}}$ *be the algebraic affine Grassmannians over* $\mathbf{C}$. *Then the map* $\mathrm{Gr}^{\mathrm{alg}}_{H,\mathbf{C}} \to \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,\mathbf{C}})$ *is an isomorphism on underlying reduced ind-schemes.*

The proof of Proposition 4.1.1 will be given shortly below, in §4.1.1. For now, we assume it and deduce a few consequences for mixed-characteristic Beilinson-Drinfeld Grassmannians.

Let I be a non-empty finite set and let S be a separated locally spatial diamond over $(\mathrm{Div}^1_X)^I$. Then we define

$$\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I} := \mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I} \times_{(\mathrm{Div}^1_X)^I} S.$$

Proposition 4.1.2. *Let* S *be a separated locally spatial diamond over* $(\mathrm{Div}^1_X)^I$. *Then the natural map*

$$\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})$$

is an isomorphism.

Proof. Since $\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$ is separated, $\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})$ is a closed subdiamond of $\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$. The map $\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})$ is a closed embedding by the argument of [SW20, Lemma 19.1.5], hence qcqs, so by [Sch22, Lemma 11.11] it is an isomorphism if and only if it induces a bijection on $\mathrm{Spa}(C,C^+)$-points for all (C,C^+). Checking this reduces to the case where $S = \mathrm{Spa}(C,C^+)$, in which case we abbreviate $\mathrm{Gr}_{G,C} := \mathrm{Gr}_{G,\mathrm{Spa}(C,C^+)/(\mathrm{Div}^1_X)^I}$ and similarly for H.

A $\mathrm{Spa}(C,C^+)$-point of $\mathrm{Gr}_{G,C}$ consists of $|I|$ untilts $\{C_i^\sharp/E\}_{i\in I}$, and for each $i \in I$ a $(C_i^\sharp, \mathcal{O}_{C_i^\sharp})$-point of $\mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}}$, the $\mathrm{B}^+_{\mathrm{dR}}$-affine Grassmannian built using $C_i^\sharp$ as discussed in [SW20, Lecture XIX]. Any choice of uniformizer ξ for $\mathrm{B}_{\mathrm{dR}}(C_i^\sharp)$ induces an isomorphism $\mathrm{B}_{\mathrm{dR}}(C_i^\sharp) \cong C_i^\sharp((\xi))$, which induces (cf. [SW20, proof of Proposition 19.2.1]) a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_H^{\mathrm{B}^+_{\mathrm{dR}}}(C_i^\sharp, \mathcal{O}_{C_i^\sharp}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Fix}(\sigma, \mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}})(C_i^\sharp, \mathcal{O}_{C_i^\sharp}) \arrow[r, hook] \arrow[d, equal] & \mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}}(C_i^\sharp, \mathcal{O}_{C_i^\sharp}) \arrow[d, equal] \\
\mathrm{Gr}^{\mathrm{alg}}_{H,C_i^\sharp}(C_i^\sharp) \arrow[r, hook] & \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,C_i^\sharp})(C_i^\sharp) \arrow[r, hook] & \mathrm{Gr}^{\mathrm{alg}}_{G,C_i^\sharp}(C_i^\sharp)
\end{tikzcd}$$

By Proposition 4.1.1, the map $\mathrm{Gr}^{\mathrm{alg}}_{H,C_i^\sharp}(C_i^\sharp) \to \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{alg}}_{G,C_i^\sharp})(C_i^\sharp)$ is a bijection, hence so is the parallel map in the top row. Thus the map

$$\mathrm{Gr}_{H,C}(C,C^+) \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,C})(C,C^+)$$

is also a bijection, as desired. $\square$

Remark 4.1.3. The same argument works for variants like $(\mathrm{Div}^1_{\mathcal{Y}})^I$ instead of $(\mathrm{Div}^1_X)^I$.

We also deduce variants for convolution Grassmannians and the "twisted" Grassmannians $\mathrm{Gr}_G^{\mathrm{tw}}$ from [SW20, §23.5].

Corollary 4.1.4. *Let* $m \in \mathbf{Z}_{\geq 1}$.

(1) *Let* $\mathbf{C}$ *be an algebraically closed field. Let* $\widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_{G,\mathbf{C}}$ *be the* m*-step convolution Grassmannian for* G*, and similarly for* H*. Then the natural map*

$$\widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_{H,\mathbf{C}} \to \mathrm{Fix}(\sigma, \widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_{G,\mathbf{C}})$$

is an isomorphism.

(2) *Let* I *be non-empty finite set and let* S *be a separated locally spatial diamond with a map* $S \to (\mathrm{Div}^1_X)^I$. *Let* $\widetilde{\mathrm{Gr}}^{(m)}_{G,S/(\mathrm{Div}^1_X)^I}$ *be the* m*-step convolution Beilinson-Drinfeld Grassmannian for* G*, and similarly for* H*. Then the natural map*

$$\widetilde{\mathrm{Gr}}^{(m)}_{H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \widetilde{\mathrm{Gr}}^{(m)}_{G,S/(\mathrm{Div}^1_X)^I})$$

is an isomorphism.

(3) *Let I be a non-empty finite set. Let $\mathrm{Gr}^{\mathrm{tw}}_{G,I} = \varinjlim_{\mu_\bullet} \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I} \mathrm{Spd}\, E_i, \leq \mu_\bullet}$ be the twisted affine Grassmannian (cf.* [SW20, Definition 23.5.1]*). Then the natural map*

$$\mathrm{Gr}^{\mathrm{tw}}_{H,I} \xrightarrow{\sim} \mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{tw}}_{G,I})$$

is an isomorphism.

Proof. (1) The map $\widetilde{\mathrm{Gr}}^{\mathrm{alg},(m)}_G \to (\mathrm{Gr}^{\mathrm{alg}}_G)^m$ sending $(\mathcal{E}^0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \ldots \dashrightarrow \mathcal{E}_m)$ to $(\mathcal{E}^0 \dashrightarrow \mathcal{E}_1, \mathcal{E}^0 \dashrightarrow \mathcal{E}_2, \ldots, \mathcal{E}^0 \dashrightarrow \mathcal{E}_m)$ is a Σ-equivariant isomorphism. Under this isomorphism, the statement transports to (the mth power of) that of Proposition 4.1.1.

(2) As in the proof of Proposition 4.1.2, the statement reduces to the case where $S = \mathrm{Spa}(C, C^+)$, and then it follows from the algebraic case over untilts $C^\sharp$ of C, which were treated in (1).

(3) As in the proof of Proposition 4.1.2, the statement reduces to the case $S = \mathrm{Spa}(C, C^+)$, where it is then a consequence of (2). □

4.2. **Borels.** Recall that a *Borus* of a reductive group G is a pair (B, T) of a Borel subgroup $B < G$ and a split maximal torus $T < B$.

Lemma 4.2.1. *Suppose Σ stabilizes a Borus (B, T) of G. Then (B^σ, T^σ) is a Borus of H.*

Proof. Suppose for the moment that we know B^σ is a Borel subgroup of H. We will argue that T^σ is a maximal torus of H. Let U be the unipotent radical of B. Applying [CGP15, Proposition A.8.10(2)], which implies that the map on Σ-fixed points of a smooth Σ-equivariant morphism is again smooth, to $B \to B/U \cong T$ shows that $B^\sigma \to (B/U)^\sigma \cong T^\sigma$ is a smooth map whose non-empty fibers are U^σ-torsors. Since the section $T^\sigma \hookrightarrow B^\sigma$ shows that $B^\sigma \to T^\sigma$ is surjective, we conclude that the map $B^\sigma/U^\sigma \to (B/U)^\sigma \cong T^\sigma$ is an isomorphism. This shows that T^σ is a maximal torus of H (and in particular is connected).

Now we return to showing that B^σ is a Borel subgroup of H. Since B^σ is clearly solvable, in order to show that it is a Borel subgroup of H it suffices by a result of Chevalley [Con, Theorem 1.3.1] to show that it is parabolic, i.e., that H/B^σ is projective. By assumption G/B is projective, hence its closed subscheme $(G/B)^\sigma$ is projective. We will prove that H/B^σ is closed and open in $(G/B)^\sigma$, which will imply that H/B^σ is projective. Evidently H acts by translation on $(G/B)^\sigma$, and it suffices to show that each H-orbit in $(G/B)^\sigma$ is open, since then the orbits are finite in number and pairwise disjoint, hence open-closed. Indeed, for any geometric point g of G representing a point in $(G/B)^\Sigma$, the orbit map $G^\sigma \to (G/B)^\sigma$ through gB is a smooth morphism by [CGP15, Proposition A.8.10(2)] again, hence has open image. This completes the proof. □

Proposition 4.2.2. *Let S be a seaparated locally spatial diamond over $(\mathrm{Div}^1_X)^I$. Suppose that G has a Σ-stable Borel subgroup B. Let $B_H := B^\sigma$, a Borel subgroup of H by Lemma 4.2.1. Then the natural map*

$$\mathrm{Gr}_{B_H, S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{B, S/(\mathrm{Div}^1_X)^I})$$

is an isomorphism.

Proof. The map $\mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Fix}(\sigma, \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I})$ is a closed embedding by the same argument as for [SW20, Lemma 19.1.5], hence qcqs, so by [Sch22, Lemma 11.11] it suffices to check that it induces a bijection on geometric points. A variation on [FS21, Proposition VI.3.1] shows that the maps $\mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I}$ and $\mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I} \to \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$ are bijections on geometric points. Hence in the commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} \arrow[r] \arrow[d] & \mathrm{Fix}(\sigma, \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I}) \arrow[d] \\
\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \arrow[r] & \mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I})
\end{tikzcd}$$

both vertical maps and the bottom horizontal map induce bijections on geometric points (the latter by Proposition 4.1.2). Therefore the upper horizontal map also induces a bijection on geometric points, concluding the proof.

□

4.3. **Proof of Proposition 4.1.1.** We write LG^{alg} for the loop group of G with respect to the loop variable t, so $LG^{\mathrm{alg}}(R) = G(R((t)))$. We write $L^+G^{\mathrm{alg}} \subset LG^{\mathrm{alg}}$ for the arc group, with functor of points $L^+G^{\mathrm{alg}}(R) = G(R[[t]])$. The algebraic affine Grassmannian is the fppf quotient $LG^{\mathrm{alg}}/L^+G^{\mathrm{alg}}$, which has a natural ind-scheme structure.

For the purpose of proving Proposition 4.1.1, we may base change G to the algebraically closed field $\mathbf{C}$. Since σ is a semisimple automorphism of G over an algebraically closed field, a theorem of Steinberg [Ste68, Theorem 7.5] implies that σ stabilizes a Borus (B, T) of G. By Lemma 4.2.1, (B^σ, T^σ) is a Borus of H.

4.3.1. *Iwahori stratification.* Let Iw^G be the Iwahori subgroup of L^+G corresponding to B. This induces a stratification by Iw^G-orbits (cf. [RW22, (4.6)])

$$\mathrm{Gr}_{G,\mathbf{C}}^{\mathrm{alg,red}} = \coprod_{\lambda \in X_*(T)} \mathrm{Gr}_{G,\lambda}^{\mathrm{alg}}.$$

Furthermore, Iw^G is stable under σ and $\mathrm{Iw}^H := (\mathrm{Iw}^G)^\sigma$ is the Iwahori subgroup of L^+H corresponding to B^σ, so we have an analogous stratification

$$\mathrm{Gr}_{H,\mathbf{C}}^{\mathrm{alg,red}} = \coprod_{\lambda \in X_*(T^\sigma)} \mathrm{Gr}_{H,\lambda}^{\mathrm{alg}}.$$

The action of σ on G (stabilizing T) induces an action on $X_*(T)$. We will show that:

(1) If λ is not fixed by σ, then $\mathrm{Gr}_{G,\lambda}^{\mathrm{alg}} \cap (\sigma \cdot \mathrm{Gr}_{G,\lambda}^{\mathrm{alg}}) = \emptyset$.
(2) If λ is fixed by σ, then $\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,\lambda}^{\mathrm{alg}}) \xleftarrow{\sim} \mathrm{Gr}_{H,\lambda}^{\mathrm{alg}}$.

The combination of (1) and (2) clearly suffices to prove Proposition 4.1.1.

Item (1) is immediate: since we arranged σ to preserve Iw^G, we have $\sigma \cdot \mathrm{Gr}_{G,\lambda} = \mathrm{Gr}_{G,\sigma(\lambda)}$. Because distinct Iw^G-orbits are disjoint, the intersection of any two distinct Iw^G-orbits is empty.

Next we turn to (2). For this we analyze the structure of the strata.

4.3.2. *Fixed points of strata.* For any root $\alpha \in \mathfrak{R}$, let $U_\alpha \subset G$ be the corresponding root subgroup. For an affine root $\alpha + m\hbar$ of G, denote by $U_{\alpha+m\hbar}$ the corresponding affine root subgroup of LG (cf. [RW22, §4.3]), which for any isomorphism $u_\alpha \colon \mathbf{G}_a \xrightarrow{\sim} U_\alpha$, identifies with the image of $x \mapsto u_\alpha(xt^m)$. Set

$$\delta_\alpha := \begin{cases} 1 & \alpha \in \mathfrak{R}^+, \\ 0 & \text{otherwise,} \end{cases}$$

and for $\lambda \in X_*(T)$ define

$$\mathrm{Iw}_{\mathrm{u}}^{G,\lambda} := \prod_{\alpha \in \mathfrak{R}} \left(\prod_{\delta_\alpha \leq m < \langle \lambda, \alpha \rangle} U_{\alpha+m\hbar} \right), \tag{4.1}$$

where the product may be taken in any order. Then as explained in the proof of [RW22, Lemma 4.4], the action of $\mathrm{Iw}_{\mathrm{u}}^{G,\lambda}$ on $t^\lambda \in \mathrm{Gr}_G(\mathbf{C})$ induces an isomorphism

$$\mathrm{Iw}_{\mathrm{u}}^{G,\lambda} \xrightarrow{\sim} \mathrm{Gr}_{G,\lambda}^{\mathrm{alg}}. \tag{4.2}$$

Suppose σ fixes λ. Then (4.2) implies that

$$\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,\lambda}^{\mathrm{alg}}) = (\mathrm{Iw}_{\mathrm{u}}^{G,\lambda})^\sigma \cdot t^\lambda.$$

Denote by $\mathrm{Iw}_{\mathrm{u}}^{G,\lambda}(\alpha)$ the factor of $\mathrm{Iw}_{\mathrm{u}}^{G,\lambda}$ in (4.1) indexed by α. If $\sigma(\alpha) \neq \alpha$, then $\mathrm{Iw}_{\mathrm{u}}^{G,\lambda}(\alpha)^\sigma = 0$. On the other hand, if $\sigma\alpha = \alpha$ then (using that σ was arranged to preserve $\mathfrak{R}^+$) we have $\mathrm{Iw}_{\mathrm{u}}^{G,\lambda}(\alpha)^\sigma = \mathrm{Iw}_{\mathrm{u}}^{H,\lambda}(\alpha)$, so that

$$(\mathrm{Iw}_{\mathrm{u}}^{G,\lambda})^\sigma \cdot t^\lambda \cong \prod_{\alpha \in \mathfrak{R}^\sigma} \mathrm{Iw}_{\mathrm{u}}^{H,\lambda}(\alpha) \cong \mathrm{Gr}_{H,\lambda}^{\mathrm{alg}}.$$

This completes the proof of Proposition 4.1.1. $\square$

4.4. **Moduli of local shtukas.** We now turn to examine the moduli spaces of local shtukas, for which a reference is Scholze's Berkeley lectures, especially [SW20, Lecture XXIII].

Let $(G, b, \{\mu_i\}_{i\in I})$ be a local shtuka datum (cf. [SW20, Definition 23.1.1], except we allow the equal characteristic case as well): G is a reductive group over E, $b \in B(G)$, and μ_i is a conjugacy class of cocharacters $\mathbf{G}_m \to G_{E^s}$. For any compact open subgroup $K \subset G(E)$ there is a *moduli space of local shtukas with K-level structure* $\mathrm{Sht}_{(G,b,\{\mu_i\}_{i\in I}),K}$, which is functorial in K. There is a moduli description of $\mathrm{Sht}_{(G,b,\{\mu_i\}_{i\in I}),K}$ given (in the mixed characteristic case) in [SW20, §23].

For $\{\mu_i\}_{i\in I} \leq \{\mu_i'\}_{i\in I}$ in the component-wise Bruhat order, there are closed embeddings $\mathrm{Sht}_{(G,b,\{\mu_i\}),K} \hookrightarrow \mathrm{Sht}_{(G,b,\{\mu_i'\}),K}$. It will also be convenient to consider the following variant without any cocharacter "bounds":

Definition 4.4.1. We define $\mathrm{Sht}_{(G,b,I),K} := \varinjlim_{\{\mu_i\}_{i\in I}} \mathrm{Sht}_{(G,b,\{\mu_i\}_{i\in I}),K}$.

4.4.1. *Grothendieck-Messing period map.* We recall some facts about the generalized Grothendieck-Messing period map

$$\pi^G_{\mathrm{GM}} \colon \mathrm{Sht}_{(G,b,I),K} \to \mathrm{Gr}^{\mathrm{tw}}_{G,I} \tag{4.3}$$

from [SW20, §23.5]. Note that we have not imposed any "bound" above; our (4.3) comes from the colimit over $\{\mu_i\}_{i\in I}$ of the map from [SW20, Corollary 23.5.3].

The image of π^G_{GM} is an open subset of $\mathrm{Gr}^{\mathrm{tw},a}_{G,I} \subset \mathrm{Gr}^{\mathrm{tw}}_{G,I}$ called the *admissible* locus. In terms of points, $\mathrm{Gr}^{\mathrm{tw}}_{G,I}$ parametrizes G-torsors $\mathcal{P}_\eta$ on $\mathcal{Y}_{(0,\infty)}$ plus additional data $\varphi_{\mathcal{P}_\eta}$ and ι_r from [SW20, Definition 23.5.1] that we do not need to reference right now. The admissible locus is cut out by the condition that the Newton point $\nu_{\mathcal{P}_\eta}$ and the Kottwitz point $\kappa_{\mathcal{P}_\eta}$ are both identically zero [SW20, Theorem 22.6.2].

Definition 4.4.2. For a basic $b \in B(G)$, denote by $\mathrm{Gr}^{\mathrm{tw},b}_{G,I}$ the subspace where $\nu_{\mathcal{P}_\eta} = \nu(b)$ and $\kappa_{\mathcal{P}_\eta} = \kappa(b)$. This is an open subspace, and when $b = 1_G$ this recovers the admissible locus, i.e., $\mathrm{Gr}^{\mathrm{tw},1_G}_{G,I} = \mathrm{Gr}^{\mathrm{tw},a}_{G,I} \subset \mathrm{Gr}^{\mathrm{tw}}_{G,I}$.

Example 4.4.3. If $b \in B(G)$ is basic, then there is a pure inner twist G_b (sometimes denoted J_b in the literature) and a canonical isomorphism

$$\mathrm{tr}_b \colon B(G) \xrightarrow{\sim} B(G_b) \tag{4.4}$$

sending $b \in B(G)$ to $1_{G_b} \in B(G_b)$. There is also a compatible isomorphism $\mathrm{Gr}^{\mathrm{tw}}_{G,I} \cong \mathrm{Gr}^{\mathrm{tw}}_{G_b,I}$ as in [FS21, §III.4.1], which takes $\mathrm{Gr}^{\mathrm{tw},b}_{G,I}$ to $\mathrm{Gr}^{\mathrm{tw},a}_{G_b,I}$. So we could have formulated the subspaces $\mathrm{Gr}^{\mathrm{tw},b}_{G,I}$ as admissible loci for pure inner twists G_b. However, we prefer to keep the distinction, for psychological reasons if nothing else.

Let $\iota \colon B(H) \to B(G)$ be the map induced by the inclusion $\iota \colon H \hookrightarrow G$. The induced map $\iota \colon \mathrm{Gr}^{\mathrm{tw}}_{H,I} \hookrightarrow \mathrm{Gr}^{\mathrm{tw}}_{G,I}$ is also a closed embedding (by the proof of [SW20, Lemma 19.1.5]), etc.

Remark 4.4.4. Since $X_*(T^\sigma_{E^s})_{\mathbf{Q}} \subset X_*(T_{E^s})_{\mathbf{Q}}$, the subset $\iota^{-1}(1_G) \subset B(H)$ consists of basic elements. For each $b' \in \iota^{-1}(1_G)$, there is an embedding

$$\iota_{b'} \colon H_{b'} \hookrightarrow G_{\iota(b')} = G.$$

As a variant of [FS21, Proposition III.4.2], this induces an embedding $\mathrm{Gr}^{\mathrm{tw}}_{H_{b'},I} \hookrightarrow \mathrm{Gr}^{\mathrm{tw}}_{G,I}$, identifying $\mathrm{Gr}^{\mathrm{tw},a}_{H_{b'},I}$ with $\mathrm{Gr}^{\mathrm{tw},b'}_{H,I}$ as sub-v-sheaves of $\mathrm{Gr}^{\mathrm{tw}}_{G,I}$.

Lemma 4.4.5. *We have*

$$\mathrm{Gr}^{\mathrm{tw},a}_{G,I} \cap \mathrm{Gr}^{\mathrm{tw}}_{H,I} = \coprod_{b' \in \iota^{-1}(1_G)} \mathrm{Gr}^{\mathrm{tw},b'}_{H,I}.$$

as subdiamonds of $\mathrm{Gr}^{\mathrm{tw}}_{G,I}$.

Proof. In Corollary 4.1.4 we have seen that $\mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{tw}}_{G,I}) = \mathrm{Gr}^{\mathrm{tw}}_{H,I}$ (with the obvious embedding). The admissible locus $\mathrm{Gr}^{\mathrm{tw},a}_{G,I} \subset \mathrm{Gr}^{\mathrm{tw}}_{G,I}$ is cut out by the condition that $\nu(\mathcal{P}_\eta) = 0$ and $\kappa(\mathcal{P}_\eta) = 0$. Hence $\mathrm{Fix}(\sigma, \mathrm{Gr}^{\mathrm{tw},a}_{G,I})$ is the pullback of the simultaneous vanishing loci of the Newton and Kottwitz maps from $\mathrm{Gr}^{\mathrm{tw}}_{G,I}$ to $\mathrm{Gr}^{\mathrm{tw}}_{H,I}$, which is precisely the expression on the RHS. $\square$

4.4.2. *Fixed points of moduli of local shtukas.* We now recall more about the structure of $\mathrm{Sht}_{(G,b,I),K}$ in terms of the Grothendieck-Messing period map. According to [SW20, Corollary 23.4.2], $\mathrm{Gr}_{G,I}^{\mathrm{tw},a}$ carries a $\underline{G(E)}$-torsor $\mathbb{P}_\eta^G$, and $\pi_{\mathrm{GM}}^G \colon \mathrm{Sht}_{(G,b,I),K} \to \mathrm{Gr}_{G,I}^{\mathrm{tw},a}$ is the étale cover parametrizing K-lattices $\mathbb{P}^G \subset \mathbb{P}_\eta^G$.

For basic $b' \in B(G)$, note that $\mathrm{Gr}_{G,I}^{\mathrm{tw},b'}$ carries a $\underline{G_b'(E)}$-torsor that we denote $\mathbb{P}_\eta^{G,b'}$, for example using the pure inner twisting procedure from [FS21, Proposition III.4.2] to reduce to the case where $b' = 1_G$ where it is $\mathbb{P}_\eta^G$.

Definition 4.4.6. Let $b' \in B(G)$ be basic and $G_{b'}$ the corresponding inner twist of G. For a compact open subgroup $K \subset G_{b'}(E)$, we denote by $\mathrm{Sht}_{(G,b,I),K}^{b'} \to \mathrm{Gr}_{G,I}^{\mathrm{tw},b'}$ the étale cover parametrizing K-lattices in $\mathbb{P}_\eta^{G,b'}$. Note that $G_b(E)$ acts on $\mathrm{Sht}_{(G,b,I),K}^{b'}$ by change of framing.

In fact, the pure inner twisting procedure of [FS21, Proposition III.4.2] gives a $G_b(E)$-equivariant isomorphism between $\mathrm{Sht}_{(G,b,I),K}^{b'}$ and $\mathrm{Sht}_{(G_{b'},b'',I),K}$ for $b'' \in B(G)$ such that the inner twist of $G_{b'}$ corresponding to b'' is isomorphic to G_b, so it was unnecessary to introduce this new notation. However, we like to use it to distinguish what arises naturally from calculations.

For $b' \in \iota^{-1}(1_G) \subset B(H)$, write $\iota_{b'}^* K \subset H_{b'}(E)$ for the pre-image of K under the map $\iota_{b'} \colon H_{b'} \hookrightarrow G$ from Remark 4.4.4. We abbreviate $\iota^* K = \iota_{1_H}^* K = K^\sigma \subset H(E)$.

Proposition 4.4.7. *Suppose $K \subset G(E)$ is a Σ-stable open compact subgroup, with prime-to-ℓ pro-order. Then we have*

$$\mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,1_G,I),K}) = \coprod_{b' \in \iota^{-1}(1_G)} \mathrm{Sht}_{(H,1_H,I),\iota_{b'}^* K}^{b'}$$

as sub-v-sheaves of $\mathrm{Sht}_{(G,1_G,I),K}$ *(with embedding on the RHS induced by $\iota_{b'}$ of Remark 4.4.4). In particular,* $\mathrm{Sht}_{(H,1_H,I),\iota^* K}$ *is an open-closed subdiamond of* $\mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,1_G,I),K})$.

Remark 4.4.8. As explained in Definition 4.4.6, the term $\mathrm{Sht}_{(H,1_H,I),\iota_{b'}^* K}^{b'}$ is $H(E)$-equivariantly isomorphic to the local Shimura variety $\mathrm{Sht}_{(H_{b'},b,I),\iota_{b'}^* K}$ where $b \in B(H_{b'})$ is such that the inner twist $(H_{b'})_b$ is isomorphic to H. Only the assertion that $\mathrm{Sht}_{(H,1_H,I),\iota^* K}$ is an open-closed subdiamond of $\mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,1_G,I),K})$, will be really crucial in future sections.

Proof. We have already calculated the σ-fixed points in the image of the Grothendieck-Messing period map π_{GM}^G in Lemma 4.4.5, and in view of the answer, it suffices to show that for all $x \in \mathrm{Gr}_{H,I}^{\mathrm{tw},b'}$, the inclusion $\mathrm{Sht}_{(H,1_H,I),\iota_{b'}^* K}^{b'} \to \mathrm{Sht}_{(G,1_G,I),K}$ sends $(\pi_{\mathrm{GM}}^H)^{-1}(x)$ isomorphically to $\mathrm{Fix}(\sigma, (\pi_{\mathrm{GM}}^G)^{-1}(x))$. By pure inner twisting procedure, as explained in Remark 4.4.4 and Remark 4.4.8, we may reduce this statement to the case where $b' = 1$. The $\underline{G(E)}$-torsor $\mathbb{P}_\eta^G$ corresponding to x has an H-structure $\mathbb{P}_\eta^H$ since x lies in the image of π_{GM}^H, and $(\pi_{\mathrm{GM}}^H)^{-1}(x)$ parametrizes $\iota^* K$-lattices in $\mathbb{P}_\eta^H$. The action of $H(E)$ on a fixed $\iota^* K$-lattice L_0 induces an isomorphism $(\pi_{\mathrm{GM}}^H)^{-1}(x) \cong H(E)/\iota^* K$.

On the other hand, $\mathrm{Fix}(\sigma, (\pi_{\mathrm{GM}}^G)^{-1}(x))$ parametrizes K-lattices in $\mathbb{P}_\eta^G$ which are stable under Σ. As $\mathbb{P}_\eta^G \cong \mathbb{P}_\eta^H \times^{H(E)} G(E)$, the action of $G(E)$ on L_0 induces an isomorphism $\mathrm{Fix}(\sigma, (\pi_{\mathrm{GM}}^G)^{-1}(x)) \cong \mathrm{Fix}(\sigma, G(E)/K)$, with σ acting in the natural way. Hence it suffices to see that

$$H(E)/K^\sigma \xrightarrow{\sim} (G(E)/K)^\sigma. \tag{4.5}$$

The long exact sequence of non-abelian cohomology of Σ reads

$$K^\sigma \hookrightarrow G(E)^\sigma \to (G(E)/K)^\sigma \to \mathrm{H}^1(\Sigma; K) \to \mathrm{H}^1(\Sigma; G(E)).$$

Since K has prime-to-ℓ pro-order by assumption, we have $\mathrm{H}^1(\Sigma; K) = 0$. This shows (4.5), completing the proof. □

5. Parity sheaf theory on p-adic affine Grassmannians

The theory of *parity sheaves*, originating in work of Juteau-Mautner-Williamson [JMW14], has driven many recent developments in geometric representation theory: see [AR15, AR16, MR18, AMRW19, Wil18] for a sampling of applications. Parity sheaves are defined somewhat similarly to perverse sheaves, but with *parity* conditions instead of inequalities on cohomological degrees.

In this section, we develop a theory of *relative* parity sheaves on affine Grassmannians arising in p-adic geometry. The usage of the adjective "relative" is in the same sense as "relative perversity" of Hansen-Scholze [HS23]: it means parity along the geometric fibers of a morphism. In practice, the morphism is usually the projection of a Beilinson-Drinfeld type affine Grassmannian to the base. Leslie-Lonergan [LL21] introduced the "Tate-parity sheaves" as the analogue of parity sheaves in Tate categories, and we also develop a theory of "relative Tate-parity sheaves" on families of affine Grassmannians.

The main results of the structure theory of relative (Tate-)parity sheaves include the calculation of Ext groups, the vanishing of maps between incongruous objects, the fact that tilting objects are relative Tate-parity, and the generation of all relative (Tate-)parity sheaves by direct sums of shifts of tilting objects. Some of the difficulties are similar to those present in the theory of perverse sheaves on p-adic affine Grassmannians: the lack of a definitive dimension theory for adic spaces, and the failure of constructibility for the sheaves of interest. There are also some new technical issues, coming from the fact that the Ext-vanishing properties underlying the structure theory of parity sheaves have an "absolute" nature, which is in tension with the "relative" nature of our definitions. Moreover, the Krull-Schmidt property, which was an important technical property underpinning [RW22] and [LL21], fails badly in the relative situation, due to the complicated nature of local systems on a general profinite set. The theory of ULA sheaves developed in [FS21] ultimately helps us to overcome these difficulties.

We now give an overview of the contents of this section. In §5.1 we define relative parity complexes, etc. on $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$ for a small v-stack S over Div^1_X and establish their structure theory (mostly for the case where S is strictly totally disconnected). In §5.2 we introduce a "small" version of the Tate category for $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$, the adjective "small" referring to that the sheaves are required to have strong finiteness properties, and a "small" version of Smith-Treumann localization. In §5.3 we define relative Tate-parity complexes, etc. on $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$ and establish their structure theory. In §5.4 we define "even maps" and prove that they preserve relative parity, and next in §5.5 we apply this to certain Demazure resolutions in order to show that relative parity sheaves exist for all strata, and that relative parity is preserved by convolution. In §5.3.3 we study the interaction of relative parity sheaves with modular reduction $\mathbb{F}$ and the operation $\mathbb{T}$. This is used in §5.6 to define the "lifting functor" à la Leslie-Lonergan, which will constitute a key step in the construction of the Brauer functor.

5.1. **Relative parity sheaves.** Let S be a small v-stack with a map $S \to \operatorname{Div}^1_X$. Then we may form $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X} := \mathrm{Gr}_{G,\operatorname{Div}^1_X} \times_{\operatorname{Div}^1_X} S$ and $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X} := \mathcal{H}\mathrm{ck}_{G,\operatorname{Div}^1_X} \times_{\operatorname{Div}^1_X} S$ as in [FS21, §VI].

5.1.1. *Schubert stratification.* Suppose $S \in \mathrm{Perf}$ is strictly totally disconnected. Then [FS21, VI.2] applies and we have a presentation

$$\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X} = \varinjlim_{\mu \in X_*(T)^+} \mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\leq\mu}$$

where T is a split maximal torus in G_{E^s}; here $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\leq\mu}$ is the subfunctor of $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X}$ parametrizing modifications which are given by some $\mu' \leq \mu$ at each geometric point of S. The open subfunctor $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\mu} \hookrightarrow \mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\leq\mu}$ is defined as the complement of $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\leq\mu'}$ for $\mu' < \mu$.

The pullback of $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\leq\mu}$ (resp. $\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X,\mu}$) to $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$ is denoted $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\leq\mu}$ (resp. $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$). Let $L^+_{S/\operatorname{Div}^1_X} G := L^+_{\operatorname{Div}^1_X} G \times_{\operatorname{Div}^1_X} S$. Then $L^+_{S/\operatorname{Div}^1_X} G$ acts on $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$ by left translation, and the orbits are the $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$.

We denote by i_μ the locally closed embedding $i_\mu \colon \mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu} \hookrightarrow \mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$. By [FS21, Proposition VI.2.4], $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$ has the structure of a fibration over the diamond of the (opposite) partial flag variety $(P^-_\mu)^\diamond_S$, with the fibers being iterated extensions of $(\operatorname{Lie} G)^\diamond_{\mu\leq m}\{m\}$ where $(\operatorname{Lie} G)_{\mu\leq m}$ is the subspace on which $\mathbf{G}_m$ acts with weights $\leq m$ via the adjoint action composed with μ. (Here $\{m\}$ is a "Breuil-Kisin twist".)

Lemma 5.1.1. *For the projection $\pi \colon \mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu} \to S$, each $\mathrm{R}^n\pi_*\Lambda$ has Λ-free geometric stalks and vanishes if n is odd.*

Proof. Both $(P^-_\mu)^\diamond$ and $(\operatorname{Lie} G)^\diamond_{\mu\leq m}$ have the property that their cohomology with constant coefficients is supported in even degrees. The preceding description shows that $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$ is an iterated fibration over $(P^-_\mu)^\diamond_S$ with fibers of the form $(\operatorname{Lie} G)^\diamond_{\mu\leq m}$ (note that the Breuil-Kisin twist is trivializable over geometric points), so the result follows from the Serre spectral sequence, which is forced to degenerate by the evenness. ☐

5.1.2. *Categories of sheaves.* We will take coefficients in a ring Λ which is an ℓ-adically complete local PID over $\mathbf{Z}_\ell$, i.e., a field of characteristic ℓ or a complete DVR with residue characteristic ℓ. In our applications of interest, Λ will be either k or $\mathbb{O} := W(k)$.

Definition 5.1.2. We define $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ to be the full subcategory of $D^{\mathrm{ULA}}_{\text{ét}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ spanned by the image of $D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ under $*$-pullback.

Remark 5.1.3 (Stability under six operations)**.** As a consequence of [FS21, Corollary VI.6.6], the category $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is stable under the operations of Verdier duality, $-\overset{\mathrm{L}}{\otimes}_\Lambda -$, $\mathcal{RH}om_\Lambda(-,-)$, $i_!i^*, \mathrm{R}i_*i^*, i_!i^!, \mathrm{R}i_*i^!$ where $i = i_\mu$ is the inclusion of any stratum. (The analogous statement does *not* hold in the "multiple legs" situation, unless the map from S factors through the locus where the legs are disjoint.)

Recall that an additive category is called *Krull-Schmidt* if each object is a finite direct sum of indecomposable objects with local endomorphism rings (in particular, this implies that an object is indecomposable if and only if its endomorphism ring is local). We will show that $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is Krull-Schmidt when $S = \mathrm{Spa}(C, C^+)$ is a geometric point.

Lemma 5.1.4. *Assume $S = \mathrm{Spa}(C, C^+)$. Then all objects of $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ admit a finite filtration by objects of the form $i_{\mu!}\Lambda$, and also a finite filtration by objects of the form $\mathrm{R}i_{\mu*}\Lambda$.*

Proof. This follows immediately from [FS21, Proposition VI.6.5]. ☐

Lemma 5.1.5. *Assume $S = \mathrm{Spa}(C, C^+)$. Then for any object $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$, $\mathrm{End}(\mathcal{K})$ is finitely generated as a Λ-module.*

Proof. Applying Lemma 5.1.4, it suffices to show that for each $\mu \in X_*(T)^+$,

$$\mathrm{Hom}(i_{\mu!}\Lambda, \mathrm{R}i_{\mu'*}\Lambda) \cong \mathrm{Hom}(i^*_{\mu'}i_{\mu!}\Lambda, \Lambda) \text{ is finitely generated as a } \Lambda\text{-module.}$$

If $\mu \neq \mu'$, then $i^*_{\mu'}i_{\mu!}\Lambda = 0$. If $\mu' = \mu$, then $i^*_{\mu'}i_{\mu!}\Lambda \cong \Lambda$, so that $\mathrm{Hom}(i^*_{\mu'}i_{\mu!}\Lambda, \Lambda) \cong \Lambda$. ☐

Lemma 5.1.6. *Assume $S = \mathrm{Spa}(C, C^+)$. Then the category $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is Krull-Schmidt.*

Proof. The usual t-structure on the category $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is bounded. Therefore, it is Karoubian (i.e., every idempotent splits) by [LC07, Theorem]. By [CYZ08, Theorem A.1], a Karoubian category such that $\mathrm{End}(\mathcal{K})$ is semiperfect (cf. [Lam01, Chapter 8] for a reference on semiperfect rings) for every $\mathcal{K}$ is Krull-Schmidt. It therefore suffices to show that $\mathrm{End}(\mathcal{K})$ is semiperfect for every $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$. For this we note that any finite Λ-algebra is semiperfect, and we showed in Lemma 5.1.5 that $\mathrm{End}(\mathcal{K})$ is a finite Λ-module. ☐

5.1.3. *Relative parity complexes.* A *pariversity* on $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$ is a function $\dagger$ from $X_*(T)^+$, thought of as the set of strata on the geometric fibers over S, to $\mathbf{Z}/2\mathbf{Z}$.

Example 5.1.7 (Dimension pariversity)**.** In this section we fix $\dagger$ to be the "dimension pariversity" $\dagger_G$, defined by

$$\dagger_G(\lambda) := \langle 2\rho, \lambda \rangle \pmod 2 \in \mathbf{Z}/2\mathbf{Z}$$

where 2ρ is the sum of the positive roots of G.

The definition below is a "relative to S" (in a sense parallel to the notion of relative perversity in [FS21, §VI.7]) version of the definition of parity sheaves in [JMW14, Definition 2.4].

Definition 5.1.8 (Parity complexes)**.** Let S be a small v-stack. For a geometric point $\overline{s} = \mathrm{Spa}(C, C^+) \to S$ and $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)^{\mathrm{bd}}$, we denote by $\mathcal{K}|_{\overline{s}}$ the $*$-restriction of $\mathcal{K}$ along $\mathrm{Gr}_{G,\overline{s}/\operatorname{Div}^1_X} \to \mathrm{Gr}_{G,S/\operatorname{Div}^1_X}$. For each $\mu \in X_*(T)^+$ we let $i_\mu \colon \mathrm{Gr}_{G,\overline{s}/\operatorname{Div}^1_X,\mu} \hookrightarrow \mathrm{Gr}_{G,\overline{s}/\operatorname{Div}^1_X}$ be the locally closed embedding of the corresponding stratum.

(1) For $? \in \{*, !\}$, we say $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is *relative ?-even* if for all geometric points $\overline{s} \to S$ and all $\mu \in X_*(T)^+$ and all $n \in \mathbf{Z}$, the cohomology sheaf $\mathcal{H}^n(i^?_\mu(\mathcal{K}|_{\overline{s}}))$ is constant and Λ-free, and vanishes for $n \not\equiv \dagger(\mu) \pmod 2$.
(2) For $? \in \{*, !\}$, we say that $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is *relative ?-odd* if $\mathcal{K}[1]$ is relative ?-even.
(3) For $? \in \{*, !\}$, we say that $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is *relative ?-parity* if $\mathcal{K}$ is either relative ?-even or relative ?-odd.
(4) We say $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is a *relative even complex* if $\mathcal{K}$ is both relative $*$-even and relative !-even. We say that $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is a *relative odd complex* if $\mathcal{K}[1]$ is even.
(5) We say $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is a *relative parity complex* if it is isomorphic to a finite direct sum of relative even and relative odd complexes. The full subcategory of $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ spanned by relative parity complexes is denoted $\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$.

The following result is part of [FS21, Corollary VI.6.6], but we state it for emphasis and take the opportunity to spell out the proof.

Lemma 5.1.9 ([FS21])**.** *Let $f\colon S' \to S$ be a map of small v-stacks over Div^1_X and $i_\mu\colon \mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu} \hookrightarrow \mathrm{Gr}_{G,S/\mathrm{Div}^1_X}$ be the locally closed immersion of a Schubert cell. Consider the diagram*

$$\begin{tikzcd}
\mathrm{Gr}_{G,S'/\mathrm{Div}^1_X,\mu} \arrow[r, "f_\mu"] \arrow[d, "i'_\mu"] & \mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu} \arrow[d, "i_\mu"] \\
\mathrm{Gr}_{G,S'/\mathrm{Div}^1_X} \arrow[r, "\widetilde{f}"] \arrow[d] & \mathrm{Gr}_{G,S/\mathrm{Div}^1_X} \arrow[d] \\
S' \arrow[r, "f"] & S
\end{tikzcd}$$

where all squares are Cartesian. Then we have a natural isomorphism $f^*_\mu(i_\mu)^! \cong (i'_\mu)^!\widetilde{f}^*$ of functors $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}} \to D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S'/\mathrm{Div}^1_X,\mu};\Lambda)^{\mathrm{bd}}$.

Proof. We abbreviate $\mathbb{D}_{/S}$ (resp. $\mathbb{D}_{/S'}$) for the relative Verdier duality over S (resp. S'). By [FS21, Proposition IV.2.15], relative Verdier duality is compatible with base change, meaning there are natural isomorphisms

$$f^*_\mu\mathbb{D}_{/S} \xrightarrow{\sim} \mathbb{D}_{/S'}f^*_\mu \quad \text{and} \quad \widetilde{f}^*\mathbb{D}_{/S} \xrightarrow{\sim} \mathbb{D}_{/S'}\widetilde{f}^*. \tag{5.1}$$

Using also that $i^!_\mu \cong \mathbb{D}_{/S}i^*_\mu\mathbb{D}_{/S}$ and $(i'_\mu)^! \cong \mathbb{D}_{/S'}(i'_\mu)^*\mathbb{D}_{/S'}$ on ULA objects (because relative Verdier duality is involutive on ULA objects [FS21, Corollary IV.2.25]), we have natural isomorphisms

$$f^*_\mu i^!_\mu \cong f^*_\mu\mathbb{D}_{/S}i^*_\mu\mathbb{D}_{/S} \overset{(5.1)}{\cong} \mathbb{D}_{/S'}f^*_\mu i^*_\mu\mathbb{D}_{/S} \cong \mathbb{D}_{/S'}(i'_\mu)^*\widetilde{f}^*\mathbb{D}_{/S} \overset{(5.1)}{\cong} \mathbb{D}_{/S'}(i'_\mu)^*\mathbb{D}_{/S'}\widetilde{f}^* \cong (i'_\mu)^!\widetilde{f}^*.$$

□

5.1.4. *Structure theory for strictly totally disconnected S.* Here we prove several structural results about $\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$, parallel to [JMW14, §2], *under the assumption that S is strictly totally disconnected.* In particular, thanks this assumption, we have the stratification of $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}$ by $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu}$, ranging over $\mu \in X_*(T)^+$, and we let i_μ be the locally closed embedding $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu} \hookrightarrow \mathrm{Gr}_{G,S/\mathrm{Div}^1_X}$. **Below, all Hom and Ext groups are taken in the category** $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$, which we omit for ease of notation.

Lemma 5.1.9 implies the following connection between relative parity as in Definition 5.1.8, and an "absolute" notion of parity.

Corollary 5.1.10. *Let $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$. For $? \in \{*, !\}$, if $\mathcal{K}$ is relative ?-even then for all $\mu \in X_*(T)^+$ and all $n \in \mathbf{Z}$, the cohomology sheaf $\mathcal{H}^n(i^?_\mu(\mathcal{K}))$ is constant and Λ-free, and vanishes for $n \not\equiv \dagger(\mu)$* (mod 2).

Proof. The constancy is automatic from the definition of $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$. The Λ-freeness of ?-restrictions can be checked after base changing to geometric points $\overline{s} \to S$, hence follows from the definition of relative ?-even, using Lemma 5.1.9 to commute base change on S with $i_\mu^!$. The fact that $\mathcal{H}^n(i_\mu^*(\mathcal{K}))$ vanishes for $n \not\equiv \dagger(\mu) \pmod 2$ follows immediately from the definition of relative ?-even, while the fact that $\mathcal{H}^n(i_\mu^!(\mathcal{K}))$ vanishes for $n \not\equiv \dagger(\mu) \pmod 2$ can be checked after base change along geometric points $\overline{s} = \mathrm{Spa}(C, C^+) \to S$, where it follows from the definition of relative ?-even after applying Lemma 5.1.9. □

Lemma 5.1.11. *Assume S is strictly totally disconnected. If $\mathcal{F} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}; \Lambda)^{\mathrm{bd}}$ is relative $*$-parity and $\mathcal{G} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}; \Lambda)^{\mathrm{bd}}$ is relative !-parity, then we have a (non-canonical) isomorphism of Λ-modules*

$$\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G}) \cong \bigoplus_{\mu \in X_*(T)^+} \mathrm{Ext}^\bullet(i_\mu^* \mathcal{F}, i_\mu^! \mathcal{G}) \tag{5.2}$$

and both sides are finite projective over $C^\infty(|S|, \Lambda) = \mathrm{Ext}^\bullet_S(\Lambda, \Lambda)$, the ring of continuous functions on $|S|$ valued in Λ.

Proof. The argument is similar to that for [JMW14, Proposition 2.6], but we have to address some issues related to the discrepancy between our relative situation and the "absolute" situation of [JMW14]. Since the statement is compatible with finite direct sums and shifts, we may assume without loss of generality that $\mathcal{F}$ is relative $*$-even and $\mathcal{G}$ is relative !-even.

We will show by induction, on the number M of μ such that $\mathrm{supp}(\mathcal{F})$ intersects nontrivially with $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$, that

(i) $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G})$ is finite projective over $C^\infty(|S|, \Lambda)$ and vanishes in odd degrees, and
(ii) satisfies the decomposition (5.2).

If $M = 1$, then $\mathcal{F} \cong i_{\mu!} i_\mu^* \mathcal{F}$ for some $\mu \in X_*(T)^+$, and we have

$$\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^\bullet(i_{\mu!} i_\mu^* \mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^\bullet(i_\mu^* \mathcal{F}, i_\mu^! \mathcal{G}).$$

This shows part (ii) of the inductive hypothesis. By Corollary 5.1.10, $i_\mu^* \mathcal{F}$ and $i_\mu^! \mathcal{G}$ are both locally constant complexes on $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$ concentrated in *even* degrees. Then part (i) of the induction hypothesis follows from Lemma 5.1.1 (here we use the assumption that S is strictly totally disconnected, so it has no Exts between locally constant sheaves).

If $M > 1$, let i be the inclusion of a closed stratum in the support of $\mathcal{F}$ and consider the excision sequence $j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}$. Applying $\mathrm{Hom}(-, \mathcal{G})$, we get a long exact sequence

$$\ldots \to \mathrm{Ext}^n(i_* i^* \mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^n(j_! j^* \mathcal{F}, \mathcal{G}) \to \ldots \tag{5.3}$$

Rewriting $\mathrm{Ext}^n(i_* i^* \mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^n(i^* \mathcal{F}, i^! \mathcal{G})$ and $\mathrm{Ext}^n(j_! j^* \mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^n(j^* \mathcal{F}, j^! \mathcal{G})$, the induction hypothesis applies to the flanking terms. In particular, (i) implies that the long exact sequence (5.3) splits, and then (i) and (ii) for the flanking terms imply (i) and (ii) for $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G})$. □

Corollary 5.1.12. *Assume S is strictly totally disconnected. If $\mathcal{F} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}; \Lambda)^{\mathrm{bd}}$ is relative $*$-even and $\mathcal{G} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}; \Lambda)^{\mathrm{bd}}$ is relative !-odd, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.*

Lemma 5.1.13. *Assume S is strictly totally disconnected. Suppose $\mathcal{F}, \mathcal{G} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X}; \Lambda)^{\mathrm{bd}}$ are either both relative even or both relative odd. If i_μ is the inclusion of a stratum which is open in the support of both $\mathcal{F}$ and $\mathcal{G}$, then the restriction map $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(i_\mu^* \mathcal{F}, i_\mu^* \mathcal{G})$ is surjective.*

Proof. Since the statement is compatible with simultaneous shifts on $\mathcal{F}$ and $\mathcal{G}$, it suffices to treat the case where $\mathcal{F}, \mathcal{G}$ are both relative even. Let i be in the inclusion of the complementary strata, so that we have an excision triangle of relative !-even complexes,

$$i_* i^! \mathcal{G} \to \mathcal{G} \to i_{\mu *} i_\mu^* \mathcal{G}.$$

Applying $\mathrm{Hom}(\mathcal{F}, -)$, we obtain a long exact sequence

$$\ldots \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(i_\mu^* \mathcal{F}, i_\mu^* \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, i_* i^! \mathcal{G}[1]) \to \ldots$$

in which $\mathrm{Hom}(\mathcal{F}, i_* i^! \mathcal{G}[1]) = 0$ by Corollary 5.1.12, because $\mathcal{G}[1]$ is relative !-odd, so the first map is surjective. □

Lemma 5.1.14. *Assume $S = \mathrm{Spa}(C, C^+)$. Let $\mathcal{F} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ be an indecomposable relative parity complex, and let $j \colon U \to \mathrm{Gr}_{G,S/\mathrm{Div}^1_X}$ be an inclusion of a union of strata open in the support of $\mathcal{F}$. Then $j^*\mathcal{F}$ is either zero or indecomposable.*

Proof. By the assumption on S, Lemma 5.1.6 applies to show that $D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is Krull-Schmidt. Hence the indecomposability of $\mathcal{F}$ implies that $\mathrm{End}(\mathcal{F})$ is local. As a quotient of a local ring is local, Lemma 5.1.13 implies that $\mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{F})$ is also local, so $j^*\mathcal{F}$ is either zero or indecomposable. $\square$

Proposition 5.1.15. *Assume $S = \mathrm{Spa}(C, C^+)$. Let $\mathcal{F} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$ be an indecomposable relative parity complex. Then $\mathcal{F}$ enjoys the following properties:*

1. *$\mathcal{F}$ has support of the form $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\leq\mu}$ for some $\mu \in X_*(T)^+$.*
2. *The restriction $i_\mu^*\mathcal{F}$ to the open stratum in $\mathrm{supp}(\mathcal{F})$ is a shifted Λ-free constant sheaf $\Lambda[d]$.*
3. *Any indecomposable relative parity complex $\mathcal{G} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$ supported on $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\leq\mu}$, such that $i_\mu^*\mathcal{G} \cong \Lambda[d]$ on $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu}$, is isomorphic to $\mathcal{F}$.*

Proof. (1) If $\mathrm{supp}(\mathcal{F})$ contains two disjoint open strata U, U' then applying Lemma 5.1.13 to each of U and U' shows that $\mathrm{End}(\mathcal{F})$ is not a local ring. But since $\mathcal{F}$ is indecomposable, the Krull-Schmidt property (Lemma 5.1.6) implies that $\mathrm{End}(\mathcal{F})$ is local.

(2) Follows from Lemma 5.1.14 and [Proposition VI.6.5]FS, using assumption that S is a geometric point.

(3) Since $\mathcal{F}$ is assumed to be indecomposable, the Krull-Schmidt property (Lemma 5.1.6) implies that $\mathrm{End}(\mathcal{F})$ is a local ring. By assumption, there are isomorphisms $\alpha \colon i_\mu^*\mathcal{F} \to i_\mu^*\mathcal{G}$ and $\beta \colon i_\mu^*\mathcal{G} \to i_\mu^*\mathcal{F}$ which are mutual inverses. By Lemma 5.1.13 we can find lifts $\widetilde{\alpha} \colon \mathcal{F} \to \mathcal{G}$ and $\widetilde{\beta} \colon \mathcal{G} \to \mathcal{F}$ such that $\widetilde{\beta} \circ \widetilde{\alpha} \in \mathrm{End}(\mathcal{F})$ does not lie in the unique maximal ideal of $\mathrm{End}(\mathcal{F})$, hence is invertible. Similarly, $\widetilde{\alpha} \circ \widetilde{\beta} \in \mathrm{End}(\mathcal{G})$ is invertible. Therefore, $\widetilde{\alpha}$ and $\widetilde{\beta}$ are isomorphisms.

$\square$

Remark 5.1.16. Lemma 5.1.14 and Proposition 5.1.15 fail badly for general strictly totally disconnected S. Indeed, if S is infinite, then any non-zero locally constant sheaf on S can be decomposed non-trivially into a direct sum by decomposing its support into a finite union of closed-open subsets. Therefore, there are no non-trivial indecomposable objects. In particular, the results of §5.1.2 are not true for general strictly totally disconnected S.[6] This represents a significant departure from how the theory works in [JMW14].

In the next section, we will establish the following facts.

- For a v-stack S, we let $\mathrm{Loc}(S;\Lambda)$ be the category of Λ-free étale local systems on S. Then for any v-stack $S \to \mathrm{Div}^1_X$, there is a symmetric monoidal equivalence (6.2)
$$\mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda) \cong \mathrm{Rep}_{\mathrm{Loc}(S;\Lambda)}(\check{G}),$$
an incarnation of the Geometric Satake equivalence relative to S. (The Satake category $\mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda) \subset D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is the full subcategory spanned by flat perverse sheaves over S.)
- For $\mu \in X_*(T)^+$, let $T_\Lambda(\mu) \in \mathrm{Rep}_\Lambda(\check{G})$ be the tilting module with highest weight μ. Assume $\ell > b(\check{G})$. Then for any $\mathcal{L} \in \mathrm{Loc}(S;\Lambda)$ and any $\mu \in X_*(T)^+$, the image $\mathcal{E}(\mu, \mathcal{L})$ of $\mathcal{L} \otimes T_\Lambda(\mu) \in \mathrm{Rep}_{\mathrm{Loc}(S;\Lambda)}(\check{G})$ under (6.2) is relative parity (Corollary 6.2.4).

We assume these facts for now.

Then we have the following generalization of Proposition 5.1.15 for strictly totally disconnected S.

Proposition 5.1.17. *Assume S is strictly totally disconnected and $\ell > b(\check{G})$.*

1. *Let $\mathcal{F} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$ and $\mu \in X_*(T)^+$ be maximal so that the support of $\mathcal{F}$ intersects $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu}$ non-trivially. If $i_\mu^*\mathcal{F} \cong \mathcal{L}[\langle 2\rho, \mu\rangle]$ on $\mathrm{Gr}_{G,S/\mathrm{Div}^1_X,\mu}$ for some $\mathcal{L} \in \mathrm{Loc}(S,\Lambda)$, then $\mathcal{E}(\mu, \mathcal{L})$ is a retract of $\mathcal{F}$.*
2. *Any $\mathcal{F} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$ is a finite direct sum of shifts of $\mathcal{E}(\mu, \mathcal{L})$ for various $\mu \in X_*(T)^+$ and $\mathcal{L} \in \mathrm{Loc}(S;\Lambda)$.*

[6] This corrects a mistake in a previous version of this paper, which was pointed out to us by David Hansen.

Proof. (1) It is immediate from the construction of $\mathcal{E}(\mu, \mathcal{L})$ that $i_\mu^* \mathcal{E}(\mu, \mathcal{L}) \cong \mathcal{L}[\langle 2\rho, \mu \rangle]$. Hence the assumption implies that there are isomorphisms $\alpha \colon i_\mu^* \mathcal{F} \to i_\mu^* \mathcal{E}(\mu, \mathcal{L})$ and $\beta \colon i_\mu^* \mathcal{E}(\mu, \mathcal{L}) \to i_\mu^* \mathcal{F}$ which are mutual inverses. By Lemma 5.1.13 we can find lifts $\widetilde{\alpha} \colon \mathcal{F} \to \mathcal{E}(\mu, \mathcal{L})$ and $\widetilde{\beta} \colon \mathcal{E}(\mu, \mathcal{L}) \to \mathcal{F}$ restricting to α and β under i_μ^*. It then suffices to show that $\widetilde{\alpha} \circ \widetilde{\beta} \in \mathrm{End}(\mathcal{E}(\mu, \mathcal{L}))$ is an isomorphism. This can be checked after pulling back along all geometric points $\mathrm{Spa}(C, C^+) \to S$, and then it follows from Proposition 5.1.15(3).

(2) We prove the statement by induction on the largest $\mu \in X_*(T)^+$ intersecting $\mathrm{supp}(\mathcal{F})$ non-trivially. We have $i_\mu^* \mathcal{F}[-\langle 2\rho, \mu \rangle] = \bigoplus \mathcal{L}_j[d_j]$ for some $\mathcal{L}_j \in \mathrm{Loc}(S; \Lambda)$ and $d_j \in \mathbf{Z}$ by [FS21, Proposition VI.6.5] and the assumption that S is strictly totally disconnected. Then part (1) implies that $\mathcal{F} \cong \bigoplus_j \mathcal{E}(\mu, \mathcal{L}_j)[d_j] \oplus \mathcal{F}'$ where the inductive hypothesis applies to $\mathcal{F}'$. $\square$

5.2. **Small Smith-Treumann localization.** We develop here a version of the Tate category and Smith-Treumann localization which is "small" in the sense that it consists only of sheaves satisfying strong finiteness conditions. It is more similar to the formalism of [Tre19, RW22], but we note that our sheaves are still not "constructible" since the Schubert stratifications are not constructible in p-adic geometry (due to failure of quasicompactness). The structure theory of ULA sheaves (relative to S) on $\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}_X^1}$ from [FS21] is what makes this small version well behaved even in the absence of quasicompactness.

5.2.1. *The small Tate category.* We continue to assume that Λ is an ℓ-adically complete PID over $\mathbf{Z}_\ell$. Let $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}} \subset D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda)^{\mathrm{bd}}$ be the full subcategory spanned by objects with finite tor-dimension over $\Lambda[\Sigma]$.

Definition 5.2.1. We define the *small Tate category* of $\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}$ to be the Verdier quotient

$$\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \mathcal{T}_\Lambda)^{\mathrm{bd}} := \frac{D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}}{\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}}$$

and the *small Tate category* of $\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}_X^1)^I}$ to be

$$\mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}_X^1)^I}; \mathcal{T}_\Lambda)^{\mathrm{bd}} := \frac{D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}}{\mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}}.$$

We make analogous definitions for $\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I,\mu}$ and $\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I,\leq\mu}$, etc.

Parallel to Lemma 3.2.4, we give an alternate characterization of $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}$; note that this one applies without assuming that $\Lambda = k$ is a field.

Lemma 5.2.2. *The subcategory* $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}} \subset D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}$ *coincides with the full subcategory spanned by objects whose stalks at all geometric points are perfect complexes over* $\Lambda[\Sigma]$.

Proof. All geometric stalks of all objects of $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}$ are pseudocoherent by [FS21, Proposition VI.6.5], and have finite tor-amplitude by [Sta20, Tag 0DJJ]. Pseudo-coherent plus finite tor-amplitude implies perfect by [Sta20, Tag 0656]; this shows one containment.

For the other containment, let $\mathcal{K} \in D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}$; we must show that if all geometric stalks of $\mathcal{K}$ are perfect complexes over $\Lambda[\Sigma]$, then $\mathcal{K}$ has finite tor-amplitude. By [Sta20, Tag 0DJJ], it suffices to exhibit a uniform bound on the tor-amplitude of the stalks. The rest of the proof is as for Lemma 3.2.4, except we use that for a perfect complex over $\Lambda[\Sigma]$, the properties of being flat and projective coincide by [Sta20, Tag 051E]. $\square$

Remark 5.2.3 (Comparison to the large Tate category)**.** The tautological fully faithful embedding

$$D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}} \hookrightarrow D^b_{\text{ét}}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])$$

carries $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])^{\mathrm{bd}}$ into $\mathrm{Flat}^b(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \Lambda[\Sigma])$ and so induces a functor

$$\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \mathcal{T}_\Lambda)^{\mathrm{bd}} \to \mathrm{Shv}(\mathrm{Gr}_{H,S/(\mathrm{Div}_X^1)^I}; \mathcal{T}_\Lambda) \tag{5.4}$$

which is conservative; similarly for the equivariant version.

Definition 5.2.4. We denote by
$$\mathbb{T}^*\colon D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I};\Lambda[\Sigma])^{\mathrm{bd}} \to \mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}^1_X)^I};\mathcal{T}_\Lambda)^{\mathrm{bd}}$$
the tautological projection.

We denote by
$$\epsilon^*\colon D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}} \to D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I};\Lambda[\Sigma])^{\mathrm{bd}}$$
the inflation via the augmentation $\Lambda[\Sigma] \to \Lambda$.

We write
$$\mathbb{T} := \mathbb{T}^*\varepsilon^*\colon D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}} \to \mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}^1_X)^I};\mathcal{T}_\Lambda)^{\mathrm{bd}}.$$
These functors are compatible with the ones having the same notations in §3.2, under (5.4). We use the same notation for the analogous operations for strata, Schubert varieties, and $\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}^1_X)^I}$, etc.

5.2.2. *The small* Psm *operation.* We now define the version of the "Smith operation" Psm from §3.6 for small Tate categories. We assume the setup of §4: G has a Σ-action and $H := H^\sigma$ is reductive, so that $\mathrm{Fix}(\sigma, \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}) = \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I}$ by Proposition 4.1.2.

Lemma 5.2.5. *Let $\iota\colon \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \hookrightarrow \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$ be the inclusion of the Σ-fixed points. For any $\mathcal{F} \in D^{\mathrm{ULA}}_{\acute{e}t}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}}$, the restriction $\iota^*\mathcal{F} \in D^{\mathrm{ULA}}_{\acute{e}t}(\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}}$ is ULA over S.*

Proof. The condition can be checked v-locally on S by [FS21, Proposition IV.2.5]. Therefore, we may assume that G has a Borus (B,T). Let $(B_H,T_H) = (B^\sigma,T^\sigma)$ be the corresponding Borus of H (cf. Lemma 4.2.1). Recall the constant term functors CT from [FS21, §VI.3]. Applying (a variant for $(\mathrm{Div}^1_X)^I$ of) [FS21, Proposition VI.6.4], it suffices to check that $\mathrm{CT}_{B_H}(\iota^*\mathcal{F})$ is ULA over S. By proper base change applied to the commutative diagram
$$\begin{tikzcd}
\mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I} & \mathrm{Gr}_{B,S/(\mathrm{Div}^1_X)^I} \arrow[l] \arrow[r] & \mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I} \\
\mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I} \arrow[u, hook, "\iota"] & \mathrm{Gr}_{B_H,S/(\mathrm{Div}^1_X)^I} \arrow[l] \arrow[r] \arrow[u, hook, "\iota"] & \mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I} \arrow[u, hook, "\iota"]
\end{tikzcd}$$
all of whose squares are Cartesian (thanks to Proposition 4.1.2 and Proposition 4.2.2), we have a natural isomorphism $\mathrm{CT}_{B_H}(\iota^*\mathcal{F}) \cong \iota^*\,\mathrm{CT}_B(\mathcal{F})$. By (a variant for $(\mathrm{Div}^1_X)^I$ of) [FS21, Proposition VI.6.4], $\mathrm{CT}_B(\mathcal{F}) \in D^{\mathrm{ULA}}_{\mathrm{\acute{e}t}}(\mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}}$ is ULA over S. Now $\iota\colon \mathrm{Gr}_{T_H,S/(\mathrm{Div}^1_X)^I} \hookrightarrow \mathrm{Gr}_{T,S/(\mathrm{Div}^1_X)^I}$ is an open embedding, hence ℓ-cohomologically smooth, so $\iota^*\,\mathrm{CT}_B(\mathcal{F})$ is ULA over S by [FS21, Proposition IV.2.13(i)]. ∎

Definition 5.2.6 (Small Smith operation)**.** The *(small) Smith operation* is the functor
$$\mathrm{Psm} := \mathbb{T}^* \circ \iota^*\colon (D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}})^{B\Sigma} \to \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/(\mathrm{Div}^1_X)^I};\mathcal{T}_\Lambda)^{\mathrm{bd}}, \tag{5.5}$$
which is well-defined by Lemma 5.2.5.

The functor Psm has an equivariant version, which we also denote
$$\mathrm{Psm}\colon (D^{\mathrm{ULA}}_{\mathrm{\acute{e}t}}(\mathcal{H}\mathrm{ck}_{G,S/(\mathrm{Div}^1_X)^I};\Lambda)^{\mathrm{bd}})^{B\Sigma} \to \mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/(\mathrm{Div}^1_X)^I};\mathcal{T}_\Lambda)^{\mathrm{bd}}.$$

5.2.3. *Compatibilities.* We now establish some compatibility statements that could be remembered under the slogan, "the Smith operation commutes with all functors".

Notation 5.2.7. Below we let Y, Y' be spaces of the form $\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}$, or the m-step convolution version thereof, or the twisted Grassmannian $\mathrm{Gr}^{\mathrm{tw}}_{G,S/(\mathrm{Div}^1_X)^I}$ or (closures of) strata thereof, and $f\colon Y \to Y'$ be a map induced by a group homomorphism $G \to G'$, or a convolution map. In all such cases, the small Tate categories of Y^σ and $(Y')^\sigma$ are defined, because of Proposition 4.1.2 and Corollary 4.1.4. The Smith operation is also defined, and will be denoted
$$\mathrm{Psm}\colon D^{\mathrm{ULA}}_?(Y;\Lambda) \to \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda)$$
where ? refers to the constructible or equivariant conditions.

Example 5.2.8. We allow G' to be the trivial group, in which case $\mathrm{Gr}_{G',S/(\mathrm{Div}^1_X)^I} \cong S$.

Let $f\colon Y \to Y'$ be a Σ-equivariant morphism of the type in Notation 5.2.7. Let $f^\sigma\colon Y^\sigma \to (Y')^\sigma$ be the induced map of fixed points. Since pullbacks preserve stalks, from Lemma 5.2.2 we see that the pullback functor $(f^\sigma)^*\colon D^{\mathrm{ULA}}_?((Y')^\sigma;\Lambda)^{B\Sigma} \to D^{\mathrm{ULA}}_?(Y^\sigma;\Lambda)^{B\Sigma}$ preserves the perfect subcategories, hence descends to a functor

$$(f^\sigma)^*\colon \mathrm{Perf}^{\mathrm{ULA}}_?((Y')^\sigma;\mathcal{T}_\Lambda) \to \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda). \tag{5.6}$$

We have the properties analogous to §3.4 in this situation. Applying relative Verdier duality to (5.6), we also get

$$(f^\sigma)^!\colon \mathrm{Perf}^{\mathrm{ULA}}_?((Y')^\sigma;\mathcal{T}_\Lambda) \to \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda). \tag{5.7}$$

Below, when we say that diagrams "canonically commute", we mean that we construct explicit natural isomorphisms.

Lemma 5.2.9. *Let $f\colon Y \to Y'$ be a Σ-equivariant morphism of the type in Notation 5.2.7. The following diagrams canonically commute:*

$$\begin{tikzcd} D^b_?(Y;\Lambda)^{B\Sigma} \arrow[d, "\mathrm{Psm}"] & D^b_?(Y';\Lambda)^{B\Sigma} \arrow[l, "f^*"'] \arrow[d, "\mathrm{Psm}"] \\ \mathrm{Perf}_?(Y^\sigma;\mathcal{T}_\Lambda) & \mathrm{Perf}_?((Y')^\sigma;\mathcal{T}_\Lambda) \arrow[l, "(f^\sigma)^*"'] \end{tikzcd} \qquad \begin{tikzcd} D^b_?(Y;\Lambda)^{B\Sigma} \arrow[d, "\mathrm{Psm}"] & D^b_?(Y';\Lambda)^{B\Sigma} \arrow[l, "f^!"'] \arrow[d, "\mathrm{Psm}"] \\ \mathrm{Perf}_?(Y^\sigma;\mathcal{T}_\Lambda) & \mathrm{Perf}_?((Y')^\sigma;\mathcal{T}_\Lambda) \arrow[l, "(f^\sigma)^!"'] \end{tikzcd}$$

Proof. For the first square, the assertion is immediate from the definitions (the point being that $*$-restrictions commute with $*$-restrictions). The assertion for the second square follows from the first square plus Lemma 3.4.3, which allows us to replace ι^* with $\iota^!$ in the definition of Psm. $\square$

Let $f\colon Y \to Y'$ be a σ-equivariant morphism of the type in Notation 5.2.7. For $f^\sigma\colon Y^\sigma \to (Y')^\sigma$ the induced map of fixed points, Lemma 3.4.5 implies that $\mathrm{R}f^\sigma_!$ and $\mathrm{R}f^\sigma_*$ preserve the perfect subcategories, hence descend to

$$\mathrm{R}f^\sigma_!\colon \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda) \to \mathrm{Perf}^{\mathrm{ULA}}_?((Y')^\sigma;\mathcal{T}_\Lambda)$$

$$\mathrm{R}f^\sigma_*\colon \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda) \to \mathrm{Perf}^{\mathrm{ULA}}_?((Y')^\sigma;\mathcal{T}_\Lambda)$$

Proposition 5.2.10. *Let $f\colon Y \to Y'$ be a Σ-equivariant morphism of the type in Notation 5.2.7. Then the following diagrams canonically commute:*

$$\begin{tikzcd} D^{\mathrm{ULA}}_?(Y;\Lambda)^{B\Sigma} \arrow[r, "\mathrm{R}f_!"] \arrow[d, "\mathrm{Psm}"] & D^{\mathrm{ULA}}_?(Y';\Lambda)^{B\Sigma} \arrow[d, "\mathrm{Psm}"] \\ \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda) \arrow[r, "\mathrm{R}f^\sigma_!"] & \mathrm{Perf}^{\mathrm{ULA}}_?((Y')^\sigma;\mathcal{T}_\Lambda) \end{tikzcd} \qquad \begin{tikzcd} D^{\mathrm{ULA}}_?(Y;\Lambda)^{B\Sigma} \arrow[r, "\mathrm{R}f_*"] \arrow[d, "\mathrm{Psm}"] & D^{\mathrm{ULA}}_?(Y';\Lambda)^{B\Sigma} \arrow[d, "\mathrm{Psm}"] \\ \mathrm{Perf}^{\mathrm{ULA}}_?(Y^\sigma;\mathcal{T}_\Lambda) \arrow[r, "\mathrm{R}f^\sigma_*"] & \mathrm{Perf}^{\mathrm{ULA}}_?((Y')^\sigma;\mathcal{T}_\Lambda) \end{tikzcd}$$

Proof. We give the argument for the first diagram, the second being similar[7]. To prove the assertion, we may (thanks to Lemma 5.2.9) base change to the Σ-fixed locus of Y', and therefore reduce to the case that Σ acts trivially on Y'. Let $i\colon Y^\sigma \hookrightarrow Y$ be the inclusion of the Σ-fixed points and $j\colon U \hookrightarrow Y$ be the complementary open embedding. For $\mathcal{F} \in D^{\mathrm{ULA}}_?(Y;\Lambda)^{B\Sigma}$ consider the exact triangle

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}.$$

Since Σ acts freely on U, Corollary 3.4.6(1) implies that $\mathrm{R}f_!j_!j^*\mathcal{F} \in \mathrm{Perf}^{\mathrm{ULA}}_?(Y';\Lambda[\Sigma])$, hence projects to 0 in $\mathrm{Perf}^{\mathrm{ULA}}_?(Y';\mathcal{T}_\Lambda)$. Therefore the map $\mathrm{R}f_!\mathcal{F} \to \mathrm{R}f_!i_*i^*\mathcal{F}$ projects to an isomorphism in $\mathrm{Perf}^{\mathrm{ULA}}_?(Y';\mathcal{T}_\Lambda)$. $\square$

5.3. **Relative Tate-parity sheaves.** We now develop an analogous theory to §5.1 in the setting of the small Tate category, inspired by work of Leslie-Lonergan [LL21] which does this for the classical affine Grassmannian. We note that it is important here to take integral coefficients in order to have any hope of parity vanishing properties, because of Example 3.7.1.

[7]Using also that $Y^\sigma \hookrightarrow Y$ is ω_0-locally extra small (cf. Example 3.1.4).

5.3.1. *Preliminary lemmas.* Let $\mathbb{O} := W(k)$, so $k = \mathbb{O}/\ell$. Note that we take $|I| = 1$ below. Recall (cf. §2.7) that we write

$$\mathbb{F}\colon D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathbb{O})^{\mathrm{bd}} \to D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; k)^{\mathrm{bd}}$$

for the change-of-coefficients functor.

The following Lemma is parallel to [LL21, Proposition 4.6.1].

Lemma 5.3.1. *Let $\mathcal{F}, \mathcal{G} \in D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathbb{O})^{\mathrm{bd}}$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}}(\mathbb{T}\mathcal{F}, \mathbb{T}\mathcal{G}) \cong \bigoplus_{i\in\mathbf{Z}} \mathrm{Hom}_{D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k)^{\mathrm{bd}}}(\mathbb{F}\mathcal{F}, \mathbb{F}\mathcal{G}[2i]). \tag{5.8}$$

Proof. We have

$$\mathcal{H}om_{\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}}(\mathbb{T}\mathcal{F}, \mathbb{T}\mathcal{G}) \cong \mathbb{T}\left(\mathcal{H}om_{D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O})^{\mathrm{bd}}}(\mathcal{F}, \mathcal{G})\right).$$

As in Example 3.7.4, for any $\mathcal{E} \in D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathbb{O})^{\mathrm{bd}}$, we have

$$\mathbb{T}\mathcal{E} \cong \mathrm{Tot}\left(\ldots \mathcal{E} \xrightarrow{\ell} \mathcal{E} \xrightarrow{0} \mathcal{E} \xrightarrow{\ell} \mathcal{E} \xrightarrow{0} \mathcal{E} \to \ldots\right) \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}.$$

Noting that

$$\mathrm{Tot}\left(\ldots \mathcal{E} \xrightarrow{\ell} \mathcal{E} \xrightarrow{0} \mathcal{E} \xrightarrow{\ell} \mathcal{E} \xrightarrow{0} \mathcal{E} \to \ldots\right) \cong \bigoplus_{i\in\mathbf{Z}} \mathbb{F}\mathcal{E}[2i],$$

the result then follows upon taking global sections. $\square$

Remark 5.3.2. The proof of Lemma 5.3.1 did not use the ULA or L^+H-constructibility hypotheses.

Lemma 5.3.3. *Assume S is strictly totally disconnected. Then any object $\mathcal{F} \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1,\mu}; \mathcal{T}_{\mathbb{O}})$ is a finite direct sum of objects of the form $\mathbb{T}(\mathcal{L})$ and $\mathbb{T}(\mathcal{L}[1])$ for $\mathcal{L} \in \mathrm{Loc}(S; \mathbb{O})$.*

Proof. The same argument as for [LL21, Theorem 5.4.1] shows that any such $\mathcal{F}$ is pulled back from the small Tate category of S. This category is generated by locally constant sheaves, since ULA complexes relative to the identity map are locally constant [FS21, Proposition IV.2.9], and S has no higher cohomology. Then we may conclude by applying the argument of [LL21, §3.5.1], with the following remark: In [LL21] the case $\ell = 2$ is excluded because of [LL21, Remark 3.5.4] which claims that $\mathrm{Ext}^1_{\mathbf{F}_2}(\mathbf{F}_2, \mathbf{F}_2) \neq 0$, however this is clearly false, so we may also take $\ell = 2$. $\square$

Lemma 5.3.4. *Assume $S = \mathrm{Spa}(C, C^+)$. Then the category $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is Krull-Schmidt.*

Proof. By [CYZ08, Theorem A.1], a Karoubian category such that the endomorphism ring of any object is semiperfect is Krull-Schmidt. We will check the semiperfect and Karoubian conditions.

First we check that $\mathrm{End}(\mathcal{K})$ is semiperfect for any $\mathcal{K} \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$, by showing that $\mathrm{End}(\mathcal{K})$ is a finite k-algebra. Indeed, it follows from Lemma 5.3.3 and Lemma 5.3.1 that for any $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$, the whole Ext-algebra $\mathrm{Ext}^\bullet(\mathcal{K}_1, \mathcal{K}_2)$ is finite-dimensional over k.

Now it suffices to show that $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is Karoubian, i.e., all idempotents are split. According to [LC07, Proposition 2.3], in a triangulated category an idempotent of a distinguished triangle which splits on any two terms splits on the third. Therefore by devissage, we may reduce to showing that for any indecomposable object in $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1,\mu}; \mathcal{T}_{\mathbb{O}})$, its endomorphism ring is local. By Lemma 5.3.3 such $\mathcal{K}$ must be isomorphic to $\mathbb{T}(\mathbb{O}^r[d])$ for some d, and then Lemma 5.3.1 computes its endomorphism algebra, which is seen to be local by inspection.

$\square$

5.3.2. *Relative Tate-parity complexes.* We define make definitions within the small Tate category analogous to those in §5.1.3.

Definition 5.3.5. Let $\mathcal{K} \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$. Fix a pariversity $\dagger\colon X_*(T_H) \to \mathbf{Z}/2\mathbf{Z}$. Below we view Tate cohomology as being indexed by $\mathbf{Z}/2\mathbf{Z}$.

(1) For $? \in \{*, !\}$, we say $\mathcal{K}$ is *relative $?$-Tate-even* if for all geometric points $\overline{s} \to S$ and all $\mu \in X_*(T_H)^+$,

$$\mathrm{T}^{\dagger(\mu)+1}(i_\mu^?(\mathcal{K}|_{\overline{s}})) = 0.$$

(2) For $? \in \{*, !\}$, we say $\mathcal{K}$ is *relative ?-Tate-odd* if $\mathcal{K}[1]$ is relative ?-Tate-even.
(3) For $? \in \{*, !\}$, we say that $\mathcal{K}$ is *relative ?-Tate-parity* if $\mathcal{K}$ is either relative ?-even or relative ?-odd.
(4) We say $\mathcal{K}$ is *relative Tate-even* (resp. *relative Tate-odd*) if $\mathcal{K}$ is both relative $*$-Tate even (resp. odd) and relative !-Tate even (resp. odd).
(5) We say $\mathcal{K}$ is a *relative Tate-parity complex* if it is isomorphic within $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ to the direct sum of a relative Tate-even complex and a relative Tate-odd complex. The full subcategory of $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ spanned by relative Tate-parity complexes (with coefficients in $\mathcal{T}_{\mathbb{O}}$) is denoted $\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})$.

5.3.3. *Modular reduction.* We now establish some properties of relative parity complexes under change-of-coefficients, parallel to [JMW14, §2.5] and [LL21, §5.6].

Lemma 5.3.6. *Recall that $\mathbb{F}$ is the change-of-coefficients functor*

$$\mathbb{F} = k \overset{L}{\otimes}_{\mathbb{O}} (-) \colon D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}} \to D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; k)^{\mathrm{bd}}.$$

Let $\mathcal{E} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}}$. Then:

(1) $\mathcal{E}$ is relative ?-even (resp. odd) if and only if $\mathbb{F}\mathcal{E}$ is relative ?-even (resp. odd).
(2) Assume S is strictly totally disconnected and $\ell > b(\check{G})$. Then for all $\mu \in X_*(T)_+$ and $\mathcal{L} \in \mathrm{Loc}(S; \mathbb{O})$, we have

$$\mathbb{F}\mathcal{E}(\mu, \mathcal{L}) \cong \mathcal{E}(\mu, \mathbb{F}L).$$

Proof. (1) It is immediate from the definitions that $i^? \mathbb{F}(\mathcal{E}) \cong \mathbb{F}(i^? \mathcal{E})$, so $\mathcal{E}$ is ?-even (resp. odd) implies that $\mathbb{F}\mathcal{E}$ is ?-even (resp. odd). The converse follows from the assumption that $\mathcal{E}$ has $\mathbb{O}$-free stalks and costalks, so the cohomology sheaves of $\mathcal{E}$ are supported in the same degrees as the cohomology sheaves of $\mathbb{F}\mathcal{E}$.

(2) follows from the definitions, using that change-of-coefficients sends tilting modules to tilting modules. □

We next explain that the functor $\mathbb{T} = \mathbb{T}^* \epsilon^*$ from Definition 5.2.4 has similar properties to modular reduction.

Proposition 5.3.7. *Let $\mathcal{E} \in D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}}$.*

(1) If $\mathcal{E} \in D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}}$ is relative even (resp. odd), then $\mathbb{T}\mathcal{E} \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is relative Tate-even (resp. odd).

(2) Assume $S = \mathrm{Spa}(C, C^+)$. Then for all $\mu \in X_*(T_H)_+$ and $\mathcal{L} \in \mathrm{Loc}(S; \mathbb{O})$, the object $\mathbb{T}\mathcal{E}(\mu, \mathcal{L})$ is relative Tate-parity and indecomposable.

Proof. As explained in §5.2.3, the operation $\mathbb{T}$ is compatible with formation of i^*_μ or $i^!_\mu$. Hence to prove (1) we reduce to showing that $\mathrm{T}^i \epsilon^* \mathbb{O}$ vanishes in odd degree, which was seen in Example 3.7.1.

Having established part (1), and using that $\mathcal{E}(\mu, L)$ is relative parity, to prove part (2) it only remains to check that $\mathbb{T}\mathcal{E}(\mu, L)$ is indecomposable. Abbreviate $\mathcal{E} := \mathcal{E}(\mu, L)$. By Lemma 5.3.4, this is equivalent to the endomorphism ring of $\mathbb{T}\mathcal{E}$ being local. Applying Lemma 5.3.1 to $\mathcal{F} = \mathcal{G} = \mathcal{E}$, we have

$$\mathrm{Hom}_{\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}}(\mathbb{T}\mathcal{E}, \mathbb{T}\mathcal{E}) = \bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k)^{\mathrm{bd}}}(\mathbb{F}\mathcal{E}, \mathbb{F}\mathcal{E}[2i]). \tag{5.9}$$

By Lemma 5.3.6(2), the ring $\mathrm{Hom}_{D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k)^{\mathrm{bd}}}(\mathbb{F}\mathcal{E}, \mathbb{F}\mathcal{E})$ is local. This shows that the subalgebra on the RHS of (5.9) indexed by $i = 0$ is local, and by perversity the summands of (5.9) indexed by $i < 0$ vanish. This implies the desired locality of the graded algebra (5.9). □

Definition 5.3.8. Assume S is strictly totally disconnected and $\ell > b(\check{H})$. For $\mu \in X_*(T)^+$ and $\mathcal{L} \in \mathrm{Loc}(S; \mathbb{O})$, we define

$$\mathcal{E}_{\mathrm{T}}(\mu, \mathcal{L}) := \mathbb{T}\mathcal{E}(\mu, \mathcal{L}) \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}}).$$

5.3.4. *Structure theory for strictly totally disconnected S.* Here we record several structural results about $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ *under the assumption S is strictly totally disconnected.* **Below, all Hom and Ext groups are formed in the category** $\mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$; we omit this for ease of notation. By arguments analogous to those in §5.1.4, we have the following results.

Lemma 5.3.9. *Assume that S is strictly totally disconnected. If $\mathcal{F}\in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is relative $*$-Tate-parity and $\mathcal{G}\in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is relative $!$-Tate-parity, then we have a (non-canonical) isomorphism of k-vector spaces*

$$\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{G})\cong \bigoplus_{\mu\in X_*(T_H)^+}\mathrm{Ext}^\bullet(i_\mu^*\mathcal{F},i_\mu^!\mathcal{G})$$

Lemma 5.3.10. *Assume that S is strictly totally disconnected. If $\mathcal{F}\in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is relative $*$-Tate-even and $\mathcal{G}\in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ is relative $!$-Tate-odd, then $\mathrm{Hom}(\mathcal{F},\mathcal{G})=0$.*

Lemma 5.3.11. *Assume that S is strictly totally disconnected. If $\mathcal{F},\mathcal{G}\in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ are either both relative Tate-even or both relative Tate-odd, and i_μ is the inclusion of a stratum which is open in the support of both $\mathcal{F}$ and $\mathcal{G}$, then the restriction map $\mathrm{Hom}(\mathcal{F},\mathcal{G})\to\mathrm{Hom}(i_\mu^*\mathcal{F},i_\mu^*\mathcal{G})$ is surjective.*

Lemma 5.3.12. *Assume $S=\mathrm{Spa}(C,C^+)$. Let $\mathcal{F}\in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$ be an indecomposable Tate-parity complex, and let $j\colon U\to \mathrm{Gr}_{H,S/\operatorname{Div}^1_X}$ be an inclusion of a union of strata open in the support of $\mathcal{F}$. Then $j^*\mathcal{F}$ is either 0 or indecomposable.*

Proposition 5.3.13. *Assume $S=\mathrm{Spa}(C,C^+)$. Let $\mathcal{F}\in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})$ be an indecomposable relative Tate-parity complex. Then $\mathcal{F}$ enjoys the following properties:*

1. *$\mathcal{F}$ has support of the form $\mathrm{Gr}_{H,S/\operatorname{Div}^1_X,\le\mu}$ for some $\mu\in X_*(T_H)^+$.*
2. *The restriction $i_\mu^*\mathcal{F}$ to the open stratum in $\mathrm{supp}(\mathcal{F})$ is isomorphic $\mathbb{T}(\mathbb{O}[d])$ for some d.*
3. *Any indecomposable relative Tate-parity complex $\mathcal{G}$ supported on $\mathrm{Gr}_{H,S/\operatorname{Div}^1_X,\le\mu}$, such that $i_\mu^*\mathcal{G}\cong\mathbb{T}(\mathbb{O}[d])$, is isomorphic to $\mathcal{F}$.*

Proposition 5.3.14. *Assume S is strictly totally disconnected and $\ell>b(\check{H})$.*

1. *Let $\mathcal{F}\in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})$ and $\mu\in X_*(T_H)^+$ be maximal so that the support of $\mathcal{F}$ intersects $\mathrm{Gr}_{H,S/\operatorname{Div}^1_X,\mu}$ non-trivially. If $i_\mu^*\mathcal{F}\cong\mathbb{T}\mathcal{L}[\langle 2\rho,\mu\rangle]$ on $\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu}$ for some $\mathcal{L}\in\mathrm{Loc}(S;\mathbb{O})$, then $\mathcal{E}_{\mathrm{T}}(\mu,\mathcal{L})$ is a retract of $\mathcal{F}$.*
2. *Any $\mathcal{F}\in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})$ is a finite direct sum of shifts of $\mathcal{E}_{\mathrm{T}}(\mu,\mathcal{L})$ for various $\mu\in X_*(T_H)^+$ and $\mathcal{L}\in\mathrm{Loc}(S;\mathbb{O})$.*

5.4. **Even maps.** We define a *stratified v-stack* to be a v-stack Y plus a decomposition of Y into locally closed strata $i_\mu\colon Y_\mu\hookrightarrow Y$, such that i_μ is shriekable for all μ. For a stratified small v-stack Y and a pariversity † (i.e., a function from the set of strata to $\mathbf{Z}/2\mathbf{Z}$), we may define (absolute) *parity complexes* analogously to Definition 5.1.8 (requiring the restriction to strata to have cohomology sheaves which are Λ-free local systems). We do not expect it to have good properties in general, but in this section we axiomatize the property that this notion of parity complex will be preserved under "even" maps, which is true in general.

We begin with a simple observation that provides a partial substitute for "Gysin isomorphisms" in p-adic geometry (which are not defined, due to the lack of a notion of local complete intersection).

Lemma 5.4.1. *Let $i\colon Z\to Y$ be a shriekable map of stratified Artin v-stacks which are ℓ-cohomologically smooth of integral ℓ-dimension (cf.* [FS21, Definition IV.1.17]*). Then $i^!$ is isomorphic to an even shift of i^* on étale local systems on Y.*

Proof. By base changing to an étale cover of Y, we reduce to showing that $i^!\Lambda$ is isomorphic to an even shift of Λ. Writing $\mathbb{D}_{(-)}$ for the dualizing sheaf and $d,d'\in\mathbf{Z}$ for the ℓ-dimensions of Y,Z respectively, we have

$$i^!\Lambda\cong i^!\mathbb{D}_Y[-2d]\cong\mathbb{D}_Z[-2d]\cong\Lambda[2d'-2d].$$

□

The following Definition is in imitation of [JMW14, Definition 2.32, Definition 2.33].

Definition 5.4.2 (Even maps)**.** Let Y and Y' be stratified small v-stacks and $f\colon Y \to Y'$ be shriekable. We say that f is *stratified* if

(1) The pre-image of any stratum $Y'_\lambda \hookrightarrow Y'$ is a union of strata of Y.
(2) For any stratum $Y_\mu \subset Y$ lying over any stratum $Y'_\lambda \subset Y'$, the restricted map of strata $f^\mu_\lambda \colon Y_\mu \to Y'_\lambda$ is ℓ-cohomologically smooth with integral ℓ-dimension.

We further say that f is *even* if for all μ, λ and any Λ-free étale local system $\mathcal{L}$ on Y'_λ, the complex $\mathrm{R}^i(f^\mu_\lambda)_*(\mathcal{L})$ is Λ-free and vanishes for odd i.

The significance of even maps lies in the fact that their direct image preserves parity, as in the following Proposition.

Proposition 5.4.3. *Let Y, Y' be stratified Artin v-stacks and let $f\colon Y \to Y'$ be an even stratified map (hence shriekable by definition). Suppose that all strata of Y, Y' are ℓ-cohomologically smooth with integral ℓ-dimension. Let $\dagger$ be a pariversity on Y', and define the pullback pariversity $\dagger_Y$ on Y via $\dagger_Y(\mu) := \dagger(\lambda)$ if f carries Y_μ to Y'_λ.*

(1) If f is proper and $\mathcal{K} \in D^b_{\acute{e}t}(Y;\Lambda)$ is a $\dagger_Y$-parity complex, then $\mathrm{R}f_*(\mathcal{K}) \in D^b_{\acute{e}t}(Y';\Lambda)$ is a $\dagger$-parity complex.

(2) If f is smooth of integral ℓ-dimension and $\mathcal{K} \in D^b_{\acute{e}t}(Y';\Lambda)$ is a $\dagger$-parity complex, then $f^*\mathcal{K} \in D^b_{\acute{e}t}(Y;\Lambda)$ is a $\dagger_Y$-parity complex.

Proof. (1) The statement and argument are the same as for [JMW14, Proposition 2.34], so we just sketch it. Using proper base change, we calculate $i^*_\lambda \mathrm{R}f_!\mathcal{K}$ and $i^!_\lambda \mathrm{R}f_*\mathcal{K}$ by stratifying the fibers by Y_μ, and calculating the cohomology of fibers in terms of cohomology of strata using excision sequences. This equips $i^*_\lambda \mathrm{R}f_!\mathcal{K}$ with a filtration whose associated graded is a direct sum of pieces of the form $\mathrm{R}(f^\mu_\lambda)_! i^*_\mu \mathcal{K}$, so its cohomology sheaves are Λ-free and vanish in the correct parity of degrees by the definition of an even map. The story is similar for $i^!_\lambda \mathrm{R}f_*\mathcal{K}$.

(2) It is immediate from the definitions that $f^*\mathcal{K}$ is $*$-parity with respect to $\dagger_Y$. Consider the commutative diagram

$$\begin{tikzcd}
Y_\mu \arrow[r, "i^\mu_\lambda"] \arrow[rr, bend left, "i_\mu"] \arrow[dr] & f^{-1}(Y'_\lambda) \arrow[r, "i'_\lambda"] \arrow[d, "f_\lambda"] & Y \arrow[d, "f"] \\
& Y'_\lambda \arrow[r, "i_\lambda"] & Y'
\end{tikzcd}$$

where the right square is Cartesian. Suppose $\mathcal{K} \in D^b_{\text{ét}}(Y';\Lambda)$ is $\dagger$-parity. Since f is smooth of integral ℓ-dimension, $(i'_\lambda)^! f^*\mathcal{K} \cong f^*_\lambda i^!_\lambda \mathcal{K}$ has locally constant Λ-free cohomology sheaves, which can only be non-zero in degrees congruent to $\dagger(\lambda)$ mod 2. By Lemma 5.4.1, the same holds for $(i^\mu_\lambda)^!(i'_\lambda)^! f^*\mathcal{K} \cong i^!_\mu f^*\mathcal{K}$, so $f^*\mathcal{K}$ is also !-parity with respect to $\dagger_Y$. $\square$

5.5. **Demazure resolutions.** Let S be a small v-stack and $S \to \mathrm{Div}^1_X$ a map factoring over $\mathrm{Spd}\, C$ where $C = \widehat{\overline{E}}$. We recall the notions of parahoric group schemes and their partial affine flag varieties in the context of $\mathrm{B}^+_{\mathrm{dR}}$-affine Grassmannians, following [FS21, §VI.5]. Let $A := W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$. For a subset J of affine simple roots there is a parahoric group scheme $\mathcal{P}_J \to G_A$, and we define the group diamond $L^+\mathcal{P}_J / \mathrm{Spd}\, \mathcal{O}_C$ by $L^+\mathcal{P}_J(R, R^+) = \mathcal{P}_J(\mathrm{B}^+_{\mathrm{dR}}(R^\sharp))$.

Now we define Bott-Samelson varieties in a parallel manner to [JMW14, §4.1]. Fix a chain $J_\bullet$ of subsets of the affine simple roots of G,

$$J_\bullet = \begin{pmatrix} I_0 & & I_1 & & \cdots & & I_n \\ & \searrow \;\; \swarrow & & \searrow \;\; \swarrow & & \searrow \;\; \swarrow & \\ & J_1 & & J_2 & \cdots & J_n & \end{pmatrix}.$$

For $(1 \le i \le k \le n)$, we define

$$\mathrm{Gr}^{(i,\ldots,k)}_{J_\bullet} = LG \times^{L^+\mathcal{P}_{I_i}} (L^+\mathcal{P}_{J_{i+1}}) \times^{L^+\mathcal{P}_{I_{i+1}}} \times \ldots \times (L^+\mathcal{P}_{J_{k-1}}) \times^{L^+\mathcal{P}_{I_{k-1}}} (L^+\mathcal{P}_{J_k})/(L^+\mathcal{P}_{I_k})$$

and the *Bott-Samelson variety*

$$\mathrm{BS}^{(i,\ldots,k)}_{J_\bullet} = (L^+\mathcal{P}_{J_i}) \times^{L^+\mathcal{P}_{I_i}} (L^+\mathcal{P}_{J_{i+1}}) \times^{L^+\mathcal{P}_{I_{i+1}}} \times \ldots \times (L^+\mathcal{P}_{J_{k-1}}) \times^{L^+\mathcal{P}_{I_{k-1}}} (L^+\mathcal{P}_{J_k})/(L^+\mathcal{P}_{I_k})$$

where the action maps are given by the same formulas as in [JMW14, p.1201]. There is an obvious closed embedding $\mathrm{BS}^{(i,\ldots,k)}_{J_\bullet} \hookrightarrow \mathrm{Gr}^{(i,\ldots,k)}_{J_\bullet}$ induced by the closed embedding $L^+\mathcal{P}_{J_i} \hookrightarrow LG$.

Example 5.5.1. If $I_i = \emptyset$ for all i, and $J_i = \{j_i\}$ is a singleton with corresponding simple reflection s_{j_i}, then $\mathrm{BS}^{(i,\ldots,k)}_{J_\bullet} = \mathrm{Dem}_{\dot{w},S}$ is the *Demazure variety* from [FS21, Definition VI.5.6] for $\dot{w} = s_{j_i} s_{j_2} \ldots s_{j_k} \in W_{\mathrm{aff}}$ (the affine Weyl group of G).

Consider $LG/L^+\mathcal{P}_I$ with the stratification by left $L^+\mathcal{P}_{I'}$-orbits for some $\mathcal{P}_{I'}$. Note that if $I \subset J$, then there is a natural map

$$\pi \colon LG/L^+\mathcal{P}_I \to LG/L^+\mathcal{P}_J$$

which is stratified and even (in particular, shriekable), as well as proper. For any I, a maximal proper choice of J gives a map from $LG/L^+\mathcal{P}_I$ to an affine Grassmannian $\mathrm{Gr}_{G,C}$, and we use it to pull back to the dimension pariversity to $LG/L^+\mathcal{P}_I$. Note that $\mathrm{Gr}^{(i,\ldots,k)}_{J_\bullet}$ maps to $\mathrm{Gr}_{G,C}$ via projection to its first factor; this map is again stratified and even, and we use it to pull back the pariversity $\dagger$. Below, "parity" will always refer to this pariversity.

Lemma 5.5.2. *If $I \subset J$, and $I' \subset J'$, the projection map*

$$\pi \colon LG/L^+\mathcal{P}_I \to LG/L^+\mathcal{P}_J$$

is an even stratified map (where the source is equipped with the stratification by left $L^+\mathcal{P}_{I'}$-orbits, and the target equipped with the stratification by left $L^+\mathcal{P}_{J'}$-orbits), which is both proper and ℓ-cohomologically smooth of integral ℓ-dimension.

Proof. The map π is a torsor for the diamond of a partial flag variety of G, hence it is proper and ℓ-cohomologically smooth (of integral ℓ-dimension). It is also evidently stratified, and even because the maps of strata are fibrations with fibers having affine pavings, hence have cohomology in even degrees. □

As in [JMW14, p.1201] there is a commutative diagram

$$\begin{tikzcd}
\mathrm{BS}^{(1,2,\ldots,n)}_{J_\bullet} \arrow[r, hook] \arrow[d] & \mathrm{Gr}^{(1,2,\ldots,n)}_{J_\bullet} \arrow[r] \arrow[d] & \mathrm{Gr}^{(2,\ldots,n)}_{J_\bullet} \arrow[r] \arrow[d] & \ldots \arrow[r] \arrow[d] & \mathrm{Gr}^{(n-1,n)}_{J_\bullet} \arrow[r] \arrow[d] & L^+G/L^+\mathcal{P}_{I_n} \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathrm{BS}^{(1,2,3)}_{J_\bullet} \arrow[r, hook] \arrow[d] & \mathrm{Gr}^{(1,2,3)}_{J_\bullet} \arrow[r] \arrow[d] & \mathrm{Gr}^{(2,3)}_{J_\bullet} \arrow[r] \arrow[d] & LG/L^+\mathcal{P}_{I_3} \arrow[r] \arrow[d] & \ldots & \\
\mathrm{BS}^{(1,2)}_{J_\bullet} \arrow[r, hook] \arrow[d] & \mathrm{Gr}^{(1,2)}_{J_\bullet} \arrow[r] \arrow[d] & LG/L^+\mathcal{P}_{I_2} \arrow[r] \arrow[d] & LG/L^+\mathcal{P}_{J_3} & & \\
\mathrm{BS}^{(1)}_{J_\bullet} \arrow[r, hook] \arrow[d] & LG/L^+\mathcal{P}_{I_1} \arrow[r] \arrow[d] & LG/L^+\mathcal{P}_{J_2} & & & \\
L^+\mathcal{P}_{J_1}/L^+\mathcal{P}_{J_1} \arrow[r, hook] & LG/L^+\mathcal{P}_{J_1} & & & &
\end{tikzcd} \tag{5.10}$$

with all squares being Cartesian; this last fact can be checked on geometric points, which then reduces as in the proof of Proposition 4.1.2 to the analogous statement for classical affine Grassmannians, which is in [JMW14, §4].

Lemma 5.5.3. *For the map $f \colon \mathrm{BS}^{(1,2,\ldots,n)}_{J_\bullet} \to L^+G/L^+\mathcal{P}_{I_n}$ which is the composite of the top row of* (5.10)*, we have $\mathrm{R}f_*\Lambda \in D_{L^+\mathcal{P}_{I_0}}(LG/L^+\mathcal{P}_{I_n})$ is a parity complex.*

Proof. Repeatedly apply proper base change to rewrite $\mathrm{R}f_*\Lambda$ in terms of pushforward and pullbacks along the right edge of (5.10). Then all the maps involved are of the kind considered in Lemma 5.5.2, so their pullbacks and pushforwards preserve parity by Proposition 5.4.3. □

Proposition 5.5.4. *Assume $S = \mathrm{Spa}(C, C^+)$. Each orbit closure of the $L^+\mathcal{P}_I$-action on $LG/L^+\mathcal{P}_J$ supports an indecomposable parity complex with full support.*

Proof. For any such orbit closure $\overline{O}$, we claim that we can find some $J_\bullet$ with $I_0 = I$ and $I_n = J$ such that the Bott-Samelson variety $f \colon \mathrm{BS}_{J_\bullet}^{(1,\ldots,n)} \to LG/\mathcal{P}_J$ has image equal to $\overline{O}$. Then by Lemma 5.5.3, $\mathrm{R}f_*\Lambda$ is a parity complex with full support on $\overline{O}$, hence it has an indecomposable summand with full support on $\overline{O}$.

Now to establish the claim: in the proof of [JMW14, Theorem 4.6], it is explained how to find such a map in the analogous situation for classical affine Grassmannians. We take the same combinatorial data $J_\bullet$. To check that the image is as desired, it suffices to check on geometric points, and then this reduces to the classical case as in the proof of Proposition 4.1.2. □

Corollary 5.5.5. *If $\mathcal{F}, \mathcal{G} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \Lambda)$ and $\mathcal{G}$ has an $L^+_{S/\mathrm{Div}^1_X}G$-equivariant structure, then the convolution $\mathcal{F} \star \mathcal{G}$ lies in $\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \Lambda)$.*

Proof. Since relative parity is a condition on the base change to points of S, the statement immediately reduces to the case where $S = \mathrm{Spa}(C, C^+)$. There the proof is similar to the proof of [JMW14, Theorem 4.8], so we will just sketch it. By the proof of Proposition 5.5.4, any indecomposable parity complex is up to shift a summand of $\mathrm{R}f_*\Lambda$ for some Bott-Samelson resolution $f \colon \mathrm{BS}_{J_\bullet}^{(1,\ldots,n)} \to LG/\mathcal{P}_J$. Therefore it suffices to show that for two such Bott-Samelson resolutions f_1, f_2 for $J^1_\bullet$ and $J^2_\bullet$, the convolution $(\mathrm{R}f_{1*}\Lambda) \star (\mathrm{R}f_{2*}\Lambda)$ is parity. But we have

$$(\mathrm{R}f_{1*}\Lambda) \star (\mathrm{R}f_{2*}\Lambda) \cong \mathrm{R}f_*\Lambda \tag{5.11}$$

where f is the Bott-Samelson resolution associated to the concatenation of $J^1_\bullet$ and $J^2_\bullet$, so it is a parity complex by Lemma 5.5.3. □

5.6. **The lifting functor.** For strictly totally disconnected S, we will construct a "lifting functor" from $\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})$ to $\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k)$ following [LL21, Theorem 5.6.6], which allows to lift relative Tate-parity complexes out of the Tate category. This will be used as part of the construction of the Brauer functor.

Definition 5.6.1 (Normalized parity sheaves)**.** Assume that S is strictly totally disconnected and $\ell > b(\check{H})$.

We denote by $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \Lambda) \subset \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \Lambda)$ the full subcategory spanned by $\mathcal{E}(\mu, \mathcal{L})$ with no shifts, for all $\mu \in X_*(T_H)^+$ and $\mathcal{L} \in \mathrm{Loc}(S; \Lambda)$. Thus, by Proposition 5.1.17, every object of $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \Lambda)$ is moreover a finite direct sum of objects of the form $\mathcal{E}(\mu, \mathcal{L})$ with no shifts.

Analogously, we denote by $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}}) \subset \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})$ the full subcategory spanned by $\mathcal{E}_{\mathrm{T}}(\mu, \mathcal{L})$ with no shifts, for all $\mu \in X_*(T_H)^+$ and $\mathcal{L} \in \mathrm{Loc}(S; \mathbb{O})$. Thus, by Proposition 5.3.14, every object of $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})$ is moreover a finite direct sum of objects of the form $\mathcal{E}_{\mathrm{T}}(\mu, \mathcal{L})$ with no shifts.

Definition 5.6.2 (The lifting functor)**.** Assume that S is strictly totally disconnected and $\ell > b(\check{H})$. Following Leslie-Lonergan, we define the *lifting functor*

$$L \colon \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}}) \to \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k)$$

which on objects sends $\mathcal{E}_{\mathrm{T}}(\mu, \mathcal{L}) \mapsto \mathbb{F}\mathcal{E}(\mu, \mathcal{L})$, and on morphisms is the augmentation to the summand indexed by $i = 0$ in (5.8). By design, the functor L fits into a commutative triangle

$$\begin{tikzcd} \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathbb{O}) \arrow[rr, "\mathbb{F}"] \arrow[dr, "\mathbb{T}"'] & & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k) \\ & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}}) \arrow[ur, dashed, "L"'] & \end{tikzcd} \tag{5.12}$$

Remark 5.6.3. Presumably, Leslie-Lonergan chose the name "lifting functor" for L because it takes objects in the Tate category, a Verdier quotient of a derived category of sheaves, to objects in a derived category of sheaves. However, note that the source of L is the integral version of the Tate category, which is k-linear (like the target of L). The triangle (5.12) suggests that L behaves like an intermediate change-of-coefficients functor.

Construction 5.6.4 (Convolution on Tate categories)**.** By Lemma 5.2.2 and Lemma 3.4.5, the subcategory of perfect objects in $D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,S/\operatorname{Div}^1_X};\mathbb{O}[\Sigma])^{\mathrm{bd}}$ is a two-sided ideal for the convolution monoidal structure. This induces a convolution monoidal structure on the quotient category $D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})^{\mathrm{bd}}$.

Assume that S is strictly totally disconnected and $\ell > b(\check{H})$. Then all objects of $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O}[\Sigma])$ are perverse by Proposition 5.1.17 (since the $\mathcal{E}(\mu,\mathcal{L})$ are perverse), hence have canonical equivariant structures, so Corollary 5.5.5 implies that convolution restricts to a monoidal structure on $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O}[\Sigma])$. This in turn induces a monoidal structure on $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})$.

Lemma 5.6.5. *Assume S is strictly totally disconnected and $\ell > b(\check{H})$. Then the functor L admits a canonical monoidal structure making* (5.12) *into a commutative diagram of monoidal functors.*

Proof. The functor $\mathbb{F}$ is monoidal, which means that there is a commutative square

$$\begin{tikzcd}
(\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O}))^{\otimes 2} \arrow[r, "\star"] \arrow[d, "\mathbb{F}^{\otimes 2}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O}) \arrow[d, "\mathbb{F}"] \\
(\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k))^{\otimes 2} \arrow[r, "\star"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k)
\end{tikzcd} \tag{5.13}$$

with the natural transformation satisfying coherence data. Also, $\mathbb{T}$ tautologically monoidal with respect to the monoidal structure of Construction 5.6.4, which gives the top commutative square in the diagram below.

$$\begin{tikzcd}
(\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O}))^{\otimes 2} \arrow[r, "\star"] \arrow[d, "\mathbb{T}^{\otimes 2}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathbb{O}) \arrow[d, "\mathbb{T}"] \\
(\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}}))^{\otimes 2} \arrow[r, "\star"] \arrow[d, "L^{\otimes 2}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}}) \arrow[d, "L"] \\
(\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k))^{\otimes 2} \arrow[r, "\star"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k)
\end{tikzcd} \tag{5.14}$$

In the top square of (5.14), the vertical arrows are essentially surjective since all $\mathcal{E}_{\mathrm{T}}(\mu,\mathcal{L})$ are in the image, and these generate under direct sums by Proposition 5.3.14. The maps on morphisms are calculated in Lemma 5.3.1. The outer square is equivalent to (5.13) by (5.12). Hence the natural isomorphisms making the upper and outer squares commute, which are part of the monoidal structures of $\mathbb{F}$ and $\mathbb{T}$, induce a natural transformation making the lower square commute. In other words, this constructs a natural isomorphism

$$L(\mathcal{F}\star\mathcal{G}) \cong L(\mathcal{F})\star L(\mathcal{G})$$

for all $\mathcal{F},\mathcal{G} \in \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};\mathcal{T}_{\mathbb{O}})$. The coherence data is similarly induced from those of $\mathbb{F}$ and $\mathbb{T}$. □

6. Tilting modules and the Geometric Satake equivalence

In this section we show that normalized relative parity complexes are relative perverse when ℓ is not too small, and establish that they correspond to *tilting modules* under the Geometric Satake equivalence. The results are parallel to those of Juteau-Mautner-Williamson in [JMW16], and we follow their proof strategy, but we also have to supply new arguments for some steps.

In fact, the majority of this section is spent on the case of tilting modules with quasi-minuscule highest weight, which occupies only six sentences in [JMW16]. This is because the singularities of quasi-minuscule Schubert varieties in the classical affine Grassmannians are understood, and smooth-locally equivalent to those of the minimal orbit in the nilpotent cone of $\mathfrak{g}$. At present we do not have such statements in p-adic geometry. We will instead degenerate to the Witt vector affine Grassmannian and study resolutions of the quasi-minuscule Schubert varieties there constructed by Zhu [Zhu17]. Using the intersection forms introduced by Juteau-Mautner-Williamson [JMW14] to analyze the failure of the Decomposition Theorem with modular coefficients, we compare the (failure of the) Decomposition Theorem for these resolutions with the parallel situation on the equicharacteristic affine Grassmannian, deducing that the behavior must be "the same" in a suitable sense.

We note that for the classical affine Grassmannians, Mautner-Riche [MR18] have proven the relevant statements, relating normalized parity sheaves to tilting modules, in the optimal generality, improving on [JMW16]. However, the proof of Mautner-Riche relies on various techniques that have not yet been developed in the setting of p-adic geometry. Hence our results here are somewhat of a "proof of concept", and we leave the optimization of technical hypotheses for future work.

We begin in §6.1 with some recollections and elaborations on the Geometric Satake equivalence of Fargues-Scholze. Then in §6.2 we formulate the main results relating parity sheaves and tilting modules through this equivalence. In §6.3 we begin the proof, reducing it to the case of quasi-minuscule highest weight. Finally, §6.4 completes the proof of this case.

Throughout this section we abbreviate $\mathrm{Gr}_{G,C} := \mathrm{Gr}_{G,\mathrm{Spa}(C,C^+)/\operatorname{Div}^1_X}$ and $d_\mu := \langle 2\rho_G, \mu\rangle$.

6.1. **The Geometric Satake equivalence.** We first record some generalities on the Geometric Satake equivalence. Let S be any small v-stack over $(\operatorname{Div}^1_X)^I$. We denote the *Satake category* $\mathrm{Sat}(\mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^I};\Lambda) \subset D^{\mathrm{ULA}}_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/(\operatorname{Div}^1_X)^I};\Lambda)^{\mathrm{bd}}$ to be the full subcategory spanned by flat perverse sheaves over S. By the same argument as for [FS21, Proposition VI.7.2], the pullback functor

$$\mathrm{Sat}(\mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^I};\Lambda) \to D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\operatorname{Div}^I_X};\Lambda)^{\mathrm{bd}}$$

is fully faithful, so we may regard $\mathrm{Sat}(\mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^I};\Lambda) \subset D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^I};\Lambda)^{\mathrm{bd}}$.

For a v-stack S, we let $\mathrm{Loc}(S;\Lambda)$ be the category of Λ-free étale local systems on S. Let $\pi_{G,S}\colon \mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^I} \to S$ be the natural projection. In [FS21, §VI.7], Fargues-Scholze establish the Geometric Satake equivalence,

$$\mathrm{Sat}(\mathrm{Gr}_{G,(\operatorname{Div}^1_X)^I};\Lambda) \cong \mathrm{Rep}_{\mathrm{Loc}((\operatorname{Div}^1_X)^I;\Lambda)}(\check{G}), \tag{6.1}$$

where the right side is the category of representations of a certain reductive group object in the category $\mathrm{Loc}((\operatorname{Div}^1_X)^I;\Lambda)$. The equivalence is symmetric monoidal, with the underlying monoidal structure given by convolution on the left and tensor product on the right, and carries the fiber functor $\bigoplus_i \mathrm{R}^i\pi_{G,\operatorname{Div}^1_X *}$ on the left to the forgetful functor on the right.

The category $\mathrm{Loc}(\operatorname{Div}^1_X;\Lambda)$ is symmetric monoidal under tensor product. Note that there is an action of $\mathrm{Loc}(\operatorname{Div}^1_X;\Lambda)$ on $\mathrm{Sat}(\mathrm{Gr}_{G,\operatorname{Div}^1_X};\Lambda)$ via pullback and tensoring. Pullback also defines a symmetric monoidal functor $\mathrm{Loc}(S;\Lambda) \to \mathrm{Sat}(\mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^1};\Lambda)$.

Lemma 6.1.1. *The natural functor*

$$\mathrm{Sat}(\mathrm{Gr}_{G,\operatorname{Div}^1_X};\Lambda) \otimes_{\mathrm{Loc}(\operatorname{Div}^1_X;\Lambda)} \mathrm{Loc}(S;\Lambda) \to \mathrm{Sat}(\mathrm{Gr}_{G,S/(\operatorname{Div}^1_X)^1};\Lambda)$$

is a monoidal equivalence with respect to the convolution product.

Proof. By descent, we may reduce to the case where G is split, say with maximal torus T. The functor is fully faithful, using that the formation of $\mathcal{RH}om$ commutes with base change [FS21, Corollary VI.6.6]. For essentially surjectivity: the Schubert stratification induces a compatible semi-orthogonal decomposition on both sides, and it suffices to see that the functor induces an equivalence on each piece of the decomposition. On the piece indexed by $\mu \in X_*(T)^+$ it takes the form

$$\underbrace{\mathrm{Sat}(\mathrm{Gr}_{G,\operatorname{Div}^1_X,\mu};\Lambda)}_{\cong \mathrm{Loc}(\operatorname{Div}^1_X;\Lambda)} \otimes_{\mathrm{Loc}(\operatorname{Div}^1_X;\Lambda)} \mathrm{Loc}(S) \to \underbrace{\mathrm{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X,\mu};\Lambda)}_{\cong \mathrm{Loc}(S;\Lambda)}$$

and the equivalence is clear by inspection. The compatibility of the monoidal structures is evident from the definition. □

Construction 6.1.2. Lemma 6.1.1 equips the convolution monoidal structure on $\mathrm{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)$ with a commutativity constraint, promoting it to a symmetric monoidal structure. We thus regard $\mathrm{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda)$ as a symmetric monoidal category.

Tensoring (6.1) with $\mathrm{Loc}(S;\Lambda)$ over $\mathrm{Loc}(\operatorname{Div}^1_X;\Lambda)$ and applying Lemma 6.1.1 gives a symmetric monoidal equivalence

$$\mathrm{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\Lambda) \cong \mathrm{Rep}_{\mathrm{Loc}(S;\Lambda)}(\check{G}), \tag{6.2}$$

carrying the fiber functor $\bigoplus_i \mathrm{R}^i\pi_{G,S*}$ on the left to the forgetful functor on the right.

Example 6.1.3. If S is strictly totally disconnected, then for $C^\infty(|S|;\Lambda)$ the ring of continuous functions from $|S|$ to Λ, we have that

$$\mathrm{Loc}(S;\Lambda) \cong \{\text{finite projective } C^\infty(|S|;\Lambda)\text{-modules}\}.$$

In particular, if $S = \mathrm{Spa}(C, C^+)$ is a geometric point then $|S|$ is a point, and (6.2) reads

$$\mathrm{Sat}(\mathrm{Gr}_{G,C};\Lambda) \cong \mathrm{Rep}_\Lambda(\check{G})$$

where $\check{G}$ is the usual Langlands dual group by [FS21, Theorem VI.11.1]. For general strictly totally disconnected S, we have

$$\mathrm{Rep}_{\mathrm{Loc}(S;\Lambda)}(\check{G}) \cong \mathrm{Rep}_\Lambda(\check{G}) \otimes_\Lambda \{\text{finite projective } C^\infty(|S|;\Lambda)\text{-modules}\}. \tag{6.3}$$

6.2. **Parity and tilting.** Recall that a representation of $\check{G}$ is called a *tilting module* if it has both a filtration by standard modules, and also a filtration by costandard modules (see [Jan03, Appendix E] for a reference). The tilting property is preserved by direct sums and tensor products; the latter case is a non-trivial theorem of Mathieu [Mat90], building on work of Wang and Donkin. Let $\mathrm{Tilt}_\Lambda(\check{G}) \subset \mathrm{Rep}_\Lambda(\check{G})$ denote the full subcategory of tilting modules. For each $\mu \in X^*(\check{T})^+$ there is a unique indecomposable tilting module of highest weight μ, which we denote $T_\Lambda(\mu)$.

Definition 6.2.1. Let $b(\check{G})$ be the supremum of $b(\check{\Phi})$ defined in Figure 1, as $\check{\Phi}$ runs over the root systems of simple factors of $\check{G}$.

By Proposition 5.5.4, there is an indecomposable relative parity complex with support $\mathrm{Gr}_{G,C,\leq\mu}$ (and coefficients in k), which we abbreviate by $\mathcal{E}(\mu)$.

Theorem 6.2.2. *If $\ell > b(\check{G})$, and $S = \mathrm{Spa}(C, C^+)$ is a geometric point, then $\mathcal{E}(\mu)$ is perverse and corresponds under the Geometric Satake equivalence to $T_k(\mu)$. Thus, the Geometric Satake equivalence restricts to an equivalence as in the diagram below.*

$$\begin{tikzcd} \mathrm{Par}^{\mathrm{ULA}}_{\mathrm{n}}(\mathrm{Gr}_{G,C};k) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Tilt}_k(\check{G}) \arrow[d, hook] \\ \mathrm{Sat}(\mathrm{Gr}_{G,C};k) \arrow[r, "\sim"] & \mathrm{Rep}_k(\check{G}) \end{tikzcd} \tag{6.4}$$

We will see that Theorem 6.2.2 implies a more general statement any strictly totally disconnected S over Div^1_X.

Notation 6.2.3. For $\mathcal{L} \in \mathrm{Loc}(S;\Lambda)$ and $\mu \in X_*(T)^+$, let $\mathcal{E}(\mu,\mathcal{L}) \in \mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$ be the image of $T_\Lambda(\mu) \otimes \mathcal{L} \in \mathrm{Rep}_{\mathrm{Loc}(S;\Lambda)}(\check{G})$ under the Geometric Satake equivalence (6.2).

Thus Theorem 6.2.2 implies that for $S = \mathrm{Spa}(C, C^+)$, we have $\mathcal{E}(\mu) = \mathcal{E}(\mu, k)$, viewing k as the constant local system on S.

Corollary 6.2.4. *Let S be strictly totally disconnected. Let $\Lambda = \mathbb{O}$ or k. If $\ell > b(\check{G})$ then $\mathcal{E}(\mu,\mathcal{L}) \in \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)$ is relative parity for all $\mu \in X_*(T)^+$ and all $\mathcal{L} \in \mathrm{Loc}(S;\Lambda)$.*

Proof. The assertion immediately reduces to the case $S = \mathrm{Spa}(C, C^+)$. If $\Lambda = k$, then $\mathcal{E}(\mu,\mathcal{L})$ is relative parity by Theorem 6.2.2. If $\Lambda = \mathbb{O}$, then $\mathbb{F}\mathcal{F}$ is relative parity by the case just handled, hence $\mathcal{F}$ is relative parity by Lemma 5.3.6. $\square$

Since $\mathrm{Sat}(\mathrm{Gr}_{G,C};k) \cong \mathrm{Rep}_k(\check{G})$ is a highest weight category, it has a notion of standard, costandard, and tilting objects. The standard objects are $\Delta_\mu := {}^{\mathfrak{p}}i_{\mu!}(\underline{k}[d_\mu])$ and the costandard objects are $\nabla_\mu := {}^{\mathfrak{p}}i_{\mu*}(\underline{k}[d_\mu])$ for all $\mu \in X_*(T)^+$, where the superscript ${}^{\mathfrak{p}}$ refers to 0th perverse cohomology. The tilting objects are those which admit both standard and costandard filtrations.

Proposition 6.2.5. *Suppose $\mathcal{F} \in \mathrm{Sat}(\mathrm{Gr}_{G,C};k)$ is relative perverse, and relative parity with respect to the dimension pariversity $\dagger_G$. Then $\mathcal{F}$ is tilting.*

Proof. The proof is similar to that of [JMW16, Proposition 3.3], substituting the results of §5 for those of [JMW14]. We need to show that $\mathcal{F}$ has a standard filtration and a costandard filtration. By a general homological algebra result of Ringel [JMW16, Theorem 3.1], $\mathcal{F}$ has a standard filtration if and only if $\mathrm{Ext}^1(\mathcal{F}, \nabla_\mu) = 0$ for all $\mu \in X_*(T)^+$, and $\mathcal{F}$ has a costandard filtration if and only if $\mathrm{Ext}^1(\Delta_\mu, \mathcal{F}) = 0$ for all $\mu \in X_*(T)^+$. By Verdier duality, it suffices to show that

$$\mathrm{Ext}^1(\mathcal{F}, \nabla_\mu) = 0 \text{ for all } \mu \in X_*(T)^+.$$

We have an exact triangle

$$\nabla_\mu \to i_{\mu *}\underline{k}[d_\mu] \to A$$

where A lies in ${}^{\mathfrak{p}}D^{>0}$ for the (relative) perverse t-structure. This gives a long exact sequence

$$\ldots \to \mathrm{Hom}(\mathcal{F}, A) \to \mathrm{Ext}^1(\mathcal{F}, \nabla_\mu) \to \mathrm{Ext}^1(\mathcal{F}, i_{\mu *}\underline{k}[d_\mu]) \to \ldots$$

The leftmost term vanishes by the axioms of a t-structure, because $\mathcal{F} \in {}^{\mathfrak{p}}D^0$ while $A \in {}^{\mathfrak{p}}D^{>0}$. The rightmost term vanishes by Corollary 5.1.12 because $i_{\mu *}\underline{k}[d_\mu + 1]$ is relative !-odd for the pariversity †, while $\mathcal{F}$ is relative $*$-even. Thus the middle term vanishes, as desired. □

6.3. **Proof of Theorem 6.2.2.** For $\lambda \in X_*(T)^+$, we let $\mathcal{T}(\mu) \in \mathrm{Sat}(\mathrm{Gr}_{G,C}; k)$ be the perverse sheaf corresponding under Geometric Satake to $T_k(\mu) \in \mathrm{Rep}_k(\check{G})$. Our goal is then to prove $\mathcal{E}(\mu) \cong \mathcal{T}(\mu)$ for all $\mu \in X_*(T)$. By Proposition 5.1.15 it suffices to show that $\mathcal{T}(\mu)$ is parity, since $\mathcal{T}(\mu)$ is indecomposable (since $T_k(\mu)$ is, by definition) and has the correct support.

6.3.1. *Group-theoretic reductions.* As in [JMW16, §3.4], by elementary reductions we may assume that G is simple, semi-simple, and adjoint. Then $\check{G}$ is simple, semi-simple, and simply connected, so it has a system of fundamental weights ω_i, as well a unique quasi-minuscule dominant root α_0 (characterized as the unique shortest dominant root of $\check{G}$).

6.3.2. *Basic cases.* We will establish the result in the minuscule and quasi-minuscule cases as basic building blocks.

Lemma 6.3.1. *If $\mu \in X_*(T)^+$ is minuscule, then $\mathcal{E}(\mu) \cong \mathcal{T}(\mu) \cong \mathrm{IC}_\mu$.*

Proof. The orbit corresponding to the minuscule coweights of G, so $\mathcal{T}(\mu) \cong \mathrm{IC}_\mu \cong k_{\mathrm{Gr}_{G,C,\mu}}[d_\mu]$. This is clearly parity, so it is isomorphic to $\mathcal{E}(\mu)$. □

Proposition 6.3.2. *Let α_0 be the quasi-minuscule root of $\check{G}$. If ℓ is a good prime for $\check{G}$ and furthermore $\ell \nmid n+1$ in type A_n, $\ell \nmid n$ in type C_n, then we have*

$$\mathcal{E}(\alpha_0) \cong \mathcal{T}(\alpha_0) \cong \nabla_{\alpha_0} \cong \Delta_{\alpha_0} \cong \mathrm{IC}_{\alpha_0}.$$

The proof of Proposition 6.3.2 will actually be quite involved (and will deviate significantly from its “classical” counterpart in [JMW16, Lemma 3.7(2)], so we will assume it for now in order to complete the proof.

6.3.3. *Fundamental weights.* Next we establish the result for the fundamental weights.

Lemma 6.3.3. *If $\ell > b(\check{G})$, then for each fundamental weight ω_i of $\check{G}$, we have $\mathcal{E}(\omega_i) \cong \mathcal{T}(\omega_i)$.*

Proof. The proof is the same as that of [JMW16, Proposition 3.8] so we just sketch it. As explained earlier, it suffices to show that $\mathcal{T}(\omega_i)$ is a parity complex.

In type A, all the fundamental weights are minuscule, so the result follows immediately from Lemma 6.3.1. We henceforth assume that $\check{G}$ is not of type A. Then under the assumption $\ell > b(\check{G})$, we have by Lemma 6.3.1 and Proposition 6.3.2 that if μ is minuscule or quasi-minuscule, then the Weyl module $W_k(\mu) = \Delta_\mu$ is tilting.

We shall want to make comparisons to characteristic zero, so we write $W_{\mathbf{C}}(\mu)$ for the Weyl module of highest weight μ in $\mathrm{Rep}_{\mathbf{C}}(\check{G})$. We may realize the Weyl module $W_{\mathbf{C}}(\omega_i)$ as a direct summand of the tensor products of Weyl modules associated to minuscule or quasi-minuscule weights as in [JMW16, §3.6]; for example, in type B_n (referring to the notation of [JMW16, §3.6.2]) the fundamental weight ϖ_n is minuscule and we have

$$W_{\mathbf{C}}(\varpi_n)^{\otimes 2} \cong W_{\mathbf{C}}(2\varpi_n) \oplus W_{\mathbf{C}}(\varpi_{n-1}) \oplus \ldots \oplus W_{\mathbf{C}}(\varpi_1) \oplus W_{\mathbf{C}}(0), \tag{6.5}$$

which realizes the other Weyl modules with fundamental highest weights $\varpi_{n-1}, \dots, \varpi_1$ as direct summands of $W_{\mathbf{C}}(\varpi_n)^{\otimes 2}$. Since the tensor product of tilting modules is tilting, the tensor products of Weyl modules over k associated to (quasi-)minuscule weights are tilting (e.g., $W_k(\varpi_1)^{\otimes 2}$ in type B_n). Then Corollary 5.5.5 implies that the corresponding perverse sheaves are parity. If each fundamental Weyl module summand in these decompositions remains simple over k (e.g., the $W_k(\varpi_i)$ in type B_n), then each is tilting and the same decomposition occurs over k as over $\mathbf{C}$. Then $\mathcal{T}(\omega_i)$ is a direct summand of a parity complex, so it is a parity complex.

It only remains to remark that for $\ell > b(\check{G})$ the fundamental Weyl modules remain simple; see [JMW16, §3.6] for references to the proofs.

□

6.3.4. *The general case.* The following Lemma completes the proof of Theorem 6.2.2 (modulo the proof of Proposition 6.3.2, which we complete in the next subsection).

Lemma 6.3.4. *If $\ell > b(\check{G})$, then for any $\mu \in X_*(T)^+$ we have $\mathcal{E}(\mu) \cong \mathcal{T}(\mu)$.*

Proof. The argument is the same as in [JMW16, §3.7], so we just sketch it. We may write $\mu = \sum n_i \varpi_i$ for $n_i \geq 0$. The representation $\bigotimes T_k(\varpi_i)^{\otimes n_i}$ is tilting, with highest weight μ, hence contains $T_k(\mu)$ as a summand. Hence $\mathcal{T}(\mu)$ is a summand of $\mathcal{T}(\varpi_1)^{\star n_1} \star \dots \star \mathcal{T}(\varpi_r)^{\star n_r}$. This convolution is parity by Corollary 5.5.5, since Lemma 6.3.3 implies that each $\mathcal{T}(\varpi_i)$ is parity. Therefore the summand $\mathcal{T}(\lambda)$ is also parity, so we must have $\mathcal{T}(\mu) \cong \mathcal{E}(\mu)$. □

6.4. **Proof of Proposition 6.3.2.** We may assume that G is split, so we fix a reductive extension $G/\mathcal{O}_F$, in which we prolong T to a split maximal torus. Let $\overline{G}, \overline{T}$, etc. denote the special fibers of G, T, etc.

6.4.1. *Reduction to the Witt vector affine Grassmannian.* As observed already, it suffices to show that $\mathcal{T}(\alpha_0)$ is parity. Since $\mathcal{T}(\alpha_0)$ is pulled back along $\mathrm{Gr}_{G,C} \to \mathcal{H}\mathrm{ck}_{G,C}$, the statement can be "degenerated" to characteristic p via [FS21, Corollary VI.6.7], which implies that it suffices to check the analogous statement for $\mathrm{Gr}_{G,\mathrm{Spd}\,\overline{\mathbf{F}}_q/\mathrm{Div}^1_{\mathcal{Y}}} \cong (\mathrm{Gr}_G^{\mathrm{Witt}})^\diamond$ where $\mathrm{Gr}_G^{\mathrm{Witt}}$ is the *Witt vector affine Grassmannian* of [Zhu17, BS17]. Then by [Sch22, §27], it suffices to show the analogous statement on $\mathrm{Gr}_G^{\mathrm{Witt}}$: the perverse sheaf $\mathcal{T}(\alpha_0)$ on $\mathrm{Gr}_G^{\mathrm{Witt}}$, corresponding to the tilting module $T_k(\alpha_0)$ under the Geometric Satake equivalence, is a parity complex.

6.4.2. *Intersection forms.* We recall the intersection forms from [JMW14, §3.1], adapted to our setting of perfect algebraic geometry. We work over an algebraically closed field $\mathbf{F}$ of characteristic p. For $Y/\operatorname{Spec}\mathbf{F}$, the *Borel-Moore homology of $\widetilde{Y}$* is $\mathrm{H}_a^{\mathrm{BM}}(Y) = \mathrm{H}^{-2a}(Y; \mathbb{D}_{Y/\mathbf{F}})$. If Y is perfectly smooth of dimension d over $\mathbf{F}$, then we have $\mathbb{D}_{Y/\mathbf{F}} \cong \underline{k}[2d]$.

If $i\colon Z \to Y$ is a closed embedding with open complement $j\colon U \hookrightarrow Y$, then we have an exact triangle

$$i_*\mathbb{D}_{Z/\mathbf{F}} \to \mathbb{D}_{Y/\mathbf{F}} \to j_*\mathbb{D}_{U/\mathbf{F}}.$$

Suppose Y is perfectly smooth of dimension d over $\mathbf{F}$, so the same holds for U. Then $\mathbb{D}_{Y/\mathbf{F}} \cong k[2d]$ and similarly for $\mathbb{D}_{U/\mathbf{F}}$. Taking cohomology, this induces an isomorphism

$$\mathrm{H}_a^{\mathrm{BM}}(Z) \cong \mathrm{H}_Z^{2d-a}(Y;k) := \mathrm{H}^{2d-a}(Y; i_*i^!k_Y).$$

The cup product on relative cohomology gives a pairing $\mathrm{H}_Z^{2d-a}(Y;\underline{k}) \otimes \mathrm{H}_Z^{2d-b}(Y;\underline{k}) \to \mathrm{H}_Z^{4d-a-b}(Y;\underline{k})$, which translates to a pairing

$$\mathrm{H}_a^{\mathrm{BM}}(Z) \otimes \mathrm{H}_b^{\mathrm{BM}}(Z) \to \mathrm{H}_{a+b-2d}^{\mathrm{BM}}(Z) \tag{6.6}$$

under the commutative diagram

$$\begin{tikzcd} \mathrm{H}_a^{\mathrm{BM}}(Z) \arrow[d, "\sim"] & \otimes & \mathrm{H}_b^{\mathrm{BM}}(Z) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}_{a+b-2d}^{\mathrm{BM}}(Z) \arrow[d, "\sim"] \\ \mathrm{H}_Z^{2d-a}(Y;\underline{k}) & \otimes & \mathrm{H}_Z^{2d-b}(Y;\underline{k}) \arrow[r] & \mathrm{H}_Z^{4d-a-b}(Y;\underline{k}) \end{tikzcd} \tag{6.7}$$

Suppose $a+b = 2d$ and Z is perfectly proper over $\mathbf{F}$. Then we have a degree map $\deg\colon \mathrm{H}_0^{\mathrm{BM}}(Z) \to k$, which when combined with (6.6) induces a pairing

$$B_Z^m\colon \mathrm{H}_{d-m}^{\mathrm{BM}}(Z) \otimes \mathrm{H}_{d+m}^{\mathrm{BM}}(Z) \to k. \tag{6.8}$$

This is the (analogue of the) *intersection form* from [JMW14, §3.1], which is in turn inspired by the work of de Cataldo and Migliorini on the Hodge theory of the Decomposition Theorem. In an analogous topological setting, it measures the intersection number of a real $(d-m)$-dimensional cycle on Z and a real $(d+m)$-dimensional cycle on Z within the (real) $2d$-dimensional space Y. We will use it to study the Decomposition Theorem with modular coefficients.

6.4.3. *Splitting at the singular point.* Let $\pi\colon \widetilde{X} \to X$ be a perfectly proper surjective map, with $\widetilde{X}$ perfectly smooth of dimension d. Let $x_0 \in X$ and form the Cartesian square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\pi"] \\ x_0 \arrow[r, "i"] & X \end{tikzcd}$$

Then by §6.4.2 (taking $Y = \widetilde{X}$ and $Z = F$), for each $0 \le m \le d$ there is an intersection form

$$B_F^m\colon \mathrm{H}^{\mathrm{BM}}_{d-m}(F) \otimes \mathrm{H}^{\mathrm{BM}}_{d+m}(F) \to k. \tag{6.9}$$

Its rank controls the splitting of $\pi_* k_{\widetilde{X}}[d]$ in the following sense.

Lemma 6.4.1. [JMW14, Proposition 3.2] *The multiplicity of $i_*\underline{k}[m]$ as a direct summand of $\pi_* k_{\widetilde{X}}[d]$ is equal to the rank of B_F^m.*

6.4.4. *Zhu's resolution.* We will apply the preceding generalities to Zhu's resolution of the quasi-minuscule Schubert variety in $\mathrm{Gr}_G^{\mathrm{Witt}}$ from [Zhu17, §2.2.2]. We briefly review this notation for the reader's convenience, and to set notation. Let $\alpha_0 \in X_*(T)^+$ be the quasi-minuscule dominant coroot of G. We abbreviate $\mathrm{Gr} := \mathrm{Gr}_G^{\mathrm{Witt}}$. Then we have the stratification

$$\mathrm{Gr}_{\le \alpha_0} = \mathrm{Gr}_{\alpha_0} \sqcup \mathrm{Gr}_0 .$$

For a root $\alpha \in X^*(T)$, let U_α denote the corresponding root subgroup of G over $\mathcal{O}$ and fix an isomorphism $U_\alpha(F)$ carrying $U_\alpha(\mathcal{O})$ isomorphically to $\mathcal{O}$. For a real number $r \in [0,1]$, let $\mathcal{G}_r$ be the parahoric group scheme over $\mathcal{O}$, with $\mathcal{O}$-points the parahoric subgroup of $G(E)$ generated by $T(\mathcal{O})$ and $\varpi^{\lceil \langle r\alpha_0^\vee, \alpha\rangle \rceil}\mathcal{O} \subset E = U_\alpha(E)$ for all roots α. Let $Q_r = L^+\mathcal{G}_r$ be the corresponding p-adic jet group.

Following Zhu, we define

$$\widetilde{\mathrm{Gr}}_{\le \alpha_0} := Q_0 \times^{Q_{1/4}} Q_{1/2}/Q_{3/4}. \tag{6.10}$$

Let

$$\pi\colon \widetilde{\mathrm{Gr}}_{\le\alpha_0} \to \mathrm{Gr}_{\le \alpha_0}$$

be the map sending (g,g') to $gg'\varpi^{\alpha_0}$. By [Zhu17, Lemma 2.12(ii)] π restricts to an isomorphism

$$\pi^\circ\colon Q_0 \times^{Q_{1/4}} (Q_{1/4}Q_{3/4})/Q_{3/4} \xrightarrow{\sim} \mathrm{Gr}_{\alpha_0}$$

over the open stratum, and to a contraction

$$\pi_0\colon (\overline{G}/\overline{P}_{\alpha_0})^{p^{-\infty}} \to \mathrm{Gr}_0 = \mathrm{pt} \tag{6.11}$$

over the singular point, where we recall that $(-)^{p^{-\infty}}$ denotes perfection and $\overline{G}$, etc. denotes the special fiber of G, etc. Note that this is exactly the setup of §6.4.3.

All of the constructions of the preceding paragraph make sense for the equal characteristic affine Grassmannian, and yield a similar picture. We will denote the analogous objects with a superscript $\natural$.

By the same argument as for Proposition 6.2.5, in order to show that $\mathcal{T}(\alpha_0) \cong \mathcal{E}(\alpha_0)$ it suffices to show that $\mathcal{E}(\alpha_0)$ is perverse. In the equal characteristic situation, we already know the analogue of Proposition 6.3.2 from [Fen24, Theorem 4.8], in particular that $\mathcal{E}(\alpha_0)^\natural \cong \mathcal{T}(\alpha_0)^\natural$ is perverse. Since $\pi^\natural$ is even, $\pi^\natural_*(k[d_{\alpha_0}])$ is a parity complex on $\mathrm{Gr}^\natural_{\le\alpha_0}$. Hence we have

$$\pi^\natural_*(k[d_{\alpha_0}]) \cong \mathcal{E}(\alpha_0)^\natural \oplus \mathcal{K}^\natural$$

where $\mathcal{K}^\natural \cong \bigoplus_{m\in\mathbf{Z}} i^\natural_* k[m]^{\oplus e_m}$ for some collection of multiplicities e_m. Since $\pi_*(k[d_{\alpha_0}])$ and $\pi^\natural_*(k[d_{\alpha_0}])$ have the same (co)stalks at the singular point (namely the cohomology of $\overline{G}/\overline{P}_{\alpha_0}$, by (6.11)), it suffices to show that for all $m \in \mathbf{Z}$, $i_*k[m]$ is a summand of $\pi_*\underline{k}[d_{\alpha_0}]$ with the same multiplicity e_m.

6.4.5. *Reduction to rank of intersection forms.* Taking $\widetilde{X} := \widetilde{\mathrm{Gr}}_{\leq \alpha_0}$ and F to be the fiber over Gr_0, and $\widetilde{X}^\natural$ and $F^\natural$ the analogous objects for the equal characteristic affine Grassmannian, it suffices by §6.4.3 to show that the intersection form

$$B_F^m : \mathrm{H}^{\mathrm{BM}}_{d-m}(F) \otimes \mathrm{H}^{\mathrm{BM}}_{d+m}(F) \to k$$

has the same rank as the intersection form

$$B_{F^\natural}^m : \mathrm{H}^{\mathrm{BM}}_{d-m}(F^\natural) \otimes \mathrm{H}^{\mathrm{BM}}_{d+m}(F^\natural) \to k.$$

For this it suffices to produce a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{\mathrm{BM}}_*(F) \arrow[r, equal] \arrow[d, "\sim"] & \mathrm{H}^{\mathrm{BM}}_*(F^\natural) \arrow[d, "\sim"] \\ \mathrm{H}^{2d-*}_F(\widetilde{X}; \underline{k}) \arrow[r, equal] & \mathrm{H}^{2d-*}_{F^\natural}(\widetilde{X}^\natural; \underline{k}) \end{tikzcd} \tag{6.12}$$

in which the bottom identification is compatible with the cup product.

6.4.6. *Equivariant formality.* Let K be a pro-(perfection of)-algebraic group and Y an ind-scheme with an action of K. We have the equivariant cohomology

$$\mathrm{H}^*_K(Y; k) \cong \mathrm{H}^*(K \backslash Y; k)$$

where on the RHS the quotient is taken in the sense of stacks. Recall that we say Y is *K-equivariantly formal* (over k) if the Leray-Serre spectral sequence $\mathrm{H}^*(Y;k) \otimes_k \mathrm{H}^*_K(\mathrm{pt};k) \implies \mathrm{H}^*_K(Y;k)$ degenerates at E_2, thus giving an isomorphism

$$\mathrm{H}^*_K(Y; k) \cong \mathrm{H}^*(Y;k) \otimes_k \mathrm{H}^*_K(\mathrm{pt}; k).$$

Lemma 6.4.2. *If $\mathrm{H}^*(Y;k)$ is concentrated in even degrees and $\mathrm{H}^*_K(\mathrm{pt};k)$ is concentrated in even degrees, then Y is K-equivariantly formal.*

Proof. The differentials in the Leray-Serre spectral sequence change parity. □

Example 6.4.3. Any K of the form $Q_r^\natural$ (resp. Q_r) has the property that the quotient by its pro-unipotent radical is (the perfection of) a reductive subgroup of G. Hence if ℓ is not a torsion prime for G, then $\mathrm{H}^*_K(\mathrm{pt};k)$ is even for any such K (cf. [JMW14, §2.6]). This is implied by the condition $\ell > b(\check{G})$.

Corollary 6.4.4. *Let $F, F^\natural, \widetilde{X}, \widetilde{X}^\natural$ be as in §6.4.5. Assume that ℓ is not a torsion prime for G.*
(1) F is Q_0-equivariantly formal over k, and $F^\natural$ is $Q_0^\natural$-equivariantly formal over k.
(2) $\widetilde{X}$ is Q_0-equivariantly formal over k, and $\widetilde{X}^\natural$ is $Q_0^\natural$-equivariantly formal over k.

Proof. Note that $F, F^\natural, X, X^\natural$ are paved by (perfections of) affine spaces, so their cohomology is concentrated in even degrees. By Example 6.4.3 the Q_0 (resp. $Q_0^\natural$) equivariant cohomology of a point is even under the hypothesis on ℓ, so the statements follow from Lemma 6.4.2. □

If Y is K-equivariantly formal, then we can recover $\mathrm{H}^*(Y;k) \cong \mathrm{H}^*_K(Y;k) \otimes_{\mathrm{H}^*_K(\mathrm{pt};k)} k$, and similarly for cohomology with supports. Hence by Corollary 6.4.4, to produce (6.12) it suffices to produce a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{\mathrm{BM}}_*(Q_0 \backslash F) \arrow[r, equal] \arrow[d, "\sim"] & \mathrm{H}^{\mathrm{BM}}_*(Q_0^\natural \backslash F^\natural) \arrow[d, "\sim"] \\ \mathrm{H}^{2d-*}_{Q_0 \backslash F}(Q_0 \backslash \widetilde{X}; k) \arrow[r, equal] & \mathrm{H}^{2d-*}_{Q_0^\natural \backslash F^\natural}(Q_0^\natural \backslash \widetilde{X}^\natural; k) \end{tikzcd} \tag{6.13}$$

where the bottom horizontal isomorphism is compatible with the cup product.

6.4.7. *Equivariant cohomology.* By definition (6.10), the Q_0-equivariant cohomology complex of $\widetilde{\mathrm{Gr}}_{\leq \alpha_0}$ is

$$\mathrm{R}\Gamma_{Q_0}(\widetilde{\mathrm{Gr}}_{\leq \alpha_0}) \cong \mathrm{R}\Gamma(Q_{1/4}\backslash Q_{1/2}/Q_{3/4}).$$

We keep track of cohomology complexes so as to be able to control the cohomology with supports. We abbreviate $R(K) := \mathrm{H}^*_K(\mathrm{pt}; k)$. Write $\mathrm{B}K$ for the classifying space $[\mathrm{pt}/K]$.

Lemma 6.4.5 (Equivariant Künneth formula)**.** *Let K be a connected pro-(perfection of)-algebraic group. Let Y_1 and Y_2 be K-equivariant ind-(perfection of)varieties such that*

$$\mathrm{Tor}^i_{R(K)}(\mathrm{H}^*_K(Y_1), \mathrm{H}^*_K(Y_2)) = 0 \textit{ for all } i.$$

Then the natural map

$$\mathrm{H}^*_K(Y_1) \otimes_{R(K)} \mathrm{H}^*_K(Y_2) \to \mathrm{H}^*_K(Y_1 \times Y_2) \tag{6.14}$$

is a k-algebra isomorphism.

Proof. The hypothesis on K implies that the existence and convergence of the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}^{*,*}_{R(K)}(\mathrm{H}^*_K(Y_1), \mathrm{H}^*_K(Y_2)) \implies \mathrm{H}^*_K(Y_1 \times Y_2).$$

The Tor-vanishing assumption then implies that this spectral sequence degenerates, concluding the proof. □

Example 6.4.6. If ℓ is not a torsion prime of G, the main theorem of [Dem73] says that $R(T)$ is flat over $R(G)$. It follows that for any Levi subgroup $L \subset G$ containing T, $R(L)$ is flat over $R(G)$ (since $R(L)$ is a direct summand of $R(T)$ as an $R(G)$-module).

We will now give descriptions of the equivariant cohomology of $F, \widetilde{\mathrm{Gr}}_{\leq \alpha_0}$, and Gr_{α_0} and their $\natural$-variants in the spirit of Soergel bimodule theory.

Example 6.4.7 (Zhu resolution)**.** Assume that ℓ is not a torsion prime of G. By definition, we have

$$Q_0\backslash\widetilde{\mathrm{Gr}}_{\leq \alpha_0} = Q_0\backslash Q_0 \times^{Q_{1/4}} Q_{1/2}/Q_{3/4} \cong Q_{1/4}\backslash Q_{1/2}/Q_{3/4}.$$

From Lemma 6.4.5 and Example 6.4.6, we have a k-algebra isomorphism

$$\mathrm{R}\Gamma(Q_0\backslash\widetilde{\mathrm{Gr}}_{\leq \alpha_0}) \cong \mathrm{R}\Gamma(\mathrm{B}Q_{1/4}) \otimes_{\mathrm{R}\Gamma(\mathrm{B}Q_{1/2})} \mathrm{R}\Gamma(\mathrm{B}Q_{3/4}). \tag{6.15}$$

Similarly for the $\natural$ version, we have a k-algebra isomorphism

$$\mathrm{R}\Gamma(Q_0^\natural\backslash\widetilde{\mathrm{Gr}}^\natural_{\leq \alpha_0}) \cong \mathrm{R}\Gamma(\mathrm{B}Q^\natural_{1/4}) \otimes_{\mathrm{R}\Gamma(\mathrm{B}Q^\natural_{1/2})} \mathrm{R}\Gamma(\mathrm{B}Q^\natural_{3/4}). \tag{6.16}$$

Example 6.4.8 (Open stratum)**.** Assume that ℓ is not a torsion prime of G. According to [Zhu17, Lemma 2.12(ii)], the open stratum has the group-theoretic description

$$Q_0\backslash \mathrm{Gr}_{\alpha_0} \cong Q_0\backslash Q_0 \times^{Q_{1/4}} (Q_{1/4}Q_{3/4})/Q_{3/4} \cong (Q_{1/4} \cap Q_{3/4})\backslash Q_{3/4}/Q_{3/4}. \tag{6.17}$$

From Lemma 6.4.5 and Example 6.4.6, we have a k-algebra isomorphism

$$\mathrm{R}\Gamma(Q_0\backslash \mathrm{Gr}_{\alpha_0}) \cong \mathrm{R}\Gamma(\mathrm{B}(Q_{1/4} \cap Q_{3/4})) \otimes_{\mathrm{R}\Gamma(\mathrm{B}Q_{3/4})} \mathrm{R}\Gamma(\mathrm{B}Q_{3/4}) \cong \mathrm{R}\Gamma(\mathrm{B}(Q_{1/4} \cap Q_{3/4})). \tag{6.18}$$

Similarly for the $\natural$ version, we have a k-algebra isomorphism

$$\mathrm{R}\Gamma(Q_0^\natural\backslash \mathrm{Gr}^\natural_{\alpha_0}) \cong \mathrm{R}\Gamma(\mathrm{B}(Q^\natural_{1/4} \cap Q^\natural_{3/4})) \otimes_{\mathrm{R}\Gamma(\mathrm{B}Q^\natural_{3/4})} R(\mathrm{B}Q^\natural_{3/4}) \cong \mathrm{R}\Gamma(\mathrm{B}(Q^\natural_{1/4} \cap Q^\natural_{3/4})). \tag{6.19}$$

Example 6.4.9 (Closed stratum)**.** According to [Zhu17, Lemma 2.12(ii)], the fiber F has the group-theoretic description

$$Q_0\backslash Q_0 \times^{Q_{1/4}} Q_{1/4}sQ_{3/4}/Q_{3/4} \cong (sQ_{1/4}s^{-1} \cap Q_{3/4})\backslash Q_{3/4}/Q_{3/4}$$

where s is the affine reflection corresponding to $1 + \alpha_0$. From Lemma 6.4.5 and Example 6.4.6, we have a k-algebra isomorphism

$$\mathrm{R}\Gamma(Q_0\backslash F) \cong \mathrm{R}\Gamma(\mathrm{B}(sQ_{1/4}s^{-1} \cap Q_{3/4})) \otimes_{\mathrm{R}\Gamma(\mathrm{B}Q_{3/4})} \mathrm{R}\Gamma(\mathrm{B}Q_{3/4}) \cong \mathrm{R}\Gamma(\mathrm{B}(sQ_{1/4}s^{-1} \cap Q_{3/4})). \tag{6.20}$$

Similarly for the $\natural$ version, we have a k-algebra isomorphism

$$\mathrm{R}\Gamma(Q_0^\natural\backslash F^\natural) \cong \mathrm{R}\Gamma(\mathrm{B}(sQ^\natural_{1/4}s^{-1} \cap Q^\natural_{3/4})) \otimes_{\mathrm{R}\Gamma(\mathrm{B}Q^\natural_{3/4})} \mathrm{R}\Gamma(\mathrm{B}Q^\natural_{3/4}) \cong \mathrm{R}\Gamma(\mathrm{B}(sQ^\natural_{1/4}s^{-1} \cap Q^\natural_{3/4})). \tag{6.21}$$

6.4.8. *Comparing mixed and equal characteristic.* Now, note that for each $r \in [0,1]$, the quotient of Q_r by its uni-potent radical Q_r^{u} is *isomorphic* to the quotient of $Q_r^\natural$ by its pro-unipotent radical $Q_r^{\natural,\mathrm{u}}$. This induces an identification $R(Q_r) \cong R(Q_r^\natural)$ via the chain of isomorphisms

$$R(Q_r) \xleftarrow{\sim} R(Q_r/Q_r^{\mathrm{u}}) = R(Q_r^\natural/Q_r^{\natural,\mathrm{u}}) \xrightarrow{\sim} R(Q_r^\natural).$$

Combining this with Examples 6.4.7, 6.4.8, and 6.4.9, we obtain a commutative diagram of identifications

$$\begin{tikzcd}
\mathrm{R}\Gamma(Q_0\backslash F; i^! k) \arrow[r, leftrightarrow, "\sim"] \arrow[d] & R\Gamma(Q_0^\natural\backslash F^\natural; i^! k) \arrow[d] \\
\mathrm{R}\Gamma(Q_0\backslash \mathrm{Gr}_{\le \alpha_0}; k) \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \mathrm{R}\Gamma(Q_0^\natural\backslash \mathrm{Gr}^\natural_{\le\alpha_0}; k) \arrow[d] \\
\mathrm{R}\Gamma(Q_0\backslash \mathrm{Gr}_{\alpha_0}; k) \arrow[r, leftrightarrow, "\sim"] & \mathrm{R}\Gamma(Q_0^\natural\backslash \mathrm{Gr}^\natural_{\alpha_0}; k)
\end{tikzcd}$$

where the vertical columns are exact triangles. Furthermore, the middle and bottom horizontal identifications are symmetric monoidal, hence they induce a symmetric monoidal isomorphism of the equivariant cohomology with supports

$$\mathrm{R}\Gamma_{Q_0\backslash F}(Q_0\backslash \mathrm{Gr}_{\le\alpha_0}) \cong \mathrm{R}\Gamma_{Q_0^\natural\backslash F^\natural}(Q_0^\natural\backslash \mathrm{Gr}^\natural_{\le\alpha_0})$$

compatibly with the identifications with the top row. This establishes the diagram (6.13), thus completing the proof of Proposition 6.3.2. □

7. The Brauer functor and the σ-dual homomorphism

Let G be a reductive group over E with an action of Σ, and let $H = G^\sigma \xrightarrow{\iota} G$. We assume that H is reductive. In this section we categorify the *normalized Brauer homomorphism* br from the spherical Hecke algebra of G to that of H, introduced in [TV16] and recalled in §7.1. This is defined via two steps: (1) a multiplicative norm that turns a function in the spherical Hecke algebra of G into a Σ-equivariant function, and then (2) a naive restriction of functions from G to H, which turns out to be multiplicative by a miracle of characteristic ℓ.

Roughly speaking, Step (1) is categorified by a monoidal norm and Step (2) is categorified by the Smith operation Psm. However, Psm lands in the Tate category while we eventually want the target of the Brauer functor to be perverse sheaves, so we want to use the lifting functor from §5.6 to lift out of the Tate category. Therefore, we need to study how Psm interacts with (Tate-)parity sheaves. This is done in §7.2 and §7.3, which together categorify Step (2).

Then in §7.4 we construct the *Brauer functor* $\widetilde{br}$ from the Satake category of G to the Satake category of H, which entails categorifying Step (1) and then also implementing certain extensions, from parity sheaves to all perverse sheaves (using the connection to tilting modules, plus results on abelian envelopes) and from strictly totally disconnected S to Div^1_X (using v-descent). In §7.5 we establish a compatibility of $\widetilde{br}$ with the constant term functor, which is needed later to show that it has good properties (e.g., additivity and exactness) and also to compute the σ-dual homomorphism in examples of interest.

In §7.6 we prove that $\widetilde{br}$ is compatible with the Tannakian structure of the Satake categories. From this we deduce the existence of the σ-dual homomorphism. Finally, in §7.7 we bootstrap to a "multi-legged" version for Beilinson-Drinfeld Grassmannians over $(\mathrm{Div}^1_X)^I$.

7.1. **Treumann-Venkatesh functoriality.** Let $K \subset G(E)$ be an open compact subgroup. The *Hecke algebra of G with respect to K with coefficients in Λ* is

$$\mathscr{H}(G,K;\Lambda) := \mathrm{Fun}_c(K\backslash G(E)/K, \Lambda), \tag{7.1}$$

the compactly supported functions on $K\backslash G(E)/K$ valued in Λ. This forms an algebra under convolution, normalized so that the indicator function $\mathbb{1}_K$ is the unit.

For a Σ-stable subgroup $K \subset G(E)$, we write $\iota^*K := \iota^{-1}(K) \subset H(E)$. We say that a compact open subgroup $K \subset G(E)$ is a *plain subgroup* if the natural map $H(E)/\iota^*K \to [G(E)/K]^\sigma$ is a bijection. We may view $\mathscr{H}(G,K;\Lambda)$ as the ring $\mathrm{Fun}_c^{G(E)}([G(E)/K] \times [G(E)/K], \Lambda)$ of $G(E)$-invariant (for the diagonal action) functions (valued in Λ) on $[G(E)/K] \times [G(E)/K]$, with compact support modulo $G(E)$, under convolution.

7.1.1. *The un-normalized Brauer homomorphism.* Now suppose that $\Lambda = k$ has characteristic ℓ. In this special situation, Treumann-Venkatesh observed that if $K \subset G(E)$ is a plain subgroup, then the *restriction map*

$$\begin{aligned}\mathscr{H}(G,K;k)^{\sigma} &= \mathrm{Fun}_c^{G(E)}([G(E)/K] \times [G(E)/K], k)^{\sigma} \\ &\xrightarrow{\text{restrict}} \mathrm{Fun}_c^{H(E)}([H(E)/\iota^* K] \times [H(E)/\iota^* K], k) = \mathscr{H}(H(E), \iota^* K; k)\end{aligned} \tag{7.2}$$

is an algebra homomorphism (cf. [Fen24, Lemma 6.5] for a proof). This map was introduced in [TV16, §4] and called the *(un-normalized) Brauer homomorphism.* We denote it

$$\mathrm{Br}\colon \mathscr{H}(G,K;k)^{\sigma} \to \mathscr{H}(H, \iota^* K; k). \tag{7.3}$$

We will categorify Br to an "un-normalized Brauer functor" $\mathcal{B}r$ in §7.3.

7.1.2. *Frobenius twist of algebras.* Let Frob be the absolute (ℓ-power) Frobenius of k. Given a commutative k-algebra A, we denote by $A^{(\ell)} := A \otimes_{k,\mathrm{Frob}} k$ its Frobenius twist. The map $\phi\colon A \to A^{(\ell)}$ sending $a \mapsto a \otimes 1$ is a Frob-semilinear (i.e., $\phi(\lambda a) = \mathrm{Frob}(\Lambda)\phi(a)$) isomorphism.

If A is equipped with an $\mathbf{F}_\ell$-structure $\varphi_0\colon A \cong A_0 \otimes_{\mathbf{F}_\ell} k$, then there is a k-linear isomorphism $f_{\varphi_0}\colon A \xrightarrow{\sim} A^{(\ell)}$, characterized by the property that it sends $A_0 \subset A$ to $A_0 \otimes 1 \subset A^{(\ell)}$ via $\mathrm{Id} \otimes 1$. We denote by $\mathrm{Frob}_{\varphi_0} = f_{\varphi_0}^{-1} \circ \phi\colon A \xrightarrow{\sim} A$; it is characterized as the unique Frob-semilinear automorphism of A which restricts to the identity on A_0.

Suppose $f\colon A \to B$ is any Frob-semilinear homomorphism (i.e., $f(\lambda a) = \lambda^\ell f(a)$ for $\lambda \in k$). Then f factors uniquely through a k-linear homomorphism

$$\begin{tikzcd} A \arrow[d, "\mathrm{Frob}_{\varphi_0}"'] \arrow[dr, "f"] & \\ A \arrow[r, dashed] & B \end{tikzcd}$$

The k-linear homomorphism $A \to B$ obtained by precomposing f with the inverse of $\mathrm{Frob}_{\varphi_0}$ (i.e., the dashed arrow in the diagram above) will be called the *linearization* of f (with respect to φ_0).

Definition 7.1.1 (The Tate diagonal)**.** Let A be a commutative k-algebra with an $\mathbf{F}_\ell$-structure φ_0. Let $\mathrm{Nm}\colon A \to \mathrm{T}^0(A)$ be the *Tate diagonal* map sending a to the class of $a \cdot ({}^{\sigma}a) \cdot \ldots \cdot ({}^{\sigma^{\ell-1}}a)$. One checks that this is a ring homomorphism, which is evidently Frob-semilinear. Let $\mathrm{Nm}^{(\ell^{-1})}\colon A \to \mathrm{T}^0(A)$ be the linearization of Nm with respect to φ_0.

7.1.3. *The normalized Brauer homomorphism.* Next suppose that $\mathscr{H}(G(E), K; k)$ and $\mathscr{H}(H(E), \iota^* K; k)$ are commutative. The Tate diagonal provides a map (cf. §3.7 for notation on Tate cohomology)

$$\mathrm{Nm}\colon \mathscr{H}(G,K;k) \xrightarrow{\sim} \mathrm{T}^0(\mathscr{H}(G,K;k)) = \frac{\mathscr{H}(G,K;k)^{\sigma}}{N \cdot \mathscr{H}(G,K;k)}.$$

One observes that the Brauer homomorphism Br factors over $\mathrm{T}^0(\mathscr{H}(G,K;k))$, hence we obtain a map

$$\mathrm{Br} \circ \mathrm{Nm}\colon \mathscr{H}(G,K;k) \to \mathscr{H}(H, \iota^* K; k).$$

This map is a Frob-semilinear ring homomorphism, and the *normalized Brauer homomorphism* is its linearization with respect to the canonical $\mathbf{F}_\ell$-structure

$$\mathscr{H}(G,K;k) \cong \mathscr{H}(G,K;\mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} k.$$

Equivalently, we may write

$$\mathrm{br} = \mathrm{Br} \circ \mathrm{Nm}^{(\ell^{-1})}\colon \mathscr{H}(G,K;k) \to \mathscr{H}(H, \iota^* K; k). \tag{7.4}$$

Below we will categorify br, under the assumption $\ell > \max\{b(\check{G}), b(\check{H})\}$, to a Tannakian functor

$$\widetilde{br}\colon \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X}; k) \to \mathrm{Sat}(\mathrm{Gr}_{H,\mathrm{Div}^1_X}; k).$$

7.2. **Good modules.** For a geometric point $S = \mathrm{Spa}(C, C^+) \to \mathrm{Div}^1_X$, we abbreviate $\mathrm{Gr}_{G,C} := \mathrm{Gr}_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^1_X}$, and similarly for $\mathrm{Gr}_{H,C}$, $\mathcal{H}\mathrm{ck}_{G,C}$, etc.

Definition 7.2.1. We say that an $\mathbb{O}[\Sigma]$-module M is *good* if it has a finite filtration as $\mathbb{O}[\Sigma]$-modules whose associated graded is a direct sum of (the trivial representation) $\mathbb{O}$ and (the regular representation) $\mathbb{O}[\Sigma]$ with arbitrary finite multiplicities.

Let T_H be a split maximal torus of H over E^s. Let $\mathcal{F} \in D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,C}; \mathbb{O}[\Sigma])^{\mathrm{bd}}$. For each $\mu \in X_*(T_H)^+$ and $n \in \mathbf{Z}$, $\mathcal{H}^n(i_\mu^* \mathcal{F})$ is a constant sheaf on some $\mathbb{O}[\Sigma]$-module, free over $\mathbb{O}$. We say that $\mathcal{F}$ is *good* if $\mathcal{H}^n(i_\mu^*\mathcal{F})$ is good in the sense of the preceding paragraph for all $\mu \in X_*(T_H)^+$ and all $n \in \mathbf{Z}$.

Let $\mathcal{F} \in (D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,C}; \mathbb{O})^{\mathrm{bd}})^{B\Sigma}$. We say that $\mathcal{F}$ is *good* if $\iota^*\mathcal{F} \in (D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,C}; \mathbb{O})^{\mathrm{bd}})^{B\Sigma} \cong D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,C}; \mathbb{O}[\Sigma])$ is good in the sense of the preceding paragraph.

Finally, we say that $\mathcal{F} \in (D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}})^{B\Sigma}$ is *good* if for all $\overline{s} = \mathrm{Spa}(C, C^+) \to S$, the $*$-restriction $\mathcal{F}|_{\overline{s}}$ is good in the sense of the preceding paragraph.

Example 7.2.2. The key example is that a permutation representation of Σ over $\mathbb{O}$ is good.

Remark 7.2.3. The significance of the definition lies in the fact that if an $\mathbb{O}[\Sigma]$-module M is good, then $\mathrm{T}^i(M)$ is concentrated in degree $i \equiv 0 \pmod 2$. This is because the Tate cohomology of $\mathbb{O}[\Sigma]$ vanishes, and the Tate cohomology of $\mathbb{O}$ lies in even degrees (cf. Example 3.7.1).

To analyze goodness, we will need to use the relationship between the (co)stalks of IC sheaves on $\mathrm{Gr}_{G,C}$ and the weight multiplicities of representations of $\check{G}$, which is documented in more "classical" settings in [BF10, Theorem 2.5] and [Zhu17, §5]. Let us formulate the necessary statement.

7.2.1. *The Brylinski-Kostant filtration.* Recall that $\check{G}$ comes equipped with a pinning. This induces a regular nilpotent element $\check{E} \in \check{\mathfrak{g}}$, which equips any $V \in \mathrm{Rep}(\check{G})$ with the *Brylinski-Kostant* (increasing) filtration

$$F_i V := \ker(\check{E}^{i+1} \colon V \to V).$$

For any $\mu \in X^*(\check{T})$, we denote by V_μ the μ-weight space of V. Then the above filtration induces a filtration $F_i(V_\mu) = V_\mu \cap F_i V$ on V_μ, whose associated graded we denote $\mathrm{gr}_i^F(V_\mu)$.

Proposition 7.2.4. *Let $\mathcal{F} \in \mathrm{Sat}(\mathrm{Gr}_{G,C}; \Lambda)$ correspond to $V \in \mathrm{Rep}_\Lambda(\check{G})$ under the Geometric Satake equivalence (cf. Example 6.1.3). Then there are natural isomorphisms of Λ-modules*

$$(i_\mu^*\mathcal{F})[-\langle 2\rho, \mu\rangle] \cong \mathrm{gr}_i^F(V_\mu).$$

Proof. The statement is equivalent to the analogous one for the Witt vector affine Grassmannian, using [FS21, Corollary VI.6.7] to degenerate from $\mathrm{Gr}_{G,C}$ to $\mathrm{Gr}_{G,\mathrm{Spd}\,\overline{\mathbf{F}}_q} \cong (\mathrm{Gr}_G^{\mathrm{Witt}})^\diamond$ and then [Sch22, §27] to transport to $\mathrm{Gr}_G^{\mathrm{Witt}}$. Then the result appears in the proof of [Zhu21, Proposition 18]: it is a small modification of the argument of [Zhu17, §5] in the equal characteristic case, replacing the purity argument in [RZ15, Lemma 5.8] by the parity argument in the middle of [Zhu17, p.452]. □

7.2.2. *The norm of Satake sheaves.*

Definition 7.2.5. Given $\mathcal{F} \in \mathrm{Sat}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}; \Lambda)$, we define

$$\mathrm{Nm}(\mathcal{F}) := \mathcal{F} \star ({}^\sigma\mathcal{F}) \star \ldots \star \left({}^{\sigma^{\ell-1}}\mathcal{F}\right) \in \mathrm{Sat}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}; \Lambda)^{B\Sigma}$$

equipped with the Σ-equivariant structure coming from the commutativity constraint (constructed in §6.1) for $(\mathrm{Sat}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}; \Lambda), \star)$. Thus Nm induces a functor

$$\mathrm{Nm} \colon \mathrm{Sat}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}; \Lambda) \to \mathrm{Sat}(\mathrm{Gr}_{G,S/(\mathrm{Div}^1_X)^I}; \Lambda)^{B\Sigma}.$$

Note that it is monoidal but neither additive nor Λ-linear.

Lemma 7.2.6. *Let S be a small v-stack over Div^1_X and let $\mathcal{F} \in \mathrm{Sat}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})$. Then $\mathrm{Nm}(\mathcal{F})$ is good.*

Proof. The assertion immediately reduces to the case $S = \mathrm{Spa}(C, C^+)$. Suppose $\mathcal{F}$ corresponds under the Geometric Satake equivalence to $V \in \mathrm{Rep}_{\mathbb{O}}(\check{G})$. Then under the Geometric Satake equivalence, $\mathrm{Nm}(\mathcal{F})$ corresponds to

$$\mathrm{Nm}(V) := V \otimes ({}^\sigma V) \otimes \ldots \otimes \left({}^{\sigma^{\ell-1}}V\right) \in \mathrm{Sat}(\mathrm{Gr}_{G,C}; \mathbb{O})^{B\Sigma}.$$

By Proposition 7.2.4, it suffices to check that for $V \in \mathrm{Rep}_{\mathbb{O}}(\check{G})$ and all $i \in \mathbf{Z}$ that the $\mathbb{O}[\Sigma]$-module $\mathrm{gr}^i((\mathrm{Nm}\, V)_\mu)$ is good, where the associated graded is for the Brylinski-Kostant filtration. Pick a basis for each V_λ adapted to the filtration $F_i V_\lambda$. This induces a basis for each $(\mathrm{Nm}\, V)_\mu$ on which Σ acts via permutation, and which is adapted to the filtration $F_i((\mathrm{Nm}\, V)_\mu)$. This realizes $\mathrm{gr}_i^F((\mathrm{Nm}\, V)_\mu)$ as a permutation representation of Σ, so each $\mathrm{gr}_i^F((\mathrm{Nm}\, V)_\mu)$ is good by Example 7.2.2. $\square$

7.2.3. *The Smith operation on good parity sheaves preserves parity.*

Proposition 7.2.7. *Suppose $\mathcal{E} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{B\Sigma}$ is a relative parity complex (with respect to the dimension pariversity $\dagger_G$) which is good. Then $\mathrm{Psm}(\mathcal{E}) \in \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})$ is relative Tate-parity with respect to the induced pariversity $\iota^*\dagger_G$.*

Proof. The assertion immediately reduces to the case $S = \mathrm{Spa}(C, C^+)$. Let $\iota \colon \mathrm{Gr}_{H,C} \hookrightarrow \mathrm{Gr}_{G,C}$ be the inclusion of Σ-fixed points. For $\mu \in X_*(T_H)^+ \subset X_*(T)^+$, we write

$$\begin{aligned} i_\mu^G &\colon \mathrm{Gr}_{G,C,\mu} \hookrightarrow \mathrm{Gr}_{G,C}, \\ i_\mu^H &\colon \mathrm{Gr}_{H,C,\mu} \hookrightarrow \mathrm{Gr}_{H,C}, \\ \iota_\mu &\colon \mathrm{Gr}_{H,C,\mu} \hookrightarrow \mathrm{Gr}_{G,C,\mu}\,. \end{aligned}$$

Without loss of generality suppose $\mathcal{E}$ is relative even, so we are given that $(i_\mu^G)^?\mathcal{E}$ has $\mathbb{O}$-free cohomology sheaves concentrated in degrees congruent to $\dagger_G(\mu)$ mod 2, where $? \in \{*, !\}$. We want to show that $(i_\mu^H)^? \mathbb{T}^*(\iota^*\mathcal{E})$ has Tate-cohomology sheaves supported in degrees congruent to $\dagger_G(\mu)$ mod 2. First we focus on the case $? = *$. Then we have

$$(i_\mu^H)^*\mathbb{T}^*(\iota^*\mathcal{E}) \cong \mathbb{T}^*(i_\mu^H)^*\iota^*\mathcal{E} \cong \mathbb{T}^*\iota_\mu^*(i_\mu^G)^*\mathcal{E}.$$

By assumption $\mathcal{E}$ is $*$-even on $\mathrm{Gr}_{G,C}$, so $\iota_\mu^*(i_\mu^G)^*\mathcal{E}$ has cohomology sheaves being $\mathbb{O}$-free modules in degrees congruent to $\dagger_G(\mu)$ mod 2. Since $\mathcal{E}$ was good by assumption, $\iota_\mu^*(i_\mu^G)^*\mathcal{E}$ is good. By Remark 7.2.3, $\mathrm{T}^j(\iota_\mu^*(i_\mu^G)^*\mathcal{E})$ is concentrated in degree $j \equiv \dagger_G(\mu) \pmod 2$.

For $? = !$, we have

$$(i_\mu^H)^! \,\mathrm{Psm}(\mathcal{E}) = (i_\mu^H)^!\mathbb{T}^*\iota^*(\mathcal{E}) \cong (i_\mu^H)^!\mathbb{T}^*\iota^!(\mathcal{E}) \cong \mathbb{T}^*(i_\mu^H)^!\iota^!(\mathcal{E}) \cong \mathbb{T}^*\iota_\mu^!(i_\mu^G)^!\mathcal{E} \tag{7.5}$$

where we used Lemma 3.4.3 in the second step. By assumption $\mathcal{E}$ is !-even on $\mathrm{Gr}_{G,C}$, so $(i_\mu^G)^!\mathcal{E}$ has cohomology sheaves being $\mathbb{O}$-free modules in degrees congruent to $\dagger_G(\mu)$ mod 2, and then the same holds for $\iota_\mu^!(i_\mu^G)^!\mathcal{E}$ by Lemma 5.4.1. Also, by Verdier duality $\iota_\mu^!(i_\mu^G)^!\mathcal{E}$ is good, so Remark 7.2.3 implies that $\mathrm{T}^j(\iota_\mu^!(i_\mu^G)^!\mathcal{E})$ is concentrated in degree $j \equiv \dagger_G(\mu) \pmod 2$, as desired. $\square$

We denote by

$$(D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}})^{B\Sigma}_{\mathrm{good}} \subset (D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{\mathrm{bd}})^{B\Sigma}$$

and

$$\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{B\Sigma}_{\mathrm{good}} \subset \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{B\Sigma}$$

the full subcategories of good objects, and we use similar notation for other variants.

7.3. **The un-normalized Brauer functor.** Assume that S is strictly totally disconnected. The pariversity $\dagger_G \colon X_*(T) \to \mathbf{Z}/2\mathbf{Z}$ from Example 5.1.7 factors canonically over $\pi_0(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X})$. Define the pariversity

$$\dagger_H^G := (\iota^* \dagger_G - \dagger_H) \colon X_*(T_H) \to \mathbf{Z}/2\mathbf{Z}.$$

We denote by

$$[\dagger_H^G] \colon \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_\Lambda)^{\mathrm{bd}} \to \mathrm{Perf}^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_\Lambda)^{\mathrm{bd}}$$

the functor given by shifting by $\dagger_H^G(c) \in \{0, 1\}$ on the connected component $c \subset \mathrm{Gr}_{H,S/\mathrm{Div}^1_X}$. We use the same notation for the equivariant version

$$[\dagger_H^G] \colon \mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_\Lambda)^{\mathrm{bd}} \to \mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_\Lambda)^{\mathrm{bd}}$$

and we note that the natural identification $[0] \cong [2]$ on $\mathrm{Perf}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_\Lambda)^{\mathrm{bd}}$ makes this functor monoidal. By Proposition 7.2.7, the composite functor $[\dagger_H^G] \circ \mathrm{Psm}$ defines a functor

$$\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{B\Sigma}_{\mathrm{good}} \to \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}})$$

where parity is with respect to the dimension pariversities (of G on the LHS and H on the RHS).

Definition 7.3.1 (Un-normalized Brauer functor)**.** Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. With L the lifting functor from §5.6, we define the functor $\mathcal{B}r := L \circ [\dagger_H^G] \circ \mathrm{Psm}$ as in the diagram

$$\begin{tikzcd}
\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{B\Sigma}_{\mathrm{good}} \arrow[rr, dashed, "\mathcal{B}r"] \arrow[dr, "{[\dagger_H^G]\circ\mathrm{Psm}}"'] & & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k) \\
& \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; \mathcal{T}_{\mathbb{O}}) \arrow[ur, "L"'] &
\end{tikzcd} \tag{7.6}$$

We regard $\mathcal{B}r$ as a categorification of the un-normalized Brauer homomorphism (7.3).

7.4. **The normalized Brauer functor.** Recall (cf. §2.7) that for an $\mathbb{O}$-linear abelian category C, we abbreviate

$$\mathsf{C} \otimes_{\mathbb{O}} k := \mathsf{C} \otimes_{\mathbb{O}-\mathrm{Mod}} (k - \mathrm{Mod}).$$

The construction below is a categorical analogue of §7.1.2.

Construction 7.4.1 (Frobenius twist of categories)**.** We summarize [Fen24, Construction 4.17]. Let Frob be the ℓ-power absolute Frobenius of k. Given a k-linear category C, the *Frobenius twist category* is $\mathsf{C}^{(\ell)} := \mathsf{C} \otimes_{k,\mathrm{Frob}} k$. Concretely, it is equivalent to the category which has the same objects as C, and morphisms

$$\mathrm{Hom}_{\mathsf{C}^{(\ell)}}(x,y) = \mathrm{Hom}_{\mathsf{C}}(x,y)^{(\ell)} := \mathrm{Hom}_{\mathsf{C}}(x,y) \otimes_{k,\mathrm{Frob}} k.$$

The tautological functor $\mathsf{C} \to \mathsf{C}^{(\ell)}$ is a Frob-semilinear equivalence.

Suppose we are given a presentation

$$\mathsf{F}_0 \colon \mathsf{C} \cong \mathsf{C}_0 \otimes_{\mathbf{F}_\ell} k \tag{7.7}$$

for some $\mathbf{F}_\ell$-linear category C_0. Then (7.7) induces another, *k-linear* equivalence $\mathsf{C} \xrightarrow{\sim} \mathsf{C}^{(\ell)}$. Combined with the tautological Frob-semilinear equivalence $\mathsf{C} \to \mathsf{C}^{(\ell)}$, (7.7) induces a Frob-semilinear equivalence $\mathrm{Frob}_{F_0} \colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$. This factors any Frob-semilinear functor $\mathsf{F} \colon \mathsf{C} \to \mathsf{D}$ over k-linear functor $\mathsf{F}^{(\ell^{-1})} \colon \mathsf{C} \to \mathsf{D}$, which we call the *linearization* of F (with respect to F_0).

7.4.1. *Initial construction.* Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. By Corollary 5.5.5 and Lemma 7.2.6, the functor Nm factors overs $\mathrm{Nm} \colon \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O}) \to \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O})^{\Sigma}_{\mathrm{good}}$ to the good subcategory. Hence we may consider the composition

$$\mathcal{B}r \circ \mathrm{Nm} \colon \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O}) \to \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k). \tag{7.8}$$

This maps to a k-linear category, hence factors uniquely over the k-linearization of the source, which is identified by the following Lemma.

Lemma 7.4.2. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. Then we have a symmetric monoidal equivalence*

$$\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbb{O}) \otimes_{\mathbb{O}} k \xrightarrow{\sim} \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; k).$$

Proof. The functor is well-defined by Lemma 5.3.6(1), and clearly monoidal. It is essentially surjective by Proposition 5.1.17, which shows that all objects are direct sums of the $\mathcal{E}(\mu, \mathcal{L})$ from Corollary 6.2.4. Finally, the description of Hom-spaces in Lemma 5.1.11 shows that it is fully faithful, completing the proof. ☐

By Lemma 7.4.2, the functor (7.8) factors uniquely through a functor

$$br^{(\ell)} \colon \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; k) \to \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k).$$

Note that $br^{(\ell)}$ is Frob-semilinear. The equivalence

$$\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; k) \cong \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; \mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} k \tag{7.9}$$

furnishes a natural $\mathbf{F}_\ell$-structure on $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; k)$, so we are in the setup to apply Construction 7.4.1.

Definition 7.4.3. Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. We define

$$br\colon \operatorname{Par}_{\mathsf{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\operatorname{Div}_X^1}; k) \to \operatorname{Par}_{\mathsf{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\operatorname{Div}_X^1}; k)$$

to be the linearization of $br^{(\ell)}$.

The functor br is an approximation to the definition of the normalized Brauer homomorphism (7.4). We still have to extend it to all perverse sheaves, and then descend to Div_X^1.

Remark 7.4.4. Parallel to (7.4), we have equivalently $br \cong (\mathcal{B}r \circ \mathrm{Nm}^{(\ell^{-1})}) \otimes_{\mathbb{O}} k$ where $\mathrm{Nm}^{(\ell^{-1})}$ is the linearization of Nm.

In §7.6 below, we will prove the following theorem.

Theorem 7.4.5. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. The functor $br\colon \operatorname{Par}_{\mathsf{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\operatorname{Div}_X^1}; k) \to \operatorname{Par}_{\mathsf{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\operatorname{Div}_X^1}; k)$ is additive, symmetric monoidal, and compatible with the fiber functor.*

Note that the compatibility with the fiber functor implies in particular that br is exact and faithful.

7.4.2. *Extending to abelian envelopes.* Recall that a tensor category over k is a k-linear rigid monoidal abelian category such that k maps isomorphically to the endomorphisms of the unit (example: $\operatorname{Rep}_k(\check{G})$) ; a pseudo-tensor category has the same definition except replacing abelian by "pseudo-abelian" (example: $\operatorname{Tilt}_k(\check{G})$). An *abelian envelope* [CEOP22, §2] of a pseudo-tensor category is a universal tensor category to which it maps via a faithful k-linear monoidal functor. Thus, a faithful monoidal functor from a pseudo-tensor category extends uniquely to its abelian envelope (if it exists).

This will be applied to the following situation. Assume $S = \operatorname{Spa}(C, C^+)$ for the moment. By Theorem 6.2.2, under the Geometric Satake equivalence (6.2) we have

$$\operatorname{Par}_{\mathrm{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{G,C}; k) \cong \operatorname{Tilt}_k(\check{G}) \quad \text{and} \quad \operatorname{Par}_{\mathrm{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{H,C}; k) \cong \operatorname{Tilt}_k(\check{H}).$$

By [CEOP22, Proposition 7.3.1], the abelian envelope of $\operatorname{Tilt}_k(\check{G})$ is $\operatorname{Rep}_k(\check{G})$, which corresponds under the Geometric Satake equivalence to $\operatorname{Sat}(\mathrm{Gr}_{G,C}; k)$. Therefore, assuming $\ell > b(\check{G})$, the abelian envelope of $\operatorname{Par}_{\mathrm{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{G,C}; k)$ is $\operatorname{Sat}(\mathrm{Gr}_{G,C}; k)$. Invoking Theorem 7.4.5 to see that br is faithful and exact, the universal property of the abelian envelope gives a unique extension of br to a functor

$$br\colon \operatorname{Sat}(\mathrm{Gr}_{G,C}; k) \to \operatorname{Sat}(\mathrm{Gr}_{H,C}; k). \tag{7.10}$$

Now suppose more generally that S is strictly totally disconnected. For a commutative ring R, let $\operatorname{Proj}^{\mathrm{f}}(R)$ be the category of finite projective R-modules. Then for $C^\infty(|S|; k)$ the ring of continuous functions on $|S|$ valued in k, we have from Example 6.1.3 an equivalence

$$\operatorname{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}_X^1}; k) \cong \operatorname{Rep}_k(\check{G}) \otimes_k \operatorname{Proj}^{\mathrm{f}}(C^\infty(|S|; k)).$$

Hence tensoring (7.10) with $\operatorname{Proj}^{\mathrm{f}}(C^\infty(|S|; k))$ gives an extension of br to the full Satake categories,

$$\begin{tikzcd} \operatorname{Par}_{\mathsf{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\operatorname{Div}_X^1}; k) \arrow[r, "br"] \arrow[d, hook] & \operatorname{Par}_{\mathsf{n}}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\operatorname{Div}_X^1}; k) \arrow[d, hook] \\ \operatorname{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}_X^1}; k) \arrow[r, dashed, "br"] & \operatorname{Sat}(\mathrm{Gr}_{H,S/\operatorname{Div}_X^1}; k) \end{tikzcd} \tag{7.11}$$

which is compatible with base change in S.

7.4.3. *Descending to* Div_X^1. Recall that $S \mapsto D_{\text{ét}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}_X^1}; k)^{\mathrm{bd}}$ satisfies v-descent, and every locally spatial diamond admits a v-cover by strictly totally disconnected spaces. Since the relative perversity condition is v-local on S by definition, we have a compatible diagram

$$\begin{tikzcd} D_{\text{ét}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,\operatorname{Div}_X^1}; k)^{\mathrm{bd}} \arrow[r, "\sim"] & \varprojlim_{S \text{ str.t.d. } \to \operatorname{Div}_X^1} D_{\text{ét}}^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{G,S/\operatorname{Div}_X^1}; k)^{\mathrm{bd}} \\ \operatorname{Sat}(\mathrm{Gr}_{G,\operatorname{Div}_X^1}; k) \arrow[u, hook] \arrow[r, "\sim"] & \varprojlim_{S \text{ str.t.d. } \to \operatorname{Div}_X^1} \operatorname{Sat}(\mathrm{Gr}_{G,S/\operatorname{Div}_X^1}; k) \arrow[u, hook] \end{tikzcd}$$

where we abbreviate "str.t.d." for "strictly totally disconnected". Comparing with the analogous diagram for H, this descends the functor br from (7.11) to a functor

$$\widetilde{br}\colon \operatorname{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};k) \to \operatorname{Sat}(\mathrm{Gr}_{H,\mathrm{Div}^1_X};k). \tag{7.12}$$

We regard $\widetilde{br}$ as the categorification of the normalized Brauer homomorphism (7.4).

Remark 7.4.6. We record for future use the following property of (7.12), which is arranged by construction: We have a natural isomorphism for all $\mathcal{F} \in \operatorname{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};k)$,

$$\mathbb{T}\widetilde{br}(\mathcal{F}) \cong [\dagger^G_H]\operatorname{Psm}(\mathrm{Nm}^{(\ell^{-1})}(\mathcal{F})) \in D^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,\mathrm{Div}^1_X};\mathcal{T}_k) \tag{7.13}$$

7.5. **Compatibility with constant terms.** Suppose G, H are split, and Σ stabilizes a Borus (B,T). Then $(B^\sigma, T^\sigma) =: (B_H, T_H)$ is a Borus of H, according to Lemma 4.2.1. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_{H,S/\mathrm{Div}^1_X} \arrow[d, hook, "\iota"] & \mathrm{Gr}_{B_H,S/\mathrm{Div}^1_X} \arrow[l, "q_H^+"'] \arrow[r, "p_H^+"] \arrow[d, hook, "\iota"] & \mathrm{Gr}_{T_H,S/\mathrm{Div}^1_X} \arrow[d, hook, "\iota"] \\
\mathrm{Gr}_{G,S/\mathrm{Div}^1_X} & \mathrm{Gr}_{B,S/\mathrm{Div}^1_X} \arrow[l, "q_G^+"'] \arrow[r, "p_G^+"] & \mathrm{Gr}_{T,S/\mathrm{Div}^1_X}
\end{tikzcd}$$

where the vertical maps are the inclusion of σ-fixed points (cf. §4.1) and each square is Cartesian. Any $\mathcal{F} \in D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}$ is monodromic (cf. [FS21, Definition IV.6.11] for the definition) for the $\mathbf{G}_m$ acting through $2\rho_G\colon \mathbf{G}_m \to G \subset L^+G$. The *Constant Term* (i.e., hyperbolic localization) functor (for G), defined in [FS21, Corollary VI.3.5], is the functor

$$\mathrm{CT}_B = \mathrm{R}(p_G^+)_!(q_G^+)^*\colon D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}} \to D^{\mathrm{ULA}}_{(L^+T)}(\mathrm{Gr}_{T,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}}.$$

We have a similar story for H with respect to the Borus (B_H, T_H).

Denote by $[\deg_G]$ the function $X_*(T) \xrightarrow{\langle 2\rho_G, -\rangle} \mathbf{Z}$, and similarly for H. Set

$$\mathrm{CT}_B[\deg_G] := \bigoplus_{\nu \in X_*(T)} \mathrm{R}(p_G^+)_!\, i_{\nu !}\, i_\nu^*\, (q_G^+)^*[\langle 2\rho_G, \nu\rangle]$$

where $i_\nu\colon S_\nu \hookrightarrow \mathrm{Gr}_{B,S/\mathrm{Div}^1_X}$ is the (open-closed) inclusion of the semi-infinite orbit through $[\nu]$; cf. [FS21, §VI.3] for more about it. Then $\mathrm{CT}_G[\deg_G]$ is t-exact and under the Geometric Satake equivalence intertwines with restriction along $\check{T} \to \check{G}$ by construction [FS21, §VI.11]. There is a completely analogous story for H.

Lemma 7.5.1. *There is a natural commutative square*

$$\begin{tikzcd}
D^{\mathrm{ULA}}_{(L^+G)}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}} \arrow[r, "\mathrm{Psm}"] \arrow[d, "\mathrm{CT}_B"] & D^{\mathrm{ULA}}_{(L^+H)}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X};\mathcal{T}_\Lambda)^{\mathrm{bd}} \arrow[d, "\mathrm{CT}_{B_H}"] \\
D^{\mathrm{ULA}}_{(L^+T)}(\mathrm{Gr}_{T,S/\mathrm{Div}^1_X};\Lambda)^{\mathrm{bd}} \arrow[r, "\mathrm{Psm}"] & D^{\mathrm{ULA}}_{(L^+T_H)}(\mathrm{Gr}_{T_H,S/\mathrm{Div}^1_X};\mathcal{T}_\Lambda)^{\mathrm{bd}}
\end{tikzcd} \tag{7.14}$$

Proof. By Lemma 5.2.9 and the Σ-fixed point calculations in Proposition 4.1.2 and Proposition 4.2.2, Psm interchanges $(q_G^+)^*$ with $(q_H^+)^*$; similarly using Proposition 5.2.10, Psm interchanges $\mathrm{R}(p_G^+)_!$ with $\mathrm{R}(p_H^+)_!$. $\square$

Lemma 7.5.2. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. There is a natural commutative square*

$$\begin{tikzcd}
\operatorname{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};k) \arrow[r, "\widetilde{br}"] \arrow[d, "\mathrm{CT}_B[\deg_G]"] & \operatorname{Sat}(\mathrm{Gr}_{H,\mathrm{Div}^1_X};k) \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] \\
\operatorname{Sat}(\mathrm{Gr}_{T,\mathrm{Div}^1_X};k) \arrow[r, "\widetilde{br}"] & \operatorname{Sat}(\mathrm{Gr}_{T_H,\mathrm{Div}^1_X};k)
\end{tikzcd} \tag{7.15}$$

Proof. By construction, it suffices to produce for each strictly totally disconnected S over Div^1_X a natural (including compatibility with base change in S) commutative square

$$\begin{tikzcd} \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\,\mathrm{Div}^1_X};k) \arrow[r, "br"] \arrow[d, "\mathrm{CT}_B[\deg_G]"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};k) \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] \\ \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T,S/\,\mathrm{Div}^1_X};k) \arrow[r, "br"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T_H,S/\,\mathrm{Div}^1_X};k) \end{tikzcd} \tag{7.16}$$

Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\,\mathrm{Div}^1_X};\mathbb{O}) \arrow[r, "\mathrm{Nm}"] \arrow[d, "\mathrm{CT}_B[\deg_G]"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\,\mathrm{Div}^1_X};\mathbb{O})^{B\Sigma}_{\mathrm{good}} \arrow[r, "{[\dagger^G_H]\circ\mathrm{Psm}}"] \arrow[d, "\mathrm{CT}_B[\deg_G]"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};\mathcal{T}_{\mathbb{O}}) \arrow[r, "L"] \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};k) \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] \\ \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T,S/\,\mathrm{Div}^1_X};\mathbb{O}) \arrow[r, "\mathrm{Nm}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T,S/\,\mathrm{Div}^1_X};\mathbb{O})^{B\Sigma}_{\mathrm{good}} \arrow[r, "\mathrm{Psm}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T_H,S/\,\mathrm{Div}^1_X};\mathcal{T}_{\mathbb{O}}) \arrow[r, "L"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T,S/\,\mathrm{Div}^1_X};k) \end{tikzcd} \tag{7.17}$$

The left square commutes because $\mathrm{CT}_B[\deg G]$ is symmetric monoidal. The middle square commutes by Lemma 7.5.1. We claim that the right square commutes. To see this, we consider the diagram

$$\begin{tikzcd} \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};\mathbb{O}) \arrow[r, "\mathbb{T}"] \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] \arrow[rr, bend left, "\mathbb{F}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};\mathcal{T}_{\mathbb{O}}) \arrow[r, "L"] \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};k) \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] \\ \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T_H,S/\,\mathrm{Div}^1_X};\mathbb{O}) \arrow[r, "\mathbb{T}"] \arrow[rr, bend right, "\mathbb{F}"'] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T_H,S/\,\mathrm{Div}^1_X};\mathcal{T}_{\mathbb{O}}) \arrow[r, "L"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T_H,S/\,\mathrm{Div}^1_X};k) \end{tikzcd} \tag{7.18}$$

The upper and lower caps commute by (5.12). It is immediate from the definition of the modular reduction functor $\mathbb{F}$ that the outer square commutes. In the left square, the horizontal arrows are essentially surjective since all $\mathcal{E}_{\mathrm{T}}(\mu,\mathcal{L})$ are in the image, and these generate under direct sums by Proposition 5.3.14. The maps on morphisms are described in §5.6, in terms of Lemma 5.3.1. From this, we see that the commutativity of the outer square of (7.18) implies commutativity of the right square.

Since the right square of (7.18) is the same as that of (7.17), we have now established that the outer rectangle in (7.17) commutes. Therefore, by definition, the diagram

$$\begin{tikzcd} \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\,\mathrm{Div}^1_X};k) \arrow[r, "br^{(\ell)}"] \arrow[d, "\mathrm{CT}_B[\deg_G]"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\,\mathrm{Div}^1_X};k) \arrow[d, "\mathrm{CT}_{B_H}[\deg_H]"] \\ \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T,S/\,\mathrm{Div}^1_X};k) \arrow[r, "br^{(\ell)}"] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{T_H,S/\,\mathrm{Div}^1_X};k) \end{tikzcd}$$

commutes. Finally, applying the Frobenius linearization process of Construction 7.4.1 completes the proof for the commutativity of (7.16). □

7.6. **The σ-dual homomorphism.** The proof of Theorem 7.4.5 is completed by combining Proposition 7.6.4, Corollary 7.6.5, and Proposition 7.6.6, which are proved below. Before embarking on these proofs, we draw a few consequences.

By the construction of $\widetilde{br}$, Theorem 7.4.5 implies the following:

Theorem 7.6.1. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. Then the functor $\widetilde{br}\colon \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X};k) \to \mathrm{Sat}(\mathrm{Gr}_{H,\mathrm{Div}^1_X};k)$ is additive, symmetric monoidal, and compatible with the fiber functor.*

Recall that

$$\mathrm{Rep}_{\mathrm{Loc}(\mathrm{Div}^1_X;\Lambda)}(\check{G}) \cong \mathrm{Rep}_{\mathrm{Rep}_\Lambda(W_E)}(\check{G}) \cong \mathrm{Rep}_\Lambda({}^L G)$$

for the *L-group* ${}^L G \cong \check{G} \rtimes W_E$. Note that this differs from Langlands' convention for the L-group by a cyclotomic twist on root groups, although the difference can be trivialized by choosing a square root of the cyclotomic character; see [FS21, §VI.11] for the precise relation.

Corollary 7.6.2. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. Then the functor*

$$\widetilde{br}\colon \operatorname{Sat}(\operatorname{Gr}_{G,\operatorname{Div}^1_X}; k) \to \operatorname{Sat}(\operatorname{Gr}_{H,\operatorname{Div}^1_X}; k)$$

corresponds under the Geometric Satake equivalence to the restriction ${}^L\psi\colon \operatorname{Rep}({}^L G) \to \operatorname{Rep}({}^L H)$ along some homomorphism ${}^L\psi\colon {}^L H \to {}^L G$.

Proof. The existence of ${}^L\psi$ with the stated property follows from the definitions of ${}^L G$ and ${}^L H$ via the Tannakian reconstruction process used in [FS21, Proposition VI.10.2]. $\square$

Remark 7.6.3. The second main theorem of Treumann-Venkatesh (see [TV16, §1.3]) is the construction of a σ-dual homomorphism when G is simply connected and H is semisimple, with three possible exceptions when G has type E_6. Their proof is based on classification of all possible examples, and then case-by-case analysis of each. By contrast, our construction is completely uniform.

However, our assumption on ℓ leaves out many interesting examples in [TV16] which are specific to small primes. This is partly due to the suboptimal hypotheses in Theorem 6.2.2, which we believe to be an artefact of the less developed state of geometric representation theory in p-adic geometry; for example, we take shortcuts in order to circumvent developing a theory of Soergel bimodules in this setting. It is also partly due to genuine problems with the theory of parity sheaves in very small characteristic; a finer investigation of perverse parity sheaves may allow us to extend our results to all ℓ. We hope to return to this in future work.

Finally, we recall that Treumann-Venkatesh pointed out [TV14, §7.8] that without the simply connected hypotheses, a σ-dual homomorphism need not exist with the "usual" L-group defined by Langlands, and they predicted that using instead the "c-group" might fix this issue. This prediction is morally consistent with our Theorem 7.6.1 since the L-group formed by taking the natural W_E-action on $\check{G}$ is exactly this c-group: see [FS21, VI.11] and [Zhu17, Remark 5.5.11].

7.6.1. *Compatibility with fiber functor.* The fiber functor on $\operatorname{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\operatorname{Gr}_{G,S/\operatorname{Div}^1_X}; k)$ (resp. $\operatorname{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\operatorname{Gr}_{H,S/\operatorname{Div}^1_X}; k)$) is given by relative cohomology over S,

$$\mathcal{F} \mapsto \bigoplus_i \mathrm{R}^i \pi_{G,S*}(\mathcal{F}) \quad (\text{resp. } \mathcal{F} \mapsto \bigoplus_i \mathrm{R}^i \pi_{H,S*}(\mathcal{F}))$$

where

$$\pi_{G,S}\colon \operatorname{Gr}_{G,S/\operatorname{Div}^1_X} \to S \quad \text{resp.} \quad \pi_{H,S}\colon \operatorname{Gr}_{H,S/\operatorname{Div}^1_X} \to S$$

are the natural projections.

Proposition 7.6.4. *The functor br is compatible with the fiber functors (cf.* (7.24) *below).*

Proof. Let $I = \operatorname{Gal}(\breve{E}^s/\breve{E})$ be the inertia subgroup of E. For $c \in \pi_1(H)_I$, there is an open-closed embedding $\operatorname{Gr}_{H,S/\operatorname{Div}^1_X}(c) \hookrightarrow \operatorname{Gr}_{H,S/\operatorname{Div}^1_X}$. For a complex $\mathcal{K} \in \operatorname{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\operatorname{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{T}_k)$, this induces a decomposition $\mathcal{K} \cong \bigoplus_{c \in \pi_1(H)_I} \mathcal{K}(c)$. We write

$$\mathrm{T}^{\dagger^G_H}(\operatorname{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{K}) := \bigoplus_{c \in \pi_1(H)_I} \mathrm{T}^{\dagger^G_H(c)}(\operatorname{Gr}_{H,S/\operatorname{Div}^1_X}; \mathcal{K}(c)),$$

i.e., we take Tate cohomology in degree $\dagger^G_H(c)$ on the connected component c.

By Proposition 5.2.10 we have a natural isomorphism

$$\mathrm{T}^0(\mathrm{R}\pi_{G,S*}(\operatorname{Nm}\mathcal{F})) \cong \mathrm{T}^0(\mathrm{R}\pi_{H,S*}(\operatorname{Psm}\circ \operatorname{Nm}\mathcal{F})). \tag{7.19}$$

Then as in Remark 7.4.6, we obtain a natural isomorphism

$$\mathrm{T}^0(\mathrm{R}\pi_{H,S*}(\operatorname{Psm}\circ \operatorname{Nm}\mathcal{F})) \cong \mathrm{T}^{\dagger^G_H}(\mathrm{R}\pi_{H,S*}(\mathbb{T}\circ br^{(\ell)}(\mathcal{F}))). \tag{7.20}$$

By Example 3.7.4, we have

$$\mathrm{T}^{\dagger^G_H}(\mathrm{R}\pi_{H,S*}(\mathbb{T}\circ br^{(\ell)}(\mathcal{F}))) \cong \bigoplus_{n\in\mathbf{Z}} \mathrm{R}^n\pi_{H,S*}(br^{(\ell)}(\mathcal{F})). \tag{7.21}$$

Below we abbreviate the fiber functor on $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};k)$ as

$$\mathsf{F}^G := \bigoplus_{n\in\mathbf{Z}} \mathrm{R}^n\pi_{G,S*}\colon \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};k)\to \mathrm{Loc}(S;k)$$

and similarly for H. Putting together equations (7.19), (7.20), and (7.21), we have produced a natural isomorphism of Frob-semilinear functors $\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};k)\to \mathrm{Loc}(S;k)$:

$$\mathrm{T}^0(\mathsf{F}^G(\mathrm{Nm}\,\mathcal{F}))\cong \mathsf{F}^H(br^{(\ell)}(\mathcal{F})). \tag{7.22}$$

By definition, we have

$$\mathsf{F}^G(\mathrm{Nm}(\mathcal{F}))\cong(\mathsf{F}^G(\mathcal{F}))\otimes\left({}^{\sigma}\mathsf{F}^G(\mathcal{F})\right)\otimes\ldots\otimes\left({}^{\sigma^{\ell-1}}\mathsf{F}^G(\mathcal{F})\right)\in \mathrm{Loc}(S;k)$$

with σ acting by cyclic rotation of the factors, so we have (cf. Example 3.7.2) a natural isomorphism

$$\mathrm{T}^0(\mathsf{F}^G(\mathrm{Nm}(\mathcal{F})))\cong \mathsf{F}^G(\mathcal{F})^{(\ell)}. \tag{7.23}$$

Putting (7.23) into (7.22), we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};k) \arrow[r, "br^{(\ell)}"] \arrow[d, "(\mathsf{F}^G)^{(\ell)}"'] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k) \arrow[d, "\mathsf{F}^H"] \\
\mathrm{Loc}(S;k) \arrow[r, equal] & \mathrm{Loc}(S;k)
\end{tikzcd}$$

Then applying linearization with respect to the $\mathbf{F}_\ell$-structure (7.9) gives the desired commutative diagram

$$\begin{tikzcd}
\mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};k) \arrow[r, "br"] \arrow[d, "\mathsf{F}^G"'] & \mathrm{Par}^{\mathrm{ULA}}_{\mathsf{n}}(\mathrm{Gr}_{H,S/\operatorname{Div}^1_X};k) \arrow[d, "\mathsf{F}^H"] \\
\mathrm{Loc}(S;k) \arrow[r, equal] & \mathrm{Loc}(S;k)
\end{tikzcd} \tag{7.24}$$

□

Corollary 7.6.5. *The functor br is additive.*

Proof. The additivity can be checked after applying the fiber functor, since the latter is faithful. Then conclude using (7.24) and the additivity of the fiber functor F^G. □

7.6.2. *Symmetric monoidality.* We complete the proof of Theorem 7.4.5 with the Proposition below.

Proposition 7.6.6. *The functor br promotes to a symmetric monoidal functor.*

Proof. First we promote br to a monoidal functor. For this it suffices to produce, naturally in $\mathcal{F}$ and $\mathcal{G}$, an isomorphism

$$\mathcal{B}r(\mathrm{Nm}^{(\ell^{-1})}(\mathcal{F}\star\mathcal{G}))\cong \mathcal{B}r(\mathrm{Nm}^{(\ell^{-1})}(\mathcal{F}))\star\mathcal{B}r(\mathrm{Nm}^{(\ell^{-1})}\,\mathcal{G}). \tag{7.25}$$

Since $\mathrm{Nm}^{(\ell^{-1})}$ has an evident symmetric monoidal structure, we rename $\mathcal{F}' := \mathrm{Nm}^{(\ell^{-1})}\,\mathcal{F}$ and $\mathcal{G}' := \mathrm{Nm}^{(\ell^{-1})}\,\mathcal{G}$, and aim to produce a natural isomorphism

$$\mathcal{B}r(\mathcal{F}'\star\mathcal{G}')\cong\mathcal{B}r(\mathcal{F}')\star\mathcal{B}r(\mathcal{G}').$$

Indeed, we have

$$\begin{aligned}
\mathcal{B}r(\mathcal{F}'\star\mathcal{G}') &= L\circ[\dagger_H^G]\operatorname{Psm}(m_!(p_0^*\mathcal{F}'\otimes p_1^*\mathcal{G}'))\\
\text{Lemma 5.2.9 + Proposition 5.2.10} \implies &\cong L\circ[\dagger_H^G]\circ m_!(p_0^*(\operatorname{Psm}\mathcal{F}')\otimes p_1^*(\operatorname{Psm}\mathcal{G}'))\\
&= L\circ[\dagger_H^G](\operatorname{Psm}\mathcal{F}'\star\operatorname{Psm}\mathcal{G}')\\
[\dagger_H^G]\text{ symmetric monoidal} \implies &= L(([\dagger_H^G]\operatorname{Psm}\mathcal{F}')\star([\dagger_H^G]\operatorname{Psm}\mathcal{G}'))\\
\text{Lemma 5.6.5} \implies &\cong L([\dagger_H^G]\operatorname{Psm}\mathcal{F}')\star L([\dagger_H^G]\operatorname{Psm}\mathcal{G}')\\
&= \mathcal{B}r(\mathcal{F}')\star\mathcal{B}r(\mathcal{G}').
\end{aligned}$$

This furnishes the monoidal structure (7.25).

Next we need to check that the monoidal functor br has the *property* of being symmetric monoidal. For this purpose, we may make a finite base change along S to reduce to the case where G is split, and has a Borus (B,T). By Lemma 4.2.1, $(B^\sigma,T^\sigma) =: (B_H,T_H)$ is a Borus of H. Since CT_{B_H} is faithful, it suffices to check the symmetric monoidality property after applying CT_{B_H}. Using Lemma 7.5.2, we are then reduced to the case where G and H are both tori. In this case, we have

$$\operatorname{Gr}_{T,S/\operatorname{Div}^1_X} = \coprod_{X_*(T)} S \quad\text{and}\quad \operatorname{Gr}_{T_H,S/\operatorname{Div}^1_X} = \coprod_{X_*(T_H)} S$$

and the commutativity constraints for G,H comes from convolution on $X_*(T), X_*(T_H)$ respectively. Then the symmetric monoidality is clear from inspection. □

The proof of Proposition 7.6.6 gives the following information about the induced map of tori.

Corollary 7.6.7. *Let $\check{T}_H,\check{T}$ be the canonical maximal tori in $\check{H},\check{G}$, respectively. The restriction $\check{\psi}\colon \check{T}_H\to\check{T}$ corresponds to the map $X^*(\check{T})\to X^*(\check{T}_H)=X^*(\check{T})^\sigma$ given by applying $N=(1+\sigma+\ldots+\sigma^{\ell-1})$.*

7.7. **Multiple legs.** For a finite non-empty set I, we abbreviate $\operatorname{Sat}^I_G(k) := \operatorname{Sat}(\operatorname{Gr}_{G,(\operatorname{Div}^1_X)^I};k)$.

Theorem 7.7.1. *There are Tannakian functors*

$$\widetilde{br}^I\colon \operatorname{Sat}^I_G(k)\to\operatorname{Sat}^I_H(k)$$

for each non-empty finite set I, with the following properties.

(1) Under the Geometric Satake equivalence (6.1) *of Fargues-Scholze, $\widetilde{br}^I$ corresponds to the restriction ${}^L\psi^*\colon \operatorname{Rep}_k({}^LG)^{\otimes I}\to\operatorname{Rep}_k({}^LH)^{\otimes I}$ induced by the σ-dual homomorphism ${}^L\psi\colon {}^LH\to{}^LG$ from Corollary 7.6.2. (In particular, $\widetilde{br}^{\{1\}}$ agress with the $\widetilde{br}$ from* (7.12)*.)*

(2) (Naturality in I) For any map of non-empty finite sets $\zeta\colon I\to J$, inducing the fusion product $\zeta\colon \operatorname{Sat}^I_G(k)\to\operatorname{Sat}^J_G(k)$ and similarly for H, there is a commutative square

$$\begin{tikzcd}
\operatorname{Sat}^I_G(k) \arrow[r, "\widetilde{br}^I"] \arrow[d, "\zeta"] & \operatorname{Sat}^I_H(k) \arrow[d, "\zeta"] \\
\operatorname{Sat}^J_G(k) \arrow[r, "\widetilde{br}^J"] & \operatorname{Sat}^J_H(k)
\end{tikzcd}$$

such that the implicit natural isomorphisms are compatible with compositions $I\to J\to K$.

(3) For $\mathcal{F}\in\operatorname{Sat}^I_G(k)$, there are natural isomorphisms

$$\mathbb{T}\widetilde{br}^I(\mathcal{F})\cong[\dagger_H^G]\operatorname{Psm}(\operatorname{Nm}^{(\ell^{-1})}\mathcal{F})\in D^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,(\operatorname{Div}^1_X)^I};\mathcal{T}_k)^{\mathrm{bd}}$$

compatible with any map of finite sets $\zeta\colon I\to J$ as in (2).

Proof. Write $I=\coprod_{i\in I}\{i\}$. We bootstrap from the case $|I|=1$ using the convolution Hecke stack

$$\widetilde{\mathcal{H}\mathrm{ck}}^I_G := \mathcal{H}\mathrm{ck}_G^{I;\{i\}_{i\in I}}\to(\operatorname{Div}^1_X)^I$$

defined in [FS21, p. 226]. We have projection maps $p_i\colon \widetilde{\mathcal{H}\mathrm{ck}}^I_G\to\widetilde{\mathcal{H}\mathrm{ck}}^{\{i\}}_{G,\operatorname{Div}^1_X}$ for each $i\in I$, as well as a convolution map $m\colon \widetilde{\mathcal{H}\mathrm{ck}}^I_G\to\mathcal{H}\mathrm{ck}_{G,(\operatorname{Div}^1_X)^I}$. The map

$$(\mathcal{K}_i)_{i\in I}\mapsto Rm_!(\otimes_{i\in I}\,p_i^*\mathcal{K}_i)\in\operatorname{Sat}^I_G(k)$$

induces an equivalence

$$\mathrm{conv}\colon \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X})^{\otimes I} \xrightarrow{\sim} \mathrm{Sat}^I_G(k).$$

The same considerations apply to H. We define $\widetilde{br}^I \colon \mathrm{Sat}^I_G(k) \to \mathrm{Sat}^I_H(k)$ by the commutative diagram

$$\begin{tikzcd}
\mathrm{Sat}_G(k)^{\otimes I} \arrow[r, "\widetilde{br}^{\otimes I}"] \arrow[d, "\sim"', "\mathrm{conv}"] & \mathrm{Sat}_H(k)^{\otimes I} \arrow[d, "\sim"', "\mathrm{conv}"] \\
\mathrm{Sat}^I_G(k) \arrow[r, dashed, "\widetilde{br}^I"] & \mathrm{Sat}^I_H(k)
\end{tikzcd}$$

Then property (1) follows from the $|I| = 1$ case, which was arranged in Corollary 7.6.2, and Tannakian reconstruction [FS21, §11].

For property (2), recall that the fusion product $\mathrm{Sat}^I_G(k) \to \mathrm{Sat}^J_G(k)$ is arranged in [FS21, §VI.9.4] so that under the identifications in the commutative diagram

$$\begin{tikzcd}
\mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X})^{\otimes I} \arrow[r] \arrow[d, "\sim"', "\mathrm{conv}"] & \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X})^{\otimes J} \arrow[d, "\sim"', "\mathrm{conv}"] \\
\mathrm{Sat}^I_G(k) \arrow[r] & \mathrm{Sat}^J_G(k)
\end{tikzcd}$$

it corresponds to

$$\star_{i\in I}\mathcal{F}_i \mapsto \otimes_{j\in J}\left(\star_{i\in\zeta^{-1}(j)}\mathcal{F}_j\right)$$

in the top row. Hence the compatibility of $\widetilde{br}^I$ in (2) is equivalent to the symmetric monoidality of $\widetilde{br}$ with respect to the fusion product, which was established in Theorem 7.6.1.

Property (3) is arranged by construction for $I = \{1\}$: see (7.13). For $|I| > 1$, we consider the diagram

$$\begin{tikzcd}
\mathrm{Sat}^{\{1\}}_G(k)^{\otimes I} \arrow[r, "{([\dagger^G_H]\circ \mathrm{Psm}\circ \mathrm{Nm}^{(\ell^{-1})})^{\otimes I}}"'] \arrow[d, "\mathrm{conv}"', "\sim"] \arrow[rr, bend left, "\widetilde{br}^{\otimes I}"] & D^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}^1_X)^I};\mathcal{T}_k)^{\otimes I} \arrow[d, "\mathrm{conv}"'] & \mathrm{Sat}^{\{1\}}_H(k)^{\otimes I} \arrow[l, "\mathbb{T}^{\otimes I}"'] \arrow[d, "\mathrm{conv}"', "\sim"] \\
\mathrm{Sat}^I_G(k) \arrow[r, "{[\dagger^G_H]\circ \mathrm{Psm}\circ \mathrm{Nm}^{(\ell^{-1})}}"'] \arrow[rr, bend right, "\widetilde{br}^I"'] & D^{\mathrm{ULA}}(\mathcal{H}\mathrm{ck}_{H,(\mathrm{Div}^1_X)^I};\mathcal{T}_k) & \mathrm{Sat}^I_H(k) \arrow[l, "\mathbb{T}"']
\end{tikzcd}$$

The outermost rectangle commutes by definition of $\widetilde{br}^I$. The top cap is the Ith tensor power of the case $|I| = 1$, so it commutes. The right square commutes by compatibility of $\mathbb{T}$ with pushforward and pullback. The left square commutes by (symmetric) monoidality of Nm, compatibility of Psm with pushforward and pullback (§5.2.3), and monoidality of $[\dagger^G_H]$. Hence the bottom cap commutes, which gives (3) for general I. □

8. Tate cohomology of moduli of local shtukas

In this section we will integrate the Brauer functor into the construction of the Fargues-Scholze correspondence (1.2) for H and for G, in order to deduce information about functoriality.

In §8.1 – §8.4 we review a construction of the Fargues-Scholze correspondence, which resembles a local version of the work of Vincent Lafforgue [Laf18] in the global setting, but (amazingly!) applies equally well in mixed characteristic. Our starting point is the *moduli space of local shtukas* $\mathrm{Sht}_{(G,b,I),K}$, defined from the input data of a reductive group G/E, an element b in the "Kottwitz set" $B(G)$, a finite non-empty set I, and a compact open subgroup $K \subset G(E)$. A generalization of the Grothendieck-Messing period map gives an étale morphism from $\mathrm{Sht}_{(G,b,I),K}$ to a "twisted" version of the Beilinson-Drinfeld affine Grassmannian

$\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\operatorname{Spd}\breve{E}}$. By pulling back sheaves along this morphism, the Geometric Satake equivalence of Fargues-Scholze supplies a functor from $\mathrm{Rep}_k(({}^LG)^I)$ to sheaves on $\mathrm{Sht}_{(G,b,I),K}$, which are "compatible with fusion" in a suitable sense. According to V. Lafforgue's paradigm, such a collection of functors gives rise to commuting *excursion operators* on the cohomology of $\mathrm{Sht}_{(G,b,I),K}$, whose simultaneous generalized eigenvalues correspond naturally to semisimple Galois representations.

To study functoriality, we link the processes outlined in the preceding paragraph for G and for H using the Brauer functor from §7. (The reader may find it helpful to consult Figure 2 again.) Thanks to the calculations in §4 we may realize $\mathrm{Sht}_{(H,b',I),\iota^*K}$ as (an open-closed subset of) the Σ-fixed points of $\mathrm{Sht}_{(G,b,I),K}$, and the Brauer functor $\widetilde{br}^I$ gives a geometric link between the sheaves on $\mathrm{Sht}_{(G,b,I),K}$ indexed by $V\in\mathrm{Rep}_k(({}^LG)^I)$ and the sheaves on $\mathrm{Sht}_{(H,b',I),\iota^*K}$ indexed by ${}^L\psi^*(V)\in\mathrm{Rep}_k(({}^LH)^I)$. We feed this link into equivariant localization for Tate cohomology, which is part of the formalism developed in §3. In §8.5 we define and study excursion operators on Tate cohomology of the moduli spaces of local shtukas. Then in §8.6, we extract functoriality relations between excursion operators for $\mathrm{Sht}_{(G,b,I),K}$ and for $\mathrm{Sht}_{(H,b',I),\iota^*K}$.

8.1. **Moduli spaces of local shtukas.** The definitions of local shtukas in p-adic geometry are developed in Scholze's Berkeley Lectures on p-adic geometry, especially [SW20, Lecture XXIII]. Properties of the cohomology of their moduli spaces are established in [FS21, §IX.3]. We recall some relevant aspects here.

8.1.1. *Setup.* Recall that G is a reductive group over E. For each finite set I, $b\in B(G)$, and compact open subgroup $K<G(E)$, we have a moduli space of local shtukas $\mathrm{Sht}_{(G,b,I),K}$. By [SW20, Theorem 23.1.4], it is an inductive limit of locally spatial diamonds with finite cohomological dimension along a countable index set. Furthermore, there is an action of $G_b(E)$ on $\mathrm{Sht}_{(G,b,I),K}$.

The space $\mathrm{Sht}_{(G,b,I),K}$ comes equipped with "leg" maps

$$f_K\colon \mathrm{Sht}_{(G,b,I),K}\to\prod_{i\in I}\operatorname{Spd}\breve{E}$$

which are partially proper and ind-shriekable, and Grothendieck-Messing period maps

$$\pi_K\colon \mathrm{Sht}_{(G,b,I),K}\to\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\operatorname{Spd}\breve{E}}\,.$$

which are étale.

8.1.2. *Satake sheaves.* As in [FS21, §IX], we choose a square root of the cyclotomic character $W_E\to k^\times$ in order to trivialize the cyclotomic twist in the Geometric Satake equivalence, giving a W_E-equivariant isomorphism $\check{G}\cong\widehat{G}$.

Let Q be a finite quotient of W_E over which the action on $\widehat{G}$-factors. For any finite set I and $W\in\mathrm{Rep}_k((\widehat{G}\rtimes Q)^I)$, the Geometric Satake equivalence gives a relative perverse sheaf $\mathcal{S}_W$ on $\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\operatorname{Spd}\breve{E}}$ (cf. [FS21, §I.9, §IX.3]) which we pull back via π_K^* to $\mathrm{Sht}_{(G,b,I),K}$, and we also denote the resulting complex by $\mathcal{S}_W\in D^b_{\text{ét}}(\mathrm{Sht}_{(G,b,I),K};k)$.

8.1.3. *Cohomology of moduli spaces of local shtukas.* By [FS21, Corollary I.7.3, Proposition IX.3.2], we may regard

$$\mathrm{R}f_{K!}\mathcal{S}_W\in D(\mathrm{Rep}^{\mathrm{sm}}_k G_b(E))^{B\prod_{i\in I}W_E} \tag{8.1}$$

as a (derived) representation of $G_b(E)$ with a commuting action of $\prod_{i\in I}W_E$. To demystify this a bit: the $G_b(E)$-action on cohomology is induced by the $G_b(E)$-action on the space $\mathrm{Sht}_{(G,b,I),K}$, and the $\prod_{i\in I}W_E$-action comes from a natural descent (using an interpretation via Hecke operators) of $\mathrm{R}f_{K!}\mathcal{S}_W$ from $\prod_{i\in I}\operatorname{Spd}\breve{E}$ to $\prod_{i\in I}[\operatorname{Spd}\breve{E}/\varphi]\cong\prod_{i\in I}[\operatorname{Spd}C/W_E]$.

Example 8.1.1. For $W=\mathbb{1}$ the trivial representation, $\mathrm{Sht}_{(G,b,I),K}$ is only non-empty for $b=1_G$. In this case, $\mathrm{R}f_{K!}\mathcal{S}_{\mathbb{1}}$ is $\text{c-Ind}_K^{G(E)}(k)$ with the obvious $G(E)$-action and the trivial W_E-action.

The functor

$$\mathrm{Rep}_k((\widehat{G}\rtimes Q)^I)\to D(\mathrm{Rep}^{\mathrm{sm}}_k G_b(E))^{B\prod_{i\in I}W_E}$$

sending $\mathcal{S}_W\mapsto\mathrm{R}f_{K!}\mathcal{S}_W$ satisfies the following *fusion compatibility.* Any map of finite non-empty sets $\zeta\colon I\to J$ induces a map $({}^LG)^J\to({}^LG)^I$. Let $\mathrm{Res}_\zeta\colon\mathrm{Rep}_k(({}^LG)^I)\to\mathrm{Rep}_k(({}^LG)^J)$ be the restriction functor along this map. By construction, Geometric Satake is arranged so that the corresponding functor $\mathrm{Sat}_G^I(k) \to \mathrm{Sat}_G^J(k)$ is the fusion product. We have commutative diagrams

$$\begin{tikzcd}
\mathrm{Sht}_{(G,b,J),K} \arrow[r] \arrow[d, "\pi_K"] & \mathrm{Sht}_{(G,b,I),K} \arrow[d, "\pi_K"] \\
\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{j\in J}\mathrm{Spd}\,\breve{E}} \arrow[r] \arrow[d, "f_K"] & \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}} \arrow[d, "f_K"] \\
\prod_{j\in J}\mathrm{Spd}\,\breve{E} \arrow[r] & \prod_{i\in I}\mathrm{Spd}\,\breve{E}
\end{tikzcd}$$

The Geometric Satake equivalence is constructed in [FS21, §VI] so that one has $\mathcal{S}_W|_{\mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{j\in J}\mathrm{Spd}\,\breve{E}}} \cong \mathcal{S}_{\mathrm{Res}_\zeta W}$, naturally in $W \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$ and compatibly under compositions of finite sets. This induces a natural isomorphism

$$\mathrm{R}f_{K!}\mathcal{S}_W \cong \mathrm{R}f_{K!}\mathcal{S}_{\mathrm{Res}_\zeta(W)} \tag{8.2}$$

of functors $\mathrm{Rep}_k((\widehat{G} \rtimes Q)^I) \to D(\mathrm{Rep}_k^{\mathrm{sm}} G_b(E))^{B\prod_{j\in J} W_E}$. Moreover, the natural isomorphism (8.2) under compositions of maps of finite sets.

8.2. **Excursion algebra.** An *excursion datum* for $\widehat{G}$ (over k) [FS21, Definition VIII.4.2] is a tuple $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ where:

- I is a finite set and $\gamma_i \in W_E$ for each $i \in I$,
- $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$ for varying Q, and $\mathbb{1} \xrightarrow{\alpha} V|_{\widehat{G}}$ and $V|_{\widehat{G}} \xrightarrow{\beta} \mathbb{1}$ are maps of $\widehat{G}$-representations. (Here $\mathbb{1}$ is the trivial representation.)

The *excursion algebra* $\mathrm{Exc}_k(W_E, \widehat{G})$ is the k-algebra on generator $S_{\mathcal{D}}$ for each excursion datum $\mathcal{D}$, with relations as in [Fen24, §2.4]. Another definition appears in [FS21, Definition VIII.3.4].

An *L-parameter* (with coefficients in k) is a class in $\mathrm{H}^1(W_E; \widehat{G}(k))$, or equivalently a section $W_E \to \widehat{G}(k) \rtimes W_E$ up to $\widehat{G}(k)$-conjugation. An L-parameter is *semisimple* if whenever it factors through a parabolic ${}^L P(k) \subset {}^L G(k)$, it also factors through a Levi ${}^L M(k) \subset {}^L P(k)$ [FS21, Definition VIII.3.1].

Combining [FS21, Proposition VIII.3.8] and the statement above [FS21, Definition VIII.3.4] that the natural map from $\mathrm{Exc}(W_E, \widehat{G})$ to the *spectral Bernstein center* $\mathcal{O}(Z^1(W_E, \widehat{G}))^{\widehat{G}}$ is a universal homeomorphism, we obtain a canonical bijection between

$$\{\text{characters } \mathrm{Exc}_k(W_E, \widehat{G}) \to k\} \longleftrightarrow \{\text{semisimple } L\text{-parameters } \rho \in \mathrm{H}^1(W_E; \widehat{G}(k))\}.$$

Suppose we are given a homomorphism ${}^L\psi \colon {}^L H \to {}^L G$. Then for any excursion datum $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ for $\widehat{G}$, we define

$${}^L\psi^*(\mathcal{D}) := (I, {}^L\psi^*(V), {}^L\psi^*(\alpha), {}^L\psi^*(\beta), (\gamma_i)_{i\in I})$$

as an excursion datum for $\widehat{H}$. The map $S_{\mathcal{D}} \mapsto S_{{}^L\psi^*(\mathcal{D})}$ defines a homomorphism

$${}^L\psi^* \colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Exc}_k(W_E, \widehat{H}). \tag{8.3}$$

On spectra, it sends (the point corresponding to) a semisimple L-parameter $\rho \in \mathrm{H}^1(W_E; \widehat{H}(k))$ to (the point corresponding to) the semisimple L-parameter ${}^L\psi \circ \rho \in \mathrm{H}^1(W_E; \widehat{G}(k))$.

8.3. **The Bernstein center.** Recall that in (7.1) we defined the Hecke algebra $\mathscr{H}(G, K; \Lambda)$ for a compact open subgroup $K \subset G(E)$. We let $\mathfrak{Z}(G, K; \Lambda) := Z(\mathscr{H}(G, K; \Lambda))$ be the center of $\mathscr{H}(G, K; \Lambda)$.

If $K \subset K'$ have prime-to-ℓ pro-order (e.g., this will be true as long as they are sufficiently small), then convolution with $\mathbb{1}_{K'}$ gives a homomorphism $\mathfrak{Z}(G, K; \Lambda) \to \mathfrak{Z}(G, K'; \Lambda)$. The *Bernstein center of G (with coefficients in Λ)* is

$$\mathfrak{Z}(G; \Lambda) := \varprojlim_K \mathfrak{Z}(G, K; \Lambda),$$

where the transition maps are as above, and the inverse limit is taken over K with prime-to-ℓ pro-order.

If $\Lambda = k$, we abbreviate $\mathscr{H}(G, K) := \mathscr{H}(G, K; k)$, $\mathfrak{Z}(G, K) := \mathfrak{Z}(G, K; k)$, and $\mathfrak{Z}(G) := \mathfrak{Z}(G; k)$.

The Bernstein center $\mathfrak{Z}(G)$ may also be identified with the ring of endomorphisms of the identity functor of the category $\mathrm{Rep}_k^{\mathrm{sm}}(G(E))$. In particular, any irreducible admissible representation Π of $G(E)$ over k induces a character of $\mathfrak{Z}(G)$.

8.4. **Review of Fargues-Scholze correspondence.** Fargues-Scholze construct in [FS21] a k-algebra homomorphism

$$\mathrm{FS}_G\colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathfrak{Z}(G;k). \tag{8.4}$$

Via (8.4), $\mathrm{Exc}_k(W_E, \widehat{G})$ acts through a character on any irreducible admissible representation $\Pi \in \mathrm{Rep}_k^{\mathrm{sm}}(G(E))$, and the corresponding semisimple L-parameter is denoted $\rho_\Pi \in \mathrm{H}^1(W_E; \widehat{G}(k))$.

We present the construction of (8.4) on [FS21, p.36]. Fix $b := 1_G \in B(G)$. Let

$$\overline{x}\colon \mathrm{Spd}\,\widehat{\overline{E}} \to \prod_{i=1}^{n} \mathrm{Spd}\,\breve{E}$$

be the geometric diagonal, and $f_K^{\Delta}\colon \mathrm{Sht}^{\Delta}_{(G,1_G,I),K} \to \mathrm{Spd}\,\widehat{\overline{E}}$ be the base change of f_K along $\overline{x}$.

Let $K \subset G(E)$ be a compact open subgroup. Recall from Example 8.1.1 that the underlying $G(E)$-representation of $\mathrm{R}f^{\Delta}_{K!}\mathcal{S}_{\mathbb{1}}$ is $\text{c-Ind}_K^{G(E)} k$. Hence for each excursion datum $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in i})$ we have a composition

$$\begin{tikzcd}
\text{c-Ind}_K^{G(E)} k \arrow[rrrrr, dashed, "\mathrm{FS}_G(S_{\mathcal{D}})"] \arrow[d, equal] & & & & & \text{c-Ind}_K^{G(E)} k \arrow[d, equal] \\
\mathrm{R}f^{\Delta}_{K!}\mathcal{S}_{\mathbb{1}} \arrow[r, "\alpha"] & \mathrm{R}f^{\Delta}_{K!}\mathcal{S}_{V|_{\widehat{G}}} \arrow[r, equal] & (\mathrm{R}f_{K!}\mathcal{S}_V)_{\overline{x}} \arrow[r, "(\gamma_i)_{i\in I}"] & (\mathrm{R}f_{K!}\mathcal{S}_V)_{\overline{x}} \arrow[r, equal] & \mathrm{R}f^{\Delta}_{K!}\mathcal{S}_{V|_{\widehat{G}}} \arrow[r, "\beta"] & \mathrm{R}f^{\Delta}_{K!}\mathcal{S}_{\mathbb{1}}
\end{tikzcd} \tag{8.5}$$

(in the middle step we used (8.1) to obtain an action of $\prod_{i\in I} W_E$ on $(\mathrm{R}f_{K!}\mathcal{S}_V)_{\overline{x}}$) which defines an element $\mathrm{FS}_G(S_{\mathcal{D}}) \in \mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)} k, \text{c-Ind}_K^{G(E)} k) \cong \mathfrak{Z}(G,K)$. As K varies, these elements are compatible under convolution, hence assemble to an element of $\mathfrak{Z}(G)$. The map (8.4) sends $S_{\mathcal{D}}$ to this element of $\mathfrak{Z}(G)$.

8.5. **Excursion operators on Tate cohomology.** Recall the notion of *Tate cohomology* from §3.7. We will now study excursion operators on the Tate cohomology of $\mathrm{Sht}_{(G,b,I),K}$.

8.5.1. *Tate cohomology of moduli of shtukas.* Recall that G has a given action of Σ. Assume that the level structure $K \subset G(E)$ is Σ-invariant, and $b \in B(G)$ is Σ-fixed. Then there is an induced Σ-action on $\mathrm{Sht}_{(G,b,I),K}$.

The given action of Σ on G induces an action $V \mapsto {}^{\sigma}V$ of Σ on $\mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$. Suppose we have a Σ-equivariant representation $W \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$. Then $\mathcal{S}_W$ has the structure of a Σ-equivariant sheaf on $\mathrm{Sht}_{(G,b,I),K}$. This equips its cohomology with a Σ-equivariant structure, so that we may regard, using (8.1),

$$\mathrm{R}f_{K!}\mathcal{S}_W \in D(\mathrm{Rep}_k^{\mathrm{sm}}\, G_b(E))^{B(\prod_{i\in I} W_E \rtimes \Sigma)}.$$

Hence we can form the jth Tate cohomology of $\mathrm{R}f_{K!}\mathcal{S}_W$ (for $j \in \mathbf{Z}/2\mathbf{Z}$), which we denote

$$\mathrm{T}^j(\mathrm{Sht}_{(G,b,I),K}; W) := \mathrm{T}^j(\mathrm{R}f_{K!}\mathcal{S}_W) \in D(\mathrm{Rep}_k^{\mathrm{sm}}\, G_b^{\sigma}(E))^{B\prod_{i\in I} W_E}. \tag{8.6}$$

as we will want to record the dependence on G, b, I.

Example 8.5.1. For $V = \mathbb{1}$, the trivial representation, $\mathrm{Sht}_{(G,b,I),K}$ is only non-empty for $b = 1_G$. In that case, we have (cf. Example 8.1.1)

$$\mathrm{T}^j(\mathrm{Sht}_{(G,1_G,I),K}; \mathbb{1}) \cong \mathrm{T}^j(\text{c-Ind}_K^{G(E)} k) \in \mathrm{Rep}_k^{\mathrm{sm}}(H(E)).$$

A similar story applies to H. Note that Σ acts trivially on $\mathrm{Sht}_{(H,b,I),K}$, so if Σ also acts trivially on $W \in \mathrm{Rep}_k((\widehat{H} \rtimes Q)^I)$, then by Example 3.7.4 and Example 3.7.1, we have for each $j \in \mathbf{Z}/2\mathbf{Z}$ a natural isomorphism

$$\mathrm{T}^j(\mathrm{Sht}_{(H,b,I),K}; W) \cong \bigoplus_{n\in\mathbf{Z}} \mathrm{H}^n(\mathrm{R}f_{K!}\mathcal{S}_W) = \mathrm{H}^*(\mathrm{R}f_{K!}\mathcal{S}_W). \tag{8.7}$$

8.5.2. *Excursion action.* The Σ action on G induces a Σ-action on $\mathrm{Exc}_k(W_E, \widehat{G})$ by transport of structure. Concretely, Σ acts on excursion data by

$$\sigma \cdot (I, V, \alpha, \beta, (\gamma_i)_{i \in I}) = (I, {}^{\sigma}V, \sigma(\alpha), \sigma(\beta), (\gamma_i)_{i \in I})$$

and then $\sigma \cdot S_{\mathcal{D}} = S_{\sigma \cdot \mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})$.

In general, given a $k[\Sigma]$-algebra A and an A-module M, there is a natural $\mathrm{T}^0(A) = A^{\sigma}/(N \cdot A)$-module structure on $\mathrm{T}^*(M)$. In particular, this equips $\mathrm{T}^j(\mathrm{Sht}_{(G,b,I),K}; W)$ with a natural $\mathrm{T}^0 \mathrm{Exc}_k(W_E, \widehat{G})$-action. We are most interested in the special case where $W = \mathbb{1}$ and $b = 1_G$, in which case we have $\mathrm{T}^j(\mathrm{Sht}_{(G,b,I),K}; \mathbb{1}) = \mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$ at the level of underlying $H(E)$-representations (cf. Example 8.5.1). If an excursion datum $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i \in I})$ is Σ-invariant, then $S_{\mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ and its action on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$ can be described more concretely using (8.5): it is given by composition

$$\begin{tikzcd}
\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k) \arrow[rrr, dashed, "S_{\mathcal{D}}"] \arrow[d, equal, "(8.5.1)"] & & & \mathrm{T}^j(\text{c-Ind}_K^{G(E)} k) \arrow[d, equal] \\
\mathrm{T}^j(\mathrm{Sht}_{(G,1,I),K}; \mathbb{1}) \arrow[r, "\alpha"] & \mathrm{T}^j(\mathrm{Sht}_{(G,1,I),K}; V) \arrow[r, "(\gamma_i)_{i \in I}"] & \mathrm{T}^j(\mathrm{Sht}_{(G,1,I),K}; V) \arrow[r, "\beta"] & \mathrm{T}^j(\mathrm{Sht}_{(G,1,I),K}; \mathbb{1})
\end{tikzcd} \tag{8.8}$$

8.5.3. *Normed excursion action.* Recall from Definition 7.1.1 that for any commutative $k[\Sigma]$-algebra A, there is the *Tate diagonal* morphism

$$A \xrightarrow{\mathrm{Nm}} \mathrm{T}^0(A)$$

sending a to the class of $\mathrm{Nm}(a) = a \cdot \sigma(a) \cdots \sigma^{\ell-1}(a)$. This is Frob-semilinear, but an $\mathbf{F}_{\ell}[\Sigma]$-structure on A induces a linearization $\mathrm{Nm}^{(\ell^{-1})} \colon A \to \mathrm{T}^0(A)$, which is a k-algebra homomorphism.

We apply this to $A := \mathrm{Exc}_k(W_E, \widehat{G})$, with the $\mathbf{F}_{\ell}$-structure coming from the fact that $\widehat{G}$ is defined over $\mathbf{F}_{\ell}$. In §8.5.2 we saw an action of $\mathrm{T}^0 \mathrm{Exc}_k(W_E, \widehat{G})$ on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$. Inflating this action through $\mathrm{Nm}^{(\ell^{-1})}$ gives an action of $\mathrm{Exc}_k(W_E, \widehat{G})$ on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$, which we call the *normed excursion action.*

8.5.4. *Native excursion action.* The Σ-invariant subalgebra $\mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ acts naturally on $\mathrm{T}^j(\mathrm{Sht}_{(G,b,I),K}; V)$ via the quotient $\mathrm{Exc}_k(W_E, \widehat{G})^{\sigma} \twoheadrightarrow \mathrm{T}^0(\mathrm{Exc}_k(W_E, \widehat{G}))$ and then the mechanism of §8.5.2. We refer to this as the *native excursion action* of $\mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ on $\mathrm{T}^j(\mathrm{Sht}_{(G,b,I),K}; V)$. This action is *not* the same as the restriction of the normed excursion action to the subalgebra $\mathrm{Exc}_k(W_E, \widehat{G})^{\sigma} \subset \mathrm{Exc}_k(W_E, \widehat{G})$; this discrepancy will be responsible for Frobenius twists which show up later.

In general, for a commutative k-algebra A the absolute Frobenius $A \to A$ is Frob-semilinear. If A has an $\mathbf{F}_{\ell}$-structure $A \cong A_0 \otimes_{\mathbf{F}_{\ell}} k$, then the absolute Frobenius may be linearized to a map $F \colon A \to A$, as explained in §7.1.2. The map F is characterized as the unique k-linear homomorphism that sends $a_0 \mapsto a_0^{\ell}$ for all $a_0 \in A_0 \subset A$.

The preceding discussion applies to $A := \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ (with the $\mathbf{F}_{\ell}$-structure coming from the fact that $\widehat{G}$ is defined over $\mathbf{F}_{\ell}$). In these terms, the normed excursion action of $\mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ is the native excursion action composed with the map $F \colon \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma} \to \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ which is the linearization of the absolute Frobenius.

8.5.5. *Norm of excursion data.* For any finite group Q over which the W_E-action on $\widehat{G}$ factors, we define a functor

$$\mathrm{Nm} \colon \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I) \to \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$

as follows:

- For a representation $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we set

$$\mathrm{Nm}(V) := V \otimes_k ({}^{\sigma}V) \otimes_k \ldots \otimes_k \left({}^{\sigma^{\ell-1}}V\right) \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$

  with the obvious Σ-equivariant structure. Note that it corresponds under Geometric Satake to Definition 7.2.5.
- Given $h \colon V \to V' \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we set

$$\mathrm{Nm}(h) := h \otimes {}^{\sigma}h \otimes \ldots \otimes {}^{\sigma^{\ell-1}}h \colon \mathrm{Nm}(V) \to \mathrm{Nm}(V').$$

Note that Nm is *not* an additive functor, nor is it even k-linear.

Definition 8.5.2. We define the *linearized norm* $\mathrm{Nm}^{(\ell^{-1})} := \mathrm{Nm} \circ \mathrm{Frob}^{-1}$ to be the linearization of Nm in the sense of Construction 7.4.1; note that Frob^{-1} is the identity on objects and on morphisms it is $(-) \otimes_{k,\mathrm{Frob}^{-1}} k$. Then the resulting functor

$$\mathrm{Nm}^{(\ell^{-1})} \colon \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I) \to \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$

is k-linear (although still not additive).

Definition 8.5.3. For $V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we denote by

$$N \cdot V \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)^{B\Sigma}$$

the σ-equivariant representation $V \oplus {}^{\sigma}V \oplus \ldots \oplus {}^{\sigma^{\ell-1}}V$, with the obvious Σ-equivariant structure.

For $h \colon V \to V' \in \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I)$, we denote by

$$N \cdot h \colon N \cdot V \to N \cdot V'$$

the σ-equivariant map $h \oplus {}^{\sigma}h \oplus \ldots \oplus {}^{\sigma^{\ell-1}}h$. Let $\Delta_\ell \colon \mathbb{1} \to \mathbb{1}^{\oplus \ell}$ denote the diagonal map and $\nabla_\ell \colon \mathbb{1}^{\oplus \ell} \to \mathbb{1}$ denote the sum map.

Definition 8.5.4. Let $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i \in I})$ be an excursion datum for $\widehat{G}$. Define the excursion data

$$\mathrm{Nm}^{(\ell^{-1})} \mathcal{D} := (I, \mathrm{Nm}^{(\ell^{-1})} V, \mathrm{Nm}^{(\ell^{-1})} \alpha, \mathrm{Nm}^{(\ell^{-1})} \beta, (\gamma_i)_{i \in I})$$

and

$$N \cdot \mathcal{D} := (I, N \cdot V, (N \cdot \alpha) \circ \Delta_\ell, \nabla_\ell \circ (N \cdot \beta), (\gamma_i)_{i \in I}),$$

which are excursion data for $\widehat{G}$.

Straightforward calculations yield:

Lemma 8.5.5 ([Fen24, Lemma 5.16])**.** *For all excursion data $\mathcal{D}$, we have*

$$\mathrm{Nm}^{(\ell^{-1})}(S_{\mathcal{D}}) = S_{\mathrm{Nm}^{(\ell^{-1})}(\mathcal{D})} \in \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$$

and

$$N \cdot S_{\mathcal{D}} = S_{N \cdot \mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}.$$

Proof. The second identity appears in [Fen24, Lemma 5.16], which also proves that

$$\mathrm{Nm}(S_{\mathcal{D}}) = S_{\mathrm{Nm}(\mathcal{D})} \in \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$$

from which the first identity follows. ☐

8.6. **Functoriality for excursion operators.** For basic $b \in B(H)$, we let H_b be the corresponding inner twist of H, as in §4.4. If $b \in B(H)$ is basic and maps to $1_G \in B(G)$, then we have $\iota_{b'} \colon H_{b'} \hookrightarrow G_{\iota(b')} = G$. Below we abbreviate $\iota^* K := \iota_{b'}^* K$ and 1 for the trivial element of $B(G)$ or $B(H_{b'})$, depending on context.

Proposition 8.6.1. *Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Let $K < G(E)$ be an open subgroup stable under Σ, and with prime-to-ℓ pro-order. Then for any $\varepsilon \in \mathbf{Z}/2\mathbf{Z}$ and each non-empty finite set I, there is a natural isomorphism*

$$\mathrm{T}^{\varepsilon}(\mathrm{Sht}_{(G,1,I),K}; \mathrm{Nm}^{(\ell^{-1})}(V)) \cong \bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^{\varepsilon}(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; {}^L\psi^*(V)) \tag{8.9}$$

of functors

$$\mathrm{Rep}_k((\widehat{G} \rtimes Q)^I) \to D(\mathrm{Rep}_k^{\mathrm{sm}} H(E))^{B \prod_{i \in I} W_E}.$$

Moreover, these natural isomorphisms are compatible with fusion along all maps of non-empty finite sets $\zeta \colon I \to J$.

The meaning of the last sentence is as follows. As explained in §8.1.3, any such $\zeta \colon I \to J$ induces a restriction functor $\mathrm{Res}_\zeta \colon \mathrm{Rep}_k((\widehat{G} \rtimes Q)^I) \to \mathrm{Rep}_k((\widehat{G} \rtimes Q)^J)$. The natural isomorphism (8.2) induces a natural isomorphism

$$\mathrm{T}^{\varepsilon}(\mathrm{Sht}_{(G,1,I),K}; W) \cong \mathrm{T}^{\varepsilon}(\mathrm{Sht}_{(G,1,J),K}; \mathrm{Res}_\zeta W) \tag{8.10}$$

compatible with compositions of maps of finite sets, and similarly for each $b' \in \iota^{-1}(1_G)$ a natural isomorphism

$$\mathrm{T}^{\varepsilon}(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; {}^L\psi^*(W)) \cong \mathrm{T}^{\varepsilon}(\mathrm{Sht}^{b'}_{(H,1_H,J),\iota^* K}; {}^L\psi^*(\mathrm{Res}_\zeta W)) \tag{8.11}$$

compatible with compositions of maps of finite sets. Then "compatible with fusion" means that for every $\zeta\colon I \to J$, the diagram below commutes:

$$\begin{tikzcd} \mathrm{T}^{\varepsilon}(\mathrm{Sht}_{(G,1,I),K};\mathrm{Nm}^{(\ell^{-1})}(V)) \arrow[d, "(8.10)"', "\sim"] & \bigoplus_{b'\in\iota^{-1}(1_G)} \mathrm{T}^{\varepsilon}(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K};{}^L\psi^*(V)) \arrow[l, "\sim"', "(8.9)"] \arrow[d, "(8.11)"', "\sim"] \\ \mathrm{T}^{\varepsilon}(\mathrm{Sht}_{(G,1,J),K};\mathrm{Nm}^{(\ell^{-1})}(\mathrm{Res}_{\zeta} V)) & \bigoplus_{b'\in\iota^{-1}(1_G)} \mathrm{T}^{\varepsilon}(\mathrm{Sht}^{b'}_{(H,1_H,J),\iota^*K};{}^L\psi^*(\mathrm{Res}_{\zeta} V)) \arrow[l, "\sim"', "(8.9)"] \end{tikzcd}$$

Proof. For each $b' \in \iota^{-1}(1_G) \subset B(H)$, the commutative diagram

$$\begin{tikzcd} \mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K} \arrow[r, "\pi^H_{\iota^*K}"] \arrow[d, "\iota"] & \mathrm{Gr}^{\mathrm{tw}}_{H,\prod_{i\in I}\mathrm{Spd}\,\breve{E}} \arrow[d, "\iota"] \\ \mathrm{Sht}_{(G,b,I),K} \arrow[r, "\pi^G_K"] & \mathrm{Gr}^{\mathrm{tw}}_{G,\prod_{i\in I}\mathrm{Spd}\,\breve{E}} \end{tikzcd}$$

induces a natural isomorphism $\iota^*(\pi^G_K)^* \cong (\pi^H_{\iota^*K})^*\iota^*$, hence natural isomorphisms

$$\mathbb{T}^*\iota^*(\pi^G_K)^*\left(\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}\right) \cong \mathbb{T}^*(\pi^H_{\iota^*K})^*\iota^*\left(\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}\right) \cong (\pi^H_{\iota^*K})^*\mathbb{T}^*\iota^*\left(\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}\right) \in \mathrm{Shv}(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K};\mathcal{T}_k), \tag{8.12}$$

compatible with fusion. Recalling that $\mathrm{Psm} = \mathbb{T}^*\iota^*$, Theorem 7.7.1 supplies natural isomorphisms

$$(\pi^H_{\iota^*K})^*\mathbb{T}^*\iota^*\left(\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}\right) \cong (\pi^H_{\iota^*K})^*\mathbb{T}^*\left(\mathcal{S}_{{}^L\psi^*(V)}\right) \in \mathrm{Shv}(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K};\mathcal{T}_k), \tag{8.13}$$

compatible with fusion.

We now consider Tate cohomology of the moduli of shtukas. To distinguish between G and H, we write

$$f^G_K\colon \mathrm{Sht}_{(G,1,I),K} \to \prod_{i\in I}\mathrm{Spd}\,\breve{E},$$
$$f^{H,b'}_K\colon \mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K} \to \prod_{i\in I}\mathrm{Spd}\,\breve{E}.$$

Note that support of any Satake sheaf on $\mathrm{Sht}_{(G,1,I),K}$ has finite dim. trg over $\prod \mathrm{Spd}\,\breve{E}$ and is extra small (cf. Example 3.1.4), and similarly for $\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K}$. As $\mathrm{Fix}(\sigma, \mathrm{Sht}_{(G,b,I),K})$ is a finite union of spaces of the form $\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^*K}$ by Proposition 4.4.7, the results of §3 apply. In particular, we may apply Proposition 3.6.2, which relates the Tate cohomology of $\mathrm{Sht}_{(G,1,I),K}$ to that of its fixed points, and Proposition 4.4.7, which identifies these fixed points; combining these with (8.12) and (8.13) gives a sequence of natural isomorphisms

$$\begin{aligned} & \mathbb{T}^*\mathrm{R}(f^G_K)_!((\pi^G_K)^*\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}) \\ \text{Propositions 3.6.2 \& 4.4.7} \implies & \cong \bigoplus_{b'\in\iota^{-1}(1_G)} \mathrm{R}(f^{H,b'}_K)_!\mathbb{T}^*\iota^*((\pi^G_K)^*\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}) \\ (8.12) \implies & \cong \bigoplus_{b'\in\iota^{-1}(1_G)} \mathrm{R}(f^{H,b'}_K)_!(\pi^H_{\iota^*K})^*\mathbb{T}^*\iota^*\left(\mathcal{S}_{\mathrm{Nm}^{(\ell^{-1})}(V)}\right) \\ (8.13) \implies & \cong \bigoplus_{b'\in\iota^{-1}(1_G)} \mathrm{R}(f^{H,b'}_K)_!(\pi^H_{\iota^*K})^*\mathbb{T}^*\left(\mathcal{S}_{{}^L\psi^*(V)}\right), \end{aligned}$$

compatible with fusion. The result then follows upon applying the Tate cohomology functor $\mathrm{T}^{\varepsilon}(-)$. □

In particular, when $V = \mathbb{1}$ is the trivial representation we have an identification

$$\mathrm{T}^0(\mathrm{Sht}_{(G,1,\{0\}),K};\mathbb{1}) \cong \bigoplus_{b'\in\iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,\{0\}),\iota^*K};\mathbb{1}) \tag{8.14}$$

Note that the $H_{b'}$ for $b' \in \iota^{-1}(1_G)$ are inner twists of each other, so their L-groups ${}^LH_{b'}$ are all canonically identified, hence we may identify each of their excursion algebras with $\mathrm{Exc}_k(W_E, \widehat{H})$.

Theorem 8.6.2. *Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Let $K < G(E)$ be a prime-to-ℓ compact open subgroup stable under Σ. Then the isomorphism* (8.14) *is equivariant for the action of* $\mathrm{Exc}_k(W_E, \widehat{G})$*, acting on the LHS via the normed excursion action (§8.5.3) and on the RHS through the homomorphism* ${}^L\psi^* \colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Exc}_k(W_E, \widehat{H})$ *from* (8.3) *followed by the native excursion action (§8.5.4).*

Proof. Let $\mathcal{D} = (I, V, \alpha, \beta, (\gamma_i)_{i\in I})$ be an excursion datum for $\widehat{G}$. By the definition of the excursion action (cf. (8.5)), what we have to show is that the action of $\mathrm{Nm}^{(\ell^{-1})} S_{\mathcal{D}} \in \mathrm{Exc}_k(W_E, \widehat{G})^{\sigma}$ on $\mathrm{T}^0(\mathrm{Sht}_{(G,1,\{0\}),K}; \mathbb{1})$ is intertwined with the action of $S_{{}^L\psi^*(\mathcal{D})} \in \mathrm{Exc}(W_E, {}^L H)$ on $\bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,\{0\}),\iota^* K}; \mathbb{1})$ under the identification (8.14).

By Lemma 8.5.5 we have $\mathrm{Nm}^{(\ell^{-1})} S_{\mathcal{D}} = S_{\mathrm{Nm}^{(\ell^{-1})}\mathcal{D}}$, whose excursion action is the composition in the left column in the diagram below:

$$\begin{tikzcd}
\mathrm{T}^0(\mathrm{Sht}_{(G,1,I),K}; \mathbb{1}) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Nm}^{(\ell^{-1})}(\alpha)"] & \bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; \mathbb{1}) \arrow[d, "{}^L\psi^*(\alpha)"] \\
\mathrm{T}^0(\mathrm{Sht}_{(G,1,I),K}; \mathrm{Nm}^{(\ell^{-1})}(V)) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "(\gamma_i)_{i\in I}"] & \bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; {}^L\psi^*(V)) \arrow[d, "(\gamma_i)_{i\in I}"] \\
\mathrm{T}^0(\mathrm{Sht}_{(G,1,I),K}; \mathrm{Nm}^{(\ell^{-1})}(V)) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Nm}^{(\ell^{-1})}(\beta)"] & \bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; {}^L\psi^*(V)) \arrow[d, "{}^L\psi^*(\beta)"] \\
\mathrm{T}^0(\mathrm{Sht}_{(G,1,I),K}; \mathbb{1}) \arrow[r, leftrightarrow, "\sim"] & \bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; \mathbb{1})
\end{tikzcd} \tag{8.15}$$

Meanwhile, $S_{{}^L\psi^*(\mathcal{D})}$ is the composition in the right column in the diagram. Proposition 8.6.1 establishes the horizontal identifications making all diagrams commutes, which gives the result. ☐

9. Derived Treumann-Venkatesh Conjecture

In this section we will reap the applications of the preceding material, especially the functoriality relations for excursion operators from Theorem 8.6.2. In §9.1 we formulate the "derived" version of Treumann-Venkatesh's Conjecture 1.2.3(2), and prove it in §9.4. In §9.2 we will construct the Treumann-Venkatesh homomorphism alluded to in (1.6), and in §9.3 we establish the commutative square (1.7). Finally, in §9.5 we prove the existence of functorial lifts along σ-dual homomorphisms, including Theorem 1.4.1.

9.1. **Formulation.** By the realization of the Bernstein center $\mathfrak{Z}(G)$ as the ring of endomorphisms of the identity functor on $D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$, there is a tautological action of $\mathfrak{Z}(G)$ on any $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$. Composing this with the Fargues-Scholze map $\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathfrak{Z}(G; k)$ discussed in §8.4, we obtain an action of $\mathrm{Exc}_k(W_E, \widehat{G})$ on any such Π. In particular, to each Π we may attach a subset

$$\mathrm{supp}_{\mathrm{Exc}_k(W_E, \widehat{G})}(\mathrm{H}^*(\Pi)) \subset \mathrm{Spec}\, \mathrm{Exc}_k(W_E, \widehat{G}),$$

which can be interpreted as the "set of semi-simple L-parameters attached to Π". For example, if $\Pi \in \mathrm{Rep}_k^{\mathrm{sm}} G(E)$ is irreducible admissible, then $\mathrm{supp}_{\mathrm{Exc}_k(W_E, \widehat{G})}(\Pi)$ is necessarily a single closed point, corresponding to the *Fargues-Scholze parameter* $\rho_\Pi \in H^1(W_E; \widehat{G}(k))$. This defines the map (1.2).

Let $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}}(G(E) \rtimes \Sigma)) = D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))^{B\Sigma}$. Then we may form the Tate cohomology $\mathrm{T}^j(\Pi)$, which is naturally an object of $\mathrm{Rep}_k^{\mathrm{sm}} H(E)$.

Theorem 9.1.1. *Let $F \colon \mathrm{Exc}_k(W_E, \widehat{G}) \to \mathrm{Exc}_k(W_E, \widehat{G})$ be the linearized Frobenius (cf. §8.5.4), which is a k-algebra homomorphism. This induces a functor F_* on $\mathrm{Exc}_k(W_E, \widehat{G})$-modules.*

Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Then for any $j \in \mathbf{Z}/2\mathbf{Z}$, $\mathrm{supp}_{\mathrm{Exc}_k(W_E, \widehat{H})}(\mathrm{T}^j(\Pi))$ *lies over* $F_* \,\mathrm{supp}_{\mathrm{Exc}_k(W_E, \widehat{G})}(\mathrm{H}^*(\Pi))$ *with respect to the map* $\mathrm{Spec}\, \mathrm{Exc}_k(W_E, \widehat{H}) \xrightarrow{{}^L\psi_*} \mathrm{Spec}\, \mathrm{Exc}_k(W_E, \widehat{G})$ *induced by the σ-dual homomorphism*

${}^L\psi\colon {}^LH \to {}^LG$. *In other words, there is a commutative diagram*

$$\begin{tikzcd} \mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{H})}(\mathrm{T}^j(\Pi)) \arrow[r, hook] \arrow[d] & \mathrm{Spec}\ \mathrm{Exc}_k(W_E,\widehat{H}) \arrow[d, "{}^L\psi_*"] \\ F_*\,\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\Pi)) \arrow[r, hook] & \mathrm{Spec}\ \mathrm{Exc}_k(W_E,\widehat{G}) \end{tikzcd}$$

Example 9.1.2 (Treumann-Venkatesh functoriality conjecture)**.** If Π is an irreducible smooth representation of $G(E)$, then $\mathrm{Exc}_k(W_E,\widehat{G})$ acts on Π through a character χ_Π. The composition $\chi_\Pi \circ F$ is the character of $\mathrm{Exc}_k(W_E,\widehat{G})$ associated to the Frobenius twist $\Pi^{(\ell)} := \Pi \otimes_{k,\mathrm{Frob}_\ell} k$, so we have

$$F_*\,\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\Pi) = \mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(F_*\Pi) = \{\chi_{\Pi^{(\ell)}}\}.$$

Suppose furthermore that $\Pi \cong {}^\sigma\Pi \in \mathrm{Rep}_k^{\mathrm{sm}}\, G(E)$. Then by [TV16, Proposition 6.1], the $G(E)$-action on Π extends uniquely to a $G(E) \rtimes \Sigma$-action. Then Theorem 9.1.1 implies, assuming $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$, that for each $j \in \mathbf{Z}/2\mathbf{Z}$ and each irreducible $H(E)$-subquotient π of $\mathrm{T}^i(\Pi)$, the semi-simple L-parameter $\rho_{\Pi^{(\ell)}} \in \mathrm{H}^1(W_E; \widehat{G}(k))$ is the image of the semi-simple L-parameter ρ_π under the map $\mathrm{H}^1(W_E;\widehat{H}(k)) \xrightarrow{{}^L\psi_*} \mathrm{H}^1(W_E;\widehat{G}(k))$. This implies Theorem 1.3.1.

Example 9.1.3. Suppose σ is an *inner* automorphism, induced by conjugation by $s \in G(E)$ (which therefore has to be an ℓ-torsion element). Then we have an isomorphism $G(E) \rtimes \Sigma \xrightarrow{\sim} G(E) \times \Sigma$ sending $(g,\sigma) \mapsto (gs^{-1},\sigma)$. Hence we get a functor

$$D^b(\mathrm{Rep}_k^{\mathrm{sm}}\, G(E)) \to D^b(\mathrm{Rep}_k^{\mathrm{sm}}(G(E)\times\Sigma)) \xrightarrow{\sim} D^b(\mathrm{Rep}_k^{\mathrm{sm}}(G(E)\rtimes\Sigma)).$$

This equips every representation $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}}\, G(E))$ with a canonical Σ-equivariant structure, where σ acts by $\Pi(s)$.

9.2. **The Treumann-Venkatesh homomorphism.** We next establish some technical lemmas which aid to study the properties of the Brauer homomorphism.

Proposition 9.2.1. *Let $K \subset G$ be a Σ-stable compact open subgroup with prime-to-ℓ pro-order. Then the natural map*

$$\iota^*K\backslash H(E)/\iota^*K \to K\backslash G(E)/K$$

is injective.

Proof. Let $a, b \in \iota^*K\backslash H(E)$ be two elements whose images in $K\backslash G(E)$ lie in the same orbit for the right translation of K. In other words, $a = b\kappa$ for some $\kappa \in K$. Applying σ to this equation, and using that a, b are fixed by σ, we obtain $a = b\sigma(\kappa)$, so $\sigma(\kappa)\kappa^{-1} \in \mathrm{Stab}_K(b)$. Since Σ is of order ℓ while K is prime-to-ℓ, we have $\mathrm{H}^1(\Sigma; \mathrm{Stab}_K(b)) = 0$. Hence there exists $y \in \mathrm{Stab}_K(b)$ such that $\sigma(\kappa)\kappa^{-1} = \sigma(y)y^{-1}$. Then $y^{-1}\kappa$ is fixed by σ, so $y^{-1}\kappa \in H(E) \cap K = \iota^*K$. But then

$$a = b\kappa = (by^{-1})\kappa = b(y^{-1}\kappa),$$

which shows that a and b lie in the same orbit for the right translation of ι^*K on $\iota^*K\backslash H(E)$. ☐

Recall that we defined a compact open subgroup $K \subset G(E)$ to be *plain* if the natural map $H(E)/\iota^*K \to [G(E)/K]^\sigma$ is an isomorphism. By a similar argument involving the vanishing of non-abelian cohomology (cf. [Fen24, Lemma 6.6]), any prime-to-ℓ subgroup $K \subset G(E)$ is plain, so the (un-normalized) Brauer homomorphism (cf. §7.1.1) $\mathrm{Br}\colon \mathscr{H}(G,K)^\sigma \to \mathscr{H}(H,\iota^*K)$ is defined for any such K.

Corollary 9.2.2. *Let $K \subset G$ be a Σ-stable prime-to-ℓ compact open subgroup. Then* Br *induces a map of centers,*

$$\mathrm{Br}\colon Z(\mathscr{H}(G,K)^\sigma) \to Z(\mathscr{H}(H,\iota^*K)). \tag{9.1}$$

Proof. Proposition 9.2.1 implies that $\mathrm{Br}\colon \mathscr{H}(G,K)^\sigma \to \mathscr{H}(H,\iota^*K)$ is a surjective k-algebra homomorphism, so it maps the center to the center. ☐

It is evident from the definition that the map (9.1) factors over the quotient $Z(\mathscr{H}(G,K)^\sigma)/N \cdot \mathfrak{Z}(G,K)$, which induces a map

$$\mathrm{T}^0\mathfrak{Z}(G,K) = \frac{\mathfrak{Z}(G,K)^\sigma}{N \cdot \mathfrak{Z}(G,K)} \hookrightarrow \frac{Z(\mathscr{H}(G,K)^\sigma)}{N \cdot \mathfrak{Z}(G,K)} \to \mathfrak{Z}(H, \iota^* K). \tag{9.2}$$

Note that we have a natural $\mathbf{F}_\ell$-structure on $\mathfrak{Z}(G,K;k)$ given by

$$\mathfrak{Z}(G,K;k) = \mathfrak{Z}(G,K;\mathbf{F}_\ell) \otimes_{\mathbf{F}_\ell} k. \tag{9.3}$$

The following is a "normalized" (cf. §7.1.3) version of (9.1), in the sense of the normalized Brauer homomorphism (7.4).

Definition 9.2.3. Let $K \subset G$ be a Σ-stable prime-to-ℓ compact open subgroup. We define *Treumann-Venkatesh homomorphism*

$$\mathfrak{Z}_{\mathrm{TV},K} \colon \mathfrak{Z}(G,K) \to \mathfrak{Z}(H, \iota^* K)$$

to be the composition of (9.2) with the map $\mathrm{Nm}^{(\ell^{-1})} \colon \mathfrak{Z}(G,K) \to \mathrm{T}^0(\mathfrak{Z}(G,K))$ from Definition 7.1.1, with respect to the $\mathbf{F}_\ell$-structure (9.3).

9.3. **Modular functoriality for Bernstein centers.** For an inclusion $K \subset K' \subset G(E)$ of Σ-stable prime-to-ℓ compact open subgroups, we have the map $e_G^{K \to K'} \colon \mathfrak{Z}(G,K) \to \mathfrak{Z}(G,K')$ given by convolution with $\mathbb{1}_{K'}$. Similarly, we have $e_H^{K\to K'} \colon \mathfrak{Z}(H,\iota^*K) \to \mathfrak{Z}(H, \iota^*K')$ given by convolution with $\mathbb{1}_{\iota^*K'}$. The diagram

$$\begin{tikzcd} \mathfrak{Z}(G,K) \arrow[r, "\mathfrak{Z}_{\mathrm{TV},K}"] \arrow[d, "e_G^{K\to K'}"] & \mathfrak{Z}(H,\iota^*K) \arrow[d, "e_H^{K\to K'}"] \\ \mathfrak{Z}(G,K') \arrow[r, "\mathfrak{Z}_{\mathrm{TV},K'}"] & \mathfrak{Z}(H,\iota^*K') \end{tikzcd}$$

commutes.

Definition 9.3.1 (Base change homomorphism for Bernstein centers)**.** We define the *Treumann-Venkatesh homomorphism* (for Bernstein centers) $\mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G) \to \mathfrak{Z}(H)$ as

$$\varprojlim_K \mathfrak{Z}_{\mathrm{TV},K} \colon \varprojlim_K \mathfrak{Z}(G,K) \to \varprojlim_K \mathfrak{Z}(H,\iota^*K)$$

where the limit is taken over Σ-stable prime-to-ℓ compact open subgroups $K \subset G(E)$.

Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Recall that in Corollary 7.6.2 we have constructed the σ-dual homomorphism ${}^L\psi \colon {}^LH \to {}^LG$ over k.

Theorem 9.3.2. *Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Exc}_k(W_E, \widehat{G}) \arrow[r, "{}^L\psi^*"] \arrow[d, "\mathrm{FS}_G"'] & \mathrm{Exc}_k(W_E, \widehat{H}) \arrow[d, "\mathrm{FS}_H"] \\ \mathfrak{Z}(G) \arrow[r, "\mathfrak{Z}_{\mathrm{TV}}"] & \mathfrak{Z}(H) \end{tikzcd} \tag{9.4}$$

Note that if K is a plain subgroup of $G(E)$, then we have

$$\mathrm{T}^0(\text{c-Ind}_K^{G(E)} k) \cong \text{c-Ind}_{\iota^*K}^{H(E)} k \in \mathrm{Rep}_k^{\mathrm{sm}} H(E). \tag{9.5}$$

(This is a special case of [BFH+, Proposition 8.12]; more general results on the interaction of Tate cohomology with compact induction are discussed later in §10.4.)

In preparation for the proof of Theorem 9.3.2, we record the following interpretation of the Brauer homomorphism.

Lemma 9.3.3. *Under the identifications $\mathscr{H}(G,K) \cong \mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)} k)$ and $\mathscr{H}(H,\iota^*K) \cong \mathrm{End}_{H(E)}(\text{c-Ind}_{\iota^*K}^{H(E)} k)$, the map $\mathrm{Br} \colon \mathrm{T}^0\mathscr{H}(G,K) \to \mathscr{H}(H,\iota^*K)$ induced by the Brauer homomorphism is identified with the map*

$$\mathrm{T}^0 \,\mathrm{End}_{G(E)}(\text{c-Ind}_K^{G(E)} k) \to \mathrm{End}_{H(E)}(\mathrm{T}^0(\text{c-Ind}_K^{G(E)} k)) \overset{(9.5)}{\cong} \mathrm{End}_{H(E)}(\text{c-Ind}_{\iota^*K}^{H(E)} k)$$

sending a Σ-equivariant endomorphism of $\text{c-Ind}_K^{G(E)} k$ *to the induced endomorphism on its Tate cohomology.*

Proof. Apply [Fen24, Lemma 6.7] (which comes from [TV16, §6.2]) with $\Pi := \text{c-Ind}_K^{G(E)} k$. $\square$

Proof of Theorem 9.3.2. For any prime-to-ℓ, Σ-stable compact open subgroup $K \subset G(E)$, we also denote by

$$\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathfrak{Z}(G, K)$$

the composition of FS_G with the projection $\mathfrak{Z}(G) \to \mathfrak{Z}(G, K)$. Applying Tate cohomology, this induces

$$\mathrm{T}^0 \,\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathrm{T}^0 \mathfrak{Z}(G, K). \tag{9.6}$$

We have identifications

$$\begin{tikzcd}
\mathrm{T}^0(\text{c-Ind}_K^{G(E)} k) \arrow[r, "\text{Ex. } 8.5.1", "\sim"'] & \mathrm{T}^0(\mathrm{Sht}_{(G,1_G,\{0\}),K}; \mathbb{1}) \arrow[r, "(8.14)", "\sim"'] & \bigoplus_{b' \in \iota^{-1}(1_G)} \mathrm{T}^0(\mathrm{Sht}^{b'}_{(H,1_H,I),\iota^* K}; \mathbb{1}) \arrow[d, "\text{Ex. } 8.5.1"', "\sim"] \\
& \text{c-Ind}_{\iota^* K}^{H(E)} k & \mathrm{T}^0(\text{c-Ind}_{\iota^* K}^{H(E)} k) \arrow[l, "\sim", "\text{Ex. } 3.7.4"']
\end{tikzcd}$$

and for any excursion datum $\mathcal{D}$ for $\widehat{G}$, Theorem 8.6.2 implies that these identifications intertwine

$$\begin{pmatrix} \text{the action of } S_{\mathrm{Nm}^{(\ell^{-1})} \mathcal{D}} \\ \text{on } \mathrm{T}^0(\text{c-Ind}_K^{G(E)} k) \end{pmatrix} = \begin{pmatrix} \text{the action of } S_{{}^L\psi^* \mathcal{D}} \\ \text{on c-Ind}_{\iota^* K}^{H(E)} k \end{pmatrix}. \tag{9.7}$$

The diagram

$$\begin{tikzcd}
\mathrm{Exc}_k(W_E, \widehat{G}) \arrow[r, "\mathrm{Nm}^{(\ell^{-1})}"] \arrow[d, "\mathrm{FS}_G"] & \mathrm{T}^0 \,\mathrm{Exc}_k(W_E, \widehat{G}) \arrow[d, "(9.6)"] \\
\mathfrak{Z}(G, K) \arrow[r, "\mathrm{Nm}^{(\ell^{-1})}"] & \mathrm{T}^0 \mathfrak{Z}(G, K)
\end{tikzcd} \tag{9.8}$$

commutes by the definition of Nm, and the fact that FS_G is defined over $\mathbf{F}_\ell$ (with respect to the $\mathbf{F}_\ell$-structures used to linearize Nm on each row). Therefore, unraveling the definition of $\mathfrak{Z}_{\mathrm{TV}}$ and using Lemma 8.5.5, it suffices to show that

$$\mathrm{Br}(\mathrm{FS}_G(S_{\mathrm{Nm}^{(\ell^{-1})} \mathcal{D}})) = \mathrm{FS}_H(S_{{}^L\psi^* \mathcal{D}}) \tag{9.9}$$

for all excursion data $\mathcal{D}$ for $\widehat{G}$. By Lemma 9.3.3, $\mathrm{Br}(\mathrm{FS}_G(S_{\mathrm{Nm}^{(\ell^{-1})} \mathcal{D}}))$ is the endomorphism of $\mathrm{T}^0(\text{c-Ind}_{\iota^* K}^{H(E)} k) = \text{c-Ind}_{\iota^* K}^{H(E)} k$ obtained by taking T^0 of the action of $S_{\mathrm{Nm}^{(\ell^{-1})} \mathcal{D}}$ on $\text{c-Ind}_K^{G(E)} k$, which according to (9.7) is the endomorphism given by $\mathrm{FS}_H(S_{{}^L\psi^* \mathcal{D}})$. $\square$

Corollary 9.3.4. *Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Then for any irreducible $H(E)$-representation π, the character*

$$\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G) \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} \mathfrak{Z}(H) \xrightarrow{\chi_\pi} k$$

has the property that for any irreducible $G(E)$-representation Π on which $\mathfrak{Z}(G)$ acts through $\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}}$, there is an isomorphism of semi-simple L-parameters $\rho_\Pi \cong {}^L\psi \circ \rho_\pi$.

9.4. **Proof of Theorem 9.1.1.** If $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$, it was seen in the proof of Theorem 9.3.2 that for any Σ-stable open compact subgroup $K < G(E)$, the normed action (cf. §8.5.3) of $\mathrm{Exc}_k(W_E, \widehat{G})$ on $\mathrm{T}^j(\text{c-Ind}_K^{G(E)} k)$ coincides under the identification (9.5) with the action obtained by composing $\mathrm{Exc}_k(W_E, \widehat{G}) \xrightarrow{\mathrm{FS}_G} \mathfrak{Z}(G) \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} \mathfrak{Z}(H)$ with the tautological action of $\mathfrak{Z}(H)$ on $\text{c-Ind}_{\iota^* K}^{H(E)} k$. Therefore, by Theorem 9.3.2, for any j the map ${}^L\psi_* \colon \mathrm{Spec}\,\mathrm{Exc}_k(W_E, \widehat{H}) \to \mathrm{Spec}\,\mathrm{Exc}_k(W_E, \widehat{G})$ carries

$$\mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E, \widehat{H})} \mathrm{T}^j(\Pi) \xrightarrow{{}^L\psi_*} \mathrm{supp}^{\mathrm{normed}}_{\mathrm{Exc}_k(W_E, \widehat{G})} \mathrm{T}^j(\Pi), \tag{9.10}$$

where the native excursion action is defined in §8.5.4.

We will compare the support of the normed and native actions of $\mathrm{Exc}_k(W_E, \widehat{G})^\sigma$ on $\mathrm{T}^j(\Pi)$. By the discussion in §8.5.3, we have for any $j \in \mathbf{Z}/2\mathbf{Z}$ an equality

$$\mathrm{supp}^{\mathrm{normed}}_{\mathrm{Exc}_k(W_E, \widehat{G})^\sigma}(\mathrm{T}^j(\Pi)) = F_* \,\mathrm{supp}^{\mathrm{native}}_{\mathrm{Exc}_k(W_E, \widehat{G})^\sigma}(\mathrm{T}^j(\Pi)) \subset \mathrm{Spec}\,\mathrm{Exc}_k(W_E, \widehat{G})^\sigma. \tag{9.11}$$

By the Tate spectral sequence $\mathrm{T}^j(\mathrm{H}^i(\Pi)) \implies \mathrm{T}^{i+j}(\Pi)$, each $\mathrm{T}^n(\Pi)$ has a finite filtration each of whose graded pieces is a subquotient of $\mathrm{H}^*(\Pi)$, hence we have

$$F_* \operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{T}^j(\Pi)) \subset F_* \operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{H}^*(\Pi)) = \operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(F_*\mathrm{H}^*(\Pi)). \tag{9.12}$$

Let $q\colon \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G}) \to \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G})^\sigma$ be the map of spectra induced by the obvious inclusion. Combining (9.11) and (9.12) with (9.10) yields

$$q \circ {}^L\psi_* \left(\operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi)\right) \subset F_* \operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})^\sigma}(\mathrm{H}^*(\Pi)). \tag{9.13}$$

The surjection $\mathrm{Exc}_k(W_E,\widehat{G})^\sigma \twoheadrightarrow \mathrm{T}^0\,\mathrm{Exc}_k(W_E,\widehat{G})$ induces a closed embedding on spectra, as in the diagram

$$\begin{tikzcd} & \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G}) \arrow[d, "q"] \\ \operatorname{Spec} \mathrm{T}^0\,\mathrm{Exc}_k(W_E,\widehat{G}) \arrow[r, hook] & \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G})^\sigma \end{tikzcd}$$

Now, [Fen24, Lemma 5.15] says that any character from $\mathrm{T}^0\,\mathrm{Exc}_k(W_E,\widehat{G})$ to any perfect field has a unique extension to $\mathrm{Exc}_k(W_E,\widehat{G})$, which implies that the map $\operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G}) \to \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G})^\sigma$ is one-to-one (and surjective) over the closed subscheme $\operatorname{Spec} \mathrm{T}^0\,\mathrm{Exc}_k(W_E,\widehat{G}) \subset \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G})^\sigma$. From Theorem 8.6.2, we see that $q \circ {}^L\psi_* \left(\operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi)\right)$ lands inside this subspace, so we may lift (9.13) to the containment

$${}^L\psi_* \left(\operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{H})} \mathrm{T}^j(\Pi)\right) \subset F_* \operatorname{supp}^{\text{native}}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\Pi), \tag{9.14}$$

as desired.

□

9.5. **Existence of functorial lifts.** We generalize the proof of [Fen24, Theorem 6.26] to prove the existence of functorial liftings along any σ-dual homomorphism.

Theorem 9.5.1. *Assume $\ell > \max\{b(\widehat{G}), b(\widehat{H})\}$. Let π be an irreducible representation of $H(E)$ over k, with Fargues-Scholze parameter $\rho_\pi \in \mathrm{H}^1(W_E; \widehat{H}(k))$. Then there is an irreducible representation Π of $G(E)$ over k such that $\rho_\Pi \cong {}^L\psi \circ \rho_\pi \in \mathrm{H}^1(W_E; \widehat{G}(k))$.*

Proof. Choose a Σ-stable pro-p compact open subgroup $K < G(E)$ such that $\pi^{\iota^* K} \neq 0$. From Theorem 9.3.2 we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathfrak{Z}(H,\iota^*K) \arrow[r, "\operatorname{Spec} \mathfrak{Z}_{\mathrm{TV}}"] \arrow[d, "\operatorname{Spec} \mathrm{FS}_H"] & \operatorname{Spec} \mathfrak{Z}(G,K) \arrow[d, "\operatorname{Spec} \mathrm{FS}_G"] \\ \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{H}) \arrow[r, "{}^L\psi_*"] & \operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G}) \end{tikzcd} \tag{9.15}$$

The representation π gives a k-point of $\operatorname{Spec} \mathfrak{Z}(H,\iota^*K)$ lying over ρ_π viewed as a k-point of $\operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{H})$. The commutativity of the diagram then tells us that there is a k-point of $\operatorname{Spec} \mathfrak{Z}(G,K)$, say with maximal ideal $\mathfrak{m}$, lying over the k-point of $\operatorname{Spec} \mathrm{Exc}_k(W_E,\widehat{G})$ corresponding to ${}^L\psi \circ \rho_\pi$.

Recall that the functor $\Pi \mapsto \Pi^K$ induces a bijection between irreducible admissible $G(E)$-representations with non-zero K-invariants and irreducible $\mathscr{H}(G,K)$-modules. It therefore suffices to construct an irreducible representation of $\mathscr{H}(G,K)$ on which the $\mathfrak{Z}(G,K)$-action factors over $\mathfrak{m}$.

By a result of Dat-Helm-Kurinczuk-Moss [DHKM24, Theorem 1.1] $\mathscr{H}(G,K)$ is a finite $\mathfrak{Z}(G,K)$-module. By the Artin-Tate Lemma, $\mathfrak{Z}(G,K)$ is finite over $\mathfrak{Z}(G,K)^\sigma$ and then $\mathscr{H}(G,K)$ is finite over $\mathfrak{Z}(G,K)^\sigma$. The localization of $\mathscr{H}(G,K)$ at $\mathfrak{m}$ is non-zero, since $\mathfrak{Z}(G,K) \hookrightarrow \mathscr{H}(G,K)$, so Nakayama's Lemma implies that the left $\mathscr{H}(G,K)$-module quotient $\mathscr{H}(G,K)/\mathscr{H}(G,K)\mathfrak{m}$ is finite-dimensional and non-zero. By design, it is supported over $\mathfrak{Z}(G,K)/\mathfrak{m}$, so it has an irreducible $\mathscr{H}(G,K)$-subquotient on which the $\mathfrak{Z}(G,K)$-action factors over $\mathfrak{m}$, as was to be showed.

□

10. Fargues-Scholze parameters of toral supercuspidals

In this section we will demonstrate how the preceding theory may be used to calculate the Fargues-Scholze parameters attached to explicitly constructed representations $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$. We will apply Theorem 9.1.1 with the automorphism σ being conjugation by a strongly regular ℓ-torsion element $s \in G(E)$. Then H is a *torus* $T < G$, so the Fargues-Scholze correspondence is completely understood for H. Using Theorem 9.1.1, we can describe $\rho_\Pi \in \mathrm{H}^1(W_E; \widehat{G}(k))$ in terms of $\mathrm{T}^j(\Pi)$ and the σ-dual homomorphism ${}^L\psi$.

We will focus our attention on $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$ related to the "(Howe-unramified) toral supercuspidals" studied by Chan-Oi [CO]. (The method has broader scope but we regard this as a sufficiently interesting proof-of-concept for now.) Then T must be taken to be an unramified elliptic torus, and we compute the σ-dual embedding in §10.2: it turns out to be the canonical L-embedding corresponding to an unramified maximal torus.

The relevant Π arise as compact inductions of representations cut out from the cohomology of so-called "deep level Deligne-Lusztig varieties" studied by Chan-Ivanov [CI21]. In §10.3 we use equivariant localization to calculate the Tate cohomology of these deep level Deligne-Lusztig varieties. We feed the answer into §10.4 in order to compute the Tate cohomology of the compact inductions, and then tie together the calculations to describe the Fargues-Scholze parameters explicitly in Theorem 10.4.1.

10.1. **Review of tori.** Let G be a connected reductive group over E.

10.1.1. *The abstract Cartan.* To G we can associate a canonical E-torus $\mathbf{T}$, which we call the "abstract Cartan" of G. (We make no claim that $\mathbf{T}$ admits an E-rational embedding into G.) If G is quasi-split, thne $\mathbf{T}$ is defined as the colimit over E-rational Borel subgroups $B < G$ of B/U, where U is the unipotent radical of B. In general, we pass to an extension of G where it becomes quasisplit, and then descend this construction.

Given an extension E'/E and a Borus (B, T) over $G_{E'}$, the composition

$$T \hookrightarrow B \twoheadrightarrow B/U \to \mathbf{T}_{E'}$$

defines an isomorphism $i_B \colon T \xrightarrow{\sim} \mathbf{T}_{E'}$.

We denote by $\mathbf{W}$ the Weyl group of $\mathbf{T}$, defined as the colimit of $N_{G_{E^s}}(T)/T$ over the category of Bori (B, T) in G_{E^s}.

10.1.2. *The cocycle of a torus.* Let $T \subset G$ be a maximal torus over E. Over E^s, we can find a Borus (B, T_{E^s}) inside G_{E^s}, which gives an identification $i_B \colon T_{E^s} \xrightarrow{\sim} \mathbf{T}_{E^s}$.

For $\gamma \in \mathrm{Gal}(E^s/E)$, γB is another Borel subgroup of G containing T_{E^s}, so we have another identification $i_{\gamma B} \colon T_{E^s} \xrightarrow{\sim} \mathbf{T}_{E^s}$.

For $\gamma \in \mathrm{Gal}(E^s/E)$, we denote by $\gamma_{\mathbf{T}}$ (resp. γ_T) the endomorphism of $\mathbf{T}_{E^s}$ (resp. T_{E^s}) induced by γ. From the definition of the abstract Cartan, we see that $\gamma_{\mathbf{T}}$ is the composition

$$\mathbf{T}_{E^s} \xrightarrow{i_B^{-1}} T_{E^s} \xrightarrow{\gamma_T} T_{E^s} \xrightarrow{i_{\gamma B}} \mathbf{T}_{E^s}.$$

For each $\gamma \in \mathrm{Gal}(E^s/E)$, the automorphism $i_B i_{\gamma B}^{-1}$ of $\mathbf{T}_{E^s}$ is given by an element of $\mathbf{W}$. Hence the function $z_{T,B} \colon \gamma \mapsto (i_B i_{\gamma B}^{-1})$ defines a cocycle in $Z^1(\mathrm{Gal}(E^s/E); \mathbf{W})$. Choosing a different B alters $z_{T,B}$ by a coboundary, hence the cohomology class $h_T := [z_{T,B}] \in \mathrm{H}^1(\mathrm{Gal}(E^s/E); \mathbf{W})$ is independent of B. Then we have

$$i_B \gamma_T i_B^{-1} = i_B (i_{\gamma B}^{-1} \gamma_{\mathbf{T}} i_B) i_B^{-1} = (i_B i_{\gamma B}^{-1}) \gamma_{\mathbf{T}}.$$

In other words, the cocycle h_T expresses the "difference" between the $\mathrm{Gal}(E^s/E)$-action on T and on $\mathbf{T}$.

10.1.3. *The canonical L-embedding.* Recall from the work of Langlands-Shelstad [LS87] that given a maximal torus $T \subset G$ together with a choice of "χ-data", there is a $\widehat{G}$-conjugacy class of admissible dual embeddings ${}^Lj \colon {}^LT \hookrightarrow {}^LG$. In general there is no distingushed choice of ${}^Lj$, but if T is *unramified* then it has a distinguished choice of χ-data, which gives a *canonical* conjugacy class of embeddings ${}^Lj \colon {}^LT \to {}^LG$. We will describe it more explicitly.

Using the $\widehat{G}$-conjugation, we can arrange that $\widehat{j}$ sends $\widehat{T}$ isomorphically to $\widehat{\mathbf{T}}$. Then ${}^Lj$ is specified by a cocycle $W_E \to \widehat{G}(k)$, which must land in the normalizer of $\widehat{T}$ in $\widehat{G}$, denoted $N_{\widehat{G}}(\widehat{T})(k)$. This cocycle will be chosen to lift $h_T \in \mathrm{H}^1(W_E; \mathbf{W})$. Since T is unramified, all the actions factor through the unramified quotient val: $W_E \twoheadrightarrow \langle\varphi\rangle \cong \mathbf{Z}$, and our lift will be chosen to be inflated from $\mathrm{H}^1(\langle\varphi\rangle; N_{\widehat{G}}(\widehat{T})(k))$. The space of such liftings is controlled by an exact sequence

$$\mathrm{H}^1(\langle\varphi\rangle; \widehat{T}(k)) \to \mathrm{H}^1(\langle\varphi\rangle; N_{\widehat{G}}(\widehat{T})(k)) \to \mathrm{H}^1(\langle\varphi\rangle; \mathbf{W}).$$

We will explicate the correct lift only in the case that T is *elliptic*, i.e., anisotropic mod center.

First suppose G is semi-simple: then T is anisotropic, so that

$$\mathrm{H}^1(\langle\varphi\rangle; \widehat{T}(k)) = \widehat{T}(k)/\varphi\text{−conj} = \{1\},$$

hence ${}^Lj$ is uniquely determined in this case by the condition of lifting h_T.

In general, consider the adjoint quotient $G \twoheadrightarrow G_{\mathrm{ad}}$. Let $\overline{T} := T/Z(G)$, so we have a pullback square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \overline{T} \arrow[d] \\ G \arrow[r] & G_{\mathrm{ad}} \end{tikzcd}$$

On the dual side we have $\widehat{G_{\mathrm{ad}}} \to \widehat{G}$ and $\widehat{\overline{T}} \to \widehat{T}$, forming a pushout square

$$\begin{tikzcd} \widehat{\overline{T}} \arrow[r] \arrow[d] & \widehat{T} \arrow[d] \\ \widehat{G_{\mathrm{ad}}} \arrow[r] & \widehat{G} \end{tikzcd}$$

Let ${}^Lj_{\mathrm{ad}} \colon {}^L\overline{T} \to {}^LG_{\mathrm{ad}}$ be the L-embedding specified in the preceding paragraph. Then from [DR09, §4.3], one sees that ${}^Lj$ is the pushout of ${}^Lj_{\mathrm{ad}}$, as in the following pushout square:

$$\begin{tikzcd} \widehat{\overline{T}} \rtimes W_E \arrow[r] \arrow[d, "{}^Lj_{\mathrm{ad}}"] & \widehat{T} \rtimes W_E \arrow[d, "{}^Lj"] \\ \widehat{G_{\mathrm{ad}}} \rtimes W_E \arrow[r] & \widehat{G} \rtimes W_E \end{tikzcd}$$

10.2. **Calculation of the σ-dual homomorphism.** Let $T \subset G$ be an unramified elliptic maximal torus defined over E.

Proposition 10.2.1. *Suppose $T(E)$ contains an element s of order ℓ which maps to a strongly regular element of $G_{\mathrm{ad}}(E)$. Let $\sigma = \mathrm{conj}_s$ be the conjugation action of s on G, so $G^\sigma = T$. Assume $\ell > b(\check{G})$. Then the σ-dual homomorphism ${}^L\psi \colon {}^LT \to {}^LG$ lies in the $\widehat{G}$-conjugacy class of the composition*

$${}^LT \xrightarrow{\mathrm{Fr}_\ell} {}^LT \xrightarrow{{}^Lj} {}^LG.$$

where Fr_ℓ is induced by the Frobenius endomorphism $\check{T} \to \check{T}$ (corresponding to multiplication by ℓ on character groups).

We begin with some preliminaries before turning to the proof. Any choice of $B \supset T_{E^s}$ gives a commutative diagram

$$\begin{tikzcd} T_{E^s} \arrow[r, equal] \arrow[d, "i_B"] & T_{E^s} \arrow[r, equal] \arrow[d] & T_{E^s} \arrow[d] \\ \mathbf{T}_{E^s} & B \arrow[l] \arrow[r] & G_{E^s} \end{tikzcd} \tag{10.1}$$

By Lemma 7.5.2 and Corollary 7.6.7, diagram (10.1) induces an isomorphism

$$br \cong (\mathrm{Fr}_\ell)^* i_B^* \, \mathrm{CT}_B \colon \mathrm{Rep}_k(\widehat{G}) \to \mathrm{Rep}_k(\widehat{T}). \tag{10.2}$$

Thus the choice of B gives an identification $\widehat{i_B}^{-1} \colon \widehat{T} \xrightarrow{\sim} \widehat{\mathbf{T}}$ in such a way that the map $\check{\psi}$, the restriction of ${}^L\psi$ to the identity component, factors as in the diagram below

$$\begin{tikzcd} \widehat{T} \arrow[r, "\check{\phi}_B"'] \arrow[rr, bend left, "\check{\psi}"] & \widehat{\mathbf{T}} \arrow[r, hook] & \check{G} \end{tikzcd} \tag{10.3}$$

where $\check{\phi}_B$ is $\widehat{i}_B^{-1}$ composed with the Frobenius endomorphism Fr_ℓ of $\widehat{T}$. As discussed in §10.1.2, the map $\widehat{i}_B$ carries the W_E-action on $\check{T}$ to the W_E-action on $\widehat{\mathbf{T}}$ twisted by the cocycle h_T, so for all $\gamma \in W_E$ we have

$$\check{\psi} \circ \gamma_{\widehat{T}} = h_T(\gamma) \cdot (\gamma_{\widehat{\mathbf{T}}} \circ \check{\phi}_B). \tag{10.4}$$

This has the following consequence. The map (10.2) carries $W_E \subset {}^L T$ to $N_{\widehat{G}}(\widehat{\mathbf{T}}) \rtimes W_E$, so its projection to the first component defines a class $\mathrm{pr}_1({}^L\psi) \in \mathrm{H}^1(W_E; N_{\widehat{G}}(\widehat{\mathbf{T}}))$. Denote by $[\mathrm{pr}_1({}^L\psi)] \in \mathrm{H}^1(W_E; \mathbf{W})$ the projection of $\mathrm{pr}_1({}^L\psi)$ along $N_{\check{G}}(\check{\mathbf{T}}) \twoheadrightarrow \mathbf{W}$. From (10.4) we conclude:

Lemma 10.2.2. *We have $h_T = [\mathrm{pr}_1({}^L\psi)] \in \mathrm{H}^1(W_E; \mathbf{W})$.*

As explained in §10.1.3, Lemma 10.2.2 already shows that Proposition 10.2.1 holds if G is semi-simple and T is an elliptic unramified torus. For the general case, note that by hypothesis s maps to a strongly regular element $s \in G_{\mathrm{ad}}$ of order ℓ, whose centralizer is $\overline{T}$, so a similar theory applies to G_{ad}. This induces a functor

$$\widetilde{br}_{\mathrm{ad}} \colon \mathrm{Sat}(\mathrm{Gr}_{G_{\mathrm{ad}},\mathrm{Div}^1_X}; k) \to \mathrm{Sat}(\mathrm{Gr}_{\overline{T},\mathrm{Div}^1_X}; k)$$

and we study its relation to the functor

$$\widetilde{br} \colon \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X}; k) \to \mathrm{Sat}(\mathrm{Gr}_{T,\mathrm{Div}^1_X}; k)$$

which corresponds under Geometric Satake to ${}^L\psi$.

We record some general properties of the Geometric Satake equivalence. If $G \to G'$ is a central isogeny, then the induced map $f \colon \mathrm{Gr}_{G,\mathrm{Div}^1_X} \to \mathrm{Gr}_{G',\mathrm{Div}^1_X}$ restricts to isomorphisms of connected components, and the restriction map $\mathrm{Rep}_k({}^L G') \to \mathrm{Rep}_k({}^L G)$ is intertwined under Geometric Satake with the (derived) pushforward functor

$$\mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X}; k) \xrightarrow{\mathrm{R}f_*} \mathrm{Sat}(\mathrm{Gr}_{G',\mathrm{Div}^1_X}; k). \tag{10.5}$$

We record the following general property of the Brauer functor.

Lemma 10.2.3. *Assume $\ell > \max\{b(\check{G}), b(\check{H})\}$. Write $\overline{H} := H/(H \cap Z(G))$ and assume that $\overline{H} = (G_{\mathrm{ad}})^\sigma$. Then the diagram*

$$\begin{tikzcd}
\mathrm{Sat}(\mathrm{Gr}_{\overline{H},\mathrm{Div}^1_X}; k) & \mathrm{Sat}(\mathrm{Gr}_{H,\mathrm{Div}^1_X}; k) \arrow[l] \\
\mathrm{Sat}(\mathrm{Gr}_{G_{\mathrm{ad}},\mathrm{Div}^1_X}; k) \arrow[u, "\widetilde{br}_{\mathrm{ad}}"] & \mathrm{Sat}(\mathrm{Gr}_{G,\mathrm{Div}^1_X}; k) \arrow[l] \arrow[u, "\widetilde{br}"']
\end{tikzcd} \tag{10.6}$$

commutes, where the horizontal maps are of the form (10.5).

Proof. Write $f \colon \mathrm{Gr}_{H,S/\mathrm{Div}^1_X} \to \mathrm{Gr}_{\overline{H},S/\mathrm{Div}^1_X}$ and $g \colon \mathrm{Gr}_{G,S/\mathrm{Div}^1_X} \to \mathrm{Gr}_{G_{\mathrm{ad}},S/\mathrm{Div}^1_X}$ for the projections induced by the quotients by $Z(G)$. By construction of $\widetilde{br}$ and $\widetilde{br}_{\mathrm{ad}}$, it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{\overline{H},S/\mathrm{Div}^1_X}; k) & \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{H,S/\mathrm{Div}^1_X}; k) \arrow[l, "\mathrm{R}f_*"'] \\
\mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G_{\mathrm{ad}},S/\mathrm{Div}^1_X}; k) \arrow[u, "br_{\mathrm{ad}}"] & \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\mathrm{Div}^1_X}; k) \arrow[l, "\mathrm{R}g_*"'] \arrow[u, "br"']
\end{tikzcd} \tag{10.7}$$

commutes for any strictly totally disconnected S over Div^1_X.

Let $\mathcal{F} \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};\mathbb{O})$ and $\mathbb{F}\mathcal{F} := \mathcal{F} \otimes_{\mathbb{O}} k \in \mathrm{Par}^{\mathrm{ULA}}(\mathrm{Gr}_{G,S/\operatorname{Div}^1_X};k)$. Then we have natural isomorphisms

$$\begin{aligned}
\mathrm{R}f_* \circ br(\mathbb{F}\mathcal{F}) &= \mathrm{R}f_* \circ L \circ [\dagger^G_H] \circ \mathrm{Psm} \circ \mathrm{Nm}^{(\ell^{-1})}(\mathbb{F}\mathcal{F}) \\
&\overset{(1)}{\cong} L \circ \mathrm{R}f_* \circ [\dagger^G_H] \circ \mathrm{Psm} \circ \mathrm{Nm}^{(\ell^{-1})}(\mathbb{F}\mathcal{F}) \\
&\overset{(2)}{\cong} L \circ [\dagger^{G_{\mathrm{ad}}}_{\overline{H}}] \circ \mathrm{R}f_* \circ \mathrm{Psm} \circ \mathrm{Nm}^{(\ell^{-1})}(\mathbb{F}\mathcal{F}) \\
&\overset{(3)}{\cong} L \circ [\dagger^{G_{\mathrm{ad}}}_{\overline{H}}] \circ \mathrm{Psm} \circ \mathrm{R}g_* \circ \mathrm{Nm}^{(\ell^{-1})}(\mathbb{F}\mathcal{F}) \\
&\overset{(4)}{\cong} L \circ [\dagger^{G_{\mathrm{ad}}}_{\overline{H}}] \circ \mathrm{Psm} \circ \mathrm{Nm}^{(\ell^{-1})} \circ \mathrm{R}g_*(\mathbb{F}\mathcal{F}) \\
&= br_{\mathrm{ad}} \circ \mathrm{R}g_*(\mathbb{F}\mathcal{F})
\end{aligned}$$

whose composition gives the desired commutativity. Here, the natural isomorphisms are justified as follows:

(1) is evident from the fact that $\mathrm{R}f_*$ preserves normalized parity sheaves, because f restricts to an isomorphism between connected components of the source and target.
(2) holds because G and G_{ad} differ by a central quotient, and similarly for H and $\overline{H}$, so $[\dagger^G_H] = [\dagger^{G_{\mathrm{ad}}}_{\overline{H}}]$; and clearly shifts commute with $\mathrm{R}f_*$.
(3) holds by Proposition 5.2.10, using Proposition 4.1.2 to see that applying Σ-fixed points to the map $g\colon \mathrm{Gr}_{G,S/\operatorname{Div}^1_X} \to \mathrm{Gr}_{G_{\mathrm{ad}},S/\operatorname{Div}^1_X}$ yields $f\colon \mathrm{Gr}_{H,S/\operatorname{Div}^1_X} \to \mathrm{Gr}_{\overline{H},S/\operatorname{Div}^1_X}$.
(4) holds because $\mathrm{R}g_*$ is symmetric monoidal.

□

Proof of Proposition 10.2.1. Combining Lemma 10.2.3 with Tannakian reconstruction, we find that the diagram

$$\begin{tikzcd}
{}^L\overline{T} \arrow[r] \arrow[d, "{}^L\psi_{\mathrm{ad}}"] & {}^LT \arrow[d, "{}^L\psi"] \\
{}^LG_{\mathrm{ad}} \arrow[r] & {}^LG
\end{tikzcd}$$

commutes. As mentioned above, Corollary 7.6.7 implies that it factors as

$$\begin{tikzcd}
{}^L\overline{T} \arrow[r] \arrow[d, "\mathrm{Fr}_\ell"] \arrow[dd, bend right, "{}^L\psi_{\mathrm{ad}}"'] & {}^LT \arrow[d, "\mathrm{Fr}_\ell"'] \arrow[dd, bend left, "{}^L\psi"] \\
{}^L\overline{T} \arrow[r] \arrow[d, "\zeta_{\mathrm{ad}}"] & {}^LT \arrow[d, "\zeta"'] \\
{}^LG_{\mathrm{ad}} \arrow[r] & {}^LG
\end{tikzcd}$$

By inspection the bottom square induces isomorphism on the cokernels of the rows, hence is a pushout square. As explained in §10.1.3, Lemma 10.2.2 implies that ζ_{ad} is $\widehat{G}$-conjugate to the canonical ${}^Lj_{\mathrm{ad}}$. Since ζ is pushed out from ζ_{ad} and ${}^Lj$ is pushed out from ${}^Lj_{\mathrm{ad}}$, we deduce that ζ is $\widehat{G}$-conjugate to ${}^Lj$, and then that ${}^L\psi$ is $\check{G}$-conjugate to ${}^Lj \circ \mathrm{Fr}_\ell$, as desired. □

10.3. **Tate cohomology of deep level Deligne-Lusztig varieties.** We briefly recall the generalized Deligne-Lusztig representations appearing in [CI21].

10.3.1. *Group-theoretic setup.* Let $T \hookrightarrow G$ be an unramified maximal torus and $x \in \mathcal{B}(G/E)$ be a point of the Bruhat-Tits building of G that lies in the apartment of T. Corresponding to x we have by Bruhat-Tits theory a parahoric group scheme $\mathcal{G}/\mathcal{O}_E$, whose generic fiber is G/E.

Recall that $\mathbf{F}_q$ is the residue field of E. By assumption, T splits over $\breve{E}$. Choose a $\breve{E}$-rational Borel subgroup of $G_{\breve{E}}$ containing $T_{\breve{E}}$, and let U be its unipotent radical.

For $r \in \mathbf{Z}_{\geq 0}$, we have group schemes $\mathbb{G}_r, \mathbb{T}_r$ over $\mathbf{F}_q$ as in [CO, §6.1][8] corresponding to subquotients of the Moy-Prasad filtration at x, such that

$$\mathbb{G}_r(\mathbf{F}_q) = G_{x,0:r+} := G_{x,0}/G_{x,r+} \quad \text{and} \quad \mathbb{T}_r(\mathbf{F}_q) = T_{0:r+} := T_{x,0}/T_{x,r+}.$$

We also have a group scheme $\mathbb{U}_r \subset (\mathbb{G}_r)_{\overline{\mathbf{F}}_q}$ corresponding to U.

10.3.2. *Deep level Deligne-Lusztig varieties.* Let Fr_q be the q-power Frobenius for schemes over $\mathbf{F}_q$. We recall certain schemes constructed in [CI21, §4]: the "deep level Deligne-Lusztig varieties"

$$S_{\mathbb{T}_r,\mathbb{U}_r} := \{x \in (\mathbb{G}_r)_{\overline{\mathbf{F}}_q} : x^{-1}\,\mathrm{Fr}_q(x) \in \mathbb{U}_r\}.$$

(The variety $S_{\mathbb{T}_r,\mathbb{U}_r}$ is called X_r in [CO].) It is a separated, smooth, finite type scheme over $\overline{\mathbf{F}}_q$, with an action of $\mathbb{G}_r(\mathbf{F}_q) \times \mathbb{T}_r(\mathbf{F}_q)$ by multiplication on the left and right.The action of $\mathbb{T}_r(\mathbf{F}_q)$ is free, and we define

$$X_{\mathbb{T}_r,\mathbb{U}_r} := S_{\mathbb{T}_r,\mathbb{U}_r}/\mathbb{T}_r(\mathbf{F}_q).$$

Example 10.3.1. When $r = 0$, the definition of $X_{\mathbb{T}_r,\mathbb{U}_r}$ specializes to that of a classical Deligne-Lusztig variety from [DL76].

Definition 10.3.2 (Deep level Deligne-Lusztig induction)**.** Let $\theta \colon \mathbb{T}_r(\mathbf{F}_q) \to k^\times$ be a character. Let $\mathcal{L}_\theta$ be the corresponding local system on $X_{\mathbb{T}_r,\mathbb{U}_r}$. We define

$$R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) := \mathrm{R}\Gamma_c^*(X_{\mathbb{T}_r,\mathbb{U}_r}; \mathcal{L}_\theta) \in D^b(\mathrm{Rep}_k\, \mathbb{G}_r(\mathbf{F}_q)).$$

10.3.3. *Calculation of Tate cohomology.* Following [CI21, §2.8], we define $W_x(T)$ to be the subgroup of $W(T,G)$ generated by vector parts of affine roots ψ of G for which $\psi(x) = 0$.

Following [CI21, Definition 5.1], we say that $s \in G_x(\mathcal{O}_{\breve{E}})$ is *unramified very regular* (with respect to x) if s is regular semisimple in $G_{\breve{E}}$, its connected centralizer $Z_G^\circ(s)$ is an $\breve{E}$-split maximal torus of $G_{\breve{E}}$, and $\alpha(s) \not\equiv 1 \pmod{\varpi_E}$ for all roots α of $Z_G^\circ(s)$. We say that $s \in \mathbb{G}_r(\overline{\mathbf{F}}_q)$ is unramified very regular if it is the image of an unramified very regular element of $G_x(\mathcal{O}_{\breve{E}})$.

Proposition 10.3.3. *Let $s \in \mathbb{T}_r(\mathbf{F}_q)$ be unramified very regular in $\mathbb{G}_r(\overline{\mathbf{F}}_q)$ and of order ℓ, and let $\sigma = \mathrm{conj}_s$ as an automorphism of G. Then for each $j \in \mathbf{Z}/2\mathbf{Z}$ we have*

$$\mathrm{T}^j(R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)) \cong \bigoplus_{w \in W_x(T)^{\mathrm{Fr}_q}} \theta^w \in D^b(\mathrm{Rep}_k\, \mathbb{T}_r(\mathbf{F}_q)). \tag{10.8}$$

where $\theta^w = \theta \circ w^{-1}$ is the translate of θ by the action of w.

Proof. By [CI21, Proposition 5.6], we have

$$\mathrm{Fix}(\sigma, X_{\mathbb{T}_r,\mathbb{U}_r}) = \bigcup_{w \in W_x(T)^{\mathrm{Fr}_q}} [\dot{w}] \tag{10.9}$$

where $[\dot{w}]$ is the coset $\dot{w}\mathbb{T}_r(\mathbf{F}_q)$ of any lift $\dot{w} \in \mathbb{G}_r(\overline{\mathbf{F}}_q)$ of w.

By [Fen24, §A.1.2] (which is the scheme-theoretic counterpart to §5.2.3), this implies that

$$\mathrm{T}^j(R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)) = \mathrm{T}^j(\mathrm{R}\Gamma_c^*(X_{\mathbb{T}_r,\mathbb{U}_r}; \mathcal{L}_\theta)) \cong \mathrm{T}^j(\bigcup_{w \in W_x(T)^{\mathrm{Fr}_q}} [\dot{w}]; \mathcal{L}_\theta) \cong \bigoplus_{w \in W_x(T)^{\mathrm{Fr}_q}} \mathcal{L}_\theta|_{[\dot{w}]}.$$

Here $\mathcal{L}|_{[\dot{w}]}$ is a $\mathbb{T}_r(\mathbf{F}_q)$-equivariant sheaf on a point, i.e., a representation of $\mathbb{T}_r(\mathbf{F}_q)$. Writing $t\dot{w} = \dot{w}(\dot{w}^{-1}t\dot{w})$, we see that $\mathcal{L}_\theta|_{[\dot{w}]} \cong \theta^w \in \mathrm{Rep}_k\, \mathbb{T}_r(\mathbf{F}_q)$.

∎

[8]The indexing seems to differ from that of [CI21, §2.5, 2.6] by 1.

10.4. **Toral compact inductions.** We may regard $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ as a (derived) smooth representation of $G_{x,0}$ by inflation. Choose some extension of θ to $T(E)$, which we use to regard $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ as a (derived) smooth representation of $T(E)G_{x,0}$. Then we define

$$\pi_{T,U,\theta} := \text{c-Ind}_{T(E)G_{x,0}}^{G(E)} R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E)).$$

Note that since σ is inner, Example 9.1.3 applies to equip any $\Pi \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))$ with a canonical Σ-equivariant structure. In particular, we use this to view $\pi_{T,U,\theta} \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} G(E))^{B\Sigma}$.

In preparation for calculating the Tate cohomology of $\pi_{T,U,\theta}$, we study the interaction between compact induction and Tate cohomology. Let $K \subset G(E)$ be any Σ-stable closed subgroup. Let $Y := G(E)/K$. Then there is a functor from finite-dimensional representations of K to $G(E)$-equivariant local systems on Y, which we denote $V \mapsto \mathcal{F}(V)$. In turn, there is a functor from $G(E)$-equivariant local systems on Y to smooth $G(E)$-representations, obtained by taking compactly supported global sections. Assuming that K is open, the composite functor

$$V \mapsto \mathrm{R}\Gamma_c(Y; \mathcal{F}(V))$$

is the compact induction from K to $G(E)$. By [TV16, §3.3] we have, for each $j \in \mathbf{Z}/2\mathbf{Z}$, a natural isomorphism

$$\mathrm{T}^j(\mathrm{R}\Gamma_c(Y;\mathcal{F})) \cong \mathrm{T}^j(\mathrm{R}\Gamma_c(Y^\sigma;\mathcal{F})) \in \mathrm{Rep}_k^{\mathrm{sm}} H(E). \tag{10.10}$$

for any $G(E)$-equivariant local system $\mathcal{F}$ on Y.

Theorem 10.4.1. *Let $T < G$ be an elliptic unramified maximal torus. Assume that $T(E)$ contains an element s of order ℓ, which is unramified very regular with respect to $x \in \mathcal{B}(G/E)$ and strongly regular in $G_{\mathrm{ad}}(E)$. Then for any $\theta\colon T(E) \to k^\times$, $\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\mathrm{H}^*(\pi_{T,U,\theta}))$ contains the point of* $\mathrm{Spec}\,\mathrm{Exc}_k(W_E,\widehat{G})$ *corresponding to the semi-simple L-parameter*

$$W_E \xrightarrow{{}^L\theta} {}^LT(k) \xrightarrow{{}^Lj} {}^LG(k) \tag{10.11}$$

where ${}^L\theta$ is the L-parameter given by class field theory, and ${}^Lj$ is canonical L-embedding of T (§10.1.3).

Proof. Let $\sigma \in \mathrm{Aut}(G)$ by the conjugation by s, so $G^\sigma = T$. Note that the hypotheses on s imply that Proposition 10.2.1 holds in this situation.

Let $K = T(E)G_{x,0}$. Since T is elliptic, we have $K = Z_G G_{x,0}$ where Z_G is the center of $G(E)$. Note that $\mathrm{H}^1(\Sigma; K)$ is finite, as it has a finite-index subgroup of the form Z_G times a pro-p subgroup and $p \neq \ell = |\Sigma|$. From the long exact sequence in non-abelian cohomology

$$1 \to K^\sigma \to G(E)^\sigma \to (G(E)/K)^\sigma \to \mathrm{H}^1(\Sigma;K) \to \mathrm{H}^1(\Sigma;G(E))$$

and the torsor-shifting discussion in [Ser94, §5.3, 5.4], we see that

$$(G(E)/K)^\sigma = \bigcup_{\xi \in \ker[\mathrm{H}^1(\Sigma;K)\to \mathrm{H}^1(\Sigma;G(E))]} G(E)^\sigma/({}_\xi K)^\sigma$$

where ${}_\xi K$ is the twist of K corresponding to the torsor ξ. Henceforth we abbreviate

$$\ker^1(\Sigma;K,G) := \ker[\mathrm{H}^1(\Sigma;K) \to \mathrm{H}^1(\Sigma;G(E))].$$

Since $({}_\xi K)^\sigma \supset ({}_\xi T(E()))^\sigma = T(E) = G(E)^\sigma$, we see that each $G(E)^\sigma/({}_\xi K)^\sigma$ is a point, which we will denote $[\xi] \in (G(E)/K)^\sigma$. From (10.10) we get, for each $j \in \mathbf{Z}/2\mathbf{Z}$, an equivariant isomorphism

$$\mathrm{T}^j(\pi_{T,U,\theta}) \cong \bigoplus_{\xi \in \ker^1(\Sigma;K,G)} \mathrm{T}^j(\mathcal{F}(R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta))_{[\xi]}) \in D^b(\mathrm{Rep}_k^{\mathrm{sm}} T(E)).$$

In particular, taking $\xi = 1 \in \mathrm{H}^1(\Sigma;K)$ to be the trivial class, we see that from Proposition 10.3.3 that a direct sum of Weyl conjugates of θ appears as a direct summand of $\mathrm{T}^j(\pi_{T,U,\theta})$, as a representation of $T(E)$.

By Theorem 9.1.1, we deduce that $F_*\,\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\pi_{T,U,\theta})$ contains the image under ${}^L\psi$ of $\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{T})}\,\theta^w$ for various $w \in W(T,G)$. For tori, the Fargues-Scholze correspondence is known to be compatible with the usual Local Langlands Correspondence, so $\mathrm{supp}_{\mathrm{Exc}_k(W_E,\widehat{T})}\,\theta^w$ is the point corresponding to the L-parameter of class field theory,

$$W_E \xrightarrow{{}^L\theta^w} {}^LT(k). \tag{10.12}$$

Note that ${}^L\psi \circ {}^L\theta^w = {}^L\psi \circ {}^L\theta \in \mathrm{H}^1(W_E; \widehat{G}(k))$ for each $w \in W$, since the Weyl conjugation becomes inner in $\widehat{G}$.

By Proposition 10.2.1 we have

$$ {}^L\psi \circ {}^L\theta = {}^L j \circ \mathrm{Fr}_\ell \circ {}^L\theta = \mathrm{Fr}_\ell \circ {}^L j \circ {}^L\theta \in \mathrm{H}^1(W_E; \widehat{G}(k)) $$

so we deduce that $F_* \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\pi_{T,U,\theta})$ contains $\mathrm{Fr}_\ell \circ {}^L j \circ {}^L\theta$. By inspection of the definition, Frobenius (un)twisting is compatible with the Fargues-Scholze correspondence in the sense that

$$ F_* \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})} \Pi = \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})} F_* \Pi = \operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})} \Pi^{(\ell)} \tag{10.13} $$

so $\operatorname{supp}_{\mathrm{Exc}_k(W_E,\widehat{G})}(\pi_{T,U,\theta})$ contains the point of $\mathrm{Exc}_k(W_E, \widehat{G})$ corresponding to ${}^L j \circ {}^L\theta \in \mathrm{H}^1(W_E; \widehat{G}(k))$, as desired. □

Corollary 10.4.2. *There is a non-zero irreducible $G(E)$-subquotient Π of $\mathrm{H}^i(\pi_{T,U,\theta})$, for some i, such that ρ_Π is* (10.11). *In particular, if $\pi_{T,U,\theta}$ is concentrated in a single cohomological degree and is irreducible, then its Fargues-Scholze parameter is* (10.11).

Remark 10.4.3. Note that Theorem 10.4.1 imposes no regularity conditions on θ. We expect that if θ is sufficiently regular, and ℓ is not too small, then $\pi_{T,U,\theta}$ should be concentrated in a single cohomological degree and irreducible, and cuspidal. There are interesting classes of examples where this is known; a notable one pertains to the *depth-zero supercuspidals* studied in [KV06, DR09], which correspond to the case where:

- $x \in \mathcal{B}(G/E)$ is a vertex (so G_x is a maximal parahoric),
- $r = 0$, so $X_{\mathbb{T}_0, \mathbb{U}_0}$ is a usual Deligne-Lusztig variety,
- T is not contained in a proper parabolic subgroup and θ is non-singular (meaning that it is not orthogonal to any coroot, cf. [DL76, Definition 5.15(1)]), so its Deligne-Lusztig induction is cuspidal.

Letting $\mathcal{L}_\theta$ be the character sheaf on $X_{\mathbb{T}_0,\mathbb{U}_0}$ associated with the non-singular character $\theta \colon T(\mathbf{F}_q) \to k^\times$, it follows from [DL76, Lemma 9.14] and [Bro90, Lemma 3.5] that in this situation $\mathrm{H}^*_c(X_{\mathbb{T}_0,\mathbb{U}_0}; \mathcal{L}_\theta)$ concentrates in a single degree; and if the prime-to-ℓ component of θ is in general position, then $\mathrm{H}^*_c(X_{\mathbb{T}_0,\mathbb{U}_0}; \mathcal{L}_\theta)$ is irreducible by [Bro90, Lemma 3.6].

We expect the *Howe-unramified toral supercuspidal representations* of [CO] to supply further examples, of arbitrary depth. This is the subject of current work-in-progress.

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