MIRROR SYMMETRY AND THE BREUIL-MÉZARD CONJECTURE

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ABSTRACT. The Breuil-Mézard Conjecture predicts the existence of hypothetical "Breuil-Mézard cycles" in the moduli space of mod p Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_q/\mathbf{Q}_q)$ that should govern congruences between mod p automorphic forms. For generic parameters, we propose a construction of Breuil-Mézard cycles in arbitrary rank, and verify that they satisfy the Breuil-Mézard Conjecture for all sufficiently generic tame types and small Hodge-Tate weights. Our method is purely local and group-theoretic, and completely distinct from previous approaches to the Breuil-Mézard Conjecture. In particular, we leverage new connections between the Breuil-Mézard Conjecture and phenomena occurring in homological mirror symmetry and geometric representation theory.

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1. INTRODUCTION

This paper introduces a new approach to the *Breuil-Mézard Conjecture*, in its refined form due to Emerton-Gee [EG23]. Before formulating this conjecture and stating our main results, we recall some context. Since much of the paper operates in realms outside number theory, the introduction will be aimed at a somewhat broader audience than usual.

1.1. **Motivation: Serre's Conjectures.** The phenomenon of *congruences between modular forms* underpins many facets of modern algebraic number theory. It is therefore natural to try to classify the possible congruences between modular forms. The mod p reductions of modular forms are organized by representation-theoretic parameters called *Serre weights*, which are irreducible representations of $\mathrm{GL}_2(\mathbf{F}_p)$ over $\overline{\mathbf{F}}_p$. The *weight part of Serre's Conjecture* [Ser87] addresses the question of when two mod p eigenforms with different Serre weights can be congruent to each other. It was proved in the early 1990s as the culmination of work by many authors; see the introduction of [GLS15] for references.

There is a much more general notion of complex-valued automorphic form on a reductive group G over a number field, wherein modular forms constitute the special case $G = \mathrm{GL}_2$ (over $\mathbf{Q}$), to which one would like to generalize Serre's Conjecture. It is subtler to formulate the notion of *mod p* automorphic form for general reductive groups, but one option is to proceed as follows. By the Eichler-Shimura relation, mod p modular forms can be interpreted as classes in the cohomology of modular curves, with coefficients in the local systems induced by Serre weights. More generally, let G be a reductive group over a number field, with good reduction at a prime p. Then the irreducible representations of $G(\mathbf{F}_p)$ are the *Serre weights* of G (at p). Each Serre weight σ induces a mod p local system on the locally symmetric space associated to G, and it is natural to regard the Hecke eigensystems in the cohomology of this local system as the analogue of mod p eigenforms on G with weight σ. One can then ask to classify the possible congruences between different weights. This problem has come to also be called the "weight part of Serre's conjectures", and beyond GL_2 it becomes much more complicated. At present, the only proposed answer that applies in general (even conjecturally) is itself contingent upon another conjecture, that we will describe next.

1.2. **The Breuil-Mézard Conjecture.** For simplicity we specialize our discussion in this Introduction to $G = \mathrm{GL}_n / \mathbf{Q}_p$. We regard $\check{G} = \mathrm{GL}_n$ as a reductive group over $\mathbf{Z}_p$.

1.2.1. *The Emerton-Gee stack.* Emerton-Gee have constructed in [EG23] a formal algebraic stack $\mathcal{X}^{\mathrm{EG}}$ over $\mathrm{Spf}\, \mathbf{Z}_p$ which is roughly meant to be a moduli stack of n-dimensional p-adic Galois representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. In particular, the reduced substack $\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}$ is an algebraic stack over $\mathbf{F}_p$, equidimensional of dimension $n(n-1)/2$, and Emerton-Gee have constructed a bijection between its irreducible components and the set of Serre weights of G. For a Serre weight σ, the corresponding irreducible component is denoted $\mathcal{C}_\sigma \subset \mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}$.

1.2.2. *Potentially crystalline substacks.* Let $\lambda \in X^*(T)^+$ be a dominant weight and τ be an *inertial parameter* for G, which is an n-dimensional representation of the inertia group of $\mathbf{Q}_p$ that extends to $\mathrm{Weil}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Then Emerton-Gee have constructed a formal p-adic substack $\mathcal{X}^{\lambda,\tau} \hookrightarrow \mathcal{X}^{\mathrm{EG}}$, whose $\overline{\mathbf{Q}}_p$ points correspond to Galois representations with potentially crystalline Hodge-Tate weights λ and Weil-Deligne inertial parameter τ (the point is that these are the local conditions that one expects to see on the local Galois representations associated to algebraic automorphic forms with infinitesimal character $\lambda - \rho$[1] and "level" τ, by p-adic Hodge theory). When λ is regular, the relative dimension of $\mathcal{X}^{\lambda,\tau}/\,\mathrm{Spf}\,\mathbf{Z}_p$ is also $n(n-1)/2$.

1.2.3. *The Breuil-Mézard Conjecture.* To each inertial parameter τ, the inertial Local Langlands correspondence associates an *inertial type* $\sigma(\tau)$, which is a smooth finite-dimensional representation of $G(\mathbf{Z}_p)$ over a finite extension of $\mathbf{Q}_p$. Fix an algebraic closure k of $\mathbf{F}_p$. For a finite-dimensional representation R of $G(\mathbf{Z}_p)$ over a finite extension of $\mathbf{Q}_p$, we let $[\overline{R}] \in K_0(\mathrm{Rep}_k(G(\mathbf{Z}_p)))$ be the class of its reduction modulo p; note that since the kernel of $G(\mathbf{Z}_p) \to G(\mathbf{F}_p)$ is pro-p, we may regard $[\overline{R}] \in K_0(\mathrm{Rep}_k(G(\mathbf{F}_p))) \xrightarrow{\sim} K_0(\mathrm{Rep}_k(G(\mathbf{Z}_p)))$. Let $W(\lambda) \in \mathrm{Rep}_{\mathbf{Q}_p}(G(\mathbf{Z}_p))$ be the Weyl module of highest weight λ.

Conjecture 1.2.1 (Geometric Breuil-Mézard Conjecture)**.** *There is a map*

$$\mathcal{Z}\colon K_0(\mathrm{Rep}_k(G(\mathbf{F}_p))) \to \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}})$$

such that for every $\lambda \in X^*(T)^+$ and every inertial parameter τ, we have

$$\mathcal{Z}[\overline{W(\lambda) \otimes \sigma(\tau)}] = [\mathcal{X}^{\lambda+\rho,\tau}_{\mathbf{F}_p}] \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}), \tag{1.2.1}$$

where $[\mathcal{X}^{\lambda+\rho,\tau}_{\mathbf{F}_p}]$ is the cycle class of $\mathcal{X}^{\lambda+\rho,\tau}_{\mathbf{F}_p}$.

[1]For a semisimple simply connected G we would take ρ to be the usual half sum of the positive roots. For a reductive group G with simply connected derived subgroup, we may take any extension of the ρ for its derived group. For $G = \mathrm{GL}_n$ we prefer the choice $\rho = (n-1, n-2, \dots, 0)$ for reasons of convention.

To connect Conjecture 1.2.1 with more typical formulations of the (refined geometric) Breuil-Mézard Conjecture, we note that the group $K_0(\mathrm{Rep}_k(G(\mathbf{F}_p)))$ is free abelian on the set $\{\sigma\}$ of Serre weights of G. Therefore, giving a map $K_0(\mathrm{Rep}_k(G(\mathbf{F}_p))) \to \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}})$ amounts to specifying cycles $\mathcal{Z}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}})$ for each Serre weight σ, such that if

$$[\overline{W(\lambda) \otimes \sigma(\tau)}] = \sum_{\sigma} n_\sigma(\lambda, \tau)[\sigma] \in K_0(\mathrm{Rep}_k(G(\mathbf{F}_p)))$$

then

$$[\mathcal{X}^{\lambda+\rho,\tau}_{\mathbf{F}_p}] = \sum_{\sigma} n_\sigma(\lambda,\tau)\mathcal{Z}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}). \tag{1.2.2}$$

This is the formulation of the Breuil-Mézard Conjecture as it appears in [EG23, §8.2.2]; we note that it goes significantly beyond the original conjectures of Breuil-Mézard from [BM02]. The hypothetical cycles $\mathcal{Z}(\sigma)$ in Conjecture 1.2.1 are called *Breuil-Mézard cycles*; they are *not* the same in general as the irreducible components $\mathcal{C}_\sigma$. Note that there are approximately p^n Breuil-Mézard cycles while there are infinitely many possibilities each for λ and τ; therefore, the conjecture can be thought of as positing the existence of a solution to a massively overdetermined system of equations.

1.2.4. *The weight part of Serre's Conjecture in higher rank.* Returning to the thread of Serre's Conjectures, Gee-Kisin [GK14] suggested using the Breuil-Mézard Conjecture to formulate the generalization of the weight part of Serre's Conjecture to general G. We will summarize this formulation as it is described in [GHS18, §6]. Suppose that the Breuil-Mézard cycles $\mathcal{Z}(\sigma)$ are given. Let $\bar{r}\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be an irreducible Galois representation. Let $\{\sigma\}_{\bar{r}}$ be the set of all Serre weights for which $\bar{r}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$ lies on the Breuil-Mézard cycle $\mathcal{Z}_\sigma$. Then one expects that $\{\sigma\}_{\bar{r}}$ *is the set of Serre weights in which the Hecke eigensystem associated to* $\bar{r}$ *occurs at some level which is good at* p. See [GHS18] and [EG23, §8] for a more detailed discussion.

The picture just described is obviously meaningless without a definition of the Breuil-Mézard cycles $\mathcal{Z}(\sigma)$, which would seem to require proving Conjecture 1.2.1, but in fact it is meaningful to construct candidate Breuil-Mézard cycles without proving the full conjecture. The point is that the cycles $\mathcal{Z}(\sigma)$ are already uniquely determined by a finite number of equations of the form (1.2.2), so one can know that a candidate construction of $\mathcal{Z}(\sigma)$ is "correct" as long as it satisfies a sufficiently large subset of such equations. The candidate cycles can then be fed into the previous paragraph to give an unconditional formulation of the weight part of Serre's Conjecture for GL_n.

1.3. **Main results.** The main results of this paper will follow along the lines just described. We will construct candidate Breuil-Mézard cycles $\mathcal{Z}(\sigma)$ for "sufficiently generic" σ, and verify that they satisfy conditions (1.2.1) whenever λ is "small" enough and τ is a "sufficiently generic" tame type. To make this more precise, we need to recall a bit of (modular) representation theory.

1.3.1. *Modular representation theory.* For $T \subset \mathrm{GL}_n$ the standard maximal torus, the irreducible algebraic representations of GL_n over k are in bijection with the dominant weights $X^*(T)^+$, with $\lambda \in X^*(T)^+$ corresponding to the highest weight representation $L(\lambda)$ of G.

- The *p-restricted weights* $X_1^*(T) \subset X^*(T)$ consist of λ such that $0 \le \langle \lambda, \alpha^\vee \rangle < p$ for all simple roots α. The simple representations (i.e., Serre weights) of $\mathrm{GL}_n(\mathbf{F}_p)$ over k are in bijection with $X_1^*(T)$ modulo an equivalence relation on central characters that we suppress, with $\lambda \in X_1^*(T)$ corresponding to $F(\lambda) := L(\lambda)|_{\mathrm{GL}_n(\mathbf{F}_p)}$.
- For $(w, \mu) \in W \times X^*(\check{T})$ there is an explicitly constructed tame type $\tau = \tau(w, \mu)$. This process gives a bijection between tame inertial types and equivalence classes of (w, μ). When $\tau = \tau(w, \mu)$, the corresponding $\sigma(\tau)$ can be taken to be a certain Deligne-Lusztig representation $R(w, \mu)$ of $\mathrm{GL}_n(\mathbf{F}_p)$[2].

(The same discussion applies to any split reductive group over $\mathbf{F}_p$ whose derived subgroup is simply connected.)

The affine hyperplanes defined by the condition $\langle -, \alpha^\vee \rangle \in p\mathbf{Z}$ divide $X^*(T)$ into alcoves, and genericity will be measured by the "distance" to the walls of these alcoves. More precisely, we say that $\lambda \in X^*(T)$ is *m-generic* (in the lowest alcove) if $m < |\langle \lambda, \alpha^\vee \rangle| < p - m$ for any root α. Note that m-generic λ only exist

[2] Here we blurred the distinction between GL_n and its dual group, and ignore subtleties of degenerate cases where $R(w, \mu)$ is not an irreducible representation.

if $p > 2m$. We say that τ *is m-generic* if $\sigma(\tau) = R(w, \mu)$ where μ is m-generic. We say that $\sigma = F(\lambda)$ *is m-generic* if $\lambda + \rho$ is m-generic. For $\lambda \in X^*(T)$, we define h_λ to be the maximum of $\langle \lambda, \alpha^\vee \rangle$ among all roots α.

1.3.2. *Main theorem.* We can now state our main theorem. We have in mind that p is large relative to n; for example, we must have $p > 2n$ for the results to be non-vacuous.

Theorem 1.3.1. *(1) There exists a collection of cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}})$ such that for each $\lambda \in X^*(T)$ and each tame inertial type τ which is $2h_{\lambda+\rho}$-generic,* (1.2.2) *is satisfied.*

(2) If Conjecture 1.2.1 is true, then the "true" $\mathcal{Z}(\sigma)$ agrees with the $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ from (1) if σ is $6h_\rho$-generic.

(3) If σ is $3h_\rho$-generic, then the multiplicities $n_{\sigma,\sigma'}$ in the decomposition

$$\mathcal{Z}^{\mathrm{EG}}(\sigma) = \sum_{\sigma'} n_{\sigma,\sigma'} \mathcal{C}_{\sigma'}$$

admit a natural microlocal interpretation in terms of $\mathrm{Rep}(\mathfrak{g}_{\mathbf{F}_p})$*.*

Statement (3) is left vague here, but will be made precise in §12.2.

Remark 1.3.2 (Optimality of the constants)**.** The constant $2h_{\lambda+\rho}$ from part (1) is almost optimal with our method. In fact, invoking stronger Galois theoretic ingredients from [LH] would improve it to $h_\lambda + 2h_\rho$, which is a natural barrier for the governing modular representation theory to behave in a stable way.

The constant $6h_\rho$ from part (2) is not optimal; the argument sacrifices optimality for simplicity by using "trivial bounds" where possible. We expect that the argument for (2) can probably be optimized to $3h_\rho$ with some more effort.

Remark 1.3.3 (Generalization to other groups)**.** We expect our arguments to easily generalize to split reductive groups over an unramified extension of $\mathbf{Q}_p$, whose derived subgroup is simply connected. In fact, the entirety of this paper is already written in that generality, except for a "homological model theorem" from [LH] (a simplified version sufficient for this paper is proven in Appendix B) which is invoked in §11. Once this result is available for more general groups, our arguments may be applied verbatim.

1.3.3. *Brief remarks on the proof.* The proof of Theorem 1.3.1 does not follow any existing approaches to the Breuil-Mézard Conjecture, which are either based on p-adic Local Langlands or automorphy lifting (hence have limited generality in the group aspect). Instead our argument is more geometric, and hearkens to the analogy between $\mathbf{Q}_p$ and a 2-manifold, which suggests in turn an analogy between the Emerton-Gee stack and the space of local systems on a 2-manifold. The latter object has a natural *symplectic* structure, and under this analogy the potentially crystalline loci correspond to Lagrangian subspaces. Kontsevich's philosophy of *homological mirror symmetry* posits (roughly) that Lagrangian subspaces of a symplectic manifold can be assembled into a Fukaya category which should then admit a mirror description in terms of coherent sheaves on a mirror variety. In particular, it predicts that Lagrangians can be indexed by mirror data of a "dual" nature, which bears a loose similarity to the Breuil-Mézard Conjecture.

In reality, the Emerton-Gee stack does not literally have a symplectic structure. However, the above metaphor can be substantiated by using p-adic Hodge theory to "approximate" the potentially crystalline substacks by explicit algebraic varieties (certain *affine Springer fibers*), which really do comprise a Lagrangian skeleton of a certain symplectic space of Higgs bundles. Then constructions of Bezrukavnikov–Boixeda Alvarez–McBreen–Yun [BBAMY23] provide the necessary "mirror symmetry" input to prescribe Lagrangians using coherent sheaves on a mirror variety A. Finally, to connect this to the Breuil-Mézard Conjecture, we "approximate" the representation theory of $G(\mathbf{F}_p)$ by representations of $\mathfrak{g}_{\mathbf{F}_p}$, which we then transform into coherent sheaves on A using the modular localization theory of Bezrukavnikov-Mirkovic-Rumynin [BMR08]. More details are given in §1.5 below.

1.4. **Comparison to other results.** We compare Theorem 1.3.1 to other results on the Breuil-Mézard Conjecture.

1.4.1. *The case $n = 2$.* For $\mathrm{GL}_2/\mathbf{Q}_p$, our understanding is that Conjecture 1.2.1 has now been proven in full, thanks mostly to work of Kisin [Kis09] and Paškūnas [Pas15], which left out cases that were completed Hu-Tang, Sander, and Tung. We refer to [EG23, §8.5] for references and more detailed descriptions. The proof is based on the p-adic Local Langlands correspondence for $\mathrm{GL}_2/\mathbf{Q}_p$, which has resisted generalization to other groups despite much effort.

For GL_2 over a finite extension $K/\mathbf{Q}_p$ and $\lambda = (0,0)$, the Breuil-Mézard Conjecture is proved by Gee-Kisin [GK14] (at the level of deformation rings) and Caraiani-Emerton-Gee-Savitt [CEGS22] (in the geometric form). The proof is based on patching arguments, and therefore is limited to groups whose automorphic theory is sufficiently well-understood, which aside from special low rank examples just leaves GL_n.

We digress to comment on the difference between the nature of the problem for GL_2 versus higher rank groups. In the case of GL_2 (and only in this case) the Serre weights all lift to Weyl modules in characteristic 0. Therefore in this case (only), Conjecture 1.2.1 already includes in its formulation the definition of $\mathcal{Z}(F(\lambda))$: it must be $[\mathcal{X}^{\lambda,\mathrm{triv}}|_{\mathbf{F}_p}]$. Furthermore, $\mathcal{Z}(\sigma)$ turns out to be relatively simple: it is simply the irreducible component $\mathcal{C}_\sigma$ for all σ which are not Steinberg.

1.4.2. *The case $n > 2$.* By contrast, the higher rank cases of the Breuil-Mézard Conjecture have a very different texture. We quote from [EG23, §8.7]:

> "We expect the situation for GL_d, $d > 2$, to be considerably more complicated than that for GL_2. Experience to date suggests that the weight part of Serre's conjecture in high dimension is consistently more complicated than is anticipated, and so it seems unwise to engage in much speculation."

For example, it is expected that $\mathcal{Z}(\sigma)$ can reducible even for very generic σ [LHLM23b, Remark 1.5.11]. In fact, it follows from our construction that the $\mathcal{Z}(\sigma)$ can be interpreted as the characteristic cycles of simples in a representation-theoretic category, and then experience in geometric representation theory suggests that their decompositions with respect to the $\mathcal{C}_\sigma$ are rather subtle in higher dimension, no matter how generic σ is.

The only general result towards the Breuil-Mézard Conjecture in arbitrary rank, prior to the present paper, was the work of [LHLM23b], which applies to $G = \mathrm{GL}_n$ and unramified $K/\mathbf{Q}_p$. Given a finite set $\Lambda = \{\lambda\}$ of dominant weights, they produce candidate Breuil-Mézard cycles $\mathcal{Z}(\sigma)$ which satisfy (1.2.2) whenever τ is very generic. Here, "very generic" depends only on Λ and n, and is of the nature that some parameters avoid some proper (possibly non-linear) subvarieties in an affine space (in particular, this is rather more stringent than familiar notions of genericity in representation theory, which are of the nature that some parameter avoids some union of hyperplanes). Worse, unlike in Theorem 1.3.1 *it seems hard to quantify this condition effectively*, so that one cannot say how large p needs to be for the statement to be non-vacuous.

Remark 1.4.1. A back-of-the-envelope estimate indicates that when $G = \mathrm{GL}_n$ [LHLM23b] verifies $O(p^n)$ of the equations (1.2.2) where the implicit constant is not effective, while Theorem 1.3.1 verifies $O(p^{2n})$ of the equations (1.2.2) where the implicit constant is effective.

The approach of [LHLM23b] is based on the patching method as in [GK14], and it thus has the added benefit of yielding global consequences such as cases of automorphy and the weight part of Serre's conjecture. On the other hand, this means that it seems unlikely to generalize to exceptional groups (for example). By contrast, our method is purely local and bypasses the need for any automorphic information. The purely group-theoretic nature means it is very likely to generalize to exceptional groups (for example).

1.5. **Discussion of proof and overview of paper.** We will now summarize our approach to Theorem 1.3.1 and how it is distributed over the various sections of the paper.

1.5.1. *Inspirations.* Our work has several inspirations, especially the work of Breuil-Hellmann-Schraen, whose [BHS19, Theorem 1.9] might be regarded as a locally analytic analogue of the Breuil-Mézard Conjecture. In their case, the locally analytic Breuil-Mézard cycles are imported via local model diagrams from the characteristic cycles of D-modules that are connected to representation theory through Beilinson-Bernstein localization. Separately, Emerton had remarked to the authors that the geometry of $\mathcal{X}^{\lambda,\tau}$ resembled that of a Lagrangian in a symplectic manifold. This led us to try to construct our Breuil-Mézard cycles by importing characteristic cycles from somewhere, although our importing process ends up being more difficult than in [BHS19], and the construction of the characteristic cycles is also much more involved.

1.5.2. *Construction of Breuil-Mézard cycles.* We will divide this discussion into two parts: the first is the construction of the Breuil-Mézard cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$, and the second is the verification that they satisfy the Breuil-Mézard relations (1.2.2).

For the first part, we want to construct (top-dimensional) cycles in $\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p} \subset \mathcal{X}^{\mathrm{EG}}_{\mathbf{F}_p}$ from representations. To begin, we construct algebraic models for $\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p}$ that we call $\mathrm{Y}^{\varepsilon=1}_{\gamma}$, where γ is cooked up from λ and τ by a recipe that we omit for now. As the name suggests, these models can be deformed in a parameter ε, and the fiber at $\mathrm{Y}^{\varepsilon=0}_{\gamma}$ is an object called an *affine Springer fiber*, which has significance in geometry representation theory. There is a "microlocal support" map from a K-group of *graded Lie algebra representations* $K_0(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ towards the Chow group of top-dimensional cycles $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_{\gamma})$, which we will explain further below. The Breuil-Mézard cycles are the images of simple representations in $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ under this sequence of transformations, summarized in the following diagram.

$$\begin{tikzcd} \text{Representation theory} \arrow[r, "\text{Part 2}"] & \text{Affine Springer fiber} \arrow[r, "\text{Part 1}"] & \text{Models} \arrow[r, "\text{Part 3}"] & \text{Emerton-Gee stack} \\ K_0(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \arrow[r, "\text{microlocal support}"] & \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_{\gamma}) \arrow[r, "\text{deform } \varepsilon"] & \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=1}_{\gamma}) \arrow[r, "p\text{-adic Hodge theory}"] & \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p}) \end{tikzcd} \tag{1.5.1}$$

We will now explain these steps in more detail, starting from the right side.

1.5.3. *Models via p-adic Hodge theory.* Kisin [Kis08] developed an approach to understanding the moduli of (potentially semistable) $\mathrm{Gal}_{\mathbf{Q}_p}$-representations using the *p-adic Hodge theory* of [Kis06], which interprets Galois representations in terms of more manageable linear algebraic data called Breuil-Kisin modules. The passage from the models to the Emerton-Gee stack in (1.5.1) builds on Kisin's ideas, as further developed in [LHLM23b]. The space $\mathrm{Y}^{\varepsilon=1}_{\gamma}$ is the special fiber of an explicit algebraic $\mathbf{Z}_p$-scheme $\mathcal{X}^{\varepsilon=1}_{\gamma}$ obtained by truncating the p-adic expansions of some equations in a moduli space of Breuil-Kisin modules. It is not clear a priori that this truncation provides a "good" approximation to $\mathcal{X}^{\lambda,\tau}$. A key point is that our strategy asks for relatively little about the quality of this approximation.

For comparison, we note that the strategy of [LHLM23b] requires a smooth local model for $\mathcal{X}^{\lambda,\tau}$ which is moreover unibranch, and this is the source of their stringent (and hard to compute) genericity conditions. Our arguments need much less, which is part of why we are able to relax the genericity hypotheses. Indeed, all we need is that $\mathcal{X}^{\varepsilon=1}_{\gamma}$ is a "homological model": (part of) its top homology accurately models the top homology of $\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p}$, the critical feature being that $\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p}$ agrees with the limit cycle of some natural locus in the generic fiber of $\mathcal{X}^{\varepsilon=1}_{\gamma}$. This fact is a consequence of the techniques in [LH], which is a substantial improvement of the theory in [LHLM23b] that overcomes the genericity barriers in the latter. In the situation relevant to this paper, the rather elaborate setup in [LH] can be simplified substantially, and in Appendix B we explain a proof of this simpler "homological model theorem"; thus *this paper does not rely on* [LH].

1.5.4. *Deformation to affine Springer fibers.* The $\mathbf{Z}_p$-scheme $\mathcal{X}^{\varepsilon=1}_{\gamma}$ can be deformed to a family $\mathcal{X}^{\varepsilon}_{\gamma}$ over $\mathbf{Z}_p[\varepsilon]$. The fiber over $\varepsilon = 0$ is a mixed characteristic degeneration of *affine Springer fibers*, whose generic fiber we denote $\mathrm{X}^{\varepsilon=0}_{\gamma}$ and whose special fiber we denote $\mathrm{Y}^{\varepsilon=0}_{\gamma}$. We remark that just as affine Springer fibers are local analogues of Hitchin spaces (moduli of Higgs bundles), the family $\mathcal{X}^{\varepsilon}_{\gamma}$ is a local analogue of moduli of "λ-connections" (with ε playing the role of λ; we reserve the notation λ for other purposes).

The entirety of Part 1 is devoted to analyzing the family $\mathcal{X}^{\varepsilon}_{\gamma}$ along with the behavior of $\mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\varepsilon}_{\gamma})$ under degeneration in ε. In §3 we define these families, construct the *affine Springer action* on them, and tabulate their top-dimensional irreducible components at specific ε. In §4 we give a coarse-grained analysis of the specialization of irreducible components of $\mathcal{X}^{\varepsilon}_{\gamma}$ from generic ε to $\varepsilon = 0$ or $\varepsilon = 1$, which is needed to carry out the deformation of cycles from $\varepsilon = 0$ to $\varepsilon = 1$ in (1.5.1). This is enough for Theorem 1.3.1(1) and (2), but for control over the finer properties of the Breuil-Mézard cycles (such as their effectivity and Theorem 1.3.1(3)) we need a more refined analysis of the specialization maps, which we undertake in §5.

1.5.5. *Microlocal support.* The "microlocal support" map from (1.5.1) is itself the composition of two steps, which are studied in Part 2.

$$\begin{tikzcd} \text{Representation theory} \arrow[r, "\S 7"] & \text{Coherent realization} \arrow[r, "\S 8"] & \text{Affine Springer fiber} \\ K_0(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \arrow[r, "\text{BMR localization}"] & K_0(\mathrm{Coh}^T_{\mathcal{B}}(\widetilde{\mathcal{N}})) \arrow[r, "\text{homological mirror symmetry}"] & \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_{\gamma}) \end{tikzcd} \tag{1.5.2}$$

The first step, explained in §7, transforms graded Lie algebra representations to coherent sheaves on the Springer resolution $\widetilde{\mathcal{N}}$ for G, with support conditions determined by central characters, via the localization functor of Bezrukavnikov-Mirkovic-Rumynin [BMR08, BM13]. It should be thought of as the characteristic

p version of Beilinson-Bernstein localization, enhanced by observations related to p-curvature that allow to describe D-modules on X by coherent sheaves on the Frobenius twist of T^*X.

The second step, explained in §8, transforms coherent sheaves into top-dimensional cycles on the affine Springer fiber $\mathrm{Y}_\gamma^{\varepsilon=0}$, using work of Bezrukavnikov–Boixeda Alvarez–McBreen–Yun [BBAMY23]. The key point here is that $\mathrm{Y}_\gamma^{\varepsilon=0}$ has a natural realization as a Lagrangian inside a symplectic space $\mathcal{M}_\psi$ of $\check{G}$-Higgs bundles (in fact $\mathrm{Y}_\gamma^{\varepsilon=0}$ is homeomorphic to a Hitchin fiber in completely integrable Hitchin system for $\mathcal{M}_\psi$), and the passage from coherent sheaves on $\widetilde{\mathcal{N}}$ to Lagrangians in $\mathcal{M}_\psi$ should be thought of as some incarnation of homological mirror symmetry. It is implemented by an instance of the geometric Langlands correspondence, relating coherent sheaves to constructible sheaves on a certain moduli space of $\check{G}$-bundles, which after forming singular support gives the "microlocal support" map alluded to above. We highlight that it is ultimately mirror symmetry which provides the passage from G to its dual group $\check{G}$ in our story.

1.5.6. *Breuil-Mézard cycles.* We consider the Lie algebra $\mathfrak{g} = (\operatorname{Lie} G)_k$ in characteristic p. The category $\operatorname{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ is the category of finitely generated $X^*(T)$-graded $\mathfrak{g}$-representations with trivial Harish-Chandra central character and nilpotent p-central character. It has well-known similarities to $\operatorname{Rep}_k(G(\mathbf{F}_p))$, under which the Serre weights σ of G correspond to certain irreducible representations $L(\sigma)$ in $\operatorname{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$. We define $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ to be the image of $[L(\sigma)]$ under (1.5.1). The details are somewhat elaborate, and are explained in §11.

1.5.7. *Verification of the Breuil-Mézard relations.* We next turn to discuss the proof that the Breuil-Mézard cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ satisfy the equations (1.2.2), which is the content of §11 (building on §9 and Appendix A). The equations concern the relation between $\operatorname{Rep}_k(G(\mathbf{F}_p))$ and the Emerton-Gee stack for $\check{G}$. By backtracking through (1.5.1), we can formulate equivalent equations relating $\operatorname{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ to the geometry of $\mathrm{Y}_\gamma^{\varepsilon=0}$, which we ultimately prove by an analysis of the microlocal support map.

We first focus on the case $\lambda = 0$, corresponding to minimal regular Hodge-Tate weights. Recall that $\mathrm{X}_\gamma^{\varepsilon=0}$ denotes the generic fiber of the scheme $\mathcal{X}_\gamma^{\varepsilon=0} \to \operatorname{Spec} \mathbf{Z}_p$. It has an irreducible component $\mathrm{X}_\gamma^{\varepsilon=0}(\rho)$, whose fundamental class we specialize to characteristic p, obtaining a cycle $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]$ on the affine Springer fiber $\mathrm{Y}_\gamma^{\varepsilon=0}$. This turns out to be the output of transferring $[\mathcal{X}_{\mathbf{F}_p}^{\rho,\tau}]$ to the model $\mathrm{Y}_\gamma^{\varepsilon=1}$ and then deforming from $\varepsilon = 1$ to $\varepsilon = 0$.

On the other hand, *Jantzen's generic decomposition formula* shows that the "similarity" between $\operatorname{Rep}_k(G(\mathbf{F}_p))$ and $\operatorname{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$, which we used to make the correspondence of irreducible representations $\sigma \leftrightarrow L(\sigma)$, takes the inertial type $\sigma(\tau)$ to the *baby Verma module* $\widehat{Z}_1(p\rho)$.

$$\begin{tikzcd}
\text{Emerton-Gee stack} & {[\mathcal{X}_{\mathbf{F}_p}^{\rho,\tau}]} \arrow[r, leftrightarrow] \arrow[d, dashed, leftrightarrow, "?"'] & \mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)] \arrow[d, leftrightarrow, "?"] & \text{Affine Springer fiber} \\
\operatorname{Rep}_k(G(\mathbf{F}_p)) & {[\sigma(\tau)]} \arrow[r, leftrightarrow] & {[\widehat{Z}_1(p\rho)]} & \operatorname{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)
\end{tikzcd}$$

From these considerations we reduce (1.2.1) in the special case $\lambda = 0$ to the statement that

$$\textit{the microlocal support of } [\widehat{Z}_1(p\rho)] \textit{ is } \mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]. \tag{1.5.3}$$

We then prove this jointly with Roman Bezrukavnikov and Pablo Boixeda Alvarez in Appendix A; it is highly non-obvious from the definition of the microlocal support map. Indeed, a major difficulty in the Breuil-Mézard Conjecture is that the geometry of a cycle obtained by degeneration (i.e., flat limit) is difficult to understand; the special fiber $\mathcal{X}_{\mathbf{F}_p}^{\rho,\tau}$ is itself defined indirectly as a flat limit of an irreducible (and reduced) space $\mathcal{X}_{\mathbf{Q}_p}^{\rho,\tau}$. From this perspective (1.5.3) appears at first glance to be as difficult as (1.2.1). We will explain however that there is now key new traction provided by the *presence of many symmetries* on $\operatorname{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma^{\varepsilon=0})$.

1.5.8. *Equivariant localization.* A key structure of $\mathrm{Y}_\gamma^{\varepsilon=1}$ and $\mathrm{Y}_\gamma^{\varepsilon=0}$, which is not present on the Emerton-Gee stack, is the existence of an action by a maximal torus $\check{T} \subset \check{G}$ that makes them equivariantly formal[3]. Then *equivariant localization* allows to compute their homology in terms of $\check{T}$-fixed points. This provides an alternative "basis" of homology in which it is easy to compute the effect of degeneration, because the degeneration of $\check{T}$-fixed points has a simple geometry. Thus, although we cannot compute $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]$ in

[3]For $\mathrm{Y}_\gamma^{\varepsilon=1}$ we need to restrict to certain bounded open subschemes to guarantee equivariant formality.

the geometrically natural basis of irreducible components, we are able to compute it in terms of equivariant localization.

1.5.9. *Two extended affine Weyl actions.* It then remains to compute the left side of (1.5.3) in terms of equivariant localization. Because of the indirect way in which microlocal support is defined, we do not know how to do this directly. Instead, we leverage more symmetries on $\mathrm{Y}_{\gamma}^{\varepsilon=0}$, which will enable us to *recognize the microlocal support of* $[\widehat{Z}_1(p\rho)]$ *by its symmetries.*

More precisely, $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_{\gamma}^{\varepsilon=0})$ has two commuting actions of the extended affine Weyl group $\widetilde{W}$ for $\check{G}$: the *centralizer-monodromy action* which we denote $(\widetilde{W}, \cdot)$ comes from the symmetries of the space $\mathrm{Y}_{\gamma}^{\varepsilon=0}$, and the *affine Springer action* $(\widetilde{W}, \bullet)$ which is subtler and does not come from an action on the space. The microlocal support map is equivariant for these actions, where on $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ the action of $(\widetilde{W}, \cdot)$ comes from change of grading, and the action of $(\widetilde{W}, \bullet)$ comes from wall-crossing functors. It turns out that the baby Verma has a very special "eigenproperty" with respect to these two actions, which allows us to characterize its microlocal support without first computing it (amusingly, this *a posteriori* gives simple formulas for the equivariant localization of microlocal supports of the baby Vermas).

1.5.10. *Variation in Hodge-Tate weights.* We have just sketched the proof of (1.2.2) for the special case $\lambda = 0$. The extension to more general λ involves an interesting new geometric observation. Using equivariant localization, we are able to express the cycle class $[\mathcal{X}_{\mathbf{F}_p}^{\lambda,\tau}]$ as a linear combination of cycle classes of the form $[\mathcal{X}_{\mathbf{F}_p}^{\rho,\tau'}]$ for certain τ', after transporting them to the model via the homological model theorem. Intriguingly, after equivariant localization the relation is just a geometric incarnation of *Weyl's character formula.* This effectively reduces to the case $\lambda = 0$, which we already handled.

The linear combination from the preceding paragraph involves the action on $(\widetilde{W}, \cdot)$ on the homology of the local model, which is "invisible to the naked eye" in the sense that it does not come from an action on the underlying space – we can only "see" it through equivariant localization. Such a reduction does not seem to have been anticipated in previous approaches to the Breuil-Mézard Conjecture, perhaps because of the subtlety of the symmetries required to express it. We note a similarity to the main result of [Bar23], which proves an upper bound in an analogous (conjectural) equality where τ is trivial. In fact, one of the main results in that work, [Bar23, Theorem 1.1], concerning cycle relations between special fibers of certain linear algebra moduli spaces, has a more conceptual interpretation in terms of equivariant localization. When combined with the appropriate homological model theorem, the inequality of cycles in [Bar23] can be promoted to actual equalities, verifying the Breuil-Mezard equations in this setting. We will return to this in future work.

1.6. **Complements.** In §12 we prove some complementary properties of the Breuil-Mézard cycles.

(1) We verify the uniqueness aspect in Theorem 1.3.1(2). En route to this, we describe a combinatorial algorithm to compute the decomposition of Breuil-Mézard cycles into irreducible components, using the equivariant cohomology of affine Springer fibers. This will be used for future empirical investigation of the geometric structure of Breuil-Mézard cycles.
(2) We match the geometric decomposition of $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ into irreducible components (with multiplicity) with that of certain characteristic cycles (i.e., singular supports) of constructible sheaves, which is the content of Theorem 1.3.1(3). This relates the geometry of Breuil-Mézard cycles with important and independently studied problems in geometric representation theory. It also reveals new patterns to their structure, because all the characteristic cycles are related by a combination of the monodromy-centralizer and affine Springer actions. The $(\widetilde{W}, \cdot)$-action comes from a geometric action, so it partitions the Breuil-Mézard cycles into orbits that have the same physical decomposition. The subtler $(\widetilde{W}, \bullet)$-action changes the geometry, but in a way that one may hope to calculate using affine Springer theory.
(3) We show that $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ is effective at least for sufficiently large p, quantified by a bound that is explicitly computable from G.

An intriguing feature of (3) is that it uses the theory of the *quantum group at a* p^{th} *root of unity.* In fact, the representation theory of quantum groups is already used at a few technical points in the proof of Theorem 1.3.1. For instance, it plays an important role in describing the support of limit cycles from the generic fibers of the models, which on the Galois theory side is equivalent to the fact that $\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p}$ has the expected underlying topological space (i.e., the "topological Breuil-Mezard Conjecture").

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2. Notation and generalities

2.1. **Notation on p-adic fields.** For $q = p^f$, we denote by $\mathbf{Z}_q$ the unramified extension of $\mathbf{Z}_p$ with residue field $\mathbf{F}_q$ and field of fractions $\mathbf{Q}_q$.

2.2. **Reductive groups.** We denote by G a split reductive group over $\mathbf{Z}_q$. Lie algebras of groups will be denoted with the corresponding lower-case fraktur letters (and will often be considered over $\mathbf{F}_q$, as indicated in the text).

We denote by $\check{G}$ the Langlands dual group of G, regarded again as a split reductive group over $\mathbf{Z}_q$. It is equipped with a canonical split maximal torus $\check{T} \subset \check{G}$, the dual torus to the "abstract Cartan" A of G. Starting in §7, we also choose a split maximal torus $T \subset G$.

Furthermore, $(\check{G}, \check{T})$ is equipped with a canonical pinning once we choose a direction of "positivity". Since different normalizations are found in the literature on this point, we will spell this out in excruciating detail. The reader may wish to ignore this on a first pass.

2.2.1. *Conventions on positivity.* Given any Borel $B = T \cdot N < G$ with unipotent radical N and Cartan subgroup T, we obtain an isomorphism $T \xrightarrow{\sim} B/N \xrightarrow{\sim} A$. *Our convention is that the roots of A on $\mathfrak{b}$ are negative*; this perhaps a less standard convention but it is consistent with the references [Jan03, BMR06, BMR08, BM13] that we will invoke.

Remark 2.2.1. Let $\mathcal{B}$ be the flag variety of G. Recall that G-equivariant lines bundles on $\mathcal{B}$ are identified with characters of A, according to the following construction. If $\mathcal{L}$ is a G-equivariant line bundle on $\mathcal{B}$, then any Borel subgroup $B < G$ corresponds to a point $[B] \in \mathcal{B}$, and acts on the fiber $\mathcal{L}|_{[B]}$ by a character, which is inflated from a character of A. For $\lambda \in X^*(A)$, we denote by $\mathcal{O}(\lambda)$ the corresponding G-equivariant line bundle on $\mathcal{B}$. Our convention on positivity is determined by the property that dominant weights of A correspond to semi-ample line bundles on $\mathcal{B}$ under this construction.

This choice equips $\check{G}$ with a canonical Borel subgroup $\check{B}$ containing $\check{T}$, such that the roots of $\check{\mathfrak{t}}$ on $\check{\mathfrak{b}}$ are *negative*. We denote by $\check{\Phi}$ be the roots of $\check{T}$ on $\check{\mathfrak{g}}$, and by $\check{\Phi}^+ \subset \check{\Phi}$ the subset of positive roots, i.e., the roots of $\check{T}$ on $\check{\mathfrak{g}}/\check{\mathfrak{b}}$. We write $\check{\Delta}$ for the simple positive roots. This induces a notion of standard parabolic, and for a coweight $\lambda \in X_*(\check{T})$ we write $\check{P}_\lambda$ for the corresponding standard parabolic subgroup of $\check{G}$.

If $\check{G}$ is semi-simple, then we denote by ρ the half sum of its positive coroots. (Later we will want to also view ρ as the half sum of positive roots for G, which explains this notation.) If $\check{G}$ is reductive, then we denote by ρ any lift of ρ from $\check{G}_{\mathrm{ad}}$ to $\check{G}$; this is ambiguous up to center, which will not affect the validity of our statements. For $\check{G} = \mathrm{GL}_n$, it is customary in the literature to choose the specific lift $\rho = (n-1, \dots, 1)$[4].

[4] It is usually called η in the literature but we reserve this notation for generic points.

Example 2.2.2. If $G = \mathrm{GL}_n$, then $\check{B}$ is the Borel subgroup of *lower* triangular matrices in $\check{G} \cong \mathrm{GL}_n$. This is the *opposite* Borel to the one chosen in [LHLM23b]; however, because we work with left cosets instead of right cosets in the affine flag variety of $\check{G}$, the upshot is that the combinatorial formulas here (e.g., for affine Weyl groups) will be *consistent* with the formulas in [LHLM23b].

Starting in §7, we will have chosen a split maximal torus $T \subset G$ and an isomorphism $X^*(T) \cong X_*(\check{T})$. Then our choice of $\check{B}$ induces a choice of Borel $B \supset T$. We let Φ be the roots of T on $\mathfrak{g}$ and take the *positive roots* $\Phi^+ \subset \Phi$ to be the roots of T on $\mathfrak{g}/\mathfrak{b}$, so the roots $\Phi^- = -\Phi^+$ of T on $\mathfrak{b}$ are *negative*. We let $\Delta \subset \Phi^+$ be the simple positive roots.

2.3. **Root systems.**

2.3.1. *Weyl groups.* Let W be the finite Weyl group of $(\check{G}, \check{T})$. We let $w_0 \in W$ be the longest element.

Let $\check{Q}^\vee \subset X_*(\check{T})$ be coroot lattice inside the coweight lattice of $\check{G}$. We let $\widetilde{W} \cong X_*(\check{T}) \rtimes W$ be the *extended affine Weyl group* (for $\check{G}$) and $W_{\mathrm{aff}} \cong \check{Q}^\vee \rtimes W$ be the *affine Weyl group*. For $\lambda \in X_*(\check{T})$, we write t^λ for the corresponding element in $\widetilde{W}$.

For $(\alpha, k) \in \check{\Phi} \times \mathbf{Z}$, the corresponding *root hyperplane* is

$$H_{\alpha,k} := \{\lambda \in X_*(\check{T})_{\mathbf{R}} \colon \langle \lambda, \alpha \rangle = k\}.$$

We abbreviate $H_\alpha := H_{\alpha,0}$. The hyperplanes $H_{\alpha,k}$ divide $X_*(\check{T})_{\mathbf{R}}$ into alcoves, which are acted upon simply transitively by W_{aff}. Let A_0 be the *dominant* base alcove anchored at 0. This choice determines the Bruhat order on $\widetilde{W}$, and a set of simple reflections for the Coxeter group W_{aff}. Given $\widetilde{w} \in \widetilde{W}$, let $\widetilde{W}_{\leq \widetilde{w}}$ be the set of elements $\leq \widetilde{w}$ in the Bruhat order. For discussions related to modular representation theory (in characteristic p), we will also need $C_0 = -\rho + pA_0$, the dominant ρ-shifted base p-aclove.

We denote by $\widetilde{W}^+$ the set of *dominant* elements, i.e., those $\widetilde{w}$ such that $\widetilde{w}(A_0)$ is dominant. For $\widetilde{w} \in \widetilde{W}$, we denote by $\widetilde{w}_{\mathrm{dom}}$ the unique element in $W\widetilde{w} \cap \widetilde{W}^+$. It is the minimal representative of $W\widetilde{w}$.

At times we will have chosen a split maximal torus $T \subset G$ and an identification of T with A, i.e., an identification of $\check{T}$ with the Langlands dual torus of T. This entails an identification $X_*(\check{T}) \cong X^*(T)$, and we will sometimes use this to interpret subsets of one side as subsets of the other. For example, this identifies the coroots $\check{\Phi}^\vee \subset X_*(\check{T})$ with the roots $\Phi \subset X^*(T)$, hence the coroot lattice $\check{Q}^\vee \subset X_*(\check{T})$ with the root lattice $Q \subset X^*(T)$. It also gives the alternate presentation $\widetilde{W} \cong X^*(T) \rtimes W$ of the extended affine Weyl group.

Let Ω be the stabilizer of the A_0 for the action of $\widetilde{W}$. We have

$$\Omega \cong X_*(\check{T})/\check{Q}^\vee \cong \widetilde{W}/W_{\mathrm{aff}}.$$

For $\lambda \in X_*(\check{T})$, we define

$$\mathrm{Adm}(\lambda) := \{\widetilde{w} \in \widetilde{W} \colon \widetilde{w} \leq t^{w(\lambda)} \text{ for some } w \in W\}.$$

Following the notation of [LHLM23b, §2.1.1], for $\alpha \in \check{\Phi}$ we define the *mth α-strip*

$$H_\alpha^{(m,m+1)} := \{x \in X_*(\check{T})_{\mathbf{R}} \colon m < \langle x, \alpha \rangle < m+1\}.$$

We say that an alcove $A \subset X^*(T)_{\mathbf{R}}$ is *regular* if it does not lie in any strips (for any α, m) containing the base alcove. We say $\widetilde{w} \in \widetilde{W}$ is *regular* if $\widetilde{w}(A_0)$ is regular. We define $\widetilde{W}^{\mathrm{reg}}$ to be the subset of regular elements in $\widetilde{W}$. We define $\mathrm{Adm}^{\mathrm{reg}}(\lambda) := \mathrm{Adm}(\lambda) \cap \widetilde{W}^{\mathrm{reg}}$.

The *fundamental box* is the intersection of all α-strips passing through A_0 where α is a simple root. It is a fundamental domain for the action of $\check{Q}^\vee$ modulo central translations, and we denote by $\widetilde{W}_1 \subset \widetilde{W}$ the subset sending A_0 to the fundamental box. Elements of $\widetilde{W}_1$ can be enumerated as follows: for each $w \in W$, there is a unique (up to central translations) element $\widetilde{w} = t^{\rho_w} w$ such that $\widetilde{w}(A_0)$ is the unique translate of $w(A_0)$ contained in the fundamental box. One can choose $\rho_w = \sum_{w^{-1}\alpha < 0} \omega_\alpha$, the sum of fundamental coweights for simple roots α such that $w^{-1}(\alpha) < 0$.

2.3.2. *Actions.* For $\widetilde{w} \in \widetilde{W}$ and $\lambda \in X_*(\check{T})$, we write $\widetilde{w}\lambda$ or $\widetilde{w} \cdot \lambda$ for the natural action of $\widetilde{w}$ on λ. We write

$$\widetilde{w} \bullet \lambda := \widetilde{w}(\lambda + \rho) - \rho$$

for the dot action.

Letting $\widetilde{w} = wt^{\nu}$, we write $\widetilde{w} \cdot_p \lambda := wt^{p\nu}\lambda$ for the natural action dilated by p on the lattice part, and $\widetilde{w} \bullet_p \lambda := wt^{p\nu} \bullet \lambda$ for the dot action dilated by p on the lattice part.

2.3.3. *Heights of weights.* For $\lambda \in X_*(\check{T}) \cong X^*(T)$, we define its *height*

$$h_\lambda := \max_\alpha |\langle \lambda, \alpha^\vee \rangle|.$$

This can be generalized for $\widetilde{w} \in \widetilde{W}$: we define $h_{\widetilde{w}}$ to be the maximum over $\alpha \in \check{\Phi}$ of the number of α root hyperplanes $H_{\alpha,k}$ separating A_0 and $\widetilde{w}A_0$.

If G is simple and simply connected, then the *Coxeter number* of G is $h_\rho + 1$. In general, let h be the maximum of the Coxeter numbers of simple factors of G_{ad}. **We assume throughout that $p > h$.**

2.3.4. *Genericity.* Let $m \in \mathbf{N}$.

- We say that $\lambda \in X_*(\check{T})$ is *m-generic* if $|\langle \lambda, \alpha^\vee \rangle + pk| > m$ for all $\alpha \in \Phi^+$ and $k \in \mathbf{Z}$. (This terminology is consistent with [LHLM23b, Definition 2.1.10].)

  Given a commutative ring R, we say that an element $\gamma \in X_*(\check{T}) \otimes_{\mathbf{Z}} R$ is *m-generic* if $\langle \gamma, \alpha^\vee \rangle + i + pk \in R^\times$ for all $k \in \mathbf{Z}$ and all $i \in \{0, \pm 1, \ldots, \pm m\}$. Taking $R = \mathbf{F}_p$, note that λ is m-generic (in the sense of the preceding paragraph) if and only if $\lambda \otimes 1 \in X_*(\check{T}) \otimes_{\mathbf{Z}} \mathbf{F}_p$ is m-generic.
- For $m \geq 0$ and $\widetilde{w} = wt^\nu \in \widetilde{W}$, we say that $\widetilde{w}$ is *m-small* if $h_\nu \leq m$.

Example 2.3.1. (1) We are interested in the examples $R_1 = \mathbf{F}_q[[t]]$ or $R_2 = \mathbf{Z}_q[\varepsilon]((t+p))$. In these cases $X_*(\check{T}) \otimes_{\mathbf{Z}} R_1 \cong \check{\mathfrak{t}}(\mathbf{F}_q)[[t]]$ or $X_*(\check{T}) \otimes_{\mathbf{Z}} R_2 \cong \check{\mathfrak{t}}(\mathbf{Z}_q)[\varepsilon]((t+p))$, and γ is an element used to construct a "deformation of an affine Springer fiber" or a "local model for a stack of potentially crystalline local Galois representations", respectively.

(2) If $\widetilde{w} \in \mathrm{Adm}(\lambda)$ then $\widetilde{w}(0)$ lies in the convex hull of $W\lambda$, hence $h_{\widetilde{w}} \leq h_\lambda$.

2.4. **Specialization for Chow groups.** Throughout this paper, we denote $\mathrm{Ch}(\mathcal{X}) := \mathrm{Ch}(\mathcal{X})_{\mathbf{Q}}$ for the *rational* Chow group of an algebraic scheme or stack $\mathcal{X}$. We will never consider integral structures on Chow groups.

Let S be a regular scheme, $i \colon Z \hookrightarrow S$ a closed regular embedding of codimension d, and $j \colon U \hookrightarrow S$ its open complement. Assume that Z is regular and the normal bundle to i has trivial top Chern class.

Let $f \colon \mathcal{X} \to S$ be a finitely presented map of schemes. Write $\mathcal{X}_Z$ for the base change of $\mathcal{X}$ to Z, and $\mathcal{X}_U$ for the base change of $\mathcal{X}$ to U. Recall the relative Chow group [Ful98, Chapter 20.1] $\mathrm{Ch}_*(\mathcal{X}_U/U)$, etc. Under the assumption, [Ful98, Chapter 20.3] constructs a *specialization map*

$$\mathfrak{sp} \colon \mathrm{Ch}_m(\mathcal{X}_U/U) \to \mathrm{Ch}_m(\mathcal{X}_Z/Z)$$

as follows. There is a refined Gysin map $i^! \colon \mathrm{Ch}_m(\mathcal{X}_U/U) \to \mathrm{Ch}_m(\mathcal{X}_Z/Z)$, and the Chern class assumption implies that $i^!$ factors over the restriction map $\mathrm{Ch}_m(\mathcal{X}/S) \to \mathrm{Ch}_m(\mathcal{X}_U/U)$, which is surjective. This factorization is by definition $\mathfrak{sp}$, as depicted in the diagram below.

$$\begin{tikzcd}
\mathrm{Ch}(X/S) \arrow[r, two heads] \arrow[dr, "i^!"'] & \mathrm{Ch}_m(\mathcal{X}_U/U) \arrow[d, dashed, "\mathfrak{sp}"] \\
& \mathrm{Ch}_m(\mathcal{X}_Z/Z)
\end{tikzcd}$$

2.4.1. *Specialization over a DVR.* Suppose S is the spectrum of a discrete valuation ring, with generic point $\eta \in S$ and special point $s \in S$. Then $\mathfrak{sp}$ can be described as follows: for the cycle class $[W_\eta] \in \mathrm{Ch}_m(\mathcal{X}_\eta)$ of some $W_\eta \subset \mathcal{X}_\eta$, let $W \subset \mathcal{X}$ be the Zariski closure of W_η. Then W is flat over S, and

$$\mathfrak{sp}([W_\eta]) = [W_s] \in \mathrm{Ch}_m(\mathcal{X}_s).$$

This description shows the following effectivity of the specialization map.

Lemma 2.4.1. *Suppose S is the spectrum of a DVR, with generic point $\eta \in S$ and special point $s \in S$. If $\alpha \in \mathrm{Ch}_m(\mathcal{X}_\eta)$ is represented by an effective cycle, then $\mathfrak{sp}(\alpha) \in \mathrm{Ch}_m(\mathcal{X}_s)$ is also represented by an effective cycle.*

2.4.2. *Iterated specialization.* Suppose S is regular scheme, and $f_1, f_2 \in \mathcal{O}(S)$ are such that:

- For each $m \in \{1, 2\}$, the closed subscheme $Z_m = V(f_m)$ is regular, realizing $i_m : Z_m \hookrightarrow S$ as a regular embedding of codimension 1.
- The closed subscheme $Z := V(f_1, f_2)$ is regular, realizing $\bar{i}_m : Z \hookrightarrow Z_m$ as a regular embedding of codimension 1.

Note that our assumption implies that for each $m = 1, 2$ the normal bundle of i_m is trivial and the normal bundle of $Z \hookrightarrow Z_m$ is also trivial.

Example 2.4.2. Let k be a field. A prototypical example is $S = k[t_1, t_2]$ or $k[[t_1, t_2]]$ with $f_i = t_i$.

We will be interested in a mixed-characteristic variant: $S = \operatorname{Spec} \mathbf{Z}_q[\varepsilon]$ or $S = \operatorname{Spec} \mathbf{Z}_q[\varepsilon]_{(\varepsilon)}$ with $f_1 = p, f_2 = \varepsilon$.

Now let $f : \mathcal{X} \to S$ be a finitely presented map. Under the assumptions above, we have specialization maps

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{X}[\frac{1}{f_1 f_2}]/S[\frac{1}{f_1 f_2}]) \arrow[r, "\mathfrak{sp}_{f_1 \to 0}"] \arrow[d, "\mathfrak{sp}_{f_2 \to 0}"] & \mathrm{Ch}(\mathcal{X}_{Z_1}[\frac{1}{f_2}]/Z_1[\frac{1}{f_2}]) \arrow[d, "\mathfrak{sp}_{f_2 \to 0}"] \\
\mathrm{Ch}(\mathcal{X}_{Z_2}[\frac{1}{f_1}]/Z_2[\frac{1}{f_1}]) \arrow[r, "\mathfrak{sp}_{f_1 \to 0}"] & \mathrm{Ch}(\mathcal{X}_Z/Z)
\end{tikzcd} \tag{2.4.1}$$

Lemma 2.4.3. *Diagram* (2.4.1) *commutes.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{X}/S) \arrow[r, two heads] \arrow[dr, "i_1^!"'] & \mathrm{Ch}(\mathcal{X}[\frac{1}{f_2}]/S[\frac{1}{f_2}]) \arrow[r, two heads] \arrow[dr, "i_1^!"'] & \mathrm{Ch}(\mathcal{X}[\frac{1}{f_1 f_2}]/S[\frac{1}{f_1 f_2}]) \arrow[d, "\mathfrak{sp}_{f_1 \to 0}"] \\
 & \mathrm{Ch}(\mathcal{X}_{Z_1}/Z_1) \arrow[r, two heads] \arrow[dr, "\bar{i}_1^!"'] & \mathrm{Ch}(\mathcal{X}_{Z_1}[\frac{1}{f_2}]/Z_1[\frac{1}{f_2}]) \arrow[d] \\
 & & \mathrm{Ch}(\mathcal{X}_Z/Z)
\end{tikzcd} \tag{2.4.2}$$

By definition of the specialization maps, the two triangles in the right column commute. By base change compatibility for the refined Gysin pullback $i_1^!$, the top left parallelogram commutes. Hence the whole diagram commutes. The right-then-down path in (2.4.1) is the right column of (2.4.2). The commutativity of (2.4.2) says that it can be computed by picking any lift in $\mathrm{Ch}_*(\mathcal{X}/S)$ and applying $\bar{i}_1^! \circ i_1^!$, which equals $i^!$ by the compositional property of the refined Gysin pullback.

Now note that $i^! : \mathrm{Ch}(\mathcal{X}/S) \to \mathrm{Ch}(\mathcal{X}_Z/Z)$ does not depend on the factorization of the embedding $i : Z \hookrightarrow S$. A symmetric argument shows that it computes the down-then-left path in (2.4.1) in the same sense, hence verifying the commutativity of (2.4.1). $\square$

2.5. **Borel-Moore homology.** Our convention is that all pullback/pushforward operations on ℓ-adic sheaves are derived, so we write $\pi_* := R\pi_*$, etc.

2.5.1. *Borel-Moore homology.* Let k be a field and $\pi : X \to \operatorname{Spec} k$ be a finite type scheme. Suppose there is a square-root of the cyclotomic character $\mathrm{Gal}_k \to \mathbf{Z}_\ell^\times$; choose one to define the half Tate-twist $(1/2)$. (This is the case if k is an algebraic extension of $\mathbf{Q}_p$ or $\mathbf{F}_p$, possibly after enlarging $\mathbf{Z}_\ell$, which covers all the cases we will consider.) Then we define the *mth (ℓ-adic) Borel-Moore homology group to be*

$$H_m^{\mathrm{BM}}(X) := H^{-2m}(X; \mathbf{D}_X(-m/2))$$

where $\mathbf{D}_X \cong \pi^! \mathbf{Q}_{\ell, \operatorname{Spec} k}$ is the dualizing sheaf on X. (We will only ever consider the case where m is even, so we will never invoke the choice of square root of cyclotomic character.)

We also define the *geometric Borel-Moore homology groups* to be

$$\mathrm{H}_m^{\mathrm{BM}}(X) := H_m^{\mathrm{BM}}(X_{\overline{k}}).$$

We emphasize the difference in font used to distinguish the geometric and absolute Borel-Moore homology. Note that the Tate twists do not affect the underlying group of $\mathrm{H}^{\mathrm{BM}}_m(X)$. Pullback defines a map $H^{\mathrm{BM}}_m(X) \to \mathrm{H}^{\mathrm{BM}}_m(X)$. We will mostly be interested in geometric Borel-Moore homology; the absolute Borel-Moore homology groups are only used as intermediate steps to make constructions with the geometric groups.

Given a proper map $f\colon X \to Y$ over k, the adjunction $f_*\mathbf{D}_X = f_!\mathbf{D}_X \to \mathbf{D}_Y$ induces a map

$$f_*\colon H^{\mathrm{BM}}_*(X) \to H^{\mathrm{BM}}_*(Y).$$

For an ind-scheme $X = \varinjlim_i X_i$, we define

$$H^{\mathrm{BM}}_m(X) := \varinjlim_i H^{\mathrm{BM}}_*(X_i)$$

with the transition maps induced by the closed embeddings $X_i \hookrightarrow X_j$ as above. Similarly we define

$$\mathrm{H}^{\mathrm{BM}}_m(X) := \varinjlim_i \mathrm{H}^{\mathrm{BM}}_*(X_i).$$

2.5.2. *Relative Borel-Moore homology.* Given a map $\pi\colon X \to S$, the *relative Borel-Moore homology* of X/S is

$$H^{\mathrm{BM}}_m(X/S) := \mathrm{H}^{-m}(X; \pi^!\mathbf{Q}_{\ell,S}(-m/2)).$$

When $S = \operatorname{Spec} k$, this recovers the previously defined (absolute) Borel-Moore homology groups.

2.5.3. *The cycle class map.* For any scheme $X \to \operatorname{Spec} k$, there is a cycle class map

$$\mathrm{Ch}_m(X) \to H^{\mathrm{BM}}_{2m}(X).$$

For a cycle Z in a space X, we write $[Z] \in \mathrm{Ch}_{\dim Z}(X)$ for its Chow class or $[Z] \in \mathrm{H}^{\mathrm{BM}}_{2\dim Z}(X)$ for the ℓ-adic realization of $[Z]$ in the *geometric* Borel-Moore homology of X, depending on context to make it clear which version we refer to.

2.6. **Specialization for Borel-Moore homology.** Let the setup be as in §2.4. Then there is a specialization map (for example, apply [DJK21, §4.5.6] with coefficients being $\mathbf{Q}_\ell$)

$$\mathfrak{sp}\colon H^{\mathrm{BM}}_*(\mathcal{X}_U/U) \to H^{\mathrm{BM}}_*(\mathcal{X}_Z/Z).$$

2.6.1. *Functoriality.* Specialization maps are functorial with respect to the following types of morphisms.

- Let $f\colon \mathcal{X} \to \mathcal{Y}$ be proper. Then there are pushforward maps

$$f_*\colon H^{\mathrm{BM}}_*(\mathcal{X}_U/U) \to H^{\mathrm{BM}}_*(\mathcal{Y}_U/U)$$
$$f_*\colon H^{\mathrm{BM}}_*(\mathcal{X}_Z/Z) \to H^{\mathrm{BM}}_*(\mathcal{Y}_Z/Z)$$

  that fit into a commutative diagram

$$\begin{tikzcd} H^{\mathrm{BM}}_*(\mathcal{X}_U/U) \arrow[r, "\mathfrak{sp}"] \arrow[d, "f_*"] & H^{\mathrm{BM}}_*(\mathcal{X}_Z/Z) \arrow[d, "f_*"] \\ H^{\mathrm{BM}}_*(\mathcal{Y}_U/U) \arrow[r, "\mathfrak{sp}"] & H^{\mathrm{BM}}_*(\mathcal{Y}_Z/Z) \end{tikzcd}$$

- Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ promotes to a quasi-smooth map of derived schemes[5] of virtual dimension $d(f)$. Then there are pullback maps

$$f^!\colon H^{\mathrm{BM}}_*(\mathcal{Y}_U/U) \to H^{\mathrm{BM}}_{*+2d(f)}(\mathcal{X}_U/U)$$
$$f^!\colon H^{\mathrm{BM}}_*(\mathcal{Y}_Z/Z) \to H^{\mathrm{BM}}_{*+2d(f)}(\mathcal{X}_Z/Z)$$

  that fit into a commutative diagram

$$\begin{tikzcd} H^{\mathrm{BM}}_{*+2d(f)}(\mathcal{X}_U/U) \arrow[r, "\mathfrak{sp}"] & H^{\mathrm{BM}}_{*+2d(f)}(\mathcal{X}_Z/Z) \\ H^{\mathrm{BM}}_*(\mathcal{Y}_U/U) \arrow[r, "\mathfrak{sp}"] \arrow[u, "f^!"] & H^{\mathrm{BM}}_*(\mathcal{Y}_Z/Z) \arrow[u, leftrightarrow, "f^!"'] \end{tikzcd}$$

[5]See [FH] for a primer on these notions, with references.

2.6.2. *Geometric specialization for DVRs.* Suppose S is a discrete valuation ring, with generic point $\eta \in S$ and special point $s \in S$. Let $f\colon \mathcal{X} \to S$ be of finite presentation. Then there is a specialization map for the *geometric* Borel-Moore homology groups,

$$\mathfrak{sp}\colon \mathrm{H}_*^{\mathrm{BM}}(\mathcal{X}_\eta) \to \mathrm{H}_*^{\mathrm{BM}}(\mathcal{X}_s).$$

It is obtained by taking the colimit of the specialization maps over finite extensions S'/S (cf. [Ful98, Example 20.3.5]).

2.6.3. *Dimensions.* We set $d := \dim \mathcal{B}$ to be the dimension of the flag variety of G. At various points we write $\mathrm{Ch}_{\mathrm{top}}$ or $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}$ for the group of top-degree classes (in Chow groups or geometric Borel-Moore homology groups), which usually (but not always) means $\mathrm{Ch}_{\mathrm{top}} = \mathrm{Ch}_d$ and $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}} = \mathrm{H}^{\mathrm{BM}}_{2d}$.

Part 1. Degeneration of local models

3. Deformations of affine Springer fibers

In this section, we define deformations of geometric objects called "affine Springer fibers". Let us give a roadmap to these constructions and their significance. We construct a family $\mathcal{X}_\gamma^\varepsilon$ over the 2-dimensional base $\mathrm{Spec}\, \mathbf{Z}_q[\varepsilon]$, depending on $\check{G}$ and a parameter $\gamma \in \check{\mathfrak{g}}(\mathbf{Z}_q)[\varepsilon]((t+p))$. To give a feel for the family $\mathcal{X}_\gamma^\varepsilon \to \mathrm{Spec}\, \mathbf{Z}_q[\varepsilon]$, we describe its fiber over various special loci in the base.

- Over the locus $(\varepsilon = 1)$, $\mathcal{X}_\gamma^\varepsilon$ is a $\mathbf{Z}_q$-scheme closely related to the local models of potentially crystalline substacks in the Emerton-Gee stack for $\check{G}$. More precisely, it is (for appropriate choices of γ) a "naive local model" of [LHLM23b, §3].
- Over the locus $(\varepsilon = 0)$, $\mathcal{X}_\gamma^\varepsilon$ is a mixed-characteristic degeneration of *affine Springer fibers*, whose geometry will be seen to be closely connected to representation theory.

The Breuil-Mézard Conjecture will be related to the locus $(\varepsilon = 1)$, but we have better traction on the locus $(\varepsilon = 0)$, thanks to geometric representation theory. The family $\mathcal{X}_\gamma^\varepsilon$ interpolates between these, and will allow us to transfer information from one to the other.

The construction of $\mathcal{X}_\gamma^\varepsilon$ occupies §3.1 and §3.2. From §3.3 – §3.5 we construct an *affine Springer action* on its relative Borel-Moore homology over the locus $p = 0$. Finally in §3.6 and §3.7 we analyze the irreducible components in the fibers of $\mathcal{X}_\gamma^\varepsilon$ over points of interest.

3.1. **Families of affine flag varieties.** Let G be a split reductive group over $\mathbf{Z}_q$, and $\check{G}$ its Langlands dual group *also* regarded as a split reductive group over $\mathbf{Z}_q$. Recalling that we have fixed a Borel subgroup $\check{B} \subset \check{G}$, we let $\check{\mathcal{G}}$ be the Bruhat-Tits group scheme over $\mathbf{A}^1_{\mathbf{Z}_q}$ obtained by dilatation of the Chevalley group scheme $\check{G}/\mathbf{A}^1_{\mathbf{Z}_q}$ along $\check{B}_{\mathbf{Z}_q} \subset \check{G}_{\mathbf{Z}_q}$ in the fiber at the origin of $\mathbf{A}^1_{\mathbf{Z}_q}$, cf. [LHLM23b, §3.1].

Remark 3.1.1 (Comparison to conventions of [LHLM23b])**.** When comparing to [LHLM23b], our $\check{B}$ will be the *opposite Borel* to the one of *loc. cit.*. This comes from our convention to view the affine Grassmanian as a left coset space rather than a right coset space. For example, when $\check{G} = \mathrm{GL}_n$ our formulas will be based on the choice of $\check{B}$ as the *lower-triangular* Borel subgroup.

Let $L\check{\mathcal{G}}$ be the functor on $\mathbf{Z}_q$-algebras R sending $R \mapsto \check{\mathcal{G}}(R((t+p)))$ and $L^+\check{\mathcal{G}}$ be the functor on $\mathbf{Z}_q$-algebras R sending $R \mapsto \check{\mathcal{G}}(R[[t+p]])$. Let $\mathrm{Gr}_{\check{\mathcal{G}}}$ be the fppf quotient $L\check{\mathcal{G}}/L^+\check{\mathcal{G}}$. It is an ind-scheme over $\mathbf{Z}_q$, with the following properties:

- The generic fiber $\mathrm{Gr}_{\check{\mathcal{G}}} \times_{\mathrm{Spec}\, \mathbf{Z}_q} \mathrm{Spec}\, \mathbf{Q}_q$ is isomorphic to $\mathrm{Gr}_{\check{G},\mathbf{Q}_q}$, the affine Grassmannian for $\check{G}_{\mathbf{Q}_q}$.
- The special fiber $\mathrm{Gr}_{\check{\mathcal{G}}} \otimes_{\mathrm{Spec}\, \mathbf{Z}_q} \mathrm{Spec}\, \mathbf{F}_q$ is isomorphic to $\mathrm{Fl}_{\check{G},\mathbf{F}_q}$, the affine flag variety for $\check{G}_{\mathbf{F}_q}$.

We therefore think of $\mathrm{Gr}_{\check{\mathcal{G}}}$ as a mixed-characteristic degeneration from the affine Grassmannian to the affine flag variety.

3.2. **Deformed affine Springer fibers.** Let $\gamma \in \check{\mathfrak{t}}(\mathbf{Z}_q)[\varepsilon]((t+p)) \subset \check{\mathfrak{g}}(\mathbf{Z}_q)[\varepsilon]((t+p))$. We will write $\gamma_0 \in \mathfrak{g}(\mathbf{Z}_q)((t))$ for the evaluation of γ at $\varepsilon = 0$.

Let $\mathbf{A}^1_{\mathbf{Z}_q} = \mathrm{Spec}\, [\varepsilon]$. Let $\mathcal{X}_\gamma^\varepsilon$ be the sub-(ind-)scheme of $\mathrm{Gr}_{\check{\mathcal{G}}} \times_{\mathbf{Z}_q} \mathbf{A}^1_{\mathbf{Z}_q}$ defined as

$$\mathcal{X}_\gamma^\varepsilon = \left\{ g \in \mathrm{Gr}_{\check{\mathcal{G}}} \times_{\mathbf{Z}_q} \mathbf{A}^1_{\mathbf{Z}_q} \colon \mathrm{Ad}_{g^{-1}}(\gamma) - \varepsilon t^2 \frac{dg^{-1}}{dt} g \in \mathrm{Lie}\, L^+\check{\mathcal{G}} \right\},$$

where the symbol $\frac{dg^{-1}}{dt}g =: d\log(g^{-1})$ is understood as in [FZ10], and is explained in [Fre07, §1.2.4]. (A small calculation is required to see that the defining equation is invariant for the right action of $L^+\check{\mathcal{G}}$, cf. [LHLM23b, Lemma 3.3.1].) We note that

$$-\frac{dg^{-1}}{dt}g = -d\log(g^{-1}) = d\log(g) = g^{-1}\frac{dg}{dt}$$

and occasionally we will use the latter form of the expression, for comparison to formulas in [LHLM23b].

Remark 3.2.1. The definition of $\mathcal{X}_\gamma^\varepsilon$ makes sense when γ lies more generally in $\check{\mathfrak{g}}(\mathbf{Z}_q)[\varepsilon]((t+p))$, but we do not need this generality and its complicates the notion of translation by the "stabilizer of γ", so we restrict our attention to γ of the stated sort.

Remark 3.2.2. More generally, there are versions of $\mathcal{X}_\gamma^\varepsilon$ with defining equation

$$\mathrm{Ad}_{g^{-1}}(\gamma) - t^r\frac{dg^{-1}}{dt}g \in \mathrm{Lie}\, L^+\check{\mathcal{G}}$$

as soon as $r \geq 2$; the versions with $r > 2$ will not be needed in this paper.

For $r \geq 2$, we introduce the notation

$$\mathrm{Ad}^{\varepsilon,r}_{g^{-1}}(\gamma) := \mathrm{Ad}_{g^{-1}}(\gamma) - \varepsilon t^r\frac{dg^{-1}}{dt}g.$$

We are primarily interested in $r = 2$, in which case we abbreviate $\mathrm{Ad}^\varepsilon := \mathrm{Ad}^{\varepsilon,2}$. A straightforward calculation shows that

$$\mathrm{Ad}^{\varepsilon,r}_{g_1g_2}(\gamma) = \mathrm{Ad}^{\varepsilon,r}_{g_1}(\mathrm{Ad}^{\varepsilon,r}_{g_2}(\gamma)). \tag{3.2.1}$$

3.2.1. *Specializations.* We introduce some notation for fibers of the family $\mathcal{X}_\gamma^\varepsilon \to \mathrm{Spec}\,\mathbf{Z}_q$ over specific loci.

- For a fixed $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{Z}_q}$ we denote by $\mathcal{X}_\gamma^{\varepsilon=\varepsilon_0}$ the fiber of $\mathcal{X}_\gamma^\varepsilon$ over ε_0.
- We denote by $\mathrm{X}_\gamma^\varepsilon := \mathcal{X}_\gamma^\varepsilon|_{\mathbf{A}^1_{\mathbf{Q}_q}}$ the fiber of $\mathcal{X}_\gamma^\varepsilon$ over $\mathrm{Spec}\,\mathbf{Q}_q$. For $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{Q}_q}$ we denote by $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}$ the fiber of $\mathrm{X}_\gamma^\varepsilon$ over ε_0.
- We denote by $\mathrm{Y}_\gamma^\varepsilon := \mathcal{X}_\gamma^\varepsilon|_{\mathbf{A}^1_{\mathbf{F}_q}}$ the fiber of $\mathcal{X}_\gamma^\varepsilon$ over $\mathrm{Spec}\,\mathbf{F}_q$. For $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{F}_q}$ we denote by $\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}$ the fiber of $\mathrm{Y}_\gamma^\varepsilon$ over ε_0.

Example 3.2.3 (The specialization $\varepsilon = 0$)**.** Consider $\varepsilon_0 = 0 \in \mathbf{A}^1_{\mathbf{Q}_q}$. Then $\mathrm{X}_\gamma^{\varepsilon=0}$ is the (spherical) *affine Springer fiber* associated to γ_0 (over $\mathbf{Q}_q$). In turn, $\mathrm{Y}_\gamma^{\varepsilon=0}$ is the Iwahori affine Springer fiber associated to γ_0 (over $\mathbf{F}_q$). These notions of affine Springer fibers were originally introduced by Kazhdan-Lusztig in [KL88]. Hence we may regard $\mathcal{X}_\gamma^\varepsilon$ as a two-parameter deformation of affine Springer fibers, over the two-dimensional base $\mathbf{A}^1_{\mathbf{Z}_q}$.

3.3. **Translation action.** For $\gamma \in \check{\mathfrak{t}}(\mathbf{Z}_q)[\varepsilon]((t+p))$, we write $\gamma_i \in \check{\mathfrak{t}}(\mathbf{Z}_q)((t+p))$ for the coefficient of ε^i in γ. The centralizer of γ_0 contains $L\check{T}$, and evidently acts on $\mathrm{X}_\gamma^{\varepsilon=0} = \mathrm{X}_{\gamma_0}^{\varepsilon=0}$ and $\mathrm{Y}_\gamma^{\varepsilon=0} = \mathrm{Y}_{\gamma_0}^{\varepsilon=0}$ by (left) translation.

At the level of underlying reduced schemes we have $L\check{T} \cong T \rtimes X_*(\check{T})$. The action of $L\check{T}$ on homology of $\mathrm{X}_\gamma^{\varepsilon=0}$ or $\mathrm{Y}_\gamma^{\varepsilon=0}$ therefore factors through an action of $X_*(\check{T})$, which we also refer to as the translation action.

There is no translation action on deformed affine Springer fibers; instead translation elements take one deformed affine Springer fiber to another. Namely, a quick computation shows that for $\gamma \in \check{\mathfrak{t}}(\mathbf{Z}_q)[\varepsilon]((t+p))$ and $h \in L\check{T}$, left multiplication by h takes $\mathcal{X}_{\gamma'}^\varepsilon$ isomorphically to $\mathcal{X}_\gamma^\varepsilon$, where

$$\gamma' := h^{-1}\gamma h - \varepsilon t^2\frac{dh^{-1}}{dt}h. \tag{3.3.1}$$

Remark 3.3.1. The reason why we consider γ having a dependence on ε is to have a class of objects preserved by translations by $L\check{T}$.

3.4. **The Grothendieck alteration.** We recall some facts from Springer theory; a reference is [Yun17, §1.2.2]. For a split reductive group H over a field, the *Grothendieck alteration* is the projection map $\pi\colon \widetilde{\mathfrak{h}} \to \mathfrak{h}$, where $\mathfrak{h}$ parametrizes pairs of $u \in \mathfrak{h}$ and a Borel subgroup $B_H \subset H$ such that $u \in \operatorname{Lie} B_H$. The map π is small, so $\pi_* \mathbf{Q}_{\ell, \widetilde{\mathfrak{h}}}$ is an intermediate extension supported on all of $\mathfrak{h}$. On the other hand, over the strongly regular semisimple locus of $\mathfrak{h}$, π is a torsor for the Weyl group W_H of H. These two facts together equip $\pi_* \mathbf{Q}_{\ell, \widetilde{\mathfrak{h}}}$ with a canonical action of W_H. Noting that $\widetilde{\mathfrak{h}}$ is smooth, so the dualizing sheaf $\mathbb{D}_{\widetilde{\mathfrak{h}}}$ is isomorphic to a shift and twist of $\mathbf{Q}_{\ell, \widetilde{\mathfrak{h}}}$, we equivalently get a W_H-action on $\mathbb{D}_{\widetilde{\mathfrak{h}}}$.

Definition 3.4.1. We say that a commutative diagram of schemes (or stacks)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is *Cartesian up to nilpotents* if the induced map on the underlying reduced subschemes from A to the fibered product $B \times_D C$ is an isomorphism.

Suppose we have a diagram

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_S"] & \widetilde{\mathfrak{h}} \arrow[d, "\pi"] \\ S \arrow[r, "f"] & \mathfrak{h} \end{tikzcd}$$

which is Cartesian up to nilpotents. Then by proper base change, the W_H-action on $\pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}}$ induces a W_H-action on

$$f^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}} \cong \pi_{S*} \widetilde{f}^! \mathbb{D}_{\widetilde{\mathfrak{h}}} \cong \pi_{S*} \mathbb{D}_{\widetilde{S}}.$$

In particular, after passing to cohomology we obtain a W_H-action on $\mathrm{H}^{\mathrm{BM}}_*(\widetilde{S})$. Actions constructed by this mechanism will generally referred to as "Springer actions".

Let us record the compatibility of Springer actions in a general situation. Given a commutative diagram

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_S"] & \widetilde{T} \arrow[r, "\widetilde{g}"] \arrow[d, "\pi_T"] & \widetilde{\mathfrak{h}} \arrow[d, "\pi"] \\ S \arrow[r, "f"] & T \arrow[r, "g"] & \mathfrak{h} \end{tikzcd}$$

in which all squares are Cartesian up to nilpotents, the W_H-action on $\pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}}$ induces also a W_H-action on $\pi_* \mathbb{D}_{\widetilde{T}}$ and $\pi_* \mathbb{D}_{\widetilde{S}}$, hence on $\mathrm{H}^{\mathrm{BM}}_*(\widetilde{T})$ and $\mathrm{H}^{\mathrm{BM}}_*(\widetilde{S})$.

Lemma 3.4.2. *(1) If f is proper, then the map $\mathrm{H}^{\mathrm{BM}}_*(\widetilde{S}) \xrightarrow{f_*} \mathrm{H}^{\mathrm{BM}}_*(\widetilde{T})$ is equivariant for the W_H-action.*

(2) If f can be promoted to a quasi-smooth map of derived schemes with virtual dimension $d(f)$,[6] *then the map $\mathrm{H}^{\mathrm{BM}}_*(\widetilde{T}) \xrightarrow{f^!} \mathrm{H}^{\mathrm{BM}}_{*+2d(f)}(\widetilde{S})$ is equivariant for the W_H-action, where $d(f)$ is the* virtual dimension *of f, i.e., the Euler characteristic of the cotangent complex of f.*

Proof. (1) By base change, the map in question is obtained by taking global sections on T of the map

$$f_* f^! g^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}} = f_! f^! g^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}} \xrightarrow{\text{counit}} g^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}},$$

and the W_H-action is induced by the Springer W_H-action on $\pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}}$. Naturality of the counit $f_! f^! \to \mathrm{Id}$ implies that the map is compatible with the W_H-action.

(2) By base change, the map in question is obtained by taking global sections on T of the composite map

$$g^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}} \xrightarrow{\text{unit}} f_* f^* g^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}} \xrightarrow{[f]} f_* f^! g^! \pi_* \mathbb{D}_{\widetilde{\mathfrak{h}}} \langle -d(f) \rangle$$

where $\langle i \rangle := [2i](2i)$ is a shift and Tate twist, and the natural transformation $[f]\colon f^* \to f^! \langle -d(f) \rangle$ is induced by the relative fundamental class of f, as explained in [FYZ, §3.4]. Naturality of $[f]$ and the unit $\mathrm{Id} \to f_* f^*$ imply that the map is compatible with the W_H-action. $\square$

[6]For example, this is the case if f is LCI.

3.5. **Affine Springer action.** Let $\check{\mathbf{I}} \subset L^+\check{G}_{\mathbf{F}_q}$ be the Iwahori subgroup corresponding to the fixed Borel subgroup $\check{B} \subset \check{G}_{\mathbf{F}_q}$.

There is an "affine Springer action" of W_{aff} on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=0})$ and on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=1})$. For the affine Springer fibers, the action was constructed by Lusztig [Lus96] and Sage [Sag97] for $W_{\mathrm{aff}} \subset \widetilde{W}$. An exposition of the construction for W_{aff} can be found in [Yun17, §2.6.3]. For the deformed affine Springer fibers at $\varepsilon = 1$, an action of W_{aff} on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=1})$ was constructed by Frenkel-Zhu [FZ10, §6], by imitating Lusztig's construction.

Remark 3.5.1 (Extended affine Springer action)**.** For $\varepsilon = 0$, the W_{aff}-action on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=0})$ was extended to an action of $\widetilde{W}$ by Yun in [Yun14, Theorem 2.5], which we also call the "affine Springer action".

We now construct an affine Springer action of W_{aff} on the relative Borel-Moore homology (cf. §2.5) of $\mathrm{Y}_\gamma^\varepsilon$ over $\mathbf{A}^1_\varepsilon$, by a slight generalization of [FZ10]. Our construction specializes to Lusztig's construction when $\varepsilon = 0$, and Frenkel-Zhu's when $\varepsilon = 1$. It works uniformly for any $r \geq 2$.

Recall that we defined
$$\mathrm{Ad}_g^{\varepsilon,r}(\gamma) := g\gamma g^{-1} - \varepsilon t^r \frac{dg}{dt} g^{-1}.$$
For each parahoric subgroup $\check{\mathbf{I}} \subset \check{\mathbf{P}} \subset L\check{G}_{\mathbf{F}_q}$, there is a corresponding affine Springer fiber
$$\mathrm{Y}^\varepsilon_{\check{\mathbf{P}},\gamma} := \{g \in L\check{G}_{\mathbf{F}_q} \colon \mathrm{Ad}^{\varepsilon,r}_{g^{-1}}(\gamma) \in \mathrm{Lie}\,\check{\mathbf{P}}\}/\check{\mathbf{P}}.$$
That $\mathrm{Y}^\varepsilon_{\check{\mathbf{P}},\gamma}$ is well-defined – i.e., the condition $\mathrm{Ad}_g^{\varepsilon,r}(\gamma) \in \mathrm{Lie}\,\check{\mathbf{P}}$ is preserved by right multiplication by $\check{\mathbf{P}}$ (for $r \geq 2$) – follows from [FZ10, Lemma 11].

Let $\check{\mathbf{P}}^u$ be the pro-unipotent radical of $\check{\mathbf{P}}$. Let $L_{\check{\mathbf{P}}}$ be the Levi quotient of $\check{\mathbf{P}}$ and $\mathfrak{l}_{\check{\mathbf{P}}} := \mathrm{Lie}\, L_{\check{\mathbf{P}}}$. Then there is an *evaluation map* sending $g\check{\mathbf{P}}^u \in \mathrm{Y}_{\check{\mathbf{P}},\gamma}$ to the reduction of $\mathrm{Ad}^{\varepsilon,r}(g^{-1})\gamma \in \mathrm{Lie}\,\check{\mathbf{P}}$ in $\mathrm{Lie}\,\check{\mathbf{P}}/\mathrm{Lie}\,\check{\mathbf{P}}^u \cong \mathfrak{l}_{\check{\mathbf{P}}}$, which is well-defined up to the adjoint action of $L_{\check{\mathbf{P}}}$, and thus defines a map
$$\mathrm{ev}\colon \mathrm{Y}^\varepsilon_{\check{\mathbf{P}},\gamma} \to [\mathfrak{l}_{\check{\mathbf{P}}}/L_{\check{\mathbf{P}}}]. \tag{3.5.1}$$

The following result is a variant of [FZ10, Proposition 12].

Lemma 3.5.2. *There is a natural (in $\check{\mathbf{P}}$) Cartesian square*
$$\begin{tikzcd} \mathrm{Y}^\varepsilon_\gamma \arrow[r, "\mathrm{ev}"] \arrow[d, "\widetilde{\pi}"] & {[\widetilde{\mathfrak{l}}_{\check{\mathbf{P}}}/L_{\check{\mathbf{P}}}]} \arrow[d, "\pi"] \\ \mathrm{Y}^\varepsilon_{\check{\mathbf{P}},\gamma} \arrow[r, "\mathrm{ev}"] & {[\mathfrak{l}_{\check{\mathbf{P}}}/L_{\check{\mathbf{P}}}]} \end{tikzcd} \tag{3.5.2}$$
where π is the Grothendieck alteration.

Proof. Let $g\check{\mathbf{P}} \in \mathrm{Y}^\varepsilon_{\check{\mathbf{P}},\gamma}$ and set
$$\gamma' := \mathrm{Ad}^{\varepsilon,r}_{g^{-1}}(\gamma) \in \mathrm{Lie}\,\check{\mathbf{P}}.$$
The fiber $\widetilde{\pi}^{-1}(g)$ consists of $gx\check{\mathbf{I}}$ such that $x \in \check{\mathbf{P}}$ and (using (3.2.1))
$$\mathrm{Ad}^{\varepsilon,r}_{(gx)^{-1}}(\gamma) = \mathrm{Ad}^{\varepsilon,r}_{x^{-1}}(\gamma') \in \mathrm{Lie}\,\check{\mathbf{I}}.$$
But
$$\mathrm{Ad}^{\varepsilon,r}_{x^{-1}}(\gamma') = \mathrm{Ad}_{x^{-1}}(\gamma') - \varepsilon t^r \frac{d(x^{-1})}{dt} x$$
and the second term belongs to $\mathrm{Lie}\,\check{\mathbf{P}}^u \subset \mathrm{Lie}\,\check{\mathbf{I}}$ by [FZ10, Proposition 11]. Thus the condition on x reduces to $\mathrm{Ad}_{x^{-1}}(\gamma') \in \mathrm{Lie}\,\check{\mathbf{I}}$, which in turn can be checked modulo $\mathrm{Lie}\,\check{\mathbf{P}}^u$. Since $\mathrm{ev}(g\check{\mathbf{P}})$ is exactly the class of γ' in $[\mathfrak{l}_{\check{\mathbf{P}}}/L_{\check{\mathbf{P}}}]$, the result follows.

□

As explained in §3.4, Lemma 3.5.2 induces an action of the Weyl group $W_{\check{\mathbf{P}}}$ associated to $\check{\mathbf{P}}$ on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}^\varepsilon_\gamma)$. Then just as in the case of affine Springer fibers [Lus96, §5.5], these actions glue to an action of W_{aff} on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}^\varepsilon_\gamma)$. It will actually be more convenient for us to normalize the action as a *right* action, using the antipode map on W_{aff}.

Remark 3.5.3 (Compatibility with translation action)**.** It is immediate from the definitions that the resulting $\widetilde{W}$-action commutes with the translation action on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=0})$.

Let us summarize the upshot of this construction.

Proposition 3.5.4 (Affine Springer action)**.** *For any $S \to \operatorname{Spec} \mathbf{F}_q[\varepsilon]$, letting $\mathrm{Y}_\gamma^\varepsilon|_S$ denote the base change of $\mathrm{Y}_\gamma^\varepsilon$ to S, there is a right action of W_{aff} on $H_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^\varepsilon|_S/S)$, with the following properties.*

(1) If s is a geometric point over $\varepsilon = 0$, then it is the usual affine Springer action of Lusztig.
(2) If s is a geometric point over $\varepsilon = 1$, then it is the action of Frenkel-Zhu [FZ10].
(3) It commutes with specialization in ε.

Proof. The first two points are clear from the construction. The third follows from the construction of the specialization map, Lemma 3.4.2, and the construction of the W_{aff}-action. □

3.6. **Parametrization of top-dimensional irreducible components: generic fiber.** Here we study certain top-dimensional irreducible components of $\mathrm{X}_\gamma^\varepsilon$ for $\gamma = (t+p)s$ where $s \in \check{\mathfrak{t}}$ is regular.

For a dominant coweight $\lambda \in X_*(\check{T})^+$, let $S^\circ(\lambda) \subset \mathrm{Gr}_{\check{G},\mathbf{Q}_q}$ be the corresponding (open) Schubert cell and $S(\lambda)$ its closure.

For $\lambda \in X_*(\check{T})$, let $\mathrm{X}_\gamma^\varepsilon(\lambda)$ be the Zariski closure of $\mathrm{X}_\gamma^\varepsilon \cap S^\circ(\lambda)$. For any $\varepsilon_0 \in \mathbf{A}_\varepsilon^1$, we also define $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda)$ as the Zariski closure of $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0} \cap S^\circ(\lambda)$.

Warning 3.6.1. It is evident that $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda)$ is a closed subscheme of the fiber at ε_0 of $\mathrm{X}_\gamma^\varepsilon(\lambda)$, which we denote $\mathrm{X}_\gamma^\varepsilon(\lambda)|_{\varepsilon_0}$, but it is not clear (at least to the authors) whether this closed embedding is an isomorphism.

Lemma 3.6.2. *Assume $\gamma_0 = (t+p)s$ with $s \in \check{\mathfrak{t}}$ regular. Then there is an open subscheme $V \subset \mathbf{A}_{\mathbf{Q}_q}^1$ containing 0 and an open subscheme $U \subset S^\circ(\rho)$ such that*

- *$(U \times V) \cap \mathrm{X}_\gamma^\varepsilon \cong \mathbf{A}^d \times V$ where we recall that $d = \dim(\check{G}/\check{B})$, and*
- *$\mathrm{X}_\gamma^\varepsilon(\rho) \setminus U \times V$ has fiberwise dimension less than d over V.*

In particular, if $\lambda < \rho$ then $\mathrm{X}_\gamma^{\varepsilon=0}(\lambda)$ has dimension strictly less than d, and $\mathrm{X}_\gamma^{\varepsilon=0}(\rho)$ is irreducible of dimension d.

Proof. We explain how this essentially follows from the computations in the proof of [LHLM23b, Proposition 3.3.4], in particular how to convert notations in *loc.cit.* to our situation.

We have a standard affine open cover of $S^\circ(\lambda)$ by translates of Iwahori orbits,

$$\{U_w(\lambda) := w^{-1} I(\lambda)(t+p)^\lambda\}_{w \in W}.$$

Here $I(\lambda)$ is the affine space of dimension $\langle \lambda, 2\rho^\vee \rangle$ given by a root group decomposition (after fixing any ordering of the roots)

$$I(\lambda) \cong \prod_{\alpha \in \check{\Phi}} \prod_{i=0}^{\langle \lambda, \alpha \rangle - 1} U_{\alpha,i},$$

where $U_{\alpha,i} \subset L\check{G}$ is the affine root group corresponding to the affine root $(\alpha, i) \in \check{\Phi} \times \mathbf{Z}$. (In the notation of the proof of [LHLM23b, Proposition 3.3.4], $I(\lambda)$ is the transpose of $(v-t)^{-\lambda}\widetilde{N}_\lambda$, noting that v, t, $\mathrm{Diag}(\mathbf{a})$ in *loc.cit.* correspond to $t, -p, s$ in our situation.)

Choosing coordinates $x_{\alpha,i}$ for the affine root groups appearing in $I(\lambda)$ identifies $(U_w(\lambda) \times \mathbf{A}_\varepsilon^1) \cap \mathrm{X}_\gamma^\varepsilon$ with the subspace of $I(\lambda)$ cut out by the equations

$$(\varepsilon i - \langle \mathrm{Ad}_w(s), \alpha \rangle) x_{\alpha,i} - p(i+1)\varepsilon x_{\alpha,i+1} = O(X_{\beta,j})$$

for $0 \le i < \langle \lambda, \alpha \rangle - 1$. Here the right-hand side is an expression in the $x_{\beta,j}$ with $0 < \beta < \alpha$. Over the locus $V \subset \mathbf{A}_\varepsilon^1$ where $\varepsilon i - \langle \mathrm{Ad}_w(s), \alpha \rangle$ is invertible, these equations cut out an affine subspace of dimension equal to $\dim(\check{G}/\check{P}_\lambda)$, with coordinates in the $x_{\alpha, \langle \lambda, \alpha \rangle - 1}$. In particular, we see that $\mathrm{X}_\gamma^\varepsilon(\lambda)|_V$ has fiberwise dimension $< d$ for $\lambda < \rho$.

Now, for each simple root $\alpha > 0$ of $\check{G}$, the coordinate $x_{\alpha,0}$ is fiberwise over V non-vanishing on $\mathrm{X}_\gamma^\varepsilon \cap (U_w(\rho) \times V)$ since $\langle \rho, \alpha \rangle = 1$, hence the intersection of $\mathrm{X}_\gamma^\varepsilon \cap (U_w(\rho) \times V)$ and $\mathrm{X}_\gamma^\varepsilon \cap (U_{s_\alpha w}(\rho) \times V)$ is fiberwise open dense in either space. It follows that $\mathrm{X}_\gamma^\varepsilon \cap (U_w(\rho) \times V)$ is fiberwise open dense in $\mathrm{X}_\gamma^\varepsilon \cap (S^\circ(\rho) \times V)$, hence the complement has fiberwise dimension less than d.

Remark 3.6.3. (1) For $\lambda \neq \rho$ regular, $\mathrm{X}_\gamma^{\varepsilon=0}(\lambda)$ is never irreducible, which is in stark contrast to the behavior of the deformed affine Springer fiber (see Lemma 3.6.4 below).

(2) By [Ngo10, Corollary 3.10.2], $\mathrm{X}_\gamma^{\varepsilon=0}$ is equidimensional, and the translation action by $L\check{T}_{\mathbf{Q}_q}$ is transitive on the set of irreducible components. In fact, one can check that $\mathrm{X}_\gamma^{\varepsilon=0} \cap S^\circ(\rho)$ is a fundamental domain for the $L\check{T}_{\mathbf{Q}_q}$-action on the regular locus of $\mathrm{X}_\gamma^{\varepsilon=0}$, and that the regular locus consists of exactly the $(L\check{T}/L^+\check{T})_{\mathbf{Q}_q} = X_*(\check{T})$ translates of $\mathrm{X}_\gamma^{\varepsilon=0} \cap S^\circ(\rho)$. This shows that for regular $\lambda \neq \rho$, $\mathrm{X}^{\varepsilon=0}(\lambda)$ is a (reducible) union of translates of $\mathrm{X}^{\varepsilon=0}(\rho)$.

□

Lemma 3.6.4. *Let $\mathrm{X}_\gamma^{\varepsilon\neq 0}$ be the fiber of $\mathrm{X}_\gamma^\varepsilon$ over $\mathbf{G}_m \subset \mathbf{A}^1_\varepsilon$. For $\lambda \in X_*(\check{T})^+$, the Bialynicki-Birula map induces an isomorphism $S^\circ(\lambda) \cap \mathrm{X}_\gamma^{\varepsilon\neq 0} \xrightarrow{\sim} \check{G}/\check{P}_\lambda \times \mathbf{G}_{m,\mathbf{Q}_q}$ as a family over $\mathbf{G}_{m,\mathbf{Q}_q} = \operatorname{Spec} \mathbf{Q}_q[\varepsilon^{\pm 1}]$.*

Proof. This is a consequence of the computation in the proof of Lemma 3.6.2. Since εp is invertible under the hypotheses, in the charts defined in the proof of Lemma 3.6.2 we can solve all the $x_{\alpha,i}$ in terms of $x_{\alpha,0}$. But the Bialynicki-Birula map on these charts is exactly the map extracting the $x_{\alpha,0}$.

□

In particular, the Bialynicki-Birula map induces an isomorphism $S^\circ(\lambda) \cap \mathrm{X}_\gamma^{\varepsilon=1} \xrightarrow{\sim} (\check{G}/\check{P}_\lambda)_{\mathbf{Q}_q}$. Therefore, the map

$$\lambda \mapsto S^\circ(\lambda) \cap \mathrm{X}_\gamma^{\varepsilon=1}$$

induces a bijection from the set of regular dominant weights $X_*(\check{T})^+$ to the top-dimensional irreducible components of $\mathrm{X}_\gamma^{\varepsilon=1}$. A similar discussion applies for $\varepsilon = \eta$.

For $\varepsilon_0 \in \mathbf{G}_{m,\mathbf{Q}_q}$, define

$$\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda) = \mathrm{X}_\gamma^{\varepsilon=\varepsilon_0} \cap S(\lambda);$$

which is a disjoint union of partial flag varieties $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda')$ for $\lambda' \leq \lambda$. We are particularly interested in $\varepsilon_0 = 1$ or $\varepsilon_0 = \eta$.

3.7. **Parametrization of top-dimensional irreducible components: special fiber.** Here we establish a combinatorial parametrization of the top-dimensional irreducible components of $\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}$ for any ε_0.

For each $\widetilde{w} \in \widetilde{W}$, let $S^\circ(\widetilde{w}) = \check{\mathbf{I}}\widetilde{w}\check{\mathbf{I}}/\check{\mathbf{I}} \subset \mathrm{Fl}_{\check{G},\mathbf{F}_q}$ be the corresponding (open) Schubert cell and $S(\widetilde{w}) \subset \mathrm{Fl}_{\check{G},\mathbf{F}_q}$ its closure.

Definition 3.7.1. For $\varepsilon_0 \in \mathbf{A}^1_\varepsilon$ and $\widetilde{w} \in \widetilde{W}$, let $\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\widetilde{w})$ be the closure of $\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0} \cap S^\circ(\widetilde{w})$ in $\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}$. (As in Warning 3.6.1, this construction may not commute with the base change in ε.)

For the statements below, recall the notation $h_{\widetilde{w}}$ from §2.3.1.

Lemma 3.7.2. *Let $\gamma = t(s + \varepsilon r)$ with $r, s \in \check{\mathfrak{t}}$. Assume s is regular semisimple.*

(1) $\mathrm{Y}_\gamma^{\varepsilon=0}(\widetilde{w})$ *is an affine space over* $\mathbf{F}_q$ *of dimension*

$$\dim \mathrm{Y}_\gamma^{\varepsilon=0}(\widetilde{w}) = d - \#\{\alpha \in \check{\Phi}^+ \mid \widetilde{w}(A_0) \subset H_\alpha^{(0,1)}\}.$$

In particular $\dim \mathrm{Y}_\gamma^{\varepsilon=0}(\widetilde{w}) = d$ *if and only if* $\widetilde{w}$ *is regular.*

(2) $\mathrm{Y}_\gamma^{\varepsilon=\eta}(\widetilde{w})$ *is an affine space over* $\mathbf{F}_q$ *of dimension*

$$\dim \mathrm{Y}_\gamma^{\varepsilon=\eta}(\widetilde{w}) = d - \#\{\alpha \in \check{\Phi}^+ \mid \widetilde{w}(A_0) \subset H_\alpha^{(0,1)}\}.$$

(3) If $s + r$ is $h_{\widetilde{w}}$-generic, then $\mathrm{Y}_\gamma^{\varepsilon=1}(\widetilde{w})$ *is an affine space over* $\mathbf{F}_q$ *of dimension*

$$\dim \mathrm{Y}_\gamma^{\varepsilon=1}(\widetilde{w}) = d - \#\{\alpha \in \check{\Phi}^+ \mid \widetilde{w}(A_0) \subset H_\alpha^{(0,1)}\}.$$

Proof. As Lemma 3.6.2, we explain how this follows from modifying the proof of [LHLM23b, Theorem 4.2.4].

The main point is that in [LHLM23b, Equation (4.6)], the coefficient $(i + \delta_{\alpha>0} + \langle \overline{\mathbf{a}}, \alpha \rangle)$ becomes

$$\varepsilon(i + \delta_{\alpha>0}) + \langle s + \varepsilon r, \alpha \rangle$$

and we need this to be invertible for all $0 \leq i < d_{\alpha,\widetilde{w}}$ and all roots α. Our hypotheses are exactly arranged for this to be the case.

□

Remark 3.7.3. The case $\varepsilon = 0$ of Lemma 3.7.2 can also be found in [BBASV22, proof of Lemma 2.6 (c)].

Example 3.7.4. If $\widetilde{w} \in \mathrm{Adm}(\lambda)$, then $h_{\widetilde{w}} \leq h_\lambda$. In this case, we find that $S^\circ(\widetilde{w}) \cap \mathrm{Y}_\gamma^{\varepsilon=1}$ has top dimension (equal to d) if and only if $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda)$.

For $\varepsilon_0 \in \mathbf{A}^1_\varepsilon$, define the "$\lambda$-admissible" part of $\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}$ to be

$$\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda) := \mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0} \cap \left(\bigcup_{\widetilde{w} \in W \cdot \lambda} S(\widetilde{w}) \right).$$

Corollary 3.7.5. *Let $\gamma = t(s + \varepsilon r)$ with $r, s \in \mathfrak{t}$.*

(1) Assume s is regular. Then for $\varepsilon_0 \in \{0, \eta\}$, the map $\widetilde{w} \mapsto \mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\widetilde{w})$ induces a bijection between $\widetilde{W}^{\mathrm{reg}}$ and the top-dimensional irreducible components of $\mathrm{Y}_\gamma^{\varepsilon=\eta}$.

(2) If $s + r$ is h_λ-generic, then the map $\widetilde{w} \mapsto \mathrm{Y}_\gamma^{\varepsilon=1}(\widetilde{w})$ induces a bijection between $\mathrm{Adm}^{\mathrm{reg}}(\lambda)$ and the top-dimensional irreducible components of $\mathrm{Y}_\gamma^{\varepsilon=1}(\leq \lambda)$.

Proof. Follows immediately from Lemma 3.7.2. □

4. Specialization of cycles

In the previous section we parametrized some irreducible components of $\mathrm{X}_\gamma^\varepsilon$ and $\mathrm{Y}_\gamma^\varepsilon$ over $\varepsilon \in \{0, 1, \eta\}$. In this section we study the behavior of these irreducible components under specialization in ε.

4.1. **Bases for top homology.** We set up some degeneration problems. We have a family $\mathcal{X}_\gamma^\varepsilon \to \mathbf{A}^1_{\mathbf{Z}_q}$,

$$\begin{tikzcd} \mathrm{X}^\varepsilon_{\gamma, \mathbf{Q}_q} \arrow[r, hook] \arrow[d] & \mathcal{X}_\gamma^\varepsilon \arrow[d] & \mathrm{Y}^\varepsilon_{\gamma, \mathbf{F}_q} \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}\ \mathbf{Q}_q \arrow[r, hook] & \mathrm{Spec}\ \mathbf{Z}_q & \mathrm{Spec}\ \mathbf{F}_q \arrow[l, hook'] \end{tikzcd}$$

Below when we say a subscheme of $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}$ is "top-dimensional", we mean that its dimension is equal to that of the ambient space $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}$, which is $d = \dim \check{G}/\check{B}$. We write $\mathrm{Ch}_{\mathrm{top}}(-)$ for the top-degree Chow group and $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(-)$ for the top-degree *geometric* Borel-Moore homology group; in the latter case the "top" degree is $2d$. Recall our notation for cycle classes from §2.5.3.

Lemma 4.1.1. *Let $\gamma = t(s + \varepsilon r)$ with $r, s \in \mathfrak{t}$.*

(1) Assume s is regular. Then for $\varepsilon_0 \in \{0, \eta\}$, the cycle classes $[\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\widetilde{w})]$ form a basis for $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0})$ as $\widetilde{w}$ ranges over $\widetilde{W}^{\mathrm{reg}}$.

(2) If $s + r$ is h_λ-generic. Then the cycle classes $[\mathrm{Y}_\gamma^{\varepsilon=1}(\widetilde{w})]$ form a basis for $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma^{\varepsilon=1}(\leq \lambda))$ as $\widetilde{w}$ ranges over $\mathrm{Adm}^{\mathrm{reg}}(\lambda)$.

Proof. Follows immediately from Corollary 3.7.5. □

4.2. **Specialization in ε.** In this subsection we analyze the behavior of specialization in ε.

4.2.1. *Generic fiber.* Below for a closed point $\varepsilon_0 \in \mathbf{A}^1_k$ over a field k, we write $\mathbf{A}^1_{(\varepsilon_0)}$ for the localization of $\mathbf{A}^1$ at ε_0, which is evidently a discrete valuation ring. For any family over $\mathbf{A}^1_{(\varepsilon_0)}$, there is then a specialization map from the Chow groups or Borel-Moore homology groups of the generic fiber to those of the special fiber (cf. §2.4 and §2.6).

Localizing the family $\mathrm{X}_\gamma^\varepsilon$ over $\mathbf{A}^1_{(0)}$ (over $\mathbf{Q}_q$), we have a specialization map

$$\mathfrak{sp}_{\varepsilon \to 0} \colon \mathrm{Ch}_{\mathrm{top}}(\mathrm{X}_\gamma^{\varepsilon=\eta}) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{X}_\gamma^{\varepsilon=0}). \tag{4.2.1}$$

Localizing the family $\mathrm{X}_\gamma^\varepsilon$ over $\mathbf{A}^1_{(1)}$ (over $\mathbf{Q}_q$), we have a specialization map

$$\mathfrak{sp}_{\varepsilon \to 1} \colon \mathrm{Ch}_{\mathrm{top}}(\mathrm{X}_\gamma^{\varepsilon=\eta}) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{X}_\gamma^{\varepsilon=1}). \tag{4.2.2}$$

Proposition 4.2.1. *Let $\gamma = (t + p)s$ with $s \in \check{\mathfrak{t}}$ regular.*

(1) The map (4.2.2) is an isomorphism and sends $[\mathrm{X}_\gamma^{\varepsilon=\eta}(\lambda)] \mapsto [\mathrm{X}_\gamma^{\varepsilon=1}(\lambda)]$ for all $\lambda \in X_*(T)^+$.

(2) The map (4.2.1) sends $[\mathrm{X}_\gamma^{\varepsilon=\eta}(\rho)] \mapsto [\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]$.

Proof. (1) This follows from Lemma 3.6.4.

(2) By Lemma 3.6.2, there are open subsets $V \subset \mathbf{A}^1_{\mathbf{Q}_q} = \operatorname{Spec} \mathbf{Q}_q[\varepsilon]$, $U \subset S^\circ(\rho)$ such that $\mathcal{U}^\varepsilon := \mathrm{X}^\varepsilon_\gamma \cap (V \times U) \cong V \times \mathbf{A}^d_{\mathbf{Q}_q}$, and such that its complement has fiberwise dimension less than d over V. We thus have a commutative diagram

$$\begin{tikzcd} \mathrm{Ch}_{\mathrm{top}}(\mathcal{U}^{\varepsilon=\eta}) \arrow[r, "\mathfrak{sp}_{\varepsilon\to 0}"] & \mathrm{Ch}_{\mathrm{top}}(\mathcal{U}^{\varepsilon=0}) \\ \mathrm{Ch}_{\mathrm{top}}(\mathrm{X}^{\varepsilon=\eta}_\gamma(\rho)) \arrow[u] \arrow[r, "\mathfrak{sp}_{\varepsilon\to 0}"] & \mathrm{Ch}_{\mathrm{top}}(\mathrm{X}^{\varepsilon=0}_\gamma(\rho)) \arrow[u] \end{tikzcd} \tag{4.2.3}$$

The bound on the fiber dimension of the complement of $\mathcal{U}^\varepsilon$ implies that the vertical restriction maps are isomorphisms. Since $\mathcal{U}^\varepsilon \cong V \times \mathbf{A}^d_{\mathbf{Q}_q}$ is the trivial family, we have $\mathfrak{sp}_{\varepsilon\to 0}[\mathcal{U}^{\varepsilon=\eta}] = [\mathcal{U}^{\varepsilon=0}]$, and the result follows.

□

4.2.2. *Special fiber.* Let $\gamma = t(s + \varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$. Assume s is regular and let $\lambda \in X_*(\check{T})^+$ be a dominant coweight such that $s + r$ is h_λ-generic.

Localizing the family $\mathrm{Y}^\varepsilon_\gamma$ over $\mathbf{A}^1_{(0)}$ (over $\mathbf{F}_q$), we have a specialization map

$$\mathfrak{sp}_{\varepsilon\to 0}\colon \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\eta}_\gamma(\leq \lambda)) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_\gamma(\leq \lambda)). \tag{4.2.4}$$

Localizing the family $\mathrm{Y}^\varepsilon_\gamma$ over $\mathbf{A}^1_{(1)}$ (over $\mathbf{F}_q$), we have a specialization map

$$\mathfrak{sp}_{\varepsilon\to 1}\colon \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\eta}_\gamma(\leq \lambda)) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=1}_\gamma(\leq \lambda)). \tag{4.2.5}$$

Lemma 4.1.1 implies that for $\varepsilon_0 \in \{0, 1, \eta\}$, the classes $[\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(\widetilde{w}')] \in \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(\leq \lambda))$ form a basis as $\widetilde{w}'$ varies over $\mathrm{Adm}^{\mathrm{reg}}(\lambda)$. Therefore, for each $\varepsilon_0 \in \{0, 1\}$ there exists a unique matrix

$$M^{\varepsilon=\varepsilon_0} := (m^{\varepsilon=\varepsilon_0}_{\widetilde{w}\widetilde{w}'} \in \mathbf{Z}_{\geq 0})_{\widetilde{w}, \widetilde{w}' \in \mathrm{Adm}^{\mathrm{reg}}(\lambda)}$$

(the non-negativity by Lemma 2.4.1) such that

$$\mathfrak{sp}_{\varepsilon\to 0}[\mathrm{Y}^{\varepsilon=\eta}_\gamma(\widetilde{w})] = \sum_{\widetilde{w}' \in \mathrm{Adm}^{\mathrm{reg}}(\lambda)} m^{\varepsilon=\varepsilon_0}_{\widetilde{w}\widetilde{w}'} [\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(\widetilde{w}')] \quad \text{for all } \widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda).$$

Remark 4.2.2 (Independence of λ)**.** The definition of $m^{\varepsilon=\varepsilon_0}_{\widetilde{w}\widetilde{w}'}$ appears to depend on λ. However, there is the following sense in which it is independent of λ as long as it is defined: if $\lambda \leq \lambda'$ and $\widetilde{w}, \widetilde{w}' \in \mathrm{Adm}^{\mathrm{reg}}(\lambda) \subset \mathrm{Adm}^{\mathrm{reg}}(\lambda')$, then each $m^{\varepsilon=\varepsilon_0}_{\widetilde{w}\widetilde{w}'}$ is the same whether defined in terms of λ or λ'. This is a consequence of pushforward functoriality (cf. §2.6.1) for the closed embedding $\mathrm{Y}^\varepsilon_\gamma(\leq \lambda) \hookrightarrow \mathrm{Y}^\varepsilon_\gamma(\leq \lambda')$.

Lemma 4.2.3. *Maintain the running assumptions on s, r and λ. Then the matrix $M^{\varepsilon=\varepsilon_0} := (m^{\varepsilon=\varepsilon_0}_{\widetilde{w}\widetilde{w}'})$ is unipotent and upper-triangular with respect to the Bruhat order on $\widetilde{W}$, with all entries non-negative and $m^{\varepsilon=\varepsilon_0}_{\widetilde{w}\widetilde{w}} = 1$ for all $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda)$.*

In particular, (4.2.4) *and* (4.2.5) *are isomorphisms.*

Proof. We prove the statement for $\varepsilon_0 = 0$, the argument for $\varepsilon_0 = 1$ being similar. The closure of $\mathrm{Y}^{\varepsilon=\eta}_\gamma(\widetilde{w})$ is a closed subset of $\mathbf{A}^1_{\mathbf{F}_q} \times S(\widetilde{w})$, hence its fiber above $\varepsilon = 0$ is a subscheme of $S(\widetilde{w})$. Since $m^{\varepsilon=0}_{\widetilde{w}\widetilde{w}'} \neq 0$ if and only if $\mathrm{Y}^{\varepsilon=0}_\gamma(\widetilde{w})$ occurs in this closure, upper triangularity follows. The fact that $m^{\varepsilon=0}_{\widetilde{w}\widetilde{w}} = 1$ follows from the fact that $\mathrm{Y}^{\varepsilon=\eta}_\gamma(\widetilde{w}) \cap (V \times S^\circ(\widetilde{w})) \cong V \times \mathbf{A}^d_{\mathbf{F}_q}$ is the trivial family for some open $V \subset \mathbf{A}^1_\varepsilon$ containing 0 (as in the proof of Proposition 4.2.1). □

Definition 4.2.4 (Deforming from $\varepsilon = 1$ to $\varepsilon = 0$)**.** Suppose γ is h_λ-generic. Then we abuse notation by defining

$$\mathfrak{sp}_{\varepsilon\to 0}\colon \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=1}_\gamma(\leq \lambda)) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_\gamma(\leq \lambda))$$

to be the composition of (4.2.4) with the inverse of (4.2.5).

5. Degeneration of irreducible components

In the previous section we established (Lemma 4.2.3) uni-triangularity of the maps $\mathfrak{sp}_{\varepsilon\to 0}$ and $\mathfrak{sp}_{\varepsilon\to 1}$ on $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon}_{\gamma})$ with respect to the bases of top-dimensional irreducible components, which are indexed by (subsets of) $\widetilde{W}$. In this section we will prove under mild technical hypotheses that these specialization maps are actually the *identity map* with respect to these bases.

The precise statement is Theorem 5.1.1 below. Its significance is to provide control over the geometry of the Breuil-Mézard cycles produced by Theorem 1.3.1. For example, it will be used to prove that (under technical assumptions) the Breuil-Mézard cycles constructed in Theorem 1.3.1 are effective, an important property that was not formulated in the original Breuil-Mézard Conjectures. In addition, it enables us to study the reducibility or reducedness of Breuil-Mézard cycles, which are properties that can be violently messed up by specialization. Later in §12.2, we will use Theorem 5.1.1 to prove that the decomposition of Breuil-Mézard cycles into irreducible components (with multiplicities) has the same behavior as certain decompositions of characteristic cycles carrying significance in mirror symmetry and geometric representation theory.

5.1. **The main statement.**

The main result of this section is the following.

Theorem 5.1.1. *Let $\gamma = t(s+\varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$. Assume s is regular. Then the specialization maps enjoy the following properties.*

(1) The map (4.2.4) sends $[\mathrm{Y}^{\varepsilon=\eta}_{\gamma}(\widetilde{w})] \mapsto [\mathrm{Y}^{\varepsilon=0}_{\gamma}(\widetilde{w})]$ for all $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$.

(2) Assume additionally that $s+r$ is $3h_\rho$-generic. Then the map (4.2.5) sends $[\mathrm{Y}^{\varepsilon=\eta}_{\gamma}(\widetilde{w})] \mapsto [\mathrm{Y}^{\varepsilon=1}_{\gamma}(\widetilde{w})]$ for all $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$.

Remark 5.1.2. In fact, our proof shows that the first item holds for all $\widetilde{w} \in \widetilde{W}^{\mathrm{reg}}$. The method of proof also shows the second item holds for $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda)$ provided that $s+r$ is $(h_\lambda + 2h_\rho)$-generic.

The remainder of this section will be devoted to the proof of Theorem 5.1.1. It can be checked in either $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_{\gamma})$ or $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_{\gamma})$ because the cycle class map is injective in the top degree; we will work with Borel-Moore homology because we want to use the affine Springer action for deformed affine Springer fibers, which was defined in terms of homology. Although we defined $\check{G}$ over $\mathbf{Z}_q$ in §2, *for the rest of the section all the geometric objects (e.g., $\check{G}, \check{\mathfrak{g}}, \mathrm{Fl}_{\check{G}}$, etc.) that come up will be base changed to* $\mathbf{F}_q$, which we assume to have characteristic $p > h$. Therefore we will suppress the subscript $\mathbf{F}_q$ for the remainder of the section.

5.2. **Recap on Springer theory.**

We review some more background about Springer theory.

5.2.1. *Springer resolution.* Let $\check{\mathcal{N}} \subset \check{\mathfrak{g}}$ be the nilpotent cone. Recall the *Springer resolution* $\pi \colon \widetilde{\check{\mathcal{N}}} \to \check{\mathcal{N}}$ where $\widetilde{\check{\mathcal{N}}}$ classifies pairs (e, B') with $e \in \check{\mathcal{N}}$ and $B' \subset \check{G}$ is a Borel subgroup such that $e \in \check{\mathfrak{n}}'$, the nilpotent radical of $\operatorname{Lie} B'$. The map π forgets the datum of B'. There is an identification $\widetilde{\check{\mathcal{N}}} \cong T^*\check{\mathcal{B}}$, where $\mathcal{B}$ is the flag variety of $\check{G}$, under which π is the moment map with respect to the identification $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$ induced by the Killing form. The obvious map from $\widetilde{\check{\mathcal{N}}}$ to $\widetilde{\check{\mathfrak{g}}}$ fits into a commutative diagram with the Grothendieck alteration

$$\begin{tikzcd} \widetilde{\check{\mathcal{N}}} \arrow[r] \arrow[d, "\pi"] & \widetilde{\check{\mathfrak{g}}} \arrow[d, "\pi"] \\ \check{\mathcal{N}} \arrow[r] & \check{\mathfrak{g}} \end{tikzcd}$$

which is Cartesian up to nilpotents (cf. Definition 3.4.1).

5.2.2. *Steinberg variety.* The *Steinberg variety* (for $\check{G}$) is $\check{\mathrm{St}} := \widetilde{\check{\mathcal{N}}} \times_{\check{\mathcal{N}}} \widetilde{\check{\mathcal{N}}}$. We let $\check{\mathrm{St}}_{\check{B}}$ be the fiber of $\check{\mathrm{St}}$ over the fixed Borel $\check{B} \in \check{\mathcal{B}}$ under the first projection to $\widetilde{\check{\mathcal{N}}}$ followed by the structure map $\widetilde{\check{\mathcal{N}}} \cong T^*\check{\mathcal{B}} \to \check{\mathcal{B}}$. Letting $\check{\mathfrak{n}}$ be the nilpotent radical of $\check{\mathfrak{b}}$, we therefore have an identification

$$\check{\mathrm{St}}_{\check{B}} \cong \check{\mathfrak{n}} \times_{\check{\mathcal{N}}} \widetilde{\check{\mathcal{N}}}.$$

Then the Borel-Moore homology $\mathrm{H}_*^{\mathrm{BM}}(\check{\mathrm{St}}_{\check{B}})$ is equipped with a natural W-action by the mechanism of §3.4, since the square

$$\begin{tikzcd} \check{\mathrm{St}}_{\check{B}} \arrow[r, hook] \arrow[d, "\pi'"] & \tilde{\check{\mathfrak{g}}} \arrow[d, "\pi"] \\ \check{\mathfrak{n}} \arrow[r, hook] & \check{\mathfrak{g}} \end{tikzcd}$$

is Cartesian up to nilpotents.

5.2.3. *Homology of the Steinberg variety.* The isomorphism $\check{G} \times^{\check{B}} \check{\mathrm{St}}_{\check{B}} \xrightarrow{\sim} \check{\mathrm{St}}$ induces a bijection between the irreducible components of $\check{\mathrm{St}}_{\check{B}}$ and of $\check{\mathrm{St}}$, which are indexed by W according to the following convention (consistent with [CG10, §3.3]): The orbits of $\check{G}$ on $\check{\mathcal{B}} \times \check{\mathcal{B}}$ stratify it into Schubert varieties $O(w) \subset \check{\mathcal{B}} \times \check{\mathcal{B}}$, ranging over $w \in W$. Then $\check{\mathrm{St}} \subset \tilde{\check{\mathcal{N}}} \times \tilde{\check{\mathcal{N}}} \cong T^*\check{\mathcal{B}} \times T^*\check{\mathcal{B}}$ can be identified with the union of the conormal bundles to the orbits $O(w)$. The irreducible component $\check{\mathrm{St}}(w)$ corresponding to $w \in W$ is the conormal bundle to $O(w)$; we denote the corresponding irreducible component of $\check{\mathrm{St}}_{\check{B}}$ by $C(w)$.

Remark 5.2.1. It follows from the definitions that $C(w)$ can be characterized as follows: $C(w)$ admits a d-dimensional open subspace parametrizing pairs $(e, B') \in \check{\mathfrak{n}} \times \check{\mathcal{B}}$ such that $e \in (\mathrm{Lie}\, B') \cap \check{\mathfrak{n}}$ and $B' \in Bw\check{B}$ has relative position w with respect to $\check{B}$.

Lemma 5.2.2. *(1) The group $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\check{\mathrm{St}})$ equipped with the Springer action is isomorphic to the regular representation of W over $\mathbf{Q}_\ell$.*

(2) There exist positive integers $\{a_w > 0\}_{w\in W}$ such that $\sum_{w\in W} a_w[\check{\mathrm{St}}(w)]$ is W-invariant.

Proof. Assertion (1) is well-known; see for example [CG10, §3.3].

For (2), choose $a_w \in \mathbf{Z}$ such that the element $\sum_{w\in W}[w] \in \mathbf{Q}_\ell[W]$ corresponds to $\sum_{w\in W} a_w[\check{\mathrm{St}}(w)] \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\check{\mathrm{St}}_{\check{B}})$ under (1). We need to check that a_w is in fact *strictly positive* for each $w \in W$. For this, we note that the Springer action on $\mathrm{H}_*^{\mathrm{BM}}(\check{\mathrm{St}})$ has an alternative definition in terms of degenerations from $\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \tilde{\check{\mathfrak{g}}}$. In this interpretation, the identification of $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\check{\mathrm{St}})$ with $\mathbf{Q}_\ell[W]$ is such that $[w]$ corresponds to the specialization of a certain irreducible cycle Λ_w in the notation of [CG10, Lemma 3.4.14], hence it is effective by Lemma 2.4.1. An immediate consequence is that the $[w]$ can be expressed in terms of the $[\check{\mathrm{St}}(w)]$ by applying some upper unipotent triangular (with respect to the Bruhat order) matrix with non-negative entries.[7] □

Lemma 5.2.3. *Let $i \colon \check{\mathrm{St}}_{\check{B}} \hookrightarrow \check{\mathrm{St}}$ be the natural inclusion. Then the maps*

$$i_* \colon \mathrm{H}_*^{\mathrm{BM}}(\check{\mathrm{St}}_{\check{B}}) \to \mathrm{H}_*^{\mathrm{BM}}(\check{\mathrm{St}}) \qquad \textit{and} \qquad i^! \colon \mathrm{H}_*^{\mathrm{BM}}(\check{\mathrm{St}}) \to \mathrm{H}_{*-2d(i)}^{\mathrm{BM}}(\check{\mathrm{St}}_{\check{B}})$$

are W-equivariant.

Proof. Apply Lemma 3.4.2 to the commutative diagram

$$\begin{tikzcd} \check{\mathrm{St}}_{\check{B}} \arrow[r] \arrow[d] & \check{\mathrm{St}} \arrow[r] \arrow[d] & \tilde{\check{\mathcal{N}}} \arrow[r] \arrow[d] & \tilde{\check{\mathfrak{g}}} \arrow[d] \\ \check{\mathfrak{n}} \arrow[r, hook] & \tilde{\check{\mathcal{N}}} \arrow[r] & \check{\mathcal{N}} \arrow[r] & \check{\mathfrak{g}} \end{tikzcd}$$

which has all squares Cartesian up to nilpotents. □

Corollary 5.2.4. *As a W-representation via the Springer action, $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\check{\mathrm{St}}_{\check{B}})$ is isomorphic to the regular representation of W over $\mathbf{Q}_\ell$. Furthermore, there exists positive integers $a_w > 0$ such that*

$$\sum_{w\in W} a_w[C(w)] \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\check{\mathrm{St}}_{\check{B}})$$

is W-invariant.

Proof. This is immediate from Lemma 5.2.2 plus the second assertion of Lemma 5.2.3. □

[7] Alternatively, one can also see this from the fact that the class correspond to $[w]$ can be interpreted as the characteristic cycle of the localization of a Verma module, by [KT84].

5.3. **Lusztig's fundamental domain.** Recall $\gamma = t(s+\varepsilon r)$ with $s,t \in \check{\mathfrak{t}}$ and s regular. In [Lus20], Lusztig proves the following.

Theorem 5.3.1 (Lusztig [Lus20])**.** *There is a locally closed subvariety $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma} \subset \mathrm{Y}^{\varepsilon=0}_{\gamma}$ such that:*

(1) $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$ is a fundamental domain for the translation action of $X_*(\check{T})$ on $\mathrm{Y}^{\varepsilon=0}_{\gamma}$.
(2) $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$ is isomorphic to $\check{\mathrm{St}}_{\check{B}}$.
(3) Let $\overline{\Omega}^{\varepsilon=0}_{\mathrm{Y},\gamma} \subset \mathrm{Y}^{\varepsilon=0}_{\gamma}$ be the closure of $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$. With respect to the affine Springer action of $\widetilde{W}$ on $\mathrm{H}^{\mathrm{BM}}_*(\mathrm{Y}^{\varepsilon=0}_{\gamma})$, the subgroup $W \subset \widetilde{W}$ preserves $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\overline{\Omega}^{\varepsilon=0}_{\mathrm{Y},\gamma}) \subset \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_{\gamma})$, and the resulting W-representation on $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\overline{\Omega}^{\varepsilon=0}_{\mathrm{Y},\gamma})$ is isomorphic to $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\check{\mathrm{St}}_{\check{B}})$ equipped with the Springer action of W.

Strictly speaking, [Lus20] proved the above theorem in the context where $\check{G}$ is a simply connected in characteristic 0. However his arguments work also in for any $\check{G}$ in characteristic $p > h+1$, as will be clear from our sketch of the construction of $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$ below.

5.3.1. *Semi-infinite stratification.* There is a stratification of $\mathrm{Gr}_{\check{G}}$ by semi-infinite orbits

$$\mathcal{S}_\lambda := L\check{N} \cdot (t^\lambda L^+\check{G}) = t^\lambda L\check{N} \cdot L^+\check{G},$$

for $\lambda \in X_*(\check{T})$, which are permuted simply transitively by the translation action of $X_*(\check{T})$.

Definition 5.3.2. Let $\mathrm{Gr}^{\varepsilon=0}_{\gamma} \subset \mathrm{Gr}_{\check{G}}$ be the affine Springer fiber of γ (defined by the same equations as $\mathrm{X}^{\varepsilon=0}_{\gamma}$ but over $\mathbf{F}_q$ instead of $\mathbf{Q}_q$). Let $\Omega^{\varepsilon=0}_{X,\gamma} := \mathcal{S}_0 \cap \mathrm{Gr}^{\varepsilon=0}_{\gamma}$, the intersection formed in $\mathrm{Gr}_{\check{G}}$. Let $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$ be the pre-image of the $\Omega^{\varepsilon=0}_{X,\gamma}$ under the natural projection map $\mathrm{Y}^{\varepsilon=0}_{\gamma} \to \mathrm{Gr}^{\varepsilon=0}_{\gamma}$.

By construction, $\Omega^{\varepsilon=0}_{X,\gamma}$ is a fundamental domain for the $X_*(\check{T})$-action on $\mathrm{X}^{\varepsilon=0}_{\gamma}$, hence $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$ is a fundamental domain for the $X_*(\check{T})$-action on $\mathrm{Y}^{\varepsilon=0}_{\gamma}$.

5.3.2. *Exponential map.* Recall we work in characteristic $p > h$, so that we have the exponential map $\exp\colon \check{\mathfrak{n}}((t)) \to L\check{N}$. Define the affine space over $\mathbf{F}_q$,

$$\check{\mathfrak{n}}((t))^- := \bigoplus_{i<0} t^i \check{\mathfrak{n}}$$

and

$$\omega^{\varepsilon=0}_{X,\gamma} := \{E \in \check{\mathfrak{n}}((t))^- :\ \mathrm{Ad}_{\exp(-E)}(\gamma) \in \check{\mathfrak{g}}[[t]]\}.$$

The *residue map* $\mathrm{Res}\colon \omega_{X,\gamma} \to \check{\mathfrak{n}}$ sends E to its coefficient of t^{-1}. The proof of [Lus20, Lemma 3] shows that $\mathrm{Res}\colon \omega^{\varepsilon=0}_{X,\gamma} \xrightarrow{\sim} \check{\mathfrak{n}}$ is an isomorphism, and that $E \mapsto \exp(E)$ defines an isomorphism $\omega^{\varepsilon=0}_{X,\gamma} \xrightarrow{\sim} \Omega^{\varepsilon=0}_{X,\gamma}$. Composing these gives an isomorphism $\phi\colon \check{\mathfrak{n}} \xrightarrow{\sim} \Omega^{\varepsilon=0}_{X,\gamma}$, as in the diagram below.

$$\begin{tikzcd}
\check{\mathfrak{n}} \arrow[r, "\sim", "\mathrm{Res}^{-1}"'] \arrow[rr, bend left, "\phi"] & \omega^{\varepsilon=0}_{X,\gamma} \arrow[r, "\sim"] & \Omega^{\varepsilon=0}_{X,\gamma} \\
e \arrow[r] & E \arrow[r] & \exp(-E)
\end{tikzcd}$$

5.3.3. *Isomorphism with Steinberg variety.* We may view $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$ as the space of $\exp(E) \in \Omega^{\varepsilon=0}_{X,\gamma}$ plus a Borel subgroup of $\check{\mathfrak{g}}$ containing the reduction of $\mathrm{Ad}_{\exp(-E)}(\gamma) \in \check{\mathfrak{g}}[[t]]$ modulo t. Lusztig shows that for $e \in \check{\mathfrak{n}}$ corresponding to $E \in \omega^{\varepsilon=0}_{X,\gamma}$, the reduction of $\mathrm{Ad}_{\exp(-E)}(\gamma) \in \check{\mathfrak{g}}[[t]]$ modulo t coincides with $-[e,s] \in \check{\mathfrak{n}} \subset \check{\mathfrak{g}}$. Since $e \mapsto -[e,s]$ is an automorphism of $\check{\mathfrak{n}}$ by the assumed regularity of s, this induces an isomorphism $\check{\mathrm{St}}_{\check{B}} \xrightarrow{\sim} \Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}$.

5.4. **Deformation of Lusztig's fundamental domain.** We now generalize Lusztig's construction to the deformed affine Springer fibers. Note that when $\varepsilon \neq 0$, we do not actually have a translation action on $\mathrm{H}^{\mathrm{BM}}_*(\mathrm{X}^{\varepsilon}_{\gamma})$, so the construction does not quite produce a fundamental domain for any lattice action. Nevertheless, it turns out that one can understand fundamental classes of certain irreducible components of deformed affine Springer fibers in terms of this construction.

5.4.1. *Intersection with negative loop orbit.* Recall that we defined

$$\mathrm{Ad}^{\varepsilon}_g(\gamma) := \mathrm{Ad}_g(\gamma) - \varepsilon t^2 \frac{dg}{dt} g^{-1} = \mathrm{Ad}_g(\gamma) - \varepsilon t^2 d\log(g).$$

For $l \geq 0$, we define the affine space over $\mathbf{F}_q$,

$$\check{\mathfrak{n}}((t))^-_{\leq l} := \bigoplus_{0<i\leq l} \check{\mathfrak{n}}\, t^{-i} \subset \check{\mathfrak{n}}((t))^-.$$

Definition 5.4.1. For $l \geq 0$, we define the subspace of $\check{\mathfrak{n}}((t))^-_{\leq l}$,

$$\omega^{\varepsilon}_{X,\gamma,\leq l} := \{E \in \check{\mathfrak{n}}((t))^-_{\leq l} \colon \mathrm{Ad}^{\varepsilon}_{\exp(-E)}(\gamma) \in \check{\mathfrak{g}}[[t]]\},$$

and abbreviate $\omega^{\varepsilon}_{X,\gamma} := \bigcup_{l\geq 0} \omega^{\varepsilon}_{X,\gamma,\leq l}$.

Recall that for $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{F}_q} = \mathrm{Spec}\, \mathbf{F}_q[\varepsilon]$ we denote by $\mathbf{A}^1_{(\varepsilon_0)}$ the localization of $\mathbf{A}^1$ at ε_0. We use this to construct specialization maps from the fiber over $\varepsilon = \eta$ to the fiber over $\varepsilon = \varepsilon_0$.

Lemma 5.4.2. *Let $\gamma = t(s + \varepsilon r)$ where $s \in \check{\mathfrak{t}}$ is regular. Assume that $p > h$.*

(1) The residue map Res *restricts to an isomorphism*

$$\omega^{\varepsilon}_{X,\gamma,\leq h_\rho}|_{\mathbf{A}^1_{(0)}} = \omega^{\varepsilon}_{X,\gamma}|_{\mathbf{A}^1_{(0)}} \xrightarrow{\sim} \check{\mathfrak{n}} \times \mathbf{A}^1_{(0)}.$$

(2) If $s + r$ is l-generic with $l \geq h_\rho$, then the residue map Res *restricts to an isomorphism*

$$\omega^{\varepsilon}_{X,\gamma,\leq h_\rho}|_{\mathbf{A}^1_{(1)}} = \omega^{\varepsilon}_{X,\gamma,\leq l}|_{\mathbf{A}^1_{(1)}} \xrightarrow{\sim} \check{\mathfrak{n}} \times \mathbf{A}^1_{(1)}.$$

Proof. Write $E \in \omega^{\varepsilon}_{X,\gamma}$ as

$$E = t^{-1}E_1 + t^{-2}E_2 + \ldots = \sum_{i>0} t^{-i}E_i. \tag{5.4.1}$$

Substituting in $\gamma = t(s + \varepsilon r)$, we have (cf. [Lus20, Proof of Lemma 3])

$$\begin{aligned}\mathrm{Ad}_{\exp(-E)}(\gamma) = t(s+\varepsilon r) + \sum_{i\geq 1} t^{-i+1}[-E_i, s+\varepsilon r] + \frac{1}{2}\sum_{i,j\geq 1} t^{-i-j+1}[-E_i,[-E_j, s+\varepsilon r]] \\ + \frac{1}{6}\sum_{i,j,k\geq 1} t^{-i-j-k+1}[-E_i,[-E_j,[-E_k, s+\varepsilon r]]] + \ldots\end{aligned}$$

Then $\mathrm{Ad}^{\varepsilon}_{\exp(-E)}(\gamma)$ is the above expression plus

$$\varepsilon t^2 \frac{d}{dt}(\log(\exp(E))) = \varepsilon E_1 + 2\varepsilon t^{-1}E_2 + 3\varepsilon t^{-2}E_3 + \ldots = \varepsilon \sum_{i>0} t^{-i+1} i E_i.$$

The defining condition of $\omega^{\varepsilon}_{X,\gamma,\leq h}$ is $\mathrm{Ad}^{\varepsilon}_{\exp(-E)}(\gamma) \in \check{\mathfrak{g}}[[t]]$. For $m > 0$, the coefficient of t^{-m+1} in $\mathrm{Ad}^{\varepsilon}_{\exp(-E)}(\gamma)$ is

$$[-E_m, s+\varepsilon r] + (\varepsilon m E_m) + \frac{1}{2}\sum_{\substack{i,j\geq 1\\ i+j=m}} [-E_i,[-E_j, s+\varepsilon r]] + \frac{1}{6}\sum_{\substack{i,j,k\geq 1\\ i+j+k=m}} [-E_i,[-E_j,[-E_k, s+\varepsilon r]]] + \ldots \tag{5.4.2}$$

and we wish to impose that this expression is 0 if $m > 1$. In particular, this condition solves for E_m in terms of $E_{m'}$ for $m' < m$ as long as $E \mapsto [E, s+\varepsilon r] + \varepsilon m E$ is an automorphism of $\check{\mathfrak{n}}$.

(1) In $\mathbf{A}^1_{(0)}$, the map $E \mapsto [E, s+\varepsilon r] + \varepsilon m E$ is an automorphism of $\check{\mathfrak{n}} \times \mathbf{A}^1_{(0)}$ under our assumption that s is regular. Hence $E \in \omega^{\varepsilon}_{X,\gamma}|_{\mathbf{A}^1_{(0)}}$ is uniquely determined by E_1, which is unconstrained.

(2) For $m \leq l$, the map $E \mapsto [E, s+\varepsilon r] + \varepsilon m E$ is an automorphism of $\check{\mathfrak{n}}$ by our hypotheses, and the equation (5.4.2) $= 0$ solves for E_m uniquely in terms of E_1. Furthermore, the recursion also shows that E_m belongs to the mth layer of the lower central series of $\check{\mathfrak{n}} \times \mathbf{A}^1_{(1)}$. Now for $m > l \geq h_\rho$, by definition of $\mathfrak{n}((t))^-_{\leq l}$ we have $E_m = 0$, and the expression in (5.4.2) is automatically 0 since the mth layer of the lower central series is trivial. Hence $E \in \omega^{\varepsilon}_{X,\gamma,\leq l}|_{\mathbf{A}^1_{(1)}}$ is uniquely determined by E_1, which is unconstrained. $\square$

5.4.2. *Deformation of the fundamental domain: affine Grassmannian version.* We now define the deformation of Lusztig's fundamental domain for the spherical version of the affine Springer fiber:

$$\Omega^{\varepsilon}_{X,\gamma} := \mathcal{S}_0 \cap \mathrm{Gr}^{\varepsilon}_{\gamma},$$

where we recall that for $\lambda \in X_*(\check{T})$, $\mathcal{S}_\lambda$ is the semi-infinite orbit through $t^\lambda L^+\check{G}$. We then define

$$\Omega^{\varepsilon}_{X,\gamma,\leq l} := \exp(\omega^{\varepsilon}_{X,\gamma,\leq l}) \subset \Omega^{\varepsilon}_{X,\gamma}.$$

If s is regular, then composing the map exp with the inverse of Res and using Lemma 5.4.2 gives an isomorphism.

$$\phi^{\varepsilon}_{(0)} \colon \check{\mathfrak{n}} \times \mathbf{A}^1_{(0)} \xrightarrow{\sim} \Omega^{\varepsilon}_{X,\gamma,\leq h_\rho}|_{\mathbf{A}^1_{(0)}}.$$

If $s+r$ is moreover h_ρ-generic, then we similarly get

$$\phi^{\varepsilon}_{(1)} \colon \check{\mathfrak{n}} \times \mathbf{A}^1_{(1)} \xrightarrow{\sim} \Omega^{\varepsilon}_{X,\gamma,\leq h_\rho}|_{\mathbf{A}^1_{(1)}}.$$

5.4.3. *Deformation of the fundamental domain: affine flag version.* We may view $\mathrm{Gr}_{\check{G}}$ as the space of $L\check{G}$-conjugates of $L_0 := \check{\mathfrak{g}}[[t]] \subset \check{\mathfrak{g}}((t))$, and $\mathrm{Fl}_{\check{G}}$ as the space of Iwahori subalgebras $\mathfrak{I} \subset \check{\mathfrak{g}}((t))$. Our fixed choice of Iwahori subgroup $\check{\mathbf{I}}$ gives a basepoint $\mathfrak{I}_0 \in \mathrm{Fl}_{\check{G}}$ lying over L_0.

For $e \in \check{\mathfrak{n}}$, write $E = \mathrm{Res}^{-1}(e) \in \omega^{\varepsilon}_{X,\gamma,\leq h_\rho}$. Then $\mathrm{Ad}^{\varepsilon}_{\exp(-E)}(\gamma) \in L_0$, so $\gamma \in \mathrm{Ad}^{\varepsilon}_{\exp(E)} L_0 =: L$. The isomorphism $\mathrm{Ad}^{\varepsilon}_{\exp(E)} \colon L_0 \to L$ induces

$$\mathrm{Ad}^{\varepsilon}_{\exp(E)} \colon \check{\mathfrak{g}} = L_0/tL_0 \xrightarrow{\sim} L/tL. \tag{5.4.3}$$

Lemma 5.4.3. *Recall that $\gamma = t(s+\varepsilon r)$. The inverse of the isomorphism* (5.4.3) *takes $\gamma \mapsto -[e, s+\varepsilon r] + \varepsilon e$.*

Proof. We compute:

$$\mathrm{Ad}^{\varepsilon}_{\exp(-E)}(\gamma) = t(s+\varepsilon r) + [-E_1, s+\varepsilon r] + \varepsilon E_1 \equiv -[E_1, s+\varepsilon r] + \varepsilon E_1 \pmod{t}.$$

□

Corollary 5.4.4. *Let $\gamma = t(s+\varepsilon r)$ where $s, r \in \check{\mathfrak{t}}$ and s is regular.*

(1) There is an isomorphism

$$\check{\mathrm{St}}_{\check{B}} \times \mathbf{A}^1_{(0)} \cong \Omega^{\varepsilon}_{\mathrm{Y},\gamma,\leq h_\rho}|_{\mathbf{A}^1_{(0)}}. \tag{5.4.4}$$

(2) If $s+r$ is h_ρ-generic, then there is an isomorphism

$$\check{\mathrm{St}}_{\check{B}} \times \mathbf{A}^1_{(1)} \cong \Omega^{\varepsilon}_{\mathrm{Y},\gamma,\leq h_\rho}|_{\mathbf{A}^1_{(1)}}. \tag{5.4.5}$$

Proof. Lemma 5.4.3 gives a bijection between Borel subgroups of $L/tL = \mathrm{Ad}^{\varepsilon}_g(L_0/tL_0)$ containing γ and Borel subgroups of $\check{\mathfrak{g}} = L_0/tL_0$ containing $-[e, s+\varepsilon r] + \varepsilon e$. Conclude by applying Lemma 5.4.2 and noting that the assumptions imply that in case (1) the map $e \mapsto -[e, s+\varepsilon r] + \varepsilon e$ is an automorphism of $\check{\mathfrak{n}} \times \mathbf{A}^1_{(0)}$, and in case (2) the map $e \mapsto -[e, s+\varepsilon r] + \varepsilon e$ is an automorphism of $\check{\mathfrak{n}} \times \mathbf{A}^1_{(1)}$. □

5.5. **Analysis of fundamental classes.** Recall $\gamma = t(s+\varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$, and assume s is regular. We will also assume that $s+r$ is at least h_ρ-generic.

5.5.1. *Inclusion into the affine flag variety.* We have an inclusion $\iota^{\varepsilon}_{\gamma} \colon \mathrm{Y}^{\varepsilon}_{\gamma} \hookrightarrow \mathrm{Fl}_{\check{G}} \times \mathbf{A}^1$ of families over $\mathbf{A}^1 = \mathrm{Spec}\, \mathbf{F}_q[\varepsilon]$. This induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\eta}_{\gamma}) & \xrightarrow{\mathfrak{sp}_{\varepsilon\to 0}} & \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=0}_{\gamma}) \\ \downarrow & & \downarrow \\ \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G},\mathbf{F}_q(\varepsilon)}) & \xrightarrow{\mathfrak{sp}_{\varepsilon\to 0}} & \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}) \end{array} \tag{5.5.1}$$

Since $\mathrm{Fl}_{\check{G}} \times \mathbf{A}^1 \to \mathbf{A}^1$ is the constant family, the bottom horizontal map can be identified with the identity map using the canonical identification $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Fl}_{\check{G},\mathbf{F}_q(\varepsilon)}) \cong \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Fl}_{\check{G}})$ coming from invariance of ℓ-adic cohomology under base change of separably closed ground fields.

For $\varepsilon_0 \in \{0, 1, \eta\}$, we have a pushforward map $\iota^{\varepsilon=\varepsilon_0}_{\gamma,*} : \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_{\gamma}) \to \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}})$. Recall that there is an affine Weyl group action on $\mathrm{H}^{\mathrm{BM}}_{*}(\mathrm{Fl}_{\check{G}})$ [Kac85, §2.7], which can also be interpreted as the affine Springer action viewing $\mathrm{Fl}_{\check{G}}$ as the affine Springer fiber for the nil-element $\gamma = 0$. It follows from the construction that the map $\iota_{\gamma,*}^{\varepsilon=\varepsilon_0}$ is equivariant for the affine Springer actions. Additionally, the map $\iota_{\gamma,*}^{\varepsilon=0}$ is equivariant for the translation action by $L\check{T}/L^+\check{T} = X_*(\check{T})$.

Remark 5.5.1. Since the translation action by the coroot lattice $\check{Q}^\vee \subset X_*(\check{T})$ on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Fl}_{\check{G}})$ is trivial (as it extends to an action of a connected group), the map $\iota_{\gamma,*}^{\varepsilon=0}$ factors through the coinvariants of $\check{Q}^\vee$ acting on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=0})$.

As a special case of Lemma 3.5.2, we have a commutative diagram

$$\begin{tikzcd} \Omega^\varepsilon_{\mathrm{Y},\gamma,\le h_\rho} \arrow[r] \arrow[d] & \mathrm{Y}^\varepsilon_\gamma \arrow[r] \arrow[d] & [\widetilde{\check{\mathfrak{g}}}/\check{G}] \arrow[d] \\ \Omega^\varepsilon_{X,\gamma,\le h_\rho} \arrow[r] & \mathrm{X}^\varepsilon_\gamma \arrow[r] & [\check{\mathfrak{g}}/\check{G}] \end{tikzcd}$$

whose squares are Cartesian. For $\varepsilon_0 \in \{0,1,\eta\}$, there is a map $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma,\le h_\rho}) \to \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma)$ sending the class of an irreducible component to the class of its closure.

Corollary 5.5.2. *Each map in the sequence*

$$\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\check{\mathrm{St}}_{\check{B}}) \cong \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\Omega^{\varepsilon=\varepsilon_0}) \longrightarrow \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma) \xrightarrow{\iota^{\varepsilon=\varepsilon_0}_{\gamma,*}} \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}})$$

is W-equivariant.

Proof. Apply Lemma 3.4.2 and Proposition 3.5.4. □

5.5.2. *Comparison of fundamental classes.* For $w \in W$, and $\varepsilon_0 \in \{0,1,\eta\}$, we have a d-dimensional irreducible component $\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma,\le h_\rho}(w)$ coming from the component $C(w)$ of $\check{\mathrm{St}}_{\check{B}}$ under its isomorphism with $\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma,\le h_\rho}$. We will abbreviate this irreducible variety as $\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma}(w)$.

On the other hand, recall from Corollary 3.7.5 that $\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma$ for $\varepsilon_0 \in \{0,\eta\}$, and $\mathrm{Y}^{\varepsilon=1}_\gamma(\le \rho)$, are d-dimensional, with top dimensional components parametrized by $\widetilde{W}^{\mathrm{reg}}$ and $\mathrm{Adm}^{\mathrm{reg}}(\rho)$ respectively. Hence for any $\varepsilon_0 \in \{0,1,\eta\}$ we have the fundamental classes $[\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(\widetilde{w})] \in \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(\le\rho))$ for all $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$.

We recall some basic properties of $\mathrm{Y}^{\varepsilon=0}_\gamma(\widetilde{w})$. The reader may need to review §2.3.1 for notation.

Lemma 5.5.3. *Let $\gamma = t(s+\varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$, s regular. Below, equalities are as subschemes of* $\mathrm{Fl}_{\check{G}}$.

(1) Left multiplication by $t^\nu w \in \widetilde{W}$ induces

$$t^\nu w \mathrm{Y}^\varepsilon_\gamma = \mathrm{Y}^\varepsilon_{w\gamma - t\varepsilon\nu}.$$

(2) Let $\widetilde{u} \in \widetilde{W}^{\mathrm{reg}}$ and factorize $\widetilde{u}$ uniquely as $\widetilde{u} = \widetilde{v}^{-1}w_0\widetilde{w}$ where $\widetilde{w} \in \widetilde{W}_1$ is restricted and $\widetilde{v} = vt^\nu \in \widetilde{W}^+$ is dominant, cf. (the proof of) [LHLM23b, Proposition 2.1.5].

(a) For $\varepsilon_0 \in \{0,\eta\}$, we have

$$\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(\widetilde{u}) = t^{-\nu}v^{-1}\mathrm{Y}^{\varepsilon=\varepsilon_0}_{v\gamma - t\varepsilon v\nu}(w_0\widetilde{w}).$$

(b) If furthermore $\widetilde{u} \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$, and both $s+r$, $s+r-\nu$ are h_ρ-generic, then we have

$$\mathrm{Y}^{\varepsilon=1}_\gamma(\widetilde{u}) = t^{-\nu}v^{-1}\mathrm{Y}^{\varepsilon=1}_{v\gamma - tv\nu}(w_0\widetilde{w}).$$

(3) Let $\widetilde{w} \in \widetilde{W}_1$ be restricted and $\sigma \in W$. For any $\varepsilon_0 \in \{0,1,\eta\}$, we have

$$\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(w_0\widetilde{w}) = \sigma^{-1}\mathrm{Y}^{\varepsilon=\varepsilon_0}_{\sigma\gamma}(w_0\widetilde{w}).$$

(4) For $\widetilde{w} = t^{\rho_w}w \in \widetilde{W}_1$ with $w \in W$,

$$\mathrm{Y}^{\varepsilon=\varepsilon_0}_\gamma(w_0\widetilde{w}) \subset (L\check{N} \cap \check{\mathbf{I}})t^{w_0\rho_w}w_0w\check{\mathbf{I}}/\check{\mathbf{I}} \subset t^{w_0\rho_w}(L\check{N})w_0w\check{\mathbf{I}}/\check{\mathbf{I}}.$$

Proof. Assertion (1) follows from a direct computation, using that

$$\mathrm{Ad}^\varepsilon_{t^\nu w}(\gamma) = w(\gamma) - t\varepsilon\nu.$$

Assertions (2) and (3) follow from the proof of [LHLM23b, Proposition 4.3.5] and [LHLM23b, Proposition 4.3.6] respectively, noting that:

- The cited proofs did not use the running assumption that $\check{G} = \mathrm{GL}_n$ in *loc.cit.*

- The role of the genericity assumptions in the cited proofs were only used to guarantee that intersecting $\mathrm{Y}_{\gamma}^{\varepsilon=\varepsilon_0}$ with certain open affine Schubert cells are affine spaces of the correct dimension. This holds for $\varepsilon_0 = 0$ (and hence also for $\varepsilon_0 = \eta$) due to the regularity of s, and holds for $\varepsilon_0 = 1$ by our assumptions, cf. Lemma 3.7.2.

Finally, (4) follows from writing standard representatives for $S^{\circ}(w_0\widetilde{w})$ and the fact that $w_0\widetilde{w}$ is anti-dominant, see for example [LHLM23b, Corollary 4.2.15].

□

The next lemma expresses components of $\mathrm{Y}_{\gamma}^{\varepsilon=\varepsilon_0}$ in terms of those of the fundamental domain:

Lemma 5.5.4. *Let $\gamma = t(s+\varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$, s regular. Let $\widetilde{u} \in \widetilde{W}^{\mathrm{reg}}$ and factorize $\widetilde{u}$ uniquely as*

$$\widetilde{u} = \widetilde{v}^{-1} w_0 \widetilde{w}$$

where $\widetilde{w} = t^{\rho_w} w \in \widetilde{W}_1$, $w \in W$, and $\widetilde{v} = vt^{\nu} \in \widetilde{W}^{+}$.

For $\varepsilon_0 \in \{0, \eta\}$, we have

$$[\mathrm{Y}_{\gamma}^{\varepsilon=\varepsilon_0}(\widetilde{u})] = [t^{-\nu+w_0\rho_w}\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma-t\varepsilon\nu+t\varepsilon w_0\rho_w}(w_0 w)] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}).$$

If $\widetilde{u}$ is furthermore in $\mathrm{Adm}^{\mathrm{reg}}(\rho)$, and $s+r$, $s+r-\nu$, $s+r-\nu+w_0\rho_w$ are h_ρ-generic, then we have

$$[\mathrm{Y}_{\gamma}^{\varepsilon=1}(\widetilde{u})] = [t^{-\nu+w_0\rho_w}\Omega^{\varepsilon=1}_{\mathrm{Y},\gamma-t\nu+tw_0\rho_w}(w_0 w)] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}).$$

Proof. By the Lemma 5.5.3(2),(3) we have

$$\mathrm{Y}_{\gamma}^{\varepsilon=\varepsilon_0}(\widetilde{u}) = t^{-\nu}v^{-1}\mathrm{Y}^{\varepsilon=\varepsilon_0}_{v\gamma-t\varepsilon v\nu}(w_0\widetilde{w}) = t^{-\nu}\mathrm{Y}^{\varepsilon=\varepsilon_0}_{\gamma-t\varepsilon\nu}(w_0\widetilde{w}).$$

Then by Lemma 5.5.3(4) we have

$$\mathrm{Y}^{\varepsilon=\varepsilon_0}_{\gamma-t\varepsilon\nu}(\widetilde{u}) \subset t^{-\nu}(L\check{N} \cap \check{\mathbf{I}})\widetilde{w}\check{\mathbf{I}}/\check{\mathbf{I}} \subset t^{-\nu+w_0\rho_w}L\check{N}w_0 w\check{\mathbf{I}}/\check{\mathbf{I}}.$$

Hence $t^{\nu-w_0\rho_w}\mathrm{Y}_{\gamma}^{\varepsilon=\varepsilon_0}(\widetilde{u})$ belongs to $\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma-t\varepsilon\nu+t\varepsilon w_0\rho_w,\le h_\rho}$. On the other hand, in our identification of $\check{\mathrm{St}}_{\check{B}}$ with $\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma-t\varepsilon\nu+t\varepsilon w_0\rho_w,\le h_\rho}$, the elements belonging to $L\check{N}w_0 w\check{\mathbf{I}}$ exactly correspond to pairs $(e, B') \subset \check{\mathfrak{n}} \times \check{B}$ such that $B' \in \check{B}w_0 w\check{B}$ has relative position $w_0 w$ with respect to $\check{B}$. Then Remark 5.2.1 shows that $t^{\nu-w_0\rho_w}\mathrm{Y}_{\gamma}^{\varepsilon=0}(\widetilde{u})$ is dense in $\Omega^{\varepsilon=\varepsilon_0}_{\mathrm{Y},\gamma-t\varepsilon\nu+t\varepsilon w_0\rho_w}(w_0 w)$. □

Corollary 5.5.5. *Let $\gamma = t(s+\varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$, and s regular. If $\widetilde{u} \in \mathrm{Adm}^{\mathrm{reg}}(\rho) \setminus \{t^{w_0\rho}\}$ then*

$$\iota^{\varepsilon=0}_{\gamma,*}[\mathrm{Y}_{\gamma}^{\varepsilon=0}(\widetilde{u})] = \iota^{\varepsilon=0}_{\gamma,*}[t^{\rho}\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}(w_0 w)] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}})$$

for some $w \neq w_0$.

Proof. Suppose the conclusion doesn't hold. Then by Lemma 5.5.4 and the fact (cf. Remark 5.5.1) that $\check{Q}^{\vee}$ acts trivially on the image of $\iota^{\varepsilon=0}_{\gamma,*}$, $\widetilde{u}$ must have the form

$$\widetilde{u} = \widetilde{v}^{-1}w_0\widetilde{w_0} = \widetilde{v}^{-1}w_0 t^{\rho} w_0 = \widetilde{v}^{-1}t^{w_0\rho}.$$

Since the second and the last factorization are reduced, we learn that $\ell(\widetilde{u}) \ge \ell(t^{w_0\rho})$. But $\widetilde{u} \in \mathrm{Adm}(\rho)$ so equality must occur, a contradiction. □

We record the following property of the affine Springer action for later use:

Lemma 5.5.6. *Let $\gamma = t(s+\varepsilon r)$ with $s, r \in \check{\mathfrak{t}}$, and s regular. For each $w \in W$ such that $w \neq w_0$, there is a simple reflection $s_\alpha \in W$ such that for the affine Springer action on $\mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}})$, we have*

$$(s_\alpha + 1)\iota^{\varepsilon=\varepsilon_0}_{\gamma,*}[\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}(w_0 w)] = 0 \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}).$$

Proof. This is probably a well-known fact about the Steinberg variety but we give a proof for completeness. The proof of Lemma 5.5.4 shows that $V := \Omega^{\varepsilon=0}_{\mathrm{Y},\gamma,\le h_\rho} \cap L\check{N}w_0 w\check{\mathbf{I}}$ is dense in $\Omega^{\varepsilon=0}_{\mathrm{Y},\gamma}(w_0 w)$. Let $g = \exp(E)w_0 w \in V$ with $E \in \omega^{\varepsilon=0}_{X,\gamma}$, and set $e = \mathrm{Res}(E)$. Then

$$\mathrm{Ad}_{\exp(-E)}(\gamma) \subset \mathrm{Ad}_{w_0 w}(\mathrm{Lie}\,\check{\mathbf{I}}) \subset \check{\mathfrak{g}}[[t]],$$

whose reduction modulo t is $-[e, s] \in \check{\mathfrak{n}} \cap \mathrm{Ad}_{w_0 w}(\check{\mathfrak{n}})$, as explained in §5.3.3.

If $w \neq w_0$, there is a simple reflection $s_\alpha \in W$ such that

$$\check{\mathfrak{n}} \cap \mathrm{Ad}_{w_0 w}(\check{\mathfrak{n}}) = \check{\mathfrak{n}} \cap \mathrm{Ad}_{w_0 w}(\check{\mathfrak{n}}) \cap \mathrm{Ad}_{w_0 w s_\alpha}(\check{\mathfrak{n}}).$$

Let $\check{\mathbf{P}}_\alpha \supsetneq \check{\mathbf{I}}$ be the minimal parahoric corresponding to s_α, and let $\pi : \mathrm{Fl}_{\check{G}} \to L\check{G}/\check{\mathbf{P}}_\alpha$ be the natural projection. Our equality above implies $\mathrm{Ad}_{(gx)^{-1}}(\gamma) \subset \mathrm{Lie}\,\check{\mathbf{I}}$ for all $x \in \check{\mathbf{P}}_\alpha$. This shows that $\pi^{-1}(\pi(V)) \subset \mathrm{Y}_\gamma^{\varepsilon=0}$. Since V is a maximal dimension locally closed subset of $\mathrm{Y}_\gamma^{\varepsilon=0}$, we learn that V is dense in $\pi^{-1}(\pi(V))$, and hence that $\overline{\Omega}_{\mathrm{Y},\gamma}^{\varepsilon=0}(w_0w) = \pi^{-1}\pi(\overline{\Omega}_{\mathrm{Y},\gamma}^{\varepsilon=0}(w_0w))$. This implies the desired conclusion by [KL88, §4, Lemma 9]. □

5.6. **Proof of Theorem 5.1.1.** Let $\varepsilon_0 \in \{0,1\}$. Suppose for some $\widetilde{u} \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$, we have

$$\left(\mathfrak{sp}_{\varepsilon\to\varepsilon_0}[\mathrm{Y}_\gamma^{\varepsilon=\eta}(\widetilde{u})]\right) - [\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\widetilde{u})] \neq 0 \in \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}),$$

so this difference is represented by a non-zero *effective* (by Lemma 4.2.3) d-cycle.

Factorize $\widetilde{u} = \widetilde{v}^{-1}w_0\widetilde{w}$ as in Lemma 5.5.3. Set $\gamma' = \gamma - t\varepsilon\nu + t\varepsilon w_0\rho_w$ so that Lemma 5.5.4 implies

$$[\mathrm{Y}_\gamma^{\varepsilon=\eta}(\widetilde{u})] = [t^{-\nu+w_0\rho_w}\Omega_{\mathrm{Y},\gamma'}^{\varepsilon=\eta}(w_0w)] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}),$$

$$[\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\widetilde{u})] = [t^{-\nu+w_0\rho_w}\Omega_{\mathrm{Y},\gamma'}^{\varepsilon=\varepsilon_0}(w_0w)] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}).$$

Note that $s + r - \nu + w_0\rho_w$ is $2h_\rho$-generic: indeed

$$\widetilde{u} = (t^{-\nu}v^{-1})w_0\widetilde{w} \geq (t^{-\nu}v^{-1})vw_0\widetilde{w} = t^{-\nu+w_0\rho_w}w_0w$$

belongs to $\mathrm{Adm}(\rho)$, hence $-\nu + w_0\rho_w \in \mathrm{Conv}(W\rho)$ is h_ρ-small.

Consider the class

$$\delta := \sum_{z\in W} a_{w_0z}\mathfrak{sp}_{\varepsilon\to\varepsilon_0}[\Omega_{Y,\gamma'}^{\varepsilon=\eta}(w_0z)] - \sum_{z\in W} a_{w_0z}[\Omega_{Y,\gamma'}^{\varepsilon=\varepsilon_0}(w_0z)] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Y}_{\gamma'}^{\varepsilon=\varepsilon_0})$$

where a_{w_0z} are strictly positive integers as in Corollary 5.2.4. We claim that δ has the following properties:

(1) δ is represented by a non-zero *effective* d-cycle in $\mathrm{Y}_{\gamma'}^{\varepsilon=\varepsilon_0}$,
(2) $\iota_{\gamma',*}^{\varepsilon=\varepsilon_0}(\delta)$ is W-invariant (for the affine Springer action), and
(3) $\iota_{\gamma',*}^{\varepsilon=\varepsilon_0}(\delta)$ belongs to the span of $\iota_{\gamma,*}^{\varepsilon=0}[\Omega_{\mathrm{Y},\gamma}^{\varepsilon=0}(w_0z)]$ for $z \neq w_0$.

Item (1) follows from Lemma 2.4.1, the positivity of a_{w_0z} and our assumption at the beginning of the proof. Item (2) follows from Corollary 5.5.2 (the fact that $\iota_{\gamma',*}^{\varepsilon=\varepsilon_0}$ is W-equivariant) and the choice of the integers a_{w_0z}.

We now check (3). For each $z \in W$, by Lemma 5.5.4 we have

$$[\Omega_{\mathrm{Y},\gamma'}^{\varepsilon=\eta}(w_0z)] = [t^{-w_0\rho_z}\mathrm{Y}_{\gamma''}^{\varepsilon=\eta}(w_0\widetilde{z})] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}),$$

$$[\Omega_{\mathrm{Y},\gamma'}^{\varepsilon=\varepsilon_0}(w_0z)] = [t^{-w_0\rho_z}\mathrm{Y}_{\gamma''}^{\varepsilon=\varepsilon_0}(w_0\widetilde{z})] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathrm{Fl}_{\check{G}}),$$

where $\widetilde{z} = t^{\rho_z}z$ and $\gamma'' = \gamma - t\varepsilon w_0\rho_z$. Note that in the case $\varepsilon_0 = 1$, Lemma 5.5.4 does apply since $s + r - \nu + w_0\rho_w$ is $2h_\rho$-generic and $w_0\rho_z$ is h_ρ-small, so $s + r - \nu + w_0\rho_w - w_0\rho_z$ is h_ρ-generic. Thus by Lemma 4.2.3,

$$\left(\mathfrak{sp}_{\varepsilon\to\varepsilon_0}[\Omega_{\mathrm{Y},\gamma'}^{\varepsilon=\eta}(w_0z)]\right) - [\Omega_{\mathrm{Y},\gamma'}^{\varepsilon=\varepsilon_0}(w_0z)]$$

lies in the span of $\{[t^{-w_0\rho_z}\mathrm{Y}_{\gamma''}^{\varepsilon=\varepsilon_0}(\widetilde{u}')]\}$ where $\widetilde{u}' \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$ such that $\widetilde{u}' < w_0\widetilde{z}$. Another invocation of Lemma 4.2.3, together with the fact that

$$\iota_{\gamma''*}^{\varepsilon=\varepsilon_0}\mathfrak{sp}_{\varepsilon\to\varepsilon_0}[\mathrm{Y}_{\gamma''}^{\varepsilon=\eta}(\widetilde{u}')] = \mathfrak{sp}_{\varepsilon\to\varepsilon_0}\iota_{\gamma''*}^{\varepsilon=\eta}[\mathrm{Y}_{\gamma''}^{\varepsilon=\eta}(\widetilde{u}')]$$

is independent of the choice $\varepsilon_0 \in \{0,1\}$ shows that $\iota_{\gamma'',*}^{\varepsilon=\varepsilon_0}[t^{-w_0\rho_z}\mathrm{Y}_{\gamma''}^{\varepsilon=\varepsilon_0}(\widetilde{u}')]$ lies in the span of $\{\iota_{\gamma'',*}^{\varepsilon=0}[t^{-w_0\rho_z}\mathrm{Y}_{\gamma''}^{\varepsilon=0}(\widetilde{u}'')]\}$ running over $\widetilde{u}$ such that $\widetilde{u}'' \leq \widetilde{u} < w_0\widetilde{z}$. Corollary 5.5.5 then justifies (3).

Now (3) together with Lemma 5.5.6 shows that under the affine Springer action

$$\left(\sum_{z\in W} z\right)\iota_{\gamma'*}^{\varepsilon=\varepsilon_0}(\delta) = 0.$$

But since $\iota_{\gamma'*}^{\varepsilon=\varepsilon_0}(\delta)$ is W-invariant, the left-hand side is $|W|\iota_{\gamma'*}^{\varepsilon=\varepsilon_0}(\delta)$, and we learn that $\iota_{\gamma'*}^{\varepsilon=\varepsilon_0}(\delta) = 0$. Finally since $\mathrm{Fl}_{\check{G}}$ is ind-projective, any effective representing the class $\delta = 0$ must be actually 0 (as can be seen, for example by computing degrees under an embedding to a large enough projective space), a contradiction. □

Part 2. Microlocal analysis

Most of this Part focuses on affine Springer fibers (and not their deformations), so we will use the abbreviations $\mathcal{X}_\gamma := \mathcal{X}_\gamma^{\varepsilon=0}$, $\mathrm{X}_\gamma := \mathrm{X}_\gamma^{\varepsilon=0}$, and $\mathrm{Y}_\gamma := \mathrm{Y}_\gamma^{\varepsilon=0}$.

Henceforth we use $\mathrm{K}(-) := K_0(-)$ for the Grothendieck group of an exact category, in order to improve the readability of the notation, because there will be many subscripts (including 0) on the categories, and we never consider higher K-groups anyways.

Recall that $h = h_\rho + 1$ is the maximum of the Coxeter numbers of the simple factors of $\check{G}$, and that we assume throughout that $p > h$.

6. Equivariant homology of affine Springer fibers

In this section we recall some notions from equivariant (co)homology, which will be applied to the (deformed) affine Springer fibers. We begin with a review of generalities such as equivariant Borel-Moore homology, equivariant formality, the equivariant localization theorem, and equivariant Euler classes in §6.1 – §6.4.

Then in §6.5 we recall the "GKM description" (named after Goresky-Kottwitz-MacPherson), which gives a combinatorial description of the equivariant Borel-Moore homology of spaces satisfying the so-called "GKM conditions". We explicate the GKM description for the affine Springer fibers Y_γ and their variants over the complex numbers. This is used in particular to fix identifications $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}_\gamma)$ for all $\gamma = ts$, where $s \in \check{\mathfrak{t}}_{\mathbf{F}_q}$ is regular, as well as to establish a bridge between the homology of affine Springers in characteristic p and the homology of their variants over $\mathbf{C}$.

Finally, in §6.6 we discuss actions on the equivariant Borel-Moore homology of the Y_γ, which in particular provide equivariant lifts of the translation action and affine Springer action defined in Part 1.

6.1. **Equivariant Borel-Moore homology.** Suppose X is a finite type scheme over a field k, equipped with the action of an algebraic group H. Recall that the H-equivariant cohomology of X, denoted $H^*_H(X)$, is the cohomology of the quotient stack $[X/H]$. We define $\mathrm{H}^*_H(X)$ to be the *geometric* cohomology of $[X/H]$.

Similarly, the *H-equivariant (ℓ-adic) Borel-Moore homology* $H^{\mathrm{BM},H}_*(X)$ is defined as the relative Borel-Moore homology of the quotient stack X/H relative to [Spec k/H],

$$H^{\mathrm{BM},H}_*(X) := H^{\mathrm{BM}}_*([X/H]/[\mathrm{Spec}\ k/H]).$$

We denote by $\mathrm{H}^{\mathrm{BM},H}_*(X)$ the *geometric* H-equivariant Borel-Moore homology, i.e., the same definition after base changing to a geometric point over Spec k.

For an ind-scheme $X = \varinjlim_i X_i$ equipped with compatible H-actions on each X_i, we define

$$\mathrm{H}^{\mathrm{BM},H}_*(X) := \varinjlim_i \mathrm{H}^{\mathrm{BM},H}_*(X_i)$$

with transition maps induced by the closed embeddings $X_i \hookrightarrow X_j$.

Example 6.1.1 (Torus actions)**.** In this paper, we will only ever consider equivariant Borel-Moore homology with respect to the action of a split torus T. We let $\mathfrak{t} := (\mathrm{Lie}\, T)_{\mathbf{Q}_\ell}$. Equivariant Borel-Moore homology has a natural module structure over equivariant cohomology. Therefore $\mathrm{H}^{\mathrm{BM},T}_*(X)$ is naturally a module over

$$\mathrm{H}^*_T(\mathrm{pt}; \mathbf{Q}_\ell) = \mathrm{Sym}_{\mathbf{Q}_\ell}(\mathfrak{t}^*) = \mathcal{O}(\mathfrak{t}) =: \mathbb{S}_T.$$

When the acting torus T is understood, we will simply abbreviate $\mathbb{S} := \mathbb{S}_T$.

From the definitions we have $\mathrm{H}^{\mathrm{BM},T}_*(\mathrm{pt}) \cong \mathrm{H}^{-*}_T(\mathrm{pt})$, so there is a natural pairing

$$\mathrm{H}^*_T(\mathrm{pt}) \otimes \mathrm{H}^{\mathrm{BM},T}_*(\mathrm{pt}) \to \mathrm{H}^0_T(\mathrm{pt}) \cong \mathbf{Q}_\ell$$

which realizes $\mathrm{H}^{\mathrm{BM},T}_*(\mathrm{pt})$ as the graded $\mathbf{Q}_\ell$-dual of $\mathrm{H}^*_T(\mathrm{pt}) \cong \mathcal{O}(\mathfrak{t})$, so we may canonically regard $\mathrm{H}^{\mathrm{BM},T}_*(\mathrm{pt})$ as $\Omega^{\wedge \dim \mathfrak{t}}_{\mathfrak{t}}$, the $\mathcal{O}(\mathfrak{t})$-module of top-dimensional differential forms on $\mathfrak{t}$. Then any choice of generator of $\omega_{\mathrm{top}} \in \Omega^{\wedge \dim \mathfrak{t}}_{\mathfrak{t}}$ over $\mathcal{O}(\mathfrak{t})$ is equivalent to a choice of trivialization of $\mathrm{H}^{\mathrm{BM},T}_*(\mathrm{pt})$ as an $\mathrm{H}^*_T(\mathrm{pt})$-module.

6.2. **Equivariant formality.** Let H be a group acting on a variety X. The graded H-equivariant Borel-Moore homology group $\mathrm{H}_*^{\mathrm{BM},H}(X)$ has a graded action action of $\mathrm{H}_H^*(\mathrm{pt})$.

There is an augmentation homomorphism $\mathrm{H}_H^*(\mathrm{pt}) \to \mathbf{Q}_\ell$. This induces a map

$$\mathrm{H}_*^{\mathrm{BM},H}(X) \otimes_{\mathrm{H}_H^*(\mathrm{pt})} \mathbf{Q}_\ell \to \mathrm{H}_*^{\mathrm{BM}}(X). \tag{6.2.1}$$

Recall that X is *H-equivariantly formal* if this map is an isomorphism.

Remark 6.2.1 (Equivariant classes in top homological degree)**.** If X is H-equivariantly formal and equidimensional of dimension d, then the map

$$\mathrm{H}_{2d}^{\mathrm{BM},H}(X) \to \mathrm{H}_{2d}^{\mathrm{BM}}(X)$$

induced by (6.2.1) is an isomorphism: "top-dimensional cycles are equipped with a canonical H-equivariant structure".

Example 6.2.2 (Equivalued affine Springer fibers)**.** According to [GKM06, Theorem 0.2], the *equivalued* affine Springer fibers admit a paving by affine spaces, and are therefore pure. Hence they are equivariantly formal with respect to any group action. This applies in particular to the affine Springer fibers Y_γ and X_γ where $\gamma = ts$ for $s \in \mathfrak{t}^*$, under the translation action of T.

6.3. **The equivariant localization theorem.** Let T be a torus acting on a variety X. We recall the localization theorem for the T-equivariant Borel-Moore homology of X. Recall that $\mathrm{H}_*^{\mathrm{BM},T}(X)$ has a natural module structure over $\mathbb{S} = \mathbb{S}_T = \mathcal{O}(\mathfrak{t})$. Hence we may regard $\mathrm{H}_*^{\mathrm{BM},T}(X)$ as a quasicoherent sheaf on $\mathfrak{t}$.

Let $\iota \colon X^T \to X$ denote the inclusion of the T-fixed points. The following is the famous *Equivariant Localization Theorem* for torus actions, which goes back to work of Atiyah-Bott; a modern reference is [AKL$^+$22, Theorem A].

Theorem 6.3.1 (Equivariant Localization Theorem)**.** *The kernel and cokernel of the map*

$$\iota_* \colon \mathrm{H}_*^{\mathrm{BM},T}(X^T) \to \mathrm{H}_*^{\mathrm{BM},T}(X)$$

are supported (as quasicoherent sheaves) on the union of $\mathrm{Lie}\, K \subset \mathfrak{t}$ *as K runs over proper stabilizer subgroups $K \subset T$. In particular, ι_* is an isomorphism after tensoring over* $\mathbb{S}$ *with* $\mathrm{Frac}(\mathbb{S})$*.*

The theorem extends to ind-varieties in the obvious way. Note that the structure of $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(X^T)$ is simple: since T acts trivially on X^T, we simply have $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(X^T) = \mathbb{S} \otimes_{\mathbf{Q}_\ell} \mathrm{H}_*^{\mathrm{BM}}(X^T)$, which is free over $\mathbb{S}$.

We give some examples below, in which we maintain the notation of Part 1.

Example 6.3.2. Let $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{Q}_q}$ be and let $X = \mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}$. Since $\gamma \in \check{\mathfrak{t}}[[t]]$, there is an action of $\check{T}$ via left translation on X (cf. §3.3), and we may identify $X^{\check{T}}$ with the discrete scheme $X_*(\check{T})$, with $\lambda \in X_*(\check{T})$ corresponding to $t^\lambda L^+ \check{G} \in \mathrm{Gr}_{\check{G},\mathbf{Q}_q}$.

Example 6.3.3. Let $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{F}_q}$ and $X = \mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}$. Since $\gamma \in \check{\mathfrak{t}}[[t]]$, there is an action of $\check{T}$ via left translation on X. Then we may identify $X^{\check{T}}$ with the discrete scheme $\widetilde{W}$, with $wt^\lambda \in \widetilde{W}$ corresponding to $wt^\lambda \check{\mathbf{I}} \in \mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}$.

6.4. **Equivariant Euler classes.** Suppose X is an ind-variety with an action of a torus T, such that X^T consists of isolated points in X. Let $x \in X^T$ be a *smooth* point of X. Decompose the tangent space $\mathrm{T}_x X$ as a representation of the torus T into a sum of characters of T,

$$\mathrm{T}_x X \cong \bigoplus_i \lambda_i, \quad \lambda_i \in X^*(T).$$

We may view $d\lambda_i$ as (linear) elements of $\mathcal{O}(\mathfrak{t})$. Then the *equivariant Euler class* of X at x is defined to be $\prod_i d\lambda_i \in \mathcal{O}(\mathfrak{t})$, and we denote its inverse by

$$e_T(x, X) := \frac{1}{\prod_i d\lambda_i} \in \mathrm{Frac}(\mathbb{S}_T).$$

If X is T-equivariantly formal of pure dimension d, then by Remark 6.2.1 the fundamental class of X admits a unique T-equivariant lift, which we denote $[X]_T \in \mathrm{H}_{2d}^{\mathrm{BM},T}(X)$. Recall that we have fixed a generator ω_{top} of $\Omega_{\mathfrak{t}}^{\wedge \dim \mathfrak{t}}$.

Lemma 6.4.1. *Suppose X is T-equivariantly and has isolated T-fixed points. If $x \in X^T$ is a smooth point of X, then the image of $[X]_T$ in*

$$\mathrm{H}_*^{\mathrm{BM},T}(X^T) \otimes_{\mathbb{S}} \mathrm{Frac}(\mathbb{S}_T) \cong \bigoplus_{x \in X^T} \mathrm{Frac}(\mathbb{S}_T)[x]$$

under (6.5.1) *has component coefficient of* $[x]$ *equal to* $e_T(x, X) \in \mathrm{Frac}(\mathbb{S})$.

Proof. This follows immediately from the Atiyah-Bott localization formula, for which a modern reference (in much more generality) is [AKL+22, Theorem D]. □

Example 6.4.2. Consider $X = \check{G}/\check{B}$ with the left translation action of $\check{T}$. Then the $\check{T}$-fixed points of X may be identified with the discrete scheme W, where $w \in W$ corresponds to the fixed point $w\check{B} := \dot{w}\check{B}$ for any lift $\dot{w} \in N(\check{T})$ of w (note that the coset $w\check{B}$ does not depend on the choice of lift). The tangent space $\mathrm{T}_{w\check{B}}X$ is $\check{T}$-equivariantly identified as

$$\mathrm{T}_{w\check{B}}X \cong \bigoplus_{\alpha \in \check{\Phi}^+} \mathfrak{g}_{w\alpha}.$$

Letting $\beta := \prod_{\alpha \in \check{\Phi}^+} d\alpha$, the equivariant fundamental class $[\check{G}/\check{B}]_{\check{T}}$ is therefore

$$\sum_{w \in W} \frac{\mathrm{sgn}(w)}{\beta}[w] \in \bigoplus_{w \in W} \mathrm{Frac}(\mathbb{S}_{\check{T}})[w].$$

6.5. **GKM description of equivariant Borel-Moore homology.** Suppose X is equivariantly formal for the action of a split torus T. Then $\mathrm{H}_*^{\mathrm{BM},T}(X) \cong \mathrm{H}_*^{\mathrm{BM}}(X) \otimes_{\mathbf{Q}_\ell} \mathbb{S}_T$ is free over $\mathbb{S}_T$, so we have an inclusion

$$\mathrm{Loc}^T \colon \mathrm{H}_*^{\mathrm{BM},T}(X) \hookrightarrow \mathrm{H}_*^{\mathrm{BM},T}(X^T) \otimes_{\mathbb{S}_T} \mathrm{Frac}(\mathbb{S}_T) \tag{6.5.1}$$

which is the dashed arrow in the diagram below

$$\begin{tikzcd}
\mathrm{H}_*^{\mathrm{BM},T}(X^T) \arrow[r, "i_*"] \arrow[d, hook] & \mathrm{H}_*^{\mathrm{BM},T}(X) \arrow[d, hook] \arrow[dl, dashed, hook, "\mathrm{Loc}^T"'] \\
\mathrm{H}_*^{\mathrm{BM},T}(X^T) \otimes_{\mathbb{S}_T} \mathrm{Frac}(\mathbb{S}_T) \arrow[r, "i_* \otimes \mathrm{Id}", "\sim"'] & \mathrm{H}_*^{\mathrm{BM},T}(X) \otimes_{\mathbb{S}_T} \mathrm{Frac}(\mathbb{S}_T)
\end{tikzcd}$$

Recall that we set $\mathfrak{t} := (\mathrm{Lie}\, T)_{\mathbf{Q}_\ell}$. Fix a generator ω_{top} of $\Omega_{\mathfrak{t}}^{\wedge \dim \mathfrak{t}}$, which induces an isomorphism $\mathrm{H}_*^{\mathrm{BM},T}(X) \cong \mathbb{S}_T$ (cf. Example 6.1.1). Using this, we have

$$\mathrm{H}_*^{\mathrm{BM},T}(X^T) \otimes_{\mathbb{S}_T} \mathrm{Frac}(\mathbb{S}_T) \cong \bigoplus_{x \in X^T} \mathrm{Frac}(\mathbb{S}_T)[x].$$

Definition 6.5.1. For $\alpha \in \mathrm{H}_*^{\mathrm{BM},T}(X)$, the *equivariant support* of α is the subset of X^T at which $\mathrm{Loc}^T(\alpha)$ has non-zero component in the direct sum decomposition (6.5.1).

Suppose now that X further satisfies the *GKM conditions*[8]: on any quasicompact subset of X, there are only finitely many T-fixed points and finitely many one-dimensional T-orbits. We review the so-called *GKM description* of $\mathrm{H}_*^{\mathrm{BM},T}(X)$, which applies under the GKM conditions. Following the formulation of [BAL21, Proposition 4.3], it says that the image of (6.5.1) is the space of

$$\sum_{x \in X^T} f_x[x] \in \bigoplus_{x \in X^T} \mathrm{Frac}(\mathbb{S}_T)[x]$$

satisfying the following conditions:

- The poles of f_x are of order ≤ 1 and contained in the union of the *singular hyperplanes*, meaning hyperplanes of the form $\ker(d\chi)$ for a character $\chi \colon T \to \mathbf{G}_m$ such that $X^{\ker \chi} \neq X^T$.
- For every *singular* character χ (meaning that $X^{\ker \chi} \neq X^T$) and every connected component $Y \subset X^{\ker(\chi)}$, we have

$$\mathrm{Res}_{\ker(d\chi)} \left(\sum_{x \in X^T \cap Y} f_x \, \omega_{\mathrm{top}} \right) = 0$$

[8]Named after Goresky, Kottwitz, and MacPherson who initiated these ideas in [GKM04, Theorem 7.5].

where we recall that ω_{top} is the fixed generator of $\Omega_{\mathfrak{t}}^{\wedge \dim \mathfrak{t}}$. Note that the collection of such Y form the 1-dimensional orbits of T on X.

Example 6.5.2 (Borel-Moore homology of equivalued unramified affine Springer fibers)**.** Let $\gamma = ts$ with $s \in \check{\mathfrak{t}}_{\mathbf{C}}$ regular. Then one can write down an analogous version of Y_γ over the complex numbers, which we denote $\mathrm{Y}_{\gamma,\mathbf{C}}$. Then $\mathrm{Y}_{\gamma,\mathbf{C}}$ satisfies the GKM conditions for the translation action of $\check{T}$, has the same fixed points as the positive characteristic version analyzed in Example 6.3.3, and the 1-dimensional $\check{T}$-orbits are calculated in [GKM04, §5.11].

The GKM description of $\mathrm{H}_*^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$ is described explicitly in [BAL21, Corollary 4.8], and is manifestly independent of γ (satisfying the hypotheses), and is used to identify $\mathrm{H}_*^{\mathrm{BM},\check{T}}(\mathrm{Y}_{\gamma,\mathbf{C}})$ for all γ satisfying the hypotheses. By equivariant formality of the $\check{T}$-action, this also identifies $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_{\gamma,\mathbf{C}})$ for all γ satisfying the hypotheses.

To bootstrap these results to positive characteristic, observe that the same analysis shows that for $\gamma = ts$ such that $s \in \check{\mathfrak{t}}_{\mathbf{F}_q}$ is regular, then Y_γ also satisfies the GKM conditions, and with the same combinatorics of fixed points, singular characters, and 1-dimensional orbits, hence $\mathrm{H}_*^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$ has the same GKM description. In particular, we use this description to identify $\mathrm{H}_*^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$ for all γ satisfying the hypotheses. By equivariant formality of the $\check{T}$-action on Y_γ, this also identifies $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma)$ for all γ satisfying the hypotheses.

Remark 6.5.3 (Deformed affine Springer fibers in the admissible region)**.** Let $\gamma = ts$ with $s \in \check{\mathfrak{t}}_{\mathbf{F}_q}$ regular and $\lambda \in X_*(\check{T})^+$ be a dominant coweight such that γ is $h_{\lambda+\rho}$-generic. Then one can deduce from [LHLM23b, Proposition 3.3.4] that:

- $\mathrm{Y}_\gamma^\varepsilon(\leq \lambda)$ satisfies the GKM description for $\check{T}$ (and in particular is $\check{T}$-equivariantly formal).
- The $\check{T}$-fixed points of the family $\mathrm{Y}_\gamma^\varepsilon(\leq \lambda) \to \operatorname{Spec} \mathbf{F}_q[\varepsilon]$ are identified with the constant family $\mathrm{Adm}(\lambda) \times \operatorname{Spec} \mathbf{F}_q[\varepsilon]$, with $wt^\mu \in \mathrm{Adm}(\lambda)$ corresponding to $[wt^\mu] \in \mathrm{Fl}_{\check{G},\mathbf{F}_q[\varepsilon]}$.
- For each $\varepsilon_0 \in \operatorname{Spec} \mathbf{F}_q[\varepsilon]$, the singular characters are independent of ε_0 and the residue conditions they induce are independent of ε_0.

Hence the GKM descriptions of $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda))$ are independent of $\varepsilon_0 \in \operatorname{Spec} \mathbf{F}_q[\varepsilon]$. This gives compatible identifications of $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda))$ for all $\varepsilon_0 \in \operatorname{Spec} \mathbf{F}_q[\varepsilon]$, with respect to which the specialization maps in ε are the identity map. In particular this gives *some* bases of $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda))$ with respect to which the specializations in ε are the identity maps (but it is completely unclear whether this is the case for the basis comprised by the cycle classes of top-dimensional irreducible components, so this does not give a cheap proof of Theorem 5.1.1.)

6.6. **Actions on equivariant Borel-Moore homology.** We define equivariant lifts of the affine Springer action and translation action on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma)$.

6.6.1. *Monodromy-centralizer action.* We will define an action of $(\widetilde{W}, \cdot)$ on $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma)$ using the GKM description from Example 6.5.2, which we call the *monodromy-centralizer action.*

Fix a generator $\omega_{\mathrm{top}} \in \Omega_{\check{\mathfrak{t}}}^{\wedge \dim \check{\mathfrak{t}}}$. Then $\mathrm{Loc}^{\check{T}}$ embeds $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma) \hookrightarrow \bigoplus_{x\in\widetilde{W}} \mathrm{Frac}(\mathbb{S}_{\check{T}})[x]$. There is a left action of $\widetilde{w} \in \widetilde{W}$ on $\bigoplus_{x\in\widetilde{W}} \mathrm{Frac}(\mathbb{S}_{\check{T}})[x]$ via

$$\widetilde{w} \cdot \sum_{x\in X^{\check{T}}} f_x[x] = \sum_{x\in X^{\check{T}}} (\widetilde{w} f_x)[\widetilde{w}x]$$

where $\widetilde{w} f_x$ refers to the natural $\widetilde{W}$-action on $\mathrm{Frac}(\mathbb{S}_{\check{T}})$. One checks from the GKM description in Example 6.5.2 that this action preserves the subspace $\mathrm{H}_*^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$.

Finally, by Remark 6.2.1, it induces an action $(\widetilde{W}, \cdot)$ on the non-equivariant Borel-Moore homology $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mathrm{Y}_\gamma)$, for which the action of $X_*(\check{T}) \subset \widetilde{W}$ agrees with the translation action of §3.3. We refer to (all these variants of) the action $(\widetilde{W}, \cdot)$ as the *monodromy-centralizer action.*

Remark 6.6.1 (Explanation of terminology)**.** This action is also defined (for $Y_{\gamma,\mathbf{C}}$) in [BAL21, §5.2], up to converting between left and right actions. As explained in [BAL21, Remark 5.4], the action of the $X_*(\check{T}) \subset \widetilde{W}$ on $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma)$ is induced by the translation action by the centralizer of γ (cf. §3.3).

The action of $W \subset \widetilde{W}$ is subtler in that it is *not* induced by an action of W on Y_γ, but could be thought of informally as the the monodromy action coming from a hypothetical local system on $\check{\mathfrak{t}}^{\mathrm{reg}}$ whose fiber over γ is $\mathrm{H}_*^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$; indeed, Example 6.5.2 suggests that the Y_γ are "$\check{T}$-equivariantly homotopy equivalent". The cleanest way we know to make this precise is to just use the GKM description, as above.

6.6.2. *Affine Springer action.* We will define a *right* action of $(\widetilde{W}, \bullet)$ on $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma)$ that lifts the affine Springer action from §3.5. The action of $\widetilde{w} \in \widetilde{W}$ on $\bigoplus_{x \in \widetilde{W}} \mathrm{Frac}(\mathbb{S}_{\check{T}})[x]$ is given by

$$\widetilde{w} \bullet \sum_{x \in X^{\check{T}}} f_x[x] = \sum_{x \in X^{\check{T}}} f_x[x\widetilde{w}].$$

One checks from the GKM description in Example 6.5.2 that this action preserves $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma)$ embedded as a subspace of $\bigoplus_{x \in \widetilde{W}} \mathrm{Frac}(\mathbb{S}_{\check{T}})[x]$ via $\mathrm{Loc}^{\check{T}}$. By equivariant formality, it induces an action on $\mathrm{H}_*^{\mathrm{BM}}(\mathrm{Y}_\gamma)$, which agrees with the affine Springer action constructed in §3.5, according to [GKM04, §14.4][9]. We refer to (all these variants of) the action $(\widetilde{W}, \bullet)$ as the *affine Springer action*.

Remark 6.6.2. It is immediate from the definitions that the actions $(\widetilde{W}, \cdot)$ and $(\widetilde{W}, \bullet)$ commute with each other.

7. Modular representation theory

Let G be a split reductive group defined over $\mathbf{F}_p$, with simply connected derived subgroup.

In this section, which has no original results due to us, we recall some facts about the representation theory of $\mathfrak{g} := \mathrm{Lie}\, G$ as well as of the *Frobenius kernel*

$$G_1 := \ker(\mathrm{Frob}_p \colon G \to G),$$

and their graded variants. Roughly speaking, these will be used in Part 3 to "approximate" the representation theory of $G(\mathbf{F}_p)$, which is more directly related to Serre weights. We also review in §7.5 the crucial theory of *Bezrukavnikov-Mirkovic-Rumynin localization*, which provides the bridge between the categories of such representations and the categories of coherent sheaves that feature into the instance of mirror symmetry which is relevant for us.

Finally, in §7.6 and §7.7 we collect some natural symmetries of these categories.

7.1. **Choice of torus.** We fix, from here until the end of the paper, a split maximal torus $T \subset G$ together with an identification $X^*(T) \cong X_*(\check{T})$, realizing $\check{T}$ as the Langlands dual group of T.

Then our choice of $\check{B}$ induces a choice of Borel $B \supset T$. Let $B^+ = w_0 B$ be the opposite Borel to B. Recall that conventions from §2.2.1 are that the positive roots Φ^+ are the roots of T on $\mathfrak{b}^+$, equivalently on $\mathfrak{g}/\mathfrak{b}$, and that ρ is the half-sum of the positive roots.

The choice of B induces an isomorphism of the flag variety $\mathcal{B}$ with G/B. Recall that our positivity conventions are normalized so that the functor $\mathrm{Rep}(T) \to \mathrm{Coh}^G(G/B)$, sending a character λ to the line bundle $\mathcal{O}(\lambda) = G \times^B \lambda$, takes dominant weights to semi-ample line bundles.

7.2. **Center of the universal enveloping algebra.** Let $k \supset \mathbf{F}_p$ be a field of characteristic p. For a variety X over k, we write $X^{(1)} := X \times_{k, \mathrm{Frob}_p} k$ for the *(first) Frobenius twist* of X. We regard $G, \mathfrak{g}$, etc. over k and write $\mathcal{U}\mathfrak{g}$ for the universal enveloping algebra of $\mathfrak{g}$ over k. By $\mathrm{Rep}(\mathfrak{g}) = \mathrm{Rep}(\mathcal{U}\mathfrak{g})$, etc. we mean the category of *finitely generated* representations of $\mathcal{U}\mathfrak{g}$.

7.2.1. *The Harish-Chandra center.* We recall some facts about the center of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. Let $\mathfrak{h} = \mathrm{Lie}\, A_k$ be the abstract Cartan Lie algebra of $\mathfrak{g}$. An argument of Harish-Chandra produces a map $k[\mathfrak{t}]^{(W, \bullet)} \hookrightarrow Z(\mathcal{U}\mathfrak{g})$. Its image is called the *Harish-Chandra center* $\mathfrak{Z}_{\mathrm{HC}}$.

[9]For the version over $\mathbf{C}$, but this translates to the version over $\mathbf{F}_q$ as in Example 6.5.2.

7.2.2. *The Frobenius center.* In the analogous characteristic zero story, the Harish-Chandra center comprises the entirety of the center of $\mathcal{U}\mathfrak{g}$. But in characteristic p, the center of $\mathcal{U}\mathfrak{g}$ is much larger: there is also the so-called "Frobenius center"

$$\mathrm{Sym}_k\, \mathfrak{g}^{(1)} \hookrightarrow Z(\mathcal{U}\mathfrak{g})$$

induced by the map sending $X \in \mathfrak{g}$ to $X^p - X^{[p]}$, whose image we denote $\mathfrak{Z}_{\mathrm{Fr}}$. Here $X \mapsto X^{[p]}$ is the p-operation on a Lie algebra in characteristic p, e.g., for $\mathfrak{g} = \mathfrak{gl}_n$ it sends a matrix to its pth power.

7.2.3. *The full center $Z(\mathcal{U}\mathfrak{g})$.* Under our assumption that $p > h$, the center of $Z(\mathcal{U}\mathfrak{g})$ is generated by the Harish-Chandra center and the Frobenius center, and has the more precise geometric description (cf. [MR99])

$$\mathrm{Spec}\ Z(\mathcal{U}\mathfrak{g}) \cong \mathfrak{h}^* /\!\!/ W \times_{\mathfrak{h}^{*(1)} /\!\!/ W} \mathfrak{g}^{*(1)}. \tag{7.2.1}$$

Here:

- The map $\mathfrak{g}^{*(1)} \to \mathfrak{h}^{*(1)} /\!\!/ W$ is the composition
$$\mathfrak{g}^{*(1)} \to \mathfrak{g}^{*(1)} /\!\!/ G^{(1)} \xrightarrow{\text{Chevalley}} \mathfrak{h}^{*(1)} /\!\!/ W.$$
- The map $\mathfrak{h}^* /\!\!/ W \to \mathfrak{h}^{*(1)} /\!\!/ W$ is induced by the "Artin-Schreier map" $t \mapsto t^p - t^{[p]}$, where $t \mapsto t^{[p]}$ is the p-operation on $\mathfrak{h}$.

7.2.4. *Representations with central conditions.* By the preceding discussion, a character of $Z(\mathcal{U}\mathfrak{g})$ is given by a compatible pair $(\lambda \in \mathfrak{h}^*, \chi \in \mathfrak{g}^{*(1)})$. For such a compatible pair (λ, χ), we define:

$$\mathcal{U}\mathfrak{g}^\lambda := (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \lambda, \quad \mathcal{U}\mathfrak{g}_\chi := (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} \chi, \quad \mathcal{U}\mathfrak{g}_\chi^\lambda := (\mathcal{U}\mathfrak{g}) \otimes_{Z(\mathcal{U}\mathfrak{g})} (\lambda, \chi).$$

We also make the following definitions.

- Define $\mathrm{Rep}^\lambda(\mathcal{U}\mathfrak{g})$ to be the full subcategory of $\mathrm{Rep}(\mathcal{U}\mathfrak{g})$ where $\mathfrak{Z}_{\mathrm{HC}}$ acts with *generalized* eigenvalue λ.
- Define $\mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g})$ to be the full subcategory of $\mathrm{Rep}(\mathcal{U}\mathfrak{g})$ where $\mathfrak{Z}_{\mathrm{Fr}}$ acts with *generalized* eigenvalue χ.
- Define $\mathrm{Rep}^\lambda(\mathcal{U}\mathfrak{g}_\chi) := \mathrm{Rep}^\lambda(\mathcal{U}\mathfrak{g}) \cap \mathrm{Rep}(\mathcal{U}\mathfrak{g}_\chi)$, and $\mathrm{Rep}^\lambda_\chi(\mathcal{U}\mathfrak{g}) := \mathrm{Rep}^\lambda(\mathcal{U}\mathfrak{g}) \cap \mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g})$, etc.

7.3. **The Frobenius kernel.** Recall that the *Frobenius kernel* G_1 is the kernel of $\mathrm{Frob}_p \colon G \to G$. Then $\mathcal{O}(G_1)$ is a finite-dimensional commutative Hopf algebra over k, which is k-dual to $\mathcal{U}\mathfrak{g}_0$ as a Hopf algebra. This induces a symmetric monoidal equivalence of categories [Jan03, I.9.6]

$$\mathrm{Rep}(G_1) \cong \mathrm{Rep}(\mathcal{U}\mathfrak{g}_0), \tag{7.3.1}$$

where we remind that "Rep" means finitely generated representations in all contexts. We will freely use this equivalence categories to transport statements between $\mathrm{Rep}(G_1)$ and $\mathrm{Rep}(\mathcal{U}\mathfrak{g}_0)$.

Recall that the simple representations of G are in bijection with $X^*(T)^+$: for each $\lambda \in X^*(T)^+$ there is a unique simple representation of G with highest weight λ, which we denote $L(\lambda)$.

Recall from §1.3.1 that a dominant weight λ is *p-restricted* if $0 \leq \langle \lambda, \alpha^\vee \rangle < p$ for all simple roots α; the set of p-restricted weights is denoted $X_1^*(T)$. The simple representations of G_1 are exactly the restrictions of simple representations $L(\lambda)$ of G with highest weight $\lambda \in X_1^*(T)$.

7.4. **Graded representations.** Now we invoke the chosen torus $T \subset G$ to define T-graded representations.

7.4.1. *Graded Lie algebra representations.* If the action of T on $\mathfrak{g}$ fixes a central character χ of $\mathfrak{Z}_{\mathrm{Fr}}$, then we define $\mathrm{Rep}(\mathcal{U}\mathfrak{g}_\chi, T)$ be the Harish-Chandra category of representations V of $\mathcal{U}\mathfrak{g}_\chi$ together with a lift of $V|_{\mathfrak{t}}$ to a representation of T. We define in an analogous way $\mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g}, T)$, $\mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g}^\lambda, T)$, etc.

Example 7.4.1 ($\chi = 0$)**.** Take $\chi = 0$. then $\mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T)$ is the category of graded $\mathcal{U}\mathfrak{g}_0$-representations in the sense of [Jan04, §D.5]. Concretely, $\mathcal{U}\mathfrak{g}_0$ has a natural $X^*(T)$-grading where $X_\alpha \in \mathfrak{g}$ has weight α, for which $\mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T)$ is the category of $X^*(T)$-graded representations of $\mathcal{U}\mathfrak{g}_0$.

Example 7.4.2 (Graded simple representations)**.** Take $\chi = 0$. The simple representations of $\mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T)$ (which are the same as the simple representations in $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}, T)$) are in bijection with $X^*(T)$, indexed by their highest weights, and we denote by $\widehat{L}(\lambda)$ the simple representation of $\mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T)$ with highest weight $\lambda \in X^*(T)$.

Example 7.4.3 (Graded baby Verma representations)**.** Take $\chi = 0$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of B. It induces an isomorphism[10] $\mathfrak{t} \xrightarrow{\sim} \mathfrak{h}$ so for $\lambda \in X^*(T)$ we may regard $d\lambda \in \mathfrak{t}^*$ as an element of $\mathfrak{h}^*$. Then $(d\lambda, \chi)$ form a character of $Z(\mathcal{U}\mathfrak{g})$. We may also regard $d\lambda$ as a character of $\mathfrak{b}$ by inflation. The *graded baby Verma module* $\widehat{Z}_{\mathfrak{b}}(\lambda) \in \mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T)$ is defined as

$$\widehat{Z}_{\mathfrak{b}}(\lambda) := \mathcal{U}\mathfrak{g}_0 \otimes_{\mathcal{U}\mathfrak{b}} d\lambda$$

where $d\lambda$ has graded weight λ and the universal enveloping algebras are equipped with their natural gradings.

7.4.2. *Graded Frobenius kernel representations.* Since the Frobenius kernel $G_1 \triangleleft G$ is normal, it generates along with T a subgroup scheme $G_1 T < G$ isomorphic to the pushout of G_1 and T along T_1. Its representation theory is studied (for example) in [Jan03, II.9]. The equivalence (7.3.1) and its version for T combine to give a monoidal equivalence of categories

$$\mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T) \cong \mathrm{Rep}(G_1 T). \tag{7.4.1}$$

We denote by $\widehat{L}_1(\lambda) \in \mathrm{Rep}(G_1 T)$ the simple representation of highest weight λ. To define baby Vermas of $G_1 T$, we must make a choice of Borel subgroup. For compatibility with the literature on $G_1 T$ representations that we will cite later, **we normalize the definition in the following way**, which is "opposite" to Example 7.4.3: we denote by $\widehat{Z}_1(\lambda) \in \mathrm{Rep}(G_1 T)$ the graded baby Verma module of highest weight λ for the Borel $B^+ = w_0 B \subset G$, i.e. $\widehat{Z}_1(\lambda)$ corresponds to

$$\widehat{Z}_{w_0\mathfrak{b}}(\lambda) := \mathcal{U}\mathfrak{g}_0 \otimes_{\mathcal{U}\mathfrak{b}^+} d\lambda \in \mathrm{Rep}(\mathcal{U}\mathfrak{g}_0, T)$$

under (7.4.1). This definition is made for compatibility with [Jan03, GHS18]: our notation agrees with [Jan03, II.9.1 equation (2)].

Recall that the *linkage class* of $\lambda \in X^*(T)$ is $W_{\mathrm{aff}} \bullet_p \lambda$, We denote by $\mathrm{Rep}^{\lambda}(G_1 T)$ the Serre subcategory generate by simples $\widehat{L}_1(\lambda')$ for $\lambda' \in W_{\mathrm{aff}} \bullet_p \lambda$. The *linkage principle* says that if $\lambda' \notin W_{\mathrm{aff}} \bullet_p \lambda$, then $\mathrm{Rep}^{\lambda}(G_1 T)$ and $\mathrm{Rep}^{\lambda'}(G_1 T)$ lie in different blocks of $\mathrm{Rep}(G_1 T)$.

Example 7.4.4 (The extended principal block)**.** The equivalence (7.3.1) intertwines

$$\mathrm{Rep}^{\lambda}(\mathcal{U}\mathfrak{g}_0, T) \cong \bigoplus_{\lambda' \in \widetilde{W} \bullet_p \lambda / W_{\mathrm{aff}} \bullet_p \lambda} \mathrm{Rep}^{\lambda'}(G_1 T),$$

which is the *extended block of* λ in $\mathrm{Rep}(G_1 T)$. When λ is regular for the action of $(\widetilde{W}, \bullet_p)$, the sum is naturally indexed over a torsor for $\Omega^\vee := X^*(T)/Q$. This applies in particular when $\lambda = 0$, which is the case of most interest to us; then the RHS is called the *extended principal block* of $\mathrm{Rep}(G_1 T)$ and abbreviated $\mathrm{Rep}^{\emptyset}(G_1 T)$.

7.5. **BMR Localization.** Let $\mathcal{B}$ be the flag variety of G. For nilpotent $\chi \in \mathfrak{g}^*$, let $\mathcal{B}_\chi$ be the inverse image of χ under the Grothendieck alteration[11] $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$ (i.e., the *Springer fiber* associated to χ). Below we a state in a special case a localization theorem for Lie algebras in positive characteristic due to Bezrukavnikov-Mirkovic. Recall that the Springer resolution $\widetilde{\mathcal{N}} := T^*\mathcal{B}$ is the cotangent bundle of the flag variety of G.

Theorem 7.5.1 ([BM13, Theorem 1.6.7])**.** *Let $\lambda \in X^*(T) \cap pA_0$ and $\chi \in \mathfrak{g}^*$ be nilpotent and fixed by T. Then there is an equivalence*

$$\gamma_\chi^\lambda \colon D^b(\mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g}^\lambda, T)) \xrightarrow{\sim} D^b(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathcal{N}}^{(1)})),$$

where $\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ denotes the category of $T^{(1)}$-equivariant coherent sheaves on $\widetilde{\mathcal{N}}^{(1)}$ with set-theoretic support on $\mathcal{B}_\chi^{(1)}$ (with $T^{(1)}$-action on $\widetilde{\mathcal{N}}^{(1)} = (T^*\mathcal{B})^{(1)}$ induced by the T-translation action on $\mathcal{B}$).

Moreover, the equivalence γ_χ^λ is t-exact for the usual t-structure on $D^b(\mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g}^\lambda, T))$ and the exotic t-structure on $D^b(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathcal{N}}^{(1)}))$.

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[10] This is (tautologically) compatible with our identifications $X^*(T) \cong X^*(A) \cong X_*(\check{T})$.

[11] In Part I we viewed the Grothendieck alteration as a map $\widetilde{\mathfrak{g}} \to \mathfrak{g}$. It is related to this one via a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, which is exists under our characteristic hypotheses (cf. [BMR08, §3.1.2]).

Theorem 7.5.1 is an equivariant enhancement of the localization theorem from [BMR08, Theorem 5.3.1], which gives an equivalence

$$D^b(\mathrm{Rep}_\chi(\mathcal{U}\mathfrak{g}^\lambda)) \xrightarrow{\sim} D^b(\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathcal{N}}^{(1)})). \tag{7.5.1}$$

Example 7.5.2. We will only apply Theorem 7.5.1 for $\chi = 0$ and $\lambda = 0$. In this case $\mathcal{B}_\chi = \mathcal{B}$ is the full flag variety, and Theorem 7.5.1 supplies an equivalence

$$\gamma_0^0 \colon D^b(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \xrightarrow{\sim} D^b(\mathrm{Coh}_{\mathcal{B}^{(1)}}^{T^{(1)}}(\widetilde{\mathcal{N}}^{(1)})). \tag{7.5.2}$$

In this case, the non-equivariant version (7.5.1) goes as follows. Since $\lambda = 0$, the characteristic p analogue of Beilinson-Bernstein localization identifies $D^b(\mathrm{Rep}(\mathcal{U}\mathfrak{g}^0))$ with the derived category of coherent $\mathcal{D}$-modules on $\mathcal{B}$. The condition that $\chi = 0$ translates into the condition that the p-curvature (of the $\mathcal{D}$-modules obtained via Beilinson-Bernstein localization) is nilpotent. The ring of differential operators on $\mathcal{B}$ pushes forward to an Azumaya algebra $\mathrm{Fr}_* \mathcal{D}_{\mathcal{B}}$ on $\mathcal{B}^{(1)}$, and $\mathcal{D}$-modules on $\mathcal{B}$ with nilpotent p-curvature push forward exactly to $\mathrm{Fr}_* \mathcal{D}_{\mathcal{B}}$-modules on $(T^*\mathcal{B})^{(1)} \cong \mathcal{N}^{(1)}$ with set-theoretic support on $\mathcal{B}^{(1)}$. Finally, the Azumaya algebra $\mathrm{Fr}_* \mathcal{D}_{\mathcal{B}}$ splits canonically on the formal neighborhood of $\mathcal{B}^{(1)}$ by Cartier descent, giving a Morita equivalence between coherent $\mathrm{Fr}_* \mathcal{D}_{\mathcal{B}}$-modules supported on $\mathcal{B}^{(1)}$ and $\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$.

The graded version is bootstrapped from the non-graded one by relating $D^b(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ to T-equivariant D-modules on $\mathcal{B}$, which follows formally from the non-equivariant version by T-equivariantization, and then tracking the equivariant structure through the Morita equivalence.

Example 7.5.3 (The trivial representation)**.** Take $\chi = 0$ and $\lambda = 0$. The equivalence (7.5.2) sends the trivial representation $\widehat{L}(0) \in \mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ to $i_*\mathcal{O}_{\mathcal{B}^{(1)}}$ where $\mathcal{O}_{\mathcal{B}^{(1)}}$ is equipped with its native $T^{(1)}$-equivariant structure induced by the translation action of $T^{(1)}$ on $\mathcal{B}^{(1)}$.

Example 7.5.4 (Graded baby Verma modules)**.** The choice of Borel $B \subset G$ induces an isomorphism of the flag variety $\mathcal{B}$ with G/B. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of B. Then from [BMR08, §3.1.4] we see that (7.5.2) sends $\widehat{Z}_{\mathfrak{b}}(2\rho)$ to $\delta_{\mathfrak{b}}$, the T-equivariant skyscraper sheaf on $\mathcal{B}^{(1)}$ supported at $\mathfrak{b}$ regarded as a point of $\mathcal{B}^{(1)}$, with its native T-equivariant structure induced by the translation action of $T^{(1)}$ on $\mathcal{B}^{(1)}$.

7.6. **Monodromy-centralizer action.** We will define an action of $\widetilde{W}$ on $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ which is parallel to the monodromy-centralizer action of §6.6 in a direct sense.

7.6.1. *Graded Lie algebra and Frobenius kernel.* Let $N(T) < G$ be the normalizer of T. Then the adjoint action of $N(T)$ on $\mathfrak{g}$ induces an action of $N(T)$ on $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$. Under the equivalence (7.3.1), this corresponds to the action of $N(T)$ on $\mathrm{Rep}^\emptyset(G_1T)$ induced by conjugation action of $N(T)$ on $G_1T < G$.

Now note that at the level of Grothendieck groups, the subgroup $T \triangleleft N(T)$ acts trivially on $\mathrm{K}(\mathrm{Rep}^\emptyset(G_1T))$ since its action is inner. Hence we obtain an action of $N(T)/T \cong W$ on $\mathrm{K}(\mathrm{Rep}^\emptyset(G_1T)) \cong \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$.

In addition, via the obvious quotient map $G_1T \twoheadrightarrow T^{(1)}$ we have an action of $\mathrm{Rep}(T^{(1)})$ on $\mathrm{Rep}^\emptyset(G_1T)$ by inflation and tensoring. On $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ this corresponds to the action of changing the grading: there is an identification $T^{(1)} \cong T$ since T is defined over $\mathbf{F}_p$, inducing $X^*(T^{(1)}) \cong X^*(T)$. With respect to this identification, the relative Frobenius $\mathrm{Fr}_T \colon T \to T^{(1)}$ induces the second map in the sequence

$$X^*(T) \cong X^*(T^{(1)}) \xrightarrow{\mathrm{Fr}_T^*} X^*(T)$$

whose composite is multiplication by p. Hence $\lambda \in X^*(T) = X^*(T^{(1)})$ acts on $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ by translating the grading by $p\lambda$.

At the level of Grothendieck groups, this gives an action of $X^*(T)$ on $\mathrm{K}(\mathrm{Rep}^\emptyset(G_1T)) \cong \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$. Together with the earlier W-action, these define an action of $\widetilde{W} \cong X^*(T) \rtimes W$ on $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$, which we call the *monodromy-centralizer action.* (This terminology is still unexplained, but is made to be parallel to §6.6.1.) We denote the action of $\widetilde{w} \in \widetilde{W}$ by $\widetilde{w} \cdot_p (-)$.

Example 7.6.1 (Graded simple modules)**.** Recall that $\widehat{L}_1(\lambda)$ is the simple representation in $\mathrm{Rep}^\emptyset(G_1T)$ with highest weight λ. For $\mu \in X^*(T)$, we have

$$t^\mu \cdot_p [\widehat{L}_1(\lambda)] = [\widehat{L}_1(\lambda + p\mu)] \in \mathrm{K}(\mathrm{Rep}^\emptyset(G_1T)). \tag{7.6.1}$$

For $w \in W$ and $\lambda \in X_1^*(T)$ (cf. §7.3 for the notation), we have

$$w \cdot_p [\widehat{L}_1(\lambda)] = [\widehat{L}_1(\lambda)] \in \mathrm{K}(\mathrm{Rep}^\emptyset(G_1T))$$

because for such λ, the simple representation $\widehat{L}_1(\lambda)$ extends to a simple representation of G, where the conjugation action of $N(T)$ is inner, hence trivial on $\mathrm{K}(\mathrm{Rep}(G))$. Together with (7.6.1), this determines the $(\widetilde{W}, \cdot)$-action on all simples, and shows that it permutes the classes of simples.

Example 7.6.2 (Graded baby Vermas)**.** Recall that $\widehat{Z}_{\mathfrak{b}}(\lambda)$ is the baby Verma representation in $\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)$ with highest weight λ and Borel $\mathfrak{b}$. Note that the central character conditions force $\lambda \in pX^*(T)$. For any $\mu \in X^*(T)$, we have

$$\mu \cdot_p [\widehat{Z}_{\mathfrak{b}}(\lambda)] = [\widehat{Z}_{\mathfrak{b}}(\lambda + p\mu)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)). \tag{7.6.2}$$

For any $w \in W$, we have according to [Jan03, §9.3]

$$w \cdot_p [\widehat{Z}_{\mathfrak{b}}(\lambda)] = [\widehat{Z}_{w\mathfrak{b}}(w\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \tag{7.6.3}$$

where $w\mathfrak{b}$ is the translate of $\mathfrak{b}$ by any $\dot{w} \in N(T)$ lifting w. Together with (7.6.2), this determines the $(\widetilde{W}, \cdot)$-action on all baby Vermas, and shows that it permutes the classes of baby Vermas.

7.6.2. *Coherent sheaves.* There is an obvious action of $\mathrm{Rep}(T^{(1)})$ on $\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ by tensoring with equivariant representations. At the level of Grothendieck groups, this induces an action of $X^*(T^{(1)}) \cong X^*(T)$ on $\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})) \cong \mathrm{K}(\mathrm{Coh}^{T^{(1)}}(\mathcal{B}^{(1)}))$.

Also, writing $\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) = \mathrm{Coh}_{\mathcal{B}^{(1)}}(T^{(1)}\backslash\widetilde{\mathcal{N}}^{(1)})$, we see that there is an action of $N(T^{(1)})$ by left multiplication. At the level of Grothendieck groups, it factors over $T^{(1)}$, inducing an action of $W \cong N(T^{(1)})/T^{(1)}$ on $\mathrm{K}(\mathrm{Coh}(T^{(1)}\backslash\mathcal{B}^{(1)}))$ by left translation.

Together, these combine into an action of $\widetilde{W} \cong X^*(T^{(1)}) \rtimes W$ on $\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}))$, which we denote $(\widetilde{W}, \cdot)$.

Lemma 7.6.3. *At the level of Grothendieck groups, the equivalence* (7.5.2) *intertwines the action of* $(\widetilde{W}, \cdot_p)$ *on* $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ *with the action of* $(\widetilde{W}, \cdot)$ *on* $\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}))$.

Proof. The action of $X^*(T^{(1)}) \cong X^*(T) \subset \widetilde{W}$ on both sides can be described as tensoring with representations of $T^{(1)}$, and it is clear from the construction that the equivalence (7.5.2) intertwines these operations.

It follows from inspecting the construction that (7.5.2) also intertwines the actions of W. Let us sketch why: the point is that the map $\mathcal{U}\mathfrak{g}^0 \to \Gamma(\mathcal{B}, \mathcal{D})$ is G-equivariant for the adjoint action of G on $\mathcal{U}\mathfrak{g}^0$ and the left translation action of G on $\mathcal{B}$. This in turn follows from the fact that the action map

$$G \times \mathcal{B} \to \mathcal{B}$$

is equivariant for the conjugation action of G on itself and the left translation action on $\mathcal{B}$. Hence a fortiori the map $\mathcal{U}\mathfrak{g}^0 \to \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}})$ is H-equivariant for any subgroup $H < G$. Apply this to $H = N(T)$ and the result follows, using that the adjoint action of $N(T)$ induced the W-action on $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$, while the translation action of $N(T)$ induced the W-action on $\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}))$.

□

7.7. **Braid action.** Recall that the *extended affine Braid group* $\widetilde{\mathbb{B}}$ has generators $T_{\widetilde{w}}$ for $\widetilde{w} \in \widetilde{W}$, and relations $T_{\widetilde{w}}T_{\widetilde{w}'} = T_{\widetilde{w}\widetilde{w}'}$ if $\mathrm{len}(\widetilde{w}\widetilde{w}') = \mathrm{len}(\widetilde{w}) + \mathrm{len}(\widetilde{w}')$. We have $\widetilde{\mathbb{B}} \cong \mathbb{B}_{\mathrm{aff}} \rtimes \Omega$, where $\mathbb{B}_{\mathrm{aff}}$ is the affine Braid group associated to the Coxeter group W_{aff}.

We will define an action of $\widetilde{W}$ on $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ which is parallel to the affine Springer action of §6.6 in a direct sense.

7.7.1. *Graded Lie algebra.* Let (χ, λ) be as in Theorem 7.5.1. In [BMR06, §2], Bezrukavnikov-Mirkovic-Rumynin constructed an action of $\widetilde{\mathbb{B}}$ on $D^b(\mathrm{Rep}^{\lambda}(\mathcal{U}\mathfrak{g}, T))$[12]. We will describe the action of $T_{\widetilde{w}}$ for $\widetilde{w} \in \widetilde{W}$. First, recall that for any $\mu, \nu \in X^*(T)$ there is a *translation functor*

$$T^{\nu}_{\mu} \colon \mathrm{Rep}^{\mu}(\mathcal{U}\mathfrak{g}, T) \to \mathrm{Rep}^{\nu}(\mathcal{U}\mathfrak{g}, T).$$

For $\omega \in \Omega$, the action of $T_\omega \in \widetilde{\mathbb{B}}$ on $D^b(\mathrm{Rep}^{\mu}(\mathcal{U}\mathfrak{g}, T))$ is via $T^{\omega\mu}_{\mu}$; in particular, it is exact.

[12]Strictly speaking, they did not consider the graded version, but the construction also goes through in that case.

If μ lies in the interior of the alcove and ν lies on a codimension-1 face, then we define the *reflection functor*

$$R_{\mu|\nu} := T^{\mu}_{\nu} \circ T^{\nu}_{\mu} \colon \operatorname{Rep}^{\mu}(\mathcal{U}\mathfrak{g}, T) \to \operatorname{Rep}^{\mu}(\mathcal{U}\mathfrak{g}, T).$$

The functors $R_{\mu|\nu}$ are naturally isomorphic for different choices of ν in the interior of the codimension-1 face, so we fix a choice and denote the reflection functor (also known as *wall-crossing functor*) by R_s where $s \in \widetilde{W}$ is the reflection through the codimension-1 face. There is a distinguished triangle of functors on $D^b(\operatorname{Rep}^{\mu}(\mathcal{U}\mathfrak{g}, T))$,

$$\mathrm{Id} \to R_s \to \mathbb{I}_s^*.$$

Then the action of $T_s \in \widetilde{\mathbb{B}}$ on $D^b(\operatorname{Rep}^{\mu}(\mathcal{U}\mathfrak{g}, T))$ is via $\mathbb{I}_s^*$.

These functors are compatible with the Frobenius-center, hence the same formulas induce a $\widetilde{\mathbb{B}}$-action on $D^b(\operatorname{Rep}_{\chi}(\mathcal{U}\mathfrak{g}, T))$.

7.7.2. *Geometric braid action on coherent sheaves.* There is also a $\widetilde{\mathbb{B}}$-action on $D^b(\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}))$ constructed in [Ric08, Theorem 1.4.1] which preserves each $D^b(\mathrm{Coh}_{\mathcal{B}^{(1)}_{\chi}}(\widetilde{\mathfrak{g}}^{(1)}))$. Then Riche shows in [Ric08, Theorem 5.4.1] that the equivalence of Theorem 7.5.1 intertwines the two $\widetilde{\mathbb{B}}$-actions. For the graded case, it is also true that the equivalence Theorem 7.5.1 intertwines the two $\widetilde{\mathbb{B}}$-actions, by the same argument as for Riche's result in the non-graded case.

7.7.3. *Converting to right actions.* Henceforth we convert the $\widetilde{\mathbb{B}}$-action to a *right* action by the anti-involution $\widetilde{\mathbb{B}} \xrightarrow{\sim} \widetilde{\mathbb{B}}^{\mathrm{opp}}$ given by the inverse map. This is for compatibility with how we normalized the affine Springer action on $\mathrm{H}^{\mathrm{BM}}_*(\mathrm{Y}_{\gamma})$ to be a right action.

7.7.4. *Steinberg action.* Let $\mathrm{St}_G := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$. This is defined in characteristic p by our convention, but there is an analogous construction over $\mathbf{C}$ that we denote $\mathrm{St}_{G,\mathbf{C}}$. Recall that the Kazhdan-Lusztig isomorphism (conjectured by Deligne-Langlands) identifies $\mathrm{K}(\mathrm{Coh}^{G \times \mathbf{G}_m}(\mathrm{St}_{G,\mathbf{C}}))$, as an algebra under convolution, with the affine Hecke algebra of $\check{G}$. This induces an isomorphism $\mathrm{K}(\mathrm{Coh}^{G}(\mathrm{St}_{G,\mathbf{C}})) \cong \mathbf{Z}[\widetilde{W}]$. A small argument, which we presently give, shows that the same holds in characteristic p.

Recall that there is a specialization map in K-theory for a Noetherian scheme X flat over a DVR (cf. [BMR08, §7.1.3]). When the DVR is $\mathbf{Z}_p$, we denote it as

$$\mathfrak{sp}_{p \to 0} \colon \mathrm{K}(\mathrm{Coh}(X_{\mathbf{Q}_p})) \to \mathrm{K}(\mathrm{Coh}(X_{\mathbf{F}_p})). \tag{7.7.1}$$

It is defined on a coherent sheaf $\mathcal{F}/X_{\mathbf{Q}_p}$ by choosing a $\mathbf{Z}_p$-lattice and then taking the (derived) tensor product with $\mathbf{F}_p$. If a split reductive group scheme $H/\mathbf{Z}_p$ acts on $X/\mathbf{Z}_p$, then there is also an equivariant version.

Let $\mathrm{St}_{G,\mathbf{Z}_p} = \widetilde{\mathcal{N}}_{\mathbf{Z}_p} \times_{\mathcal{N}_{\mathbf{Z}_p}} \widetilde{\mathcal{N}}_{\mathbf{Z}_p}$ and define $\mathrm{St}_{G,\mathbf{Q}_p}$, etc. analogously.

Lemma 7.7.1. *The map*

$$\mathfrak{sp}_{p \to 0} \colon \mathrm{K}(\mathrm{Coh}^{G \times \mathbf{G}_m}(\mathrm{St}_{G,\mathbf{Q}_p})) \to \mathrm{K}(\mathrm{Coh}^{G \times \mathbf{G}_m}(\mathrm{St}_{G,\mathbf{F}_p})) \tag{7.7.2}$$

induced by $\mathrm{St}_{G,\mathbf{Z}_p}$ *is an algebra isomorphism (both sides being equipped with the convolution product).*

Proof. Over $\mathbf{Z}_p$, the identification

$$\widetilde{\mathcal{N}}_{\mathbf{Z}_p} \times_{\mathrm{Spec}\, \mathbf{Z}_p} \widetilde{\mathcal{N}}_{\mathbf{Z}_p} \cong T^*(\mathcal{B}_{\mathbf{Z}_p}) \times_{\mathrm{Spec}\, \mathbf{Z}_p} T^*(\mathcal{B}_{\mathbf{Z}_p})$$

realizes the Steinberg variety St_G as the conormal space (relative to $\mathbf{Z}_p$) to the Bruhat stratification of $\mathcal{B}_{\mathbf{Z}_p} \times \mathcal{B}_{\mathbf{Z}_p}$ by diagonal $G_{\mathbf{Z}_p}$-orbits. In particular, $\mathrm{St}_{G,\mathbf{Z}_p}$ admits a stratification into affine spaces over $\mathbf{Z}_p$. This equips both sides of (7.7.2) with filtrations such that each graded is the specialization map for an affine space, which is an isomorphism (cf. the Cellular Fibration Lemma [CG10, §5.5]). Therefore, $\mathfrak{sp}_{p \to 0}$ is an isomorphism of groups.

The algebra structure on either side is given by convolution, which is defined because over any field $\widetilde{\mathcal{N}}$ is smooth and the projection $\widetilde{\mathcal{N}} \to \mathcal{N}$ is proper. Since $\widetilde{\mathcal{N}}_{\mathbf{Z}_p}$ is smooth over $\mathbf{Z}_p$ and the projection $\widetilde{\mathcal{N}}_{\mathbf{Z}_p} \to \mathcal{N}_{\mathbf{Z}_p}$ is proper, the operations constituting convolution are compatible with $\mathfrak{sp}_{p \to 0}$. Therefore the map (7.7.2) is compatible with the convolution. □

Note that the flat base change from $\mathbf{Q}_p$ to $\mathbf{C}$ induces an isomorphism

$$\mathrm{K}(\mathrm{Coh}^{G\times\mathbf{G}_m}(\mathrm{St}_{G,\mathbf{Q}_p})) \xrightarrow{\sim} \mathrm{K}(\mathrm{Coh}^{G\times\mathbf{G}_m}(\mathrm{St}_{G,\mathbf{C}})),$$

again because of the stratification into affine spaces. We deduce that $\mathrm{K}(\mathrm{Coh}^{G\times\mathbf{G}_m}(\mathrm{St}_{G,\mathbf{F}_p}))$ is also isomorphic to the affine Hecke algebra for $\check{G}$.

We resume working over a field k of characteristic p, and omit the subscripts indicating the base field. For any nilpotent $\chi \in \mathfrak{g}^*$, there is a natural (right) *Steinberg action* of $\mathrm{K}(\mathrm{Coh}^{G\times\mathbf{G}_m}(\mathrm{St}_G))$ on $\mathrm{K}(\mathrm{Coh}^T(\mathcal{B}_\chi))$, the K-theory of the corresponding Springer fiber, by convolution on the right. On the other hand, the T-equivariant version of the construction of [Ric08, Theorem 1.4.1] induces a right action of $\widetilde{\mathbb{B}}$ on $D^b(\mathrm{Coh}^T_{\mathcal{B}_\chi}(\widetilde{\mathcal{N}}))$. At the level of Grothendieck groups, according to [BM13, Theorem 1.3.2], if $\chi \in \mathfrak{g}^*$ is nilpotent then the action of $\widetilde{\mathbb{B}}$ on $\mathrm{K}(\mathrm{Coh}^T_{\mathcal{B}_\chi}(\widetilde{\mathcal{N}})) \cong \mathrm{K}(\mathrm{Coh}^T(\mathcal{B}_\chi))$ factors through the Steinberg action of $\widetilde{W}$.

Repeating this discussion with appropriate Frobenius twists, we obtain a Steinberg action of $\widetilde{W}$ on $\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}))$, which we denote $(\widetilde{W}, \bullet)$. It follows from (the graded version of) Riche's Theorem that:

Lemma 7.7.2. *The action of* $\widetilde{\mathbb{B}}$ *on* $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ *induced by §7.7.1 factors through an action of* $\widetilde{W}$ *that we denote* $(\widetilde{W}, \bullet_p)$.

Furthermore, at the level of Grothendieck groups the equivalence (7.5.2) *intertwines the action of* $(\widetilde{W}, \bullet_p)$ *on* $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ *with the action of* $(\widetilde{W}, \bullet)$ *on* $\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}))$.

7.7.5. *Comparison to characteristic zero.* We want to compare the K-group and actions in Lemma 7.7.2 to the analogous situation in characteristic zero. The flat family $\mathcal{B}_{\mathbf{Z}_p}/\mathbf{Z}_p$ induces a specialization map in equivariant K-theory,

$$\mathfrak{sp}_{p\to 0} \colon \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{Q}_p}}(\mathcal{B}_{\mathbf{Q}_p})) \to \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{F}_p}}(\mathcal{B}_{\mathbf{F}_p})). \tag{7.7.3}$$

Lemma 7.7.3. *The map* (7.7.3) *is an isomorphism. It has the following properties:*

1. *It is equivariant for the action of* $(\widetilde{W}, \cdot)$ *and also for the Steinberg action of* $(\widetilde{W}, \bullet)$.
2. *It takes* $[\mathcal{O}_{\mathcal{B}_{\mathbf{Q}_p}}]$ *to* $[\mathcal{O}_{\mathcal{B}_{\mathbf{F}_p}}]$.
3. *For each Borel subgroup* $B < G$ *defined over* $\mathbf{Z}_p$*, it takes the skyscraper class* $[\delta_{B_{\mathbf{Q}_p}}] \in \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{Q}_p}}(\mathcal{B}_{\mathbf{Q}_p}))$ *to the skyscraper class* $[\delta_{B_{\mathbf{F}_p}}] \in \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{F}_p}}(\mathcal{B}_{\mathbf{F}_p}))$.

Proof. The Bruhat stratification decomposes $\mathcal{B}_{\mathbf{Z}_p}$ into affine spaces over $\mathbf{Z}_p$. Therefore (7.7.1) is an isomorphism by the same cellular fibration argument as in the proof of Lemma 7.7.1.

The compatibility with the action $(\widetilde{W}, \cdot)$ is evident from the definitions. The compatibility with the action $(\widetilde{W}, \bullet)$ follows from that the fact that the action maps come from tensor products on smooth ambient spaces or pushforward along proper morphisms defined over $\mathbf{Z}_p$, like in the proof of Lemma 7.7.1, which are therefore compatible with specialization.

The computation of $\mathfrak{sp}$ on the structure sheaf and skyscrapers is evident from the definition. □

We note again that the flat base change from $\mathcal{B}_{\mathbf{Q}_p}$ to $\mathcal{B}_{\mathbf{C}}$ induces an isomorphism of K-groups, by the Bruhat stratification into affine spaces.

Remark 7.7.4. The specialization map $\mathfrak{sp}_{p\to 0}$ and base change from $\mathbf{Q}_p$ to $\mathbf{C}$ are analyzed for the K-theory of more general Springer fibers (without the T-equivariance) in [BMR08, Proposition 7.1.7].

7.8. **Upshot.** Summarizing, we have:

Theorem 7.8.1. *There is an isomorphism*

$$\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \to \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}}))$$

which has the following properties:

1. *It intertwines the left action* $(\widetilde{W}, \cdot_p)$ *on the LHS with the left action* $(\widetilde{W}, \cdot)$ *on the RHS.*
2. *It intertwines the right action* $(\widetilde{W}, \bullet_p)$ *on the LHS with the right action* $(\widetilde{W}, \bullet)$ *on the RHS.*
3. *It sends* $[\widehat{L}(0)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ *to* $[\mathcal{O}_{\mathcal{B}_{\mathbf{C}}}] \in \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}}))$.
4. *It sends* $[\widehat{Z}_{\mathfrak{b}}(2\rho)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ *to* $[\delta_{B_{\mathbf{C}}}] \in \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}}))$.

Proof. Combine Lemma 7.7.3, Lemma 7.7.2, and Lemma 7.6.3. □

8. Shadows of mirror symmetry

Our goal is to compare the representation-theoretic information of G measured in $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ with the geometric information of $\check{G}$ measured in $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$. In the preceding section we explained a more geometrical incarnation of $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ in terms of coherent sheaves on the flag variety for G. In this section, we will connect this with $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$. This connection may be viewed as some manifestation of homological mirror symmetry, which relates Lagrangians on a symplectic manifold and coherent sheaves on a mirror variety. *This provides in particular the passage from G to its dual group $\check{G}$.*

Proposition 8.0.1 (Bezrukavnikov–Boixeda Alvarez–McBreen–Yun [BBAMY23])**.** *Let $s \in \mathfrak{t}$ be regular semisimple and $\gamma = ts \in \mathfrak{t}[[t]]$. There is a map*

$$\mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}_{\mathcal{B}_{\mathbf{C}}}(\widetilde{\mathcal{N}}_{\mathbf{C}})) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma), \tag{8.0.1}$$

with the following properties:

1. *It intertwines the actions $(\widetilde{W}, \cdot)$ defined in §7.7 for the LHS and §6.6 for the RHS.*
2. *It intertwines the actions $(\widetilde{W}, \bullet)$ defined in §7.7 for the LHS and in §6.6 for the RHS.*
3. *It sends $[\mathcal{O}_{\mathcal{B}_{\mathbf{C}}}] \in \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}_{\mathcal{B}_{\mathbf{C}}}(\widetilde{\mathcal{N}}_{\mathbf{C}}))$ to the fundamental class of the unique (top-dimensional) irreducible component of Y_γ which is the pre-image of $[t^0] \in \mathrm{Gr}_{G,\mathbf{F}_q}$ under the projection map $\mathrm{Fl}_{G,\mathbf{F}_q} \to \mathrm{Gr}_{G,\mathbf{F}_q}$.*

Remark 8.0.2. The left side of (8.0.1) does not depend on γ while the right side seems to depend on it. Recall however that in Example 6.5.2 that $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$ is "independent of γ" in a suitable sense. With this said, we may choose γ to come from a regular semisimple element of $\mathfrak{t}_{\mathbf{Z}_p}[[t]]$, and by Example 6.5.2 again it is equivalent to prove the analogous statement with $\mathrm{Y}_{\gamma,\mathbf{C}}$, the complex version of the affine Springer fiber, in place of Y_γ.

Proposition 8.0.1 is a consequence of constructions in [BBAMY23]. For the sake of being self-contained, we sketch the relevant constructions below, while emphasizing that they are entirely due to [BBAMY23]; *we do not claim any original results in this section.*

Remark 8.0.3. To explain the title of this section: the map (8.0.1) is the shadow of an instance of *homological mirror symmetry* which predicts in some form that given a symplectic manifold M with a Lagrangian skeleton L, there should be an equivalence between a "Fukaya category"[13] of microlocal sheaves $\mu\mathrm{Sh}_L(M)$ with supports on L, and the derived category of coherent sheaves on a mirror algebraic variety. We do not attempt to be precise, because the technicalities are complicated and prevent currently existing general conjectures from covering the case at hand. The paper [BBAMY23] establishes an equivalence of categories which should be interpreted as "homological mirror symmetry" in this instance. In this case the coherent category is the obvious one; on the other side Y_γ will be seen to be Lagrangian in a certain "de Rham moduli space" $\mathcal{M}_\psi$ studied in [BBAMY22], and the relevant "Fukaya category" is a certain subcategory of microlocal sheaves on $\mathcal{M}_\psi$ supported on Y_γ. In particular, this will imply that the maps in (8.0.1) are isomorphisms after tensoring with $\mathbf{Q}$.

We emphasize that *our applications do not require* the deep categorical equivalences forthcoming in [BBAMY23]; we require only the construction of a functor (explained below), which is relatively easy (given existing technology). Interestingly, although Proposition 8.0.1 is a completely decategorified statement, we do not know how to produce the map except by categorical considerations.

The next two subsections, §8.1 and §8.2, will not be logically used in the rest of this paper. They consist solely of summarizing certain excepts from [BBAMY23], in order to elucidate the nature of Proposition 8.0.1.

8.1. **Constructible realization.** We work over the complex numbers $\mathbf{C}$. We will define a "constructible realization" of $\mathrm{Coh}^T_{\mathcal{B}}(\widetilde{\mathcal{N}})$.

Abbreviate $\check{\mathbf{K}} = L^+G$ for the arc group of G and let $\check{\mathbf{I}} \subset \check{\mathbf{K}}$ be the Iwahori subgroup corresponding to $\check{B} \subset \check{G}$.

[13] This incarnation of the Fukaya category is in the spirit of [NS20] rather than Kontsevich's original formulation.

8.1.1. *Equivariant sheaves.* Let $L_1^-\check{G}$ be the first congruence subgroup of the negative loop group, whose value on a $\mathbf{C}$-algebra R is

$$L_1^-\check{G}(R) := \ker(\check{G}(R[t^{-1}]) \to \check{G}(R)).$$

Fix a regular semi-simple $s \in \check{\mathfrak{t}}$, and write $\gamma := ts \in \check{\mathfrak{g}}[[t]]$. Using the Killing form, we may also view s as an element of $\check{\mathfrak{g}}^*$. There is a filtration of $L_1^-\check{G}$ by congruence subgroups, and the quotient by the second congruence subgroup gives a surjection $L_1^-\check{G} \twoheadrightarrow \check{\mathfrak{g}}$. We write ψ for the additive character of $L_1^-\check{G}$ induced by inflating $s \in \check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$ along this map. We write Ψ for the exponential D-module on $\mathbf{A}^1$ pulled back to a character sheaf on $L_1^-\check{G}$.

For a space with an action of $L_1^-\check{G}$, there is a derived category of sheaves equivariant with respect to $(L_1^-\check{G}, \Psi)$. More precisely, we write $\mathcal{D}(\dots)$ for presentable stable ∞-categories and $D(\dots)$ for the corresponding homotopy category, and $\mathcal{D}_{(L_1^-\check{G},\Psi)}(\dots)$ or $D_{(L_1^-\check{G},\Psi)}(\dots)$ for the respective equivariant derived categories. We apply these considerations to $\mathrm{Gr}_{\check{G}} := L\check{G}/\check{\mathbf{K}}$ and $\mathrm{Fl}_{\check{G}} := L\check{G}/\check{\mathbf{I}}$. More generally, for any parahoric subgroup $\check{\mathbf{I}} \subset \check{\mathbf{P}} \subset \check{\mathbf{K}}$ we consider the equivariant derived categories

$$\mathcal{D}_{\psi,\check{\mathbf{P}}} := \mathcal{D}_{(L_1^-\check{G},\Psi)}(L\check{G}/\check{\mathbf{P}}).$$

We will construct a functor

$$\mathcal{D}\,\mathrm{Coh}^{\check{T}}_{\mathcal{B}}(\widetilde{\mathcal{N}}) \to \mathcal{D}_{\psi,\check{\mathbf{I}}} \tag{8.1.1}$$

following [BBAMY23, §6].

8.1.2. *Ingredients.* To begin, we tabulate some categorical equivalences.

(1) Bezrukavnikov-Finkelberg constructed in [BF08] a monoidal equivalence

$$\mathcal{D}_{\check{\mathbf{K}}}(\mathrm{Gr}_{\check{G}}) \cong \mathcal{D}\,\mathrm{Coh}^{G}(0 \overset{\mathrm{L}}{\times}_{\mathfrak{g}} 0) \tag{8.1.2}$$

where the monoidal structure is given by convolution on both sides. Here and throughout, $\overset{\mathrm{L}}{\times}$ means the derived fibered product.

(2) Bezrukavnikov constructed in [Bez16] a monoidal equivalence

$$\mathcal{D}_{\check{\mathbf{I}}}(\mathrm{Fl}_{\check{G}}) \cong \mathcal{D}\,\mathrm{Coh}^{G}(\widetilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \widetilde{\mathcal{N}}) \tag{8.1.3}$$

where the monoidal structure is given by convolution on both sides.

(3) Arkhipov-Bezrukavnikov-Ginzburg constructed in [ABG04] an equivalence

$$\mathcal{D}_{\check{\mathbf{K}}}(\mathrm{Fl}_{\check{G}}) \cong \mathcal{D}\,\mathrm{Coh}^{G}(0 \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \widetilde{\mathcal{N}}). \tag{8.1.4}$$

Moreover, the LHS carries a left convolution action $\mathcal{D}_{\check{\mathbf{K}}}(\mathrm{Gr}_{\check{G}})$ and a right convolution action by $\mathcal{D}_{\check{\mathbf{I}}}(\mathrm{Fl}_{\check{G}})$, while the RHS carries a left convolution action by $\mathcal{D}\,\mathrm{Coh}^{G}(0 \overset{\mathrm{L}}{\times}_{\mathfrak{g}} 0)$ and a right convolution action by $\mathcal{D}\,\mathrm{Coh}^{G}(\widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}})$. The work [Bez16] shows that the equivalence (8.1.4) respects these actions via (8.1.2) and (8.1.3).

(4) Bezrukavnikov–Boixeda Alvarez–McBreen–Yun construct in [BBAMY23, Theorem 4.1.3] an equivalence

$$\mathcal{D}_{\psi,\check{\mathbf{K}}} \xrightarrow{\sim} \mathcal{D}(\mathrm{Rep}(T)). \tag{8.1.5}$$

Moreover, the LHS carries a lattice translation action on the left by $X_*(\check{T})$ and a right convolution action by $\mathcal{D}_{\check{\mathbf{K}}}(\mathrm{Gr}_{\check{G}})$, while the RHS carries a tensoring action by $\mathrm{Rep}(T)$ on the left and a right convolution action by $\mathcal{D}\,\mathrm{Coh}^{G}(0 \overset{\mathrm{L}}{\times}_{\mathfrak{g}} 0)$. In [BBAMY23, §4.2] it is proved that the equivalence (8.1.5) respects the left actions under the identification between $\mathrm{Rep}(T)$ and $X^*(T) \cong X_*(\check{T})$-graded vector spaces, and in [BBAMY23, §4.3] it is proved that the equivalence (8.1.5) respects the right actions under (8.1.2).

8.1.3. *Convolution.* Convolution induces a functor

$$\mathcal{D}_{\psi,\check{\mathbf{K}}} \otimes_{\mathcal{D}_{\check{\mathbf{K}}}(\mathrm{Gr}_{\check{G}})} \mathcal{D}_{\check{\mathbf{K}}}(\mathrm{Fl}_{\check{G}}) \hookrightarrow \mathcal{D}_{\psi,\check{\mathbf{I}}}. \tag{8.1.6}$$

(In fact, this is fully faithful because $\mathrm{Gr}_{\check{G}}$ and $\mathrm{Fl}_{\check{G}}$ are ind-proper, but we will not need this.)

Convolution induces a fully faithful functor

$$\mathcal{D}(\mathrm{Rep}(T)) \otimes_{\mathcal{D}\,\mathrm{Coh}^G(0 \overset{\mathrm{L}}{\times}_{\mathfrak{g}} 0)} \mathcal{D}\,\mathrm{Coh}^G(0 \times_{\mathfrak{g}} \widetilde{\mathcal{N}}) \to \mathcal{D}\,\mathrm{Coh}^{\check{T}}(\widetilde{\mathcal{N}}),$$

whose essential image is precisely $\mathcal{D}\,\mathrm{Coh}^T_{\mathcal{B}}(\widetilde{\mathcal{N}})$. Combining this with the equivalences in (1), (2), (3) and using (8.1.6) gives the desired functor (8.1.1). Moreover, by construction the functor is equivariant for the left action of $\mathrm{Rep}(T)$, and right convolution action of (8.1.3).

Remark 8.1.1 (Relation to Geometric Langlands)**.** The category $\mathcal{D}_{\psi,\check{\mathbf{I}}}$ is denoted $\widetilde{\mathcal{D}}_{\psi}$ in [BBAMY23]. There is a pullback functor $\mathcal{D}_{\psi,\check{\mathbf{K}}} \to \mathcal{D}_{\psi,\check{\mathbf{I}}}$. Define $\mathcal{D}_{\psi} \subset \mathcal{D}_{\psi,\check{\mathbf{I}}}$ to be the full subcategory generated by $\mathcal{D}_{\psi,\check{\mathbf{K}}}$ under the right convolution action of $\mathcal{D}_{\check{\mathbf{I}}}(\mathrm{Fl}_{\check{G}})$. Then [BBAMY23, Theorem 6.2.1] shows that the functor (8.1.1) is an equivalence onto $\mathcal{D}_{\psi}$. The category $\mathcal{D}_{\psi}$ may be interpeted as a certain category of sheaves on the moduli stack of $\check{G}$-bundles on $\mathbf{P}^1$ with Iwahori level structure at 0 and certain wild ramification at ∞, while the category $\mathcal{D}\,\mathrm{Coh}^{\check{T}}_{\mathcal{B}}(\widetilde{\mathcal{N}})$ may be interpreted as a category of coherent sheaves on a corresponding moduli space of (Betti) local systems. As such, the equivalence (8.1.1) may be viewed as proving a certain wildly ramified instance of global *Geometric Langlands* – see [BBAMY22, §5].

8.2. **Microlocalization.** We continue to work over the complex numbers $\mathbf{C}$. Let $\gamma = ts$ for regular semisimple $s \in \check{\mathfrak{t}}$ be as in the previous subsection, and $\mathrm{Fl}_{\gamma} := Y_{\gamma,\mathbf{C}}$. The upshot of the constructible realization is a map

$$\mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}_{\mathcal{B}_{\mathbf{C}}}(\widetilde{\mathcal{N}}_{\mathbf{C}})) \to \mathrm{K}(D^b_{(L_1^-\check{G},\Psi)}(\mathrm{Fl}_{\check{G}})). \tag{8.2.1}$$

To obtain the map of Proposition 8.0.1, we compose (8.2.1) with the *singular support* map. In this situation the singular support will be a Lagrangian in the "twisted cotangent bundle" $T^*_{\psi}(L_1^-\check{G}\backslash \mathrm{Fl}_{\check{G}})$, and it turns out that the affine Springer fiber Fl_{γ} is essentially such a Lagrangian.

8.2.1. *Moduli stack of bundles.* We will now be more precise, following [BBAMY23, §3]. We fix the curve $\mathbf{P}^1$ over $\mathbf{C}$. Let $\check{\mathbf{I}}_0 = \check{\mathbf{I}}$ be the Iwahori group at 0 and $\check{\mathbf{K}}^2_{\infty}$ be the second congruence subgroup at ∞. We consider $\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty})$ with S-points $(\mathcal{E}, \tau_{\infty}, \tau_{0,\check{B}})$ where:

- $\mathcal{E}$ is a $\check{G}$-bundle over $\mathbf{P}^1_S$.
- τ_{∞} is a trivialization of $\mathcal{E}$ along the divisor $2\infty_S \hookrightarrow \mathbf{P}^1_S$.
- $\tau_{0,\check{B}}$ is a $\check{B}$-reduction of $\mathcal{E}$ along the divisor $0_S \hookrightarrow \mathbf{P}^1_S$.

In particular, the group $\check{\mathbf{K}}^1_{\infty}/\check{\mathbf{K}}^2_{\infty} \cong \check{\mathfrak{g}}$ acts on $\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty})$ through changing the level structure τ_{∞}, inducing a moment map

$$\mu\colon T^*\,\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty}) \to \check{\mathfrak{g}}^*.$$

We write $\psi \in \check{\mathfrak{g}}^*$ for the image of $s \in \check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$ given by the Killing form. We write Ψ for the additive character sheaf on $\check{\mathfrak{g}}$ pulled back from the exponential D-module via ψ.

8.2.2. *Hitchin stack.* Define the Hitchin stack $\mathcal{M}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty}) := T^*\,\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty})$, viewed with the canonical symplectic structure coming from its nature as a cotangent bundle. Then define $\mathcal{M}_{\psi}$ as the symplectic reduction

$$\mathcal{M}_{\psi} := \big(\mathcal{M}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty}) /\!\!/_{\psi}\, \check{\mathfrak{g}}\big) := [\mu^{-1}(\psi)/\check{\mathfrak{g}}].$$

In preparation for describing its functor of points, we trivialize $\omega_{\mathbf{P}^1}([\infty]+[0])$ with the differential form dt/t, thus identifying $\omega_{\mathbf{P}^1}(2[\infty]+[0]) \cong \omega_{\mathbf{P}^1}([\infty])$. Then $\mathcal{M}_{\psi}$ has S-points the groupoid of $(\mathcal{E}, \tau_{\infty}, \tau_{0,\check{B}}, \varphi)$ where $(\mathcal{E}, \tau_{\infty}, \tau_{0,B}) \in \mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_{\infty})(S)$, and $\varphi \in H^0(\mathbf{P}^1_S, \mathrm{Ad}^*\,\mathcal{E} \otimes \omega_{\mathbf{P}^1}([\infty]))$ such that

- Around ∞_S, $\varphi = \psi dt$ plus higher order terms under the trivialization τ_{∞}.
- $\mathrm{Res}_{0_S}(\varphi) \in \mathfrak{n}$ under the trivialization $\tau_{0,\check{B}}$.

8.2.3. *Hitchin fibration.* By its nature as a symplectic reduction, $\mathcal{M}_{\psi}$ carries a natural symplectic structure. There is a Hitchin fibration $f_{\psi}\colon \mathcal{M}_{\psi} \to \mathcal{A}_{\psi}$, which is a completely integrable system [BBAMY23, Lemma 3.2.3] and a $\mathbf{G}_m$-action on A_{ψ} contracting it to a central point $a_{\psi} \in \mathcal{A}_{\psi}$. The central *Hitchin fiber* $f_{\psi}^{-1}(a_{\psi})$ is Lagrangian in $\mathcal{M}_{\psi}$.

8.2.4. *Microlocalization.* By the uniformization of $\mathrm{Bun}_{\check{G}}$ for $\mathbf{P}^1$, there is a canonical equivalence

$$\mathcal{D}_{\psi,\check{\mathbf{I}}} \xrightarrow{\sim} \mathcal{D}_{(\mathfrak{g},\Psi)}(\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_\infty)).$$

Then the formation of singular support induces a map

$$\mathrm{K}(D^b_{(\mathfrak{g},\Psi)}(\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_\infty)) \to \mathrm{Ch}_{\mathrm{top}}(f_\psi^{-1}(a_\psi)). \tag{8.2.2}$$

Remark 8.2.1. The map (8.2.2) is categorified by the *microlocalization functor*

$$\mu\mathrm{Loc}\colon \mathcal{D}_{(\mathfrak{g},\Psi)}(\mathrm{Bun}_G(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_\infty)) \to \mu\mathrm{Sh}_{f_\psi^{-1}(a_\psi)}(\mathcal{M}_\psi),$$

where the right hand side is the category of *microlocal sheaves with support in* $f_\psi^{-1}(a_\psi)$. The singular support of $\mathcal{F} \in D^b_{(\mathfrak{g},\Psi)}(\mathrm{Bun}_{\check{G}}(\check{\mathbf{I}}_0, \check{\mathbf{K}}^2_\infty))$ is the naive support of $\mu\mathrm{Loc}(\mathcal{F})$.

8.2.5. *Relation to affine Springer fiber.* By [BBAMY23, Proposition 3.4.1], there is a canonical homeomorphism $\mathrm{Fl}_\gamma \to f_\psi^{-1}(a_\psi)$, which in particular induces an isomorphism

$$\mathrm{Ch}_{\mathrm{top}}(f_\psi^{-1}(a_\psi)) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Fl}_\gamma) \tag{8.2.3}$$

compatible with the two actions of $\widetilde{W}$. Finally, composing (8.2.2) with (8.2.3) and (8.2.1) gives the desired map of Proposition 8.0.1.

8.3. **The microlocal support map.** We may now define a map that casts representations of G_1T onto cycles in $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$.

Definition 8.3.1. We define the *microlocal support*[14] map

$$\mathrm{SS}\colon \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$$

to be the composition of the maps from Theorem 7.5.1 and Proposition 8.0.1.

Note that we have

$$\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \xleftarrow{\sim} \mathrm{K}(\mathrm{Rep}(\mathcal{U}\mathfrak{g}^0_0, T)) \xrightarrow{\sim} \mathrm{K}(\mathrm{Rep}^0(\mathcal{U}\mathfrak{g}_0, T)) \cong \mathrm{K}(\mathrm{Rep}^\emptyset(G_1T)),$$

where the first two isomorphisms come from the fact that the centers act by scalars on simple representations, and the last isomorphism comes from (7.3.1). Therefore we will also view SS as being defined on any of these other groups. In Part 3, we will mostly view it as a map

$$\mathrm{SS}\colon \mathrm{K}(\mathrm{Rep}^\emptyset(G_1T)) \to \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma).$$

Since $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma) \subset \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}_\gamma)$, we will also regard SS as having target in $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mathrm{Y}_\gamma)$ at times. By Theorem 7.5.1 and Proposition 8.0.1, we know that SS has the following properties.

(1) It intertwines the action of $(\widetilde{W}, \cdot_p)$ on the LHS (defined in §7.6) with the action of $(\widetilde{W}, \cdot)$ on the RHS (defined in §6.6).
(2) It intertwines the action of $(\widetilde{W}, \bullet_p)$ on the LHS (defined in §7.7) with the action of $(\widetilde{W}, \bullet)$ on the RHS (defined in §6.6).
(3) It sends $[\widehat{L}(0)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ to the fundamental class of the unique (top-dimensional) irreducible component of Y_γ which is the pre-image of $[t^0] \in \mathrm{Gr}_{\check{G},\mathbf{F}_q}$ under the projection map $\mathrm{Fl}_{\check{G},\mathbf{F}_q} \to \mathrm{Gr}_{\check{G},\mathbf{F}_q}$.

[14]The notation "SS" is an abbreviation for "singular support". This is synonymous with "microlocal support" which is often denoted μsupp; we do not use the latter notation because it would render certain equations too long.

Since the map SS is essential, and its definition meandered through a rather serpentine construction, we recapitulate it in the diagram below, where the left (resp. right) column pertains to G (resp. $\check{G}$).

$$\begin{tikzcd}
\mathrm{K}(\mathrm{Rep}^{\emptyset}(G_1T)) \arrow[d, equal] \arrow[rrd, dashed, "\mathrm{SS}"] & & \\
\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \arrow[rr, dashed, "\mathrm{SS}"'] \arrow[d, "\text{BMR localization}"'] & & \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma) \\
\mathrm{K}(\mathrm{Coh}^{T^{(1)}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})) \arrow[d, "(\mathfrak{sp}_{p\to 0})^{-1}"'] & & \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_{\gamma,\mathbf{C}}) \arrow[u, "\mathfrak{sp}_{p\to 0}"'] \\
\mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}_{\mathcal{B}_{\mathbf{C}}}(\widetilde{\mathcal{N}}_{\mathbf{C}})) \arrow[rru, "\text{mirror symmetry}"] \arrow[rr, "\text{geometric Langlands}"'] & & \mathrm{K}(\mathcal{D}_\psi) \arrow[u, "\text{microlocalization}"']
\end{tikzcd}$$

Remark 8.3.2. The identification $\mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_{\gamma,\mathbf{C}}) \xrightarrow{\sim} \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$ from Example 6.5.2 was not defined via $\mathfrak{sp}_{p\to 0}$. However, from the observation that the degeneration from characteristic zero to characteristic p is constant on $\check{T}$-fixed points, it is clear that the identification using the GKM description coincides with $\mathfrak{sp}_{p\to 0}$.

9. Degeneration of affine Springer fibers

In this section we assemble the ingredients from the preceding sections in order to finally do geometric calculations related to the Breuil-Mézard Conjecture (although the precise connection will not be explained until Part 3). Our goal here is to "understand" the limit cycle $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma(\lambda)] \in \mathrm{Ch}_{\mathrm{top}}(\mathrm{Y}_\gamma)$. We will express this cycle in terms of representation theory via the microlocal support map from Definition 8.3.1.

The first difficulty is that the specialization process is a priori mysterious, at least in terms of the basis of irreducible components, since specialization does not (in general) interact well with the properties of being reduced or irreducible. A key tool for us is equivariant localization, which allows to calculate the specialization in terms of torus-fixed points instead. We illustrate this in §9.1, where we calculate explicitly the equivariant fundamental class of $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma(\lambda)]$.

We apply this in §9.1 to express $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma(\lambda)]$ in terms of the case $\lambda = \rho$. In Part 3 this calculation will be used to reduce the Breuil-Mézard Conjecture for all sufficiently small λ to the fundamental case $\lambda = \rho$.

At the next stage we want to identify $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma(\rho)]$ with the microlocal support of a particular baby Verma module in $\mathrm{Rep}^0(G_1T)$. Although we know an explicit formula for the equivariant fundamental class of the former object, it does not seem to be easy to compute the latter object in these terms. Hence we have to take a more indirect approach. In §9.3 we prove a "Recognition Principle" that characterizes the class $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma(\lambda)]$ in somewhat more conceptual terms. Then in §9.4 and Appendix A, jointly with Bezrukavnikov and Boixeda Alvarez, we check that this Recognition Principle applies to the desired baby Verma.

9.1. **Equivariant class of limit cycles.**

We fix, once and for all, a generator ω_{top} of $\Omega_{\check{\mathfrak{t}}}^{\wedge \dim \check{\mathfrak{t}}}$. We also abbreviate $\mathbb{S} := \mathbb{S}_{\check{T}}$. Define

$$\beta := \prod_{\alpha \in \check{\Phi}^+} d\alpha \in \mathrm{Sym}_{\mathbf{Q}_\ell}(\check{\mathfrak{t}}^*) \cong \mathbb{S}. \tag{9.1.1}$$

Recall from Example 6.3.2 that there is an isomorphism $X_*(\check{T}) \xrightarrow{\sim} (\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0})^{\check{T}}$ sending $\lambda \mapsto t^\lambda$, which identifies (implicitly using ω_{top})

$$\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}((\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0})^{\check{T}}) \cong \bigoplus_{\lambda \in X_*(\check{T})} \mathbb{S}[t^\lambda]. \tag{9.1.2}$$

Recall that if X is $\check{T}$-equivariantly formal, then for $\alpha \in \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(X)$ we write $\alpha_{\check{T}} \in \mathrm{H}^{\mathrm{BM},\check{T}}_{\mathrm{top}}(X)$ for its equivariant lift, which exists and is unique by Remark 6.2.1. This applies to $X = \mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}$, whose $\check{T}$-fixed points may be identified with $X_*(\check{T})$ as in Example 6.3.2.

Lemma 9.1.1. *Let $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{Q}_q}$ be non-zero and let $\lambda \in X_*(\check{T})^+$ be regular. Then, with respect to* (9.1.2)*, we have*

$$\mathrm{Loc}^{\check{T}}([\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda)]_{\check{T}}) = \frac{1}{\beta} \sum_{w \in W} \mathrm{sgn}(w)[t^{w\lambda}] \in \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}((\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0})^{\check{T}}) \otimes_{\mathbb{S}} \mathrm{Frac}(\mathbb{S}). \tag{9.1.3}$$

Proof. Since λ is regular, by Lemma 3.6.4 the Bialynicki-Birula map takes $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda)$ isomorphically to $\check{G}/\check{B}$. Therefore, the $\check{T}$-fixed points of $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda)$ are identified with $\{[t^{w\lambda}]\}_{w\in W} \subset X_*(\check{T}) = (\mathrm{Gr}_{\check{G},\mathbf{Q}_q})^{\check{T}}$.

As explained in Example 6.4.2, the component at $wB \in (\check{G}/\check{B})^{\check{T}}$ of $\mathrm{Loc}^{\check{T}}([\check{G}/\check{B}]_T) \in \mathrm{H}_{2d}^{\mathrm{BM},\check{\mathrm{T}}}(\check{G}/\check{B})$ is $(\frac{\mathrm{sgn}(w)}{\beta})$. Then we conclude using that $wB \in \check{G}/\check{B}$ corresponds to the fixed point $[t^{w\lambda}] \in \mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda)$ under the Bialynicki-Birula isomorphism $\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda) \xrightarrow{\sim} \check{G}/\check{B}$. ☐

Corollary 9.1.2. *With β defined as* (9.1.1)*, we have*

$$\mathrm{Loc}^{\check{T}}([\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]_{\check{T}}) = \frac{1}{\beta}\sum_{w\in W} \mathrm{sgn}(w)[t^{w\rho}] \in \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}((\mathrm{X}_\gamma^{\varepsilon=0})^{\check{T}}) \otimes_{\mathbb{S}} \mathrm{Frac}(\mathbb{S}).$$

Proof. This follows from Lemma 9.1.1, (the equivariant version of) Proposition 4.2.1, and the fact that $\mathfrak{sp}_{\varepsilon\to 0}$ is the identity map in the common GKM description of $\mathrm{H}_*^{\mathrm{BM},T}(\mathrm{X}_\gamma^\varepsilon)$. ☐

9.2. **General Schubert cells.** In this part we consider $\mathrm{X}_\gamma^{\varepsilon=\eta}(\lambda+\rho) \hookrightarrow \mathrm{X}_\gamma^{\varepsilon=\eta}$, the closure of $\mathrm{X}_\gamma^{\varepsilon\neq 0} \cap S^\circ(\lambda+\rho)$ indexed by general $\lambda \in X_*(T)^+$. We will calculate the fundamental class $[\mathrm{X}_\gamma^{\varepsilon=\eta}(\lambda+\rho)]$ in terms of the "basic" case $\lambda = 0$, which was analyzed in Lemma 9.1.1. This will have significance for the Breuil-Mézard Conjecture with Hodge-Tate weights $\lambda+\rho$.

By the Equivariant Localization Theorem and Example 6.3.2, we have

$$\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}) \otimes_{\mathbb{S}} \mathrm{Frac}(\mathbb{S}) \cong \bigoplus_{\mu\in X_*(\check{T})} \mathrm{Frac}(\mathbb{S})[t^\mu]. \tag{9.2.1}$$

This has an obvious (left) action of $X_*(\check{T})$, through left translation on the indexing set.

Theorem 9.2.1. *Let $\varepsilon_0 \in \mathbf{A}^1_{\mathbf{Q}_q}$ be non-zero and $\lambda \in X_*(\check{T})^+ \cong X^*(T)^+$. Then we have*

$$\mathrm{Loc}^{\check{T}}[\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda+\rho)]_{\check{T}} = \sum_{\mu\le\lambda} m_\mu(\lambda) t^\mu \cdot \mathrm{Loc}^{\check{T}}[\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\rho)]_{\check{T}} \in \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}) \otimes_{\mathbb{S}} \mathrm{Frac}(\mathbb{S}).$$

where $m_\mu(\lambda)$ is the multiplicity of the weight μ in the highest weight representation V_λ of $G_{\mathbf{Q}_q}$. Here the action $t^\mu\cdot$ is the one defined just above, through (9.2.1).

Proof. According to Lemma 9.1.1, we have

$$\mathrm{Loc}^{\check{T}}([\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\lambda+\rho)]_{\check{T}}) = \frac{1}{\beta}\sum_{w\in W} \mathrm{sgn}(w)[t^{w(\lambda+\rho)}] \in \bigoplus_{\mu\in X^*(T)} \mathrm{Frac}(\mathbb{S})[t^\mu] \tag{9.2.2}$$

and

$$\mathrm{Loc}^{\check{T}}([\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\rho)]_{\check{T}}) = \frac{1}{\beta}\sum_{w\in W} \mathrm{sgn}(w)[t^{w\rho}] \in \bigoplus_{\mu\in X^*(T)} \mathrm{Frac}(\mathbb{S})[t^\mu]. \tag{9.2.3}$$

From (9.2.3) we find that

$$\sum_{\mu\le\lambda} m_\mu(\lambda) t^\mu \cdot [\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}(\rho)]_{\check{T}} = \frac{1}{\beta}\sum_{\mu\le\lambda} m_\mu(\lambda) t^\mu \cdot \left(\sum_{w\in W} \mathrm{sgn}(w)[t^{w\rho}]\right) \in \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{X}_\gamma^{\varepsilon=\varepsilon_0}) \otimes_{\mathbb{S}} \mathrm{Frac}(\mathbb{S}). \tag{9.2.4}$$

Below we recall the Weyl character formula. We regard the characters of representations of G as elements of the group ring $\mathbf{Q}_\ell[X^*(T)] = \mathbf{Q}_\ell[X_*(\check{T})]$. When writing characters, we use $e^\lambda \in \mathbf{Q}_\ell[X^*(T)]$ to represent the group element $\lambda \in X^*(T) \cong X_*(\check{T})$. The Weyl character formula says

$$\sum_{\mu\le\lambda} m_\mu(\lambda) e^\mu = \frac{\sum_{w\in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} \in \mathbf{Q}_\ell[X^*(T)]$$

and the Weyl denominator formula says

$$\prod_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in W} \mathrm{sgn}(w) e^{w\rho} \in \mathbf{Q}_\ell[X^*(T)].$$

Combining them, we find that

$$\sum_{w\in W} \operatorname{sgn}(w)e^{w(\lambda+\rho)} = \sum_{\mu\leq\lambda} m_\mu(\lambda)e^\mu \left(\sum_{w\in W}\operatorname{sgn}(w)e^{w\rho}\right) \in \mathbf{Q}_\ell[X^*(T)].$$

Hence we have an identity

$$\sum_{w\in W} \operatorname{sgn}(w)[t^{w(\lambda+\rho)}] = \sum_{\mu\leq\lambda} m_\mu(\lambda)[t^\mu] \left(\sum_{w\in W}\operatorname{sgn}(w)[t^{w\rho}]\right) \in \bigoplus_{\mu\in X_*(\check{T})} \operatorname{Frac}(\mathbb{S})[t^\mu]. \tag{9.2.5}$$

Now the desired equality follows from comparing (9.2.2), (9.2.4), and (9.2.5). □

Below, for ease of notation we abbreviate $C_{\widetilde{w}}$ for the irreducible component $\mathrm{Y}_\gamma(\widetilde{w})$ of Y_γ from Corollary 3.7.5, for all $\widetilde{w}\in\widetilde{W}^{\mathrm{reg}}$. As usual, $[C_{\widetilde{w}}]$ is its cycle class and $[C_{\widetilde{w}}]_{\check{T}}$ its $\check{T}$-equivariant lift.

Lemma 9.2.2. *The irreducible component $C_{t^{w\rho}} \subset \mathrm{Y}_\gamma(\leq\rho)$ has $t^{w\rho}$ as a smooth point, and the equivariant Euler class of $[C_{t^{w\rho}}]_{\check{T}}$ at $t^{w\rho}$ is*

$$e_T(t^{w\rho}, C_{t^{w\rho}}) = \frac{\operatorname{sgn} w}{\beta}.$$

Proof. This follows from the proof of Lemma 3.7.2 and the explicit description of the affine space chart around $t^{w\rho}$. □

9.3. **Recognition principle.** Let $\mathbb{S} := \mathbb{S}_{\check{T}}$. We will refer to the GKM description of $\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma)$ inside

$$\mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma^{\check{T}})\otimes_{\mathbb{S}}\operatorname{Frac}(\mathbb{S}) \cong \bigoplus_{w\in\widetilde{W}} \operatorname{Frac}(\mathbb{S})[\widetilde{w}].$$

Proposition 9.3.1 (Recognition principle)**.** *There is a unique class $[Z]\in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$ which in the GKM description has the following properties:*

- *(1) (Eigenclass) $[Z]$ has equivariant support contained in $X_*(\check{T})\subset\widetilde{W}$.*
- *(2) (Support bound) $[Z]$ has equivariant support contained in the admissible set*

$$\operatorname{Adm}(\rho) := \{\widetilde{w}\in\widetilde{W} \colon \widetilde{w}\leq t^{w\rho} \quad \textit{for some } w\in W\}.$$

- *(3) (Normalization) The component of $[Z]$ at t^ρ is $1/\beta\in\operatorname{Frac}(\mathbb{S})$.*

Remark 9.3.2. Condition (1) can be formulated alternatively as an eigenproperty (cf. Proposition A.1.3 below) for the lattice parts of the two actions $(\widetilde{W},\cdot)$ and $(\widetilde{W},\bullet)$, which can be thought of as "Hecke actions" (because they are literally given by convolving with Hecke operators in the constructible realization of §8.1). This explains why we call (1) an "eigenclass" property.[15]

Proof. Let $[Z]$ and $[Z']$ be two elements of $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$ satisfying all of these conditions. Then $[Z]-[Z']$ is a class $\delta=\sum a_{\widetilde{w}}[\widetilde{w}]\in \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma)$ with equivariant support concentrated in $X_*(\check{T})\cap\operatorname{Adm}(\rho)$, but not on t^ρ.

Note that as $C_{\widetilde{w}}$ is the closure of $C_{\widetilde{w}}\cap S^\circ(\widetilde{w})$, $[C_{\widetilde{w}}]_{\check{T}}$ has equivariant support only on elements $\leq\widetilde{w}$. We have

$$\delta = \sum_{\widetilde{w}\in\widetilde{W}} m_{\widetilde{w}}[C_{\widetilde{w}}]_{\check{T}}$$

for some integers $m_{\widetilde{w}}$. If $\widetilde{w}$ is a maximal element such that $m_{\widetilde{w}}\neq 0$, then δ has a non-trivial coefficient at $[\widetilde{w}]$, so $\widetilde{w}\in\operatorname{Adm}(\rho)$. We conclude that only $\widetilde{w}\in\operatorname{Adm}^{\mathrm{reg}}(\rho)$ contributes in the above sum.

Let $w\in W$, and α be a simple root of $\check{T}$ such that $ws_\alpha > w$, and set $\widetilde{u}=wt^\rho s_\alpha w^{-1}$. Then:

- (cf. [BAL21, Proposition 4.6][16]) The 1-dimensional $\check{T}$-orbits of $\mathrm{Fl}_{\check{G},\mathbf{F}_q}$ joining $t^{w\rho}$ and $\widetilde{u}$, resp. $t^{ws_\alpha\rho}$ and $\widetilde{u}$ belong to $\mathrm{Y}_\gamma(\leq\rho)$. Furthermore, both these orbits are associated to the character $w\alpha\in X^*(\check{T})$.

[15] In an earlier draft of this paper, the main result of this section was proved using a different Recognition Principle formulated in terms of a more abstract eigenproperty, before we realized that we could instead work with a more explicit characterization in terms of translation elements.

[16] This is a reference to [GKM04, §5.11], which focuses on the version of the affine Springer fiber over $\mathbf{C}$, but the same analysis applies in characteristic p, as explained in Example 6.5.2.

- By [LHLM23a, Proposition 2.2.6], the connected component of $\mathrm{Y}_\gamma^{\ker w\alpha}$ passing through $t^{w\rho}$ intersects $\mathrm{Y}_\gamma(\leq \rho)^{\check{T}} = \mathrm{Adm}(\rho)$ at exactly $t^{w\rho}, \widetilde{u}$ and $t^{ws_\alpha\rho}$.

The GKM description thus gives

$$\mathrm{Res}_{dw\alpha}(a_{t^{w\rho}}) + \mathrm{Res}_{dw\alpha}(a_{\widetilde{u}}) + \mathrm{Res}_{dw\alpha}(a_{t^{ws_\alpha\rho}}) = 0.$$

But our hypothesis implies that the middle term vanishes (since $\widetilde{u}$ is not a translation), hence the two outer terms must either both vanish or both not vanish. Since $t^{w\rho}$ is maximal in $\mathrm{Adm}(\rho)$, $[C_{t^{w\rho}}]_{\check{T}}$ is the only irreducible component that contributes to $a_{t^{ws_\alpha\rho}}$. Hence by Lemma 9.2.2 we have

$$a_{t^{w\rho}} = m_{t^{w\rho}} \frac{\operatorname{sgn} w}{\beta},$$

and we have an analogous formula for $a_{t^{ws_\alpha\rho}}$. Combining this with the previous observation, we learn that $m_{t^{w\rho}}$ and $m_{t^{ws_\alpha\rho}}$ are either both zero or both non-zero. Since we also have $m_{t^\rho} = 0$, this gives $m_{t^{w\rho}} = 0$ for all $w \in W$, so $m_{t^{w\rho}} = m_{t^{ws_\alpha\rho}} = 0$.

Finally, if $\delta \neq 0$, then there must be a maximal element $\widetilde{w}$ such that $m_{\widetilde{w}} \neq 0$. Then δ has non-trivial coefficient at $[\widetilde{w}]$, so $\widetilde{w}$ is a translation in $\mathrm{Adm}^{\mathrm{reg}}(\rho)$. However, Lemma 9.3.3 below shows that such a translation must be of the form $t^{w\rho}$, contradicting what we showed in the previous paragraph that $m_{t^{w\rho}} = 0$ for all $w \in W$.

□

Lemma 9.3.3. *We have*

$$\mathrm{Adm}^{\mathrm{reg}}(\rho) \cap X_*(\check{T}) = \{t^{w\rho} \colon w \in W\}.$$

Proof. Suppose $t^\mu \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$. Then by [LHLM23b, Corollary 2.1.7] we have

$$t^\mu = \widetilde{w}_2^{-1} w_0 \widetilde{w}_1$$

where

- $\widetilde{w}_1 \in \widetilde{W}_1$ is restricted, i.e., $\widetilde{w}_1(A_0)$ belongs to the fundamental box (cf. §2.3.1),
- $\widetilde{w}_2 \in \widetilde{W}^+$ is dominant, i.e., $\widetilde{w}_2(A_0)$ belongs to the dominant cone,
- $\widetilde{w}_2 \uparrow w_0 t^{-\rho} \widetilde{w}_1$. (See [Jan03, II.6] for the definition of the $\uparrow$ order.)

The fact that $\widetilde{w}_2^{-1} w_0 \widetilde{w}_1$ is a translation in $X_*(\check{T})$ shows that

$$t^\nu w_0 t^{-\rho} \widetilde{w}_1 = \widetilde{w}_2 \quad \text{for some } \nu \in X_*(\check{T}).$$

Since any dominant alcove is uniquely a dominant translation of a restricted alcove, ν is dominant. But then $w_0 t^{-\rho} \widetilde{w}_1 \uparrow \widetilde{w}_2$, so the third bullet point above forces $\nu = 0$, and

$$\widetilde{w}_2^{-1} w_0 \widetilde{w}_1 = (w_0 t^{-\rho} \widetilde{w}_1)^{-1} w_0 \widetilde{w}_1 = t^{w_1^{-1}\rho},$$

where w_1 is the image of $\widetilde{w}_1$ in W. □

Remark 9.3.4. The proof of Proposition 9.3.1 applies also to deformed affine Springer fibers as long as they satisfy the GKM conditions. By Lemma 3.7.2, this holds for $\gamma = ts$ when s is regular semi-simple and $\varepsilon = \eta$, and if s is furthermore h_ρ-generic then it holds for $\mathrm{Y}_\gamma^{\varepsilon=1}(\leq \rho)$. The only step in the proof that requires additional commentary is the calculation of 1-dimensional $\check{T}$-orbits, which in the deformed case is done in the proof of [LHLM23b, Proposition 4.3.].

Lemma 9.3.5. *The class*

$$\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]_{\check{T}} \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\check{T}}(\mathrm{Y}_\gamma)$$

satisfies the conditions of Proposition 9.3.1.

Proof. By §2.6.1 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{X}_\gamma^{\check{T}}) & \xrightarrow{\mathfrak{sp}_{p\to 0}} & \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma^{\check{T}}) \\ \downarrow & & \downarrow \\ \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{X}_\gamma) & \xrightarrow{\mathfrak{sp}_{p\to 0}} & \mathrm{H}_*^{\mathrm{BM},\check{\mathrm{T}}}(\mathrm{Y}_\gamma) \end{array}$$

From this and Corollary 9.1.2 we conclude

$$\mathrm{Loc}^{\check{T}}\left(\mathfrak{sp}_{p\to 0}[\mathrm{X}_{\gamma}^{\varepsilon=0}(\rho)]_{\check{T}}\right) = \frac{1}{\beta}\sum_{w\in W}\mathrm{sgn}(w)[t^{w\rho}], \tag{9.3.1}$$

which visibly satisfies the conditions of Proposition 9.3.1. □

9.4. **The microlocal support of baby Verma modules.** Recall the microlocal support map SS from Definition 8.3.1. Henceforth we mostly focus on $\mathrm{K}(\mathrm{Rep}^{\emptyset}(G_1T))$ instead of $\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$; although these are canonically isomorphic we remind that they carry different normalizations for their baby Verma modules. The following result, obtained jointly with Bezrukavnikov and Boixeda Alvarez, is established in the Appendix.

Theorem 9.4.1 (Joint with Bezrukavnikov–Boixeda Alvarez)**.** *We have*

$$\mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}} = \mathfrak{sp}_{p\to 0}[\mathrm{X}_{\gamma}^{\varepsilon=0}(\rho)]_{\check{T}} \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\check{T}}(\mathrm{Y}_{\gamma})$$

and under $\mathrm{Loc}^{\check{T}}$ *they are given by* $\frac{1}{\beta}\sum_{w\in W}\mathrm{sgn}(w)[t^{w\rho}]$.

For the proof of Theorem 9.4.1, and also for other purposes later, we need the following technical lemma.

Lemma 9.4.2. *Let* $\widetilde{w}\in\widetilde{W}_1$ *be restricted (cf. §2.3.1 for notation). Then* $\mathrm{SS}[\widehat{L}_1(\widetilde{w}\bullet_p 0)]_{\check{T}}$ *has equivariant support in* $\widetilde{W}_{\le w_0\widetilde{w}} = \{\widetilde{u}|\widetilde{u}\le w_0\widetilde{w}\}$. *Furthermore,* $W\widetilde{w}$ *occurs in the equivariant support.*

Proof. We induct on $\ell(\widetilde{w})$, the case $\ell(\widetilde{w})=0$ being true by the normalization condition (3) of §8.3.

If $\ell(\widetilde{w})>0$, we can find a simple affine reflection s such that $\widetilde{w}s\in\widetilde{W}_1$ and $\widetilde{w}s<\widetilde{w}$. By [Jan03, II.7.15–21, II.9.22] (for description of the wall-crossing functors for G and then their variants for G_rT, respectively), the wall-crossing functor R_s $(=\Theta_s$ in the notation of *loc. cit.*) satisfies

$$[\widehat{L}_1(\widetilde{w}\bullet_p 0)] = [R_s(\widehat{L}_1(\widetilde{w}s\bullet_p 0))] + \sum_{\substack{\widetilde{u}\in\widetilde{W}^+:\\ \widetilde{u}\le\widetilde{w}s}} m_{\widetilde{u}}[\widehat{L}_1(\widetilde{u}\bullet_p 0)] \in \mathrm{K}(\mathrm{Rep}^0(G_1T))$$

for some $m_{\widetilde{u}}\in\mathbf{Z}$ (note that $\widetilde{u}$ need not be restricted). We will show that the equivariant support of SS applied to each of the terms on the right-hand side belongs to $W_{\le w_0\widetilde{w}}$:

1. We have $\mathrm{SS}[R_s(\widehat{L}_1(\widetilde{w}s\bullet_p 0))]_{\check{T}} = \mathrm{SS}[\widehat{L}_1(\widetilde{w}s\bullet_p 0)]_{\check{T}}\bullet_p s + \mathrm{SS}[\widehat{L}_1(\widetilde{w}s\bullet_p 0)]_{\check{T}}$. By induction, the equivariant support of $\mathrm{SS}[\widehat{L}_1(\widetilde{w}s\bullet_p 0)]_{\check{T}}$ belongs to $\widetilde{W}_{\le w_0\widetilde{w}s}$, and the equivariant support of $\mathrm{SS}[\widehat{L}_1(\widetilde{w}s\bullet_p 0)]_{\check{T}}\bullet_p s$ belongs to $\widetilde{W}_{\le w_0\widetilde{w}s}s\subset\widetilde{W}_{\le w_0\widetilde{w}}$, since $w_0\widetilde{w}=(w_0\widetilde{w}s)s$ is a reduced factorization and s is simple. Furthermore, by induction $w_0\widetilde{w}$ does occur in the equivariant support of this term.
2. For $\mathrm{SS}[\widehat{L}_1(\widetilde{u}\bullet_p 0)]_{\check{T}}$: Decompose $\widetilde{u}=t^{\nu}\widetilde{v}$ with $\widetilde{v}\in\widetilde{W}_1$ and ν dominant. Note that $\widetilde{v}(A_0)\uparrow t^{\nu}\widetilde{v}(A_0)\uparrow\widetilde{w}s(A_0)$, so $\ell(\widetilde{v})\le\ell(\widetilde{w}s)<\ell(\widetilde{w})$ (note however that $\widetilde{v}$ and $\widetilde{w}$ may be incomparable since ν may fail to be in Q). By induction, the equivariant support belongs to
$$t^{\nu}\widetilde{W}_{\le w_0\widetilde{v}}\subset\widetilde{W}_{\le w_0\widetilde{w}s}\subset\widetilde{W}_{\le w_0\widetilde{w}}$$
where the first inclusion is Lemma 9.4.3 below. We also note that $w_0\widetilde{w}$ cannot occur in the equivariant support of this term, since $\ell(\widetilde{v})<\ell(\widetilde{w})$.

Finally, note that we have shown that $w_0\widetilde{w}$ does occur in the equivariant support. Example 7.6.1 then shows that all of $W\widetilde{w}$ occurs in the equivariant support. □

Lemma 9.4.3. *Let* $\widetilde{w}\in\widetilde{W}_1$, $\nu\in X^*(T)^+$, *and* $\widetilde{v}\in\widetilde{W}_1$ *such that* $t^{\nu}\widetilde{v}\le\widetilde{w}$. *Then we have*

$$t^{\nu}\widetilde{W}_{\le w_0\widetilde{v}}\subset\widetilde{W}_{\le w_0\widetilde{w}}.$$

Proof. Let $\widetilde{u}\le w_0\widetilde{v}$, so $\widetilde{u}_{\mathrm{dom}}\le\widetilde{v}$. Let $\sigma\in W$ be the unique element such that $\sigma t^{\nu}\widetilde{u}$ is dominant. It suffices to show $\sigma t^{\nu}\widetilde{u}\le\widetilde{w}$.

Since ν is dominant, we have $\nu-\sigma\nu\ge 0$, and

$$\sigma t^{\nu}\widetilde{u}\uparrow t^{-\sigma\nu+\nu}\sigma t^{\nu}\widetilde{u} = t^{\nu}\sigma\widetilde{u}.$$

We also have $\sigma\widetilde{u}\uparrow\widetilde{u}_{\mathrm{dom}}$, hence

$$\sigma t^{\nu}\widetilde{u}\uparrow t^{\nu}\widetilde{u}_{\mathrm{dom}}.$$

But for dominant elements of $\widetilde{W}$, the $\uparrow$ order and the Bruhat order coincide, hence

$$\sigma t^{\nu}\widetilde{u} \leq t^{\nu}\widetilde{u}_{\mathrm{dom}} \leq t^{\nu}\widetilde{v} \leq \widetilde{w},$$

as desired. $\square$

Let

$$X^0(T) := \{\lambda \in \mathrm{X}^*(T) \colon \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Delta\}.$$

Decompose

$$[\widehat{Z}_1(p\rho)] = \sum_{u \in \widetilde{W}_1/X^0(T)} \sum_{\nu \in X^*(T)} m_{u,\nu}[\widehat{L}_1(u \bullet_p 0 + p\nu)].$$

Then it follows from [GHS18, Lemma 10.1.5] that $m_{u,\nu} > 0$ if and only if

$$(t^{-\nu})_{\mathrm{dom}} \leq w_0 t^{-\rho} u.$$

The map taking $(u, \nu) \mapsto \widetilde{w} := (t^{-\nu})_{\mathrm{dom}}^{-1} w_0 u$ is a bijection from $\{(u,\nu) \colon m_{u,\nu} \neq 0\}$ to $\mathrm{Adm}^{\mathrm{reg}}(\rho)$. It follows from Lemma 9.4.3 that $\mathrm{SS}[\widehat{L}_1(u \bullet_p 0 + p\nu)]$ must contain $[\mathrm{Y}_\gamma(\widetilde{w})]$ in its support. In particular, if we knew that all the $\mathrm{SS}[\widehat{L}_1(u \bullet_p 0 + p\nu)]$ were effective, then we would learn that $\mathrm{SS}[\widehat{Z}_1(p\rho)]$ contains $[\mathrm{Y}_\gamma(\widetilde{w})]$ for all $\widetilde{w} \in \mathrm{Adm}^{\mathrm{reg}}(\rho)$, i.e., its support contains all the top dimensional irreducible components of $\mathrm{Y}_\gamma(\leq \rho)$. While we will only be able to prove this effectivity property for $p \gg 0$, we can get the last conclusion already for $p > h + 1$:

Proposition 9.4.4. *The Zariski closure of $\mathrm{X}_\gamma^{\varepsilon=0}(\rho)$ contains all top dimensional irreducible components of $\mathrm{Y}_\gamma^{\varepsilon=0}(\leq \rho)$. If $\gamma = ts$ with $s \in \check{\mathfrak{t}}_{\mathbf{F}_q}$ being h_ρ-generic, the same statement also holds for $\mathrm{X}_\gamma^{\varepsilon=1}(\rho)$. More generally, both statements hold when ρ is replaced by any regular $\lambda \in X^*(T)^+$ (under the hypothesis that s is h_λ-generic for the $\varepsilon = 1$ statement).*

Remark 9.4.5. The conclusion of the Proposition for $\varepsilon = 1$ was also established in [LHLM23b, Theorem 7.4.2] for $\check{G} = \mathrm{GL}_n$ and some special choices of γ (with worse genericity), using global automorphic inputs.

Proof. For this proof, we need to refer to some notions from the representation theory of quantum groups recalled in §12.3 below. We first prove the statement for $\varepsilon = 0$. By Theorem 9.4.1, it suffices to check that $\mathrm{SS}[\widehat{Z}_1(p\rho)]$ contains all top dimensional irreducible components of $\mathrm{Y}_\gamma(\leq \rho)$. By construction, this reduces to the same statement for the quantum group version $\mathrm{SS}[{}^{\mathfrak{q}}\widehat{Z}_{w_0\mathfrak{b}}(p\rho)]$, cf. Lemma 12.3.4. However, our above discussion applies equally well to the quantum simples $\mathrm{SS}[{}^{\mathfrak{q}}\widehat{L}(u \bullet_p 0 + p\nu)]$, which are actually effective by Corollary 12.3.7. The statement for general λ follows by taking translations of the statement for ρ.

The statement for $\varepsilon = 1$ now follows from combining the $\varepsilon = 0$ case with Theorem 5.1.1(1) and Lemma 4.2.3. $\square$

Part 3. Applications to the Breuil-Mézard Conjecture

Recall that G, B, T were defined over $\mathbf{Z}_q$. In this section we write

$$(\underline{G}, \underline{B}, \underline{T}) := \mathrm{Res}_{\mathbf{Z}_q/\mathbf{Z}_p}(G, B, T).$$

In general, for S over $\mathbf{Z}_q, \mathbf{Q}_q$, or $\mathbf{F}_q$ we will denote by $\underline{S}$ the Weil restriction of S to $\mathbf{Z}_p, \mathbf{Q}_p$, or $\mathbf{F}_p$. Note that the (deformed) affine Springer fibers $\underline{\mathrm{X}}_\gamma^\varepsilon$, $\underline{\mathrm{Y}}_\gamma^\varepsilon$, etc. may either be regarded as Weil restrictions or as defined by the same formulas as $\mathrm{X}_\gamma^\varepsilon, \mathrm{Y}_\gamma^\varepsilon$, etc. with $\underline{G}$ in place of G.

Fix an algebraic closure $\overline{\mathbf{Q}}_p$ and let $\mathcal{J} := \mathrm{Hom}_{\mathbf{Q}_p}(\mathbf{Q}_q, \overline{\mathbf{Q}}_p)$ be the set of $\mathbf{Q}_p$-algebra embeddings of $\mathbf{Q}_q$ into $\overline{\mathbf{Q}}_p$. Although $\underline{G}$ is not split, so that it lies outside the scope of Parts 1 and 2, the base change

$$\underline{G} \otimes_{\mathbf{Z}_p} \mathbf{Z}_q \cong \prod_{j \in \mathcal{J}} G$$

is split. We will only be applying the results of the previous Parts after base changing to (an extension of) $\mathbf{Z}_q$, so that objects are just the $\mathcal{J}$-fold product of their correspond constructions for G.

Let F be the relative Frobenius of $\underline{G}_{\overline{\mathbf{F}}_p}$, so $\underline{G}^F \cong \underline{G}(\mathbf{F}_p) = G(\mathbf{F}_q)$. Let π be the finite order automorphism of $(\underline{G}_{\overline{\mathbf{F}}_p}, \underline{B}_{\overline{\mathbf{F}}_p}, \underline{T}_{\overline{\mathbf{F}}_p})$ from [GHS18, Proof of Lemma 9.2.4]. It is characterized by the property that $F \circ \pi^{-1}$ is the relative Frobenius for the $\mathbf{F}_p$-structure on $\underline{G}_{\overline{\mathbf{F}}_p}$ induced by the split group $\prod_{j \in \mathcal{J}} G_{\overline{\mathbf{F}}_p}$.

We write

$$\mathrm{X}^*(\underline{T}) := X^*(\underline{T} \times_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p) \cong X^*(\underline{T} \times_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)$$

for the geometric character group of $\underline{T}$. We write $\underline{\Phi} \subset \mathrm{X}^*(\underline{T})$ for the geometric roots, $\underline{\Phi}^+ \subset \underline{\Phi}$ for the positive geometric roots, and $\underline{\Delta} \subset \underline{\Phi}^+$ for the simple geometric roots, etc. One **exception** is that we abuse notation by writing $\rho := \sum_{\alpha \in \underline{\Phi}^+} \alpha$ for the half sum of positive roots of $\underline{G}$, without an underline.[17]

The automorphism π induces an automorphism of $\mathrm{X}^*(\underline{T}) \cong \mathrm{X}_*(\check{\underline{T}})$ which we denote by the same name. Explicitly, for $\lambda = (\lambda_j)_{j \in \mathcal{J}} \in \mathrm{X}^*(\underline{T})$ and φ the Frobenius automorphism of $\mathbf{Z}_q$, then the component of $\pi(\lambda)$ at the embedding $j \in \mathcal{J}$ is $\lambda_{j \circ \varphi^{-1}}$.

Note that when $p = q$ the underlines can all be omitted and π is trivial, so the entirety of the preceding discussion can be ignored; none of the main ideas are lost when specializing to this case.

10. The Breuil-Mézard Conjecture

In this section we set up the formulation of the geometric, refined version of the Breuil-Mézard Conjecture. In §10.1 we discuss the parametrization of Serre weights. In §10.2 we define tame inertial types and their parametrization by Deligne-Lusztig representations. In §10.3 we recall the Emerton-Gee stack and state Emerton-Gee's formulation of the Breuil-Mézard Conjecture.

10.1. **Serre weights.** Fix an algebraic closure $k = \overline{\mathbf{F}}_p$. The simple representations of $G(\mathbf{F}_q) = \underline{G}(\mathbf{F}_p)$ over k are called the *Serre weights* of $\underline{G}$.

For $\lambda \in \mathrm{X}^*(T)$, write $L(\lambda)$ for the simple representation of $\underline{G}/k$ with highest weight λ.

10.1.1. *Parametrization of Serre weights.* The *p-restricted weights* $\mathrm{X}_1^*(\underline{T}) \subset \mathrm{X}^*(\underline{T})^+$ are defined as

$$\mathrm{X}_1^*(\underline{T}) := \{\lambda \in \mathrm{X}^*(\underline{T})^+ \colon 0 \leq \langle \lambda, \alpha^\vee \rangle < p-1 \text{ for all } \alpha \in \underline{\Delta}\}.$$

Note that this is consistent with §7.3 in the case $q = p$. Let

$$\mathrm{X}^0(\underline{T}) := \{\lambda \in \mathrm{X}^*(\underline{T}) \colon \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \underline{\Delta}\}.$$

Then $\mathrm{X}_1^*(\underline{T})$ is a finite union of $\mathrm{X}^0(\underline{T})$-cosets. For $\lambda \in \mathrm{X}_1^*(\underline{T})$, we write $F(\lambda) := L(\lambda)|_{\underline{G}(\mathbf{F}_p)}$. Then $F(\lambda)$ is a simple representation of $\underline{G}(\mathbf{F}_p)$, and the map $\lambda \mapsto F(\lambda)$ induces a bijection

$$\frac{\mathrm{X}_1^*(\underline{T})}{(F-1)\mathrm{X}^0(\underline{T})} \leftrightarrow \{\text{Serre weights of } \underline{G}\}.$$

10.1.2. *The Frobenius kernel.* We write $\underline{G}_1 := \ker(\underline{G} \xrightarrow{F} \underline{G})$ for the Frobenius kernel of $\underline{G}$ and $\underline{G}_1\underline{T} < \underline{G}$ for the subgroup scheme generated by $\underline{G}_1$ and $\underline{T}$. For $\lambda \in \mathrm{X}^*(T)$, let $\widehat{L}_1(\lambda)$ be the simple representation of $\underline{G}_1\underline{T}$ with highest weight λ. (As a representation of $\underline{G}_1$, we have $\widehat{L}_1(\lambda) := L(\lambda_0)|_{\underline{G}_1}$ where λ_0 is the unique $p\mathrm{X}^*(\underline{T})$-translate of λ into the fundamental box.) Then $\widehat{L}_1(\lambda)$ is simple and the map $\lambda \mapsto \widehat{L}_1(\lambda)$ induces a bijection

$$\mathrm{X}^*(\underline{T}) \leftrightarrow \{\text{simple } \underline{G}_1\underline{T}\text{-representations}\}.$$

10.1.3. *Reparametrization of Serre weights.* Recall the following notations:

- $\underline{A}_0 \subset \mathrm{X}^*(T)$ is the dominant base alcove anchored at 0, and $\underline{C}_0 := -\rho + p\underline{A}_0$ is the dominant ρ-shifted base p-alcove.
- The *dominant* affine Weyl group elements are

$$\widetilde{\underline{W}}^+ := \{\widetilde{w} \in \widetilde{\underline{W}} \colon \widetilde{w} \bullet_p \underline{C}_0 \text{ is dominant}\}$$

  and the *dominant p-restricted* affine Weyl group elements are

$$\widetilde{\underline{W}}_1 := \{\widetilde{w} \in \widetilde{\underline{W}}^+ \colon \widetilde{w} \bullet_p \underline{C}_0 \text{ is } p\text{-restricted (and dominant)}\}.$$

We will occasionally use the reparametrization from [LHLM23b, (2.5)] of Serre weights by pairs $\widetilde{w}_1 \in \widetilde{\underline{W}}_1^+$ and $\omega \in \mathrm{X}^*(\underline{T})$ such that $\omega - \rho \in \underline{C}_0$:

$$F_{(\widetilde{w}_1, \omega)} := F(\pi^{-1}(\widetilde{w}_1) \bullet_p (\omega - \rho)).$$

[17] The notation $\underline{\rho}$ was contemplated, but made later formulas too horrific.

10.2. **Tame inertial types.** We now assume that $G = \mathrm{GL}_n / \mathbf{Q}_q$. We recall some notation regarding tame inertial parameters from [LHLM23b].

10.2.1. *Fundamental characters.* Let E be a sufficiently large coefficient field and $\mathcal{O}$ its ring of integers. Let K be a finite unramified extension of $\mathbf{Q}_p$ of degree d. Fix a separable closure $\overline{K}$ of K. Choose $u \in \overline{K}$ such that $u^{p^d-1} = -p$ and define $\omega_K \colon \mathrm{Gal}_K \to \mathcal{O}_K^\times$ to be the character given by

$$g(u) = \omega_K(g)u \text{ for all } g \in \mathrm{Gal}_K .$$

Note that since u is well-defined up to multiplication by a (p^d-1)st root of unity, ω_K is independent of the choice of u.

10.2.2. *Tame inertial parameters.* The notion of *tame inertial L-parameter* is defined as in [LHLM23b, §2.4]. We parametrize them combinatorially as follows. For $(w, \mu) \in \underline{W} \times \mathrm{X}_*(\underline{\check{T}})$, we let $\tau(w,\mu) \colon I_{\mathbf{Q}_p} \to \check{T}(E)$ be the tame inertial L-parameter

$$\left(\sum_{i=0}^{d-1} (F^* \circ w^{-1})^i(\mu)\right)(\omega_d)$$

where

- F^* is the endomorphism $p\pi^{-1}$ on $\mathrm{X}_*(\underline{\check{T}})$, and
- $d \geq 1$ is an integer such that $(F^* \circ w^{-1})^d = p^d$, and
- ω_d is the composition of $\omega_{\mathbf{Q}_{p^d}}$ with a fixed inclusion $\mathbf{Q}_{p^d} \hookrightarrow E$.

This parametrization is highly redundant (see [LHLM23b, Proposition 2.4.5] for the equivalence relation). Any 1-generic tame inertial L-parameter τ admits a *lowest alcove presentation*, meaning that $\tau \cong \tau(w,\mu)$ where $\mu \in \underline{C}_0$ lies in the fundamental alcove.

A *tame Weil-Deligne inertial type* τ is a conjugacy class of pairs (r_τ, N_τ), where r_τ is a tame inertial L-parameter and N_τ in a nilpotent element of $\mathrm{Lie}\, \underline{\check{G}}(E)$, which can be extended to a Weil-Deligne representation.[18]

There are also wild Weil-Deligne inertial types, but we never deal with them in this paper and therefore do not discuss them any further.

10.2.3. *Tame inertial types.* To a Weil-Deligne inertial type τ the inertial Local Langlands correspondence associates a smooth irreducible $G(\mathbf{Z}_p)$-representation $\sigma(\tau)$ called the *inertial type* of τ, with properties explained in [LHLM23b, Theorem 2.5.4].

We focus on the case where τ is tame, and $\sigma(\tau)$ is then called a *tame inertial type.* Let $(w,\mu) \in \underline{W} \times \mathrm{X}^*(\underline{T})$ be a *good pair* in the sense of [LLHL19, §2.2]. Then we have a Deligne-Lusztig representation $R(w,\mu)$ [GHS18, Proposition 9.2.1 and 9.2.2]. For 1-generic μ, we can take $R(w,\mu) = \sigma(\tau(w,\mu))$, according to [LHLM23b, Proposition 2.5.5].

10.3. **The Emerton-Gee stack.** Recall that we are assuming $G = \mathrm{GL}_n / \mathbf{Q}_q$, so $\check{G} = \mathrm{GL}_n$. Emerton-Gee have constructed a moduli stack $\mathcal{X}^{\mathrm{EG}}$ of rank n (φ, Γ)-modules, whose groupoid of points in any finite $\mathbf{Z}_p$-algebra A is naturally equivalent to the groupoid of continuous representations $\mathrm{Gal}_{\mathbf{Q}_q} \to \mathrm{GL}_n(A)$. Therefore, we refer to $\mathcal{X}^{\mathrm{EG}}$ as the *moduli stack of n-dimensional representations of* $\mathrm{Gal}_{\mathbf{Q}_q}$.

The underlying reduced substack $\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}$ is of finite type over $\mathbf{F}_q$, equidimensional of dimension $fn(n-1)/2$, where we recall that $q = p^f$. For a pair (λ, τ) where $\lambda \in \mathrm{X}_*(\check{T})$ and τ is a tame inertial type, Emerton-Gee constructed a substack $\mathcal{X}^{\mathrm{crys},\lambda,\tau} \subset \mathcal{X}^{\mathrm{EG}}$ which is finite type and flat over $\mathrm{Spf}\, \mathbf{Z}_p$, and then determined by the property that for any finite flat $\mathbf{Z}_p$-algebra A°, $\mathcal{X}^{\mathrm{crys},\lambda,\tau}(A^\circ)$ is the groupoid of A°-lattices in potentially crystalline $\mathrm{Gal}_{\mathbf{Q}_q}$-representations with Hodge-Tate weights λ and inertial type τ. When λ is regular, the special fiber $\mathcal{X}^{\mathrm{crys},\lambda,\tau}|_{\mathbf{F}_p}$ is an algebraic stack of dimension $fn(n-1)/2 = \dim \mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}$; when λ is irregular, the dimension of its special fiber is strictly smaller.

For a Serre weight σ, let $W(\lambda)$ be the Weyl module with highest weight λ and

$$n_\sigma(\lambda, \tau) := [\overline{W(\lambda) \otimes \sigma(\tau)} : \sigma]$$

[18] Using the notation τ for both tame inertial L-parameters and tame Weil-Deligne inertial types looks abusive, but we may regard any tame inertial L-parameter as a tame Weil-Deligne inertial by taking $N_\tau = 0$. (See [LHLM23b, Remark 2.5.2] for elaboration.)

be the Jordan-Hölder multiplicity of σ in a mod p reduction of $W(\lambda) \otimes \sigma(\tau)$. The *geometric Breuil-Mézard Conjecture* due to Emerton-Gee, predicts:

Conjecture 10.3.1 (cf. [EG23, Conjecture 8.2.2])**.** *There is a collection of effective cycles $\mathcal{Z}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}})$ indexed by Serre weights σ of $\mathrm{GL}_n(\mathbf{F}_q)$, such that for all $\lambda \in \mathrm{X}^*(\underline{T})^+$ and inertial types τ, we have*

$$[\mathcal{X}^{\mathrm{crys},\lambda+\rho,\tau}|_{\mathbf{F}_p}] = \sum_{\sigma} n_\sigma(\lambda,\tau)\mathcal{Z}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}).$$

Remark 10.3.2. The effectivity is not part of the original formulations in [BM02, EG23], but it has become part of the conjectural picture relating the Breuil-Mézard Conjecture to patching, and is conjectured explicitly in [EG20, §9.4].

Remark 10.3.3 (Generalization to other groups)**.** Generalizing Conjecture 10.3.1 beyond $G = \mathrm{GL}_n$ requires:

(1) The construction of an Emerton-Gee stack for $\check{G}$.
(2) The construction of substacks of potentially crystalline representations (with given Hodge-Tate weights and inertial parameter).
(3) An inertial Local Langlands correspondence for G.

None of these ingredients was available when we began writing this paper, but the situation has since changed rapidly. The Emerton-Gee stack is now available in large generality thanks to [Lin23b]; the potentially crystalline substacks are available in large generality thanks to [Lin23c]; and the inertial Local Langlands is mostly unavailable, but if one restricts to *tame* τ then it is available in large generality thanks to [Lin23a].

We also note the work [Lee23] which obtains these ingredients in the special case $G = \mathrm{GSp}_4/\mathbf{Q}_q$ (using the Local Langlands Correspondence for GSp_4 established by Gan-Takeda).

11. Existence of Breuil-Mézard cycles

Let $G = \mathrm{GL}_n/\mathbf{Q}_q$ and $\mathcal{X}^{\mathrm{EG}}$ be the Emerton-Gee stack of representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_q/\mathbf{Q}_q)$ into $\check{G}$. In this section we will prove the following Theorem, which gives partial evidence to Conjecture 10.3.1.

Theorem 11.0.1. *There exists a collection $\{\mathcal{Z}^{\mathrm{EG}}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}})\}$, indexed by Serre weights σ for $G(\mathbf{F}_q)$, such that for all $\lambda \in \mathrm{X}^*(T)$ and $\tau = \tau(w,\mu)$ satisfying $\mu \in \underline{C}_0$ is $2h_{\lambda+\rho}$-generic, we have*

$$[\mathcal{X}^{\mathrm{crys},\lambda+\rho,\tau}|_{\mathbf{F}_p}] = \sum_{\sigma} n_\sigma(\lambda,\tau)\mathcal{Z}^{\mathrm{EG}}(\sigma), \tag{11.0.1}$$

where $n_\sigma(\lambda,\tau)$ is defined as in Conjecture 10.3.1.

Remark 11.0.2. Later in §12, we will discuss properties of the cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ such as their effectivity, uniqueness, and decomposition into irreducible components.

Here is the outline of this section. In §11.1 we discuss the relationship between the deformed affine Springer fibers $\underline{\mathcal{X}}^{\varepsilon=1}_\gamma$ and the potentially crystalline substacks $\mathcal{X}^{\mathrm{crys},\lambda,\tau}$ of $\mathcal{X}^{\mathrm{EG}}$. Roughly speaking, we cook up γ from τ so that $\underline{\mathcal{X}}^{\varepsilon=1}_\gamma(\leq \lambda)$ is a "model" for $\bigcup_{\lambda' \leq \lambda} \mathcal{X}^{\mathrm{crys},\lambda',\tau}$, at least at the level of top homology. In §11.2 we discuss the comparison of the special fibers $\underline{\mathrm{Y}}^{\varepsilon=1}_\gamma$ as γ varies. In §11.3 we construct the incarnations $\mathcal{Z}^{\varepsilon=1}_\gamma(\sigma)$ of Breuil-Mézard cycles on the model spaces $\underline{\mathrm{Y}}^{\varepsilon=1}_\gamma$, and in §11.4 we prove that they satisfy relations which should be thought of as corresponding on the model to (11.0.1). Finally in §11.5 we show how to produce the $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ of Theorem 11.0.1 from the cycles $\mathcal{Z}^{\varepsilon=1}_\gamma(\sigma)$ on the models, and then in §11.6 we show how to deduce that they satisfy the relations (11.0.1).

11.1. **Modeling potentially crystalline loci.** Let $\tau = \tau(w,\mu)$ be a lowest alcove presentation of a tame inertial parameter. We define $\gamma(w,\mu) = ts$ where $s \in \check{\mathfrak{t}}$ is the regular semisimple element associated to (w,μ) from [LHLM23b]: explicitly, it is given by the term "Diag(...)" from the equation below [LHLM23b, (7.6)], whose notation is explained in [LHLM23b, Example 2.4.1]. *The only thing that we need to take away from this formula is that modulo p we have $\gamma = t(w^{-1}\mu)$.* Below we tacitly use the equivalence between representations from $\mathrm{Gal}_{\mathbf{Q}_q}$ into $\check{G}$ and L-parameters from $\mathrm{Gal}_{\mathbf{Q}_p}$ into $\underline{\check{G}}$.

The deformed affine Springer fiber $\mathcal{X}^{\varepsilon=1}_\gamma$ provides a homological model for the union of potentially crystalline loci with regular Hodge-Tate weights in the following sense:

Theorem 11.1.1. *(Homological model theorem) Let $\tau = \tau(w, \mu)$, $\lambda \in X^*(\underline{T})^+$. Suppose μ is $h_{2(\lambda+\rho)}$-generic, and set $\gamma = t(w^{-1}\mu)$. Then there is an isomorphism*

$$\mathrm{transfer}_\gamma \colon \mathrm{Ch}_{\mathrm{top}}(\underline{Y}_\gamma^{\varepsilon=1}(\leq \lambda+\rho)) \xrightarrow{\sim} \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}_p}^{\leq \lambda+\rho,\tau})$$

with the following properties.

(1) *For each $\lambda' \leq \lambda + \rho$,*

$$\mathrm{transfer}_\gamma \left(\mathfrak{sp}_{p\to 0}[\underline{X}_\gamma^{\varepsilon=1}(\lambda')]\right) = [\mathcal{X}_{\mathbf{F}_p}^{\lambda',\tau}] \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}_p}^{\leq\lambda+\rho,\tau}).$$

(2) *Let $\sigma \in \mathrm{JH}(\overline{W(\lambda) \otimes R(w,\mu)})$. There is a unique pair $(\widetilde{u}, \widetilde{v}) \in \underline{\widetilde{W}}_1 \times \underline{\widetilde{W}}^+$ with $\widetilde{w} = \widetilde{v}^{-1} w_0 \widetilde{u} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda+\rho)$ such that $\sigma = F(\pi^{-1}(\widetilde{u}) \bullet_p (t^\mu w \widetilde{v}^{-1}(0) - \rho))$. Then*

$$\mathrm{transfer}_\gamma \left([\underline{Y}_\gamma^{\varepsilon=1}(\widetilde{w})]\right) = [\mathcal{C}_\sigma] \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}_p}^{\leq\lambda+\rho,\tau}),$$

where we recall that $\mathcal{C}_\sigma$ is the irreducible component of $\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}}$ corresponding to σ, cf. [EG23, Theorem 1.2.1, Theorem 6.5.1].

The proof appears in Appendix B.

Remark 11.1.2. (1) A slightly more sophisticated version of the argument in Appendix B, which will appear in [LH], establishes the theorem for μ just about $h_{\lambda+\rho}$-generic, which is expected to be optimal.

(2) When τ is very generic relative to λ in the sense that μ avoids some universal closed subvariety in $(\mathbf{A}^n_{\mathbf{F}_p})^{\mathcal{J}}$ depending on λ, the first part of Theorem 11.1.1 can be deduced from [LHLM23b, Theorem 7.3.2]. However, the nature of this very generic condition prevents us from making this deduction when λ has a dependency on p.

When μ is around $\max\{h_{\lambda+\rho} + 4h_\rho, 2h_{\lambda+\rho}\}$-generic the second part of theorem can be deduced from [LHLM23b, Theorem 7.4.2], but the argument in *loc.cit.* relies on input from Taylor-Wiles patching.

(3) Combing Theorem 11.1.1 with Corollary 9.4.4, we immediately deduce

$$\mathcal{X}^{\lambda+\rho,\tau}_{\mathbf{F}_q,\mathrm{red}} = \bigcup_{\sigma\in\mathrm{JH}(W(\lambda)\otimes\sigma(\tau))} \mathcal{C}_\sigma$$

whenever τ is $h_{2(\lambda+2\rho)}$-generic.

We want to define candidate Breuil-Mézard cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ on $\mathcal{X}^{\mathrm{EG}}$, as σ ranges over Serre weights. We will do this by choosing some auxiliary tame type τ that contains σ as a Jordan-Hölder factor and transferring a cycle $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ from the model $\underline{Y}_\gamma^{\varepsilon=1}$. In turn, the cycle $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ will be constructed by applying the inverse of $\mathfrak{sp}_{\varepsilon\to 0} \colon \mathrm{Ch}_{\mathrm{top}}(\underline{Y}_\gamma^{\varepsilon=1}) \to \mathrm{Ch}_{\mathrm{top}}(\underline{Y}_\gamma^{\varepsilon=0})$ to the microlocal support of simple representations of $\underline{G}_1\underline{T}$.

For this to work, we need to know that the recipe produces cycles that are independent of the choice of the auxiliary tame type τ (or essentially equivalently, γ). This requires understanding how the intersection of $\mathcal{X}_{\mathbf{F}_p}^{\lambda,\tau}$ and $\mathcal{X}_{\mathbf{F}_p}^{\lambda',\tau'}$ behaves under the two transfer maps (i.e., a "change-of-type" formula).

Write $\delta := \mu - \mu'$ and let $\gamma := \gamma(w,\mu)$ and $\gamma' := \gamma(w',\mu')$. By the first part of Lemma 5.5.3, left translation by $(w')^{-1}t^\delta w$ maps $\underline{Y}_\gamma^{\varepsilon=1}$ isomorphically to $\underline{Y}_{\gamma'}^{\varepsilon=1}$ as subschemes of $\mathrm{Fl}_{\underline{\check{G}}}$,

$$\mathrm{tr}_{(w')^{-1}t^\delta w} \colon \underline{Y}_\gamma^{\varepsilon=1} \xrightarrow{\sim} \underline{Y}_{\gamma'}^{\varepsilon=1}. \tag{11.1.1}$$

This induces an isomorphism

$$\mathrm{tr}_{(w')^{-1}t^\delta w} \colon \mathrm{Ch}_{\mathrm{top}}(\underline{Y}_\gamma^{\varepsilon=1}) \xrightarrow{\sim} \mathrm{Ch}_{\mathrm{top}}(\underline{Y}_{\gamma'}^{\varepsilon=1}) \tag{11.1.2}$$

and similarly on Borel-Moore homology. Given $\lambda, \lambda' \in X^*(\underline{\check{T}})^+$, (11.1.2) induces a partially defined map

$$\mathrm{Ch}_{\mathrm{top}}(\underline{Y}_\gamma^{\varepsilon=1}(\leq\lambda)) \overset{\mathrm{tr}_{(w')^{-1}t^\delta w}}{\dashrightarrow} \mathrm{Ch}_{\mathrm{top}}(\underline{Y}_{\gamma'}^{\varepsilon=1}(\leq\lambda')) \tag{11.1.3}$$

where the domain of definition is generated by the classes of top-dimensional irreducible components contained in $\underline{Y}_\gamma^{\varepsilon=1}(\leq\lambda)$ which are mapped by $\mathrm{tr}_{(w')^{-1}t^\delta w}$ to top-dimensional irreducible components lying in $\underline{Y}_{\gamma'}^{\varepsilon=1}(\leq\lambda')$.

Proposition 11.1.3 (Change-of-type fomula)**.** *Let* $\lambda \in \mathrm{X}^*(\underline{T})^+$*,* $\lambda' \in \mathrm{X}^*(\underline{T})^+$*, and* $\delta := \mu - \mu'$*. Let* $\tau = \tau(w, \mu)$*, and* $\tau' = \tau(w', \mu')$ *be two lowest alcove presentations such that* μ *is* $h_{2(\lambda+\rho)}$*-generic and* μ' *is* $h_{2(\lambda'+\rho)}$*-generic. Then the diagram*

$$\begin{tikzcd} \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\leq \lambda+\rho)) \arrow[rr, dashed, "\mathrm{tr}_{(w')^{-1}t^{\delta}w}"] \arrow[dr, "\mathrm{transfer}_{\gamma}"'] & & \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma'}(\leq \lambda+\rho)) \arrow[dl, "\mathrm{transfer}_{\gamma'}"] \\ & \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}) & \end{tikzcd}$$

commutes.

Proof. From Theorem 11.1.1 we have

$$\mathrm{transfer}_{\gamma}\left([\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\widetilde{w})]\right) = [\mathcal{C}_{\sigma}]$$

where $\widetilde{w} = \widetilde{v}^{-1}w_0\widetilde{u} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda+\rho)$ and $\sigma = F(\pi^{-1}(\widetilde{u})\bullet_p(t^{\mu}w\widetilde{v}^{-1}(0)-\rho))$. The condition that $\mathrm{tr}_{(w')^{-1}t^{\delta}w}[\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\widetilde{w})]$ is defined guarantees that $\sigma \in \mathrm{JH}(\overline{W(\lambda')\otimes R(w,\mu')})$. By Theorem 11.1.1(2), there is $\widetilde{w}' = (\widetilde{v}')^{-1}w_0\widetilde{u}' \in \mathrm{Adm}^{\mathrm{reg}}(\lambda'+\rho)$ such that

$$\mathrm{transfer}_{\gamma}\left([\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma'}(\widetilde{w}')]\right) = [\mathcal{C}_{\sigma}],$$

hence $F(\pi^{-1}(\widetilde{u}) \bullet_p (t^{\mu}w\widetilde{v}^{-1}(0)-\rho)) = F(\pi^{-1}(\widetilde{u}') \bullet_p (t^{\mu'}w'(\widetilde{v}')^{-1}(0)-\rho))$. Up to adjusting by $\underline{\Omega}$ it suffices to consider the case $\widetilde{u} = \widetilde{u}'$, hence

$$t^{\mu}w\widetilde{v}^{-1}(0) = t^{\mu'}w'(\widetilde{v}')^{-1}(0)$$

so that

$$t^{\mu}w\widetilde{v}^{-1} = t^{\mu'}w'(\widetilde{v}')^{-1}s$$

for some $s \in \underline{W}$. But by Lemma 5.5.3, for an appropriate γ_1 we have

$$t^{\mu}w\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\widetilde{w}) = t^{\mu}w\widetilde{v}^{-1}\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma_1}(w_0\widetilde{u}) = t^{\mu'}w'(\widetilde{v}')^{-1}s\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma_1}(w_0\widetilde{u}) = t^{\mu'}w'(\widetilde{v}')^{-1}\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma_1}(w_0\widetilde{u}) = t^{\mu'}w'\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma'}(\widetilde{w}')$$

so that

$$\mathrm{tr}_{(w')^{-1}t^{\delta}w}[\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\widetilde{w})] = [\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma'}(\widetilde{w}')]$$

as required. □

11.2. **Gluing homology.** As has been discussed, the spaces $\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}$ are models for the special fibers $\bigcup_{\lambda}\mathcal{X}^{\mathrm{crys},\lambda,\tau}_{\gamma}|_{\mathbf{F}_p} \subset \mathcal{X}^{\mathrm{EG}}|_{\mathbf{F}_p}$, where $\gamma = tw^{-1}\mu$. As τ varies, the special fibers of potentially crystalline substacks overlap. In this section, we will see how to implement the corresponding overlapping for the irreducible components of $\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}$.

Recall that in Example 6.5.2, we used the GKM description to fix (in particular) an identification

$$\mathrm{H}^{\mathrm{BM},\check{T}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}) \xrightarrow[\sim]{\mathrm{GKM}} \mathrm{H}^{\mathrm{BM},\check{T}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'}) \tag{11.2.1}$$

characterized by the commutativity of the diagram (where $\mathbb{S} := \mathbb{S}_{\check{\underline{T}}}$)

$$\begin{tikzcd} \mathrm{H}^{\mathrm{BM},\check{\underline{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}) \arrow[r, "\mathrm{GKM}", "\sim"'] \arrow[d, hook] & \mathrm{H}^{\mathrm{BM},\check{\underline{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'}) \arrow[d, hook] \\ \mathrm{Frac}(\mathbb{S})\otimes_{\mathbb{S}}\mathrm{H}^{\mathrm{BM},\check{\underline{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}) \arrow[r, "\sim"] & \mathrm{Frac}(\mathbb{S})\otimes_{\mathbb{S}}\mathrm{H}^{\mathrm{BM},\check{\underline{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'}) \\ \mathrm{Frac}(\mathbb{S})\otimes_{\mathbb{S}}\mathrm{H}^{\mathrm{BM},\check{\underline{T}}}_{\mathrm{top}}((\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma})^{\check{T}}) \arrow[u, "\sim"] \arrow[r, equal] \arrow[d, equal] & \mathrm{Frac}(\mathbb{S})\otimes_{\mathbb{S}}\mathrm{H}^{\mathrm{BM},\check{\underline{T}}}_{\mathrm{top}}((\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'})^{\check{T}}) \arrow[u, "\sim"'] \arrow[d, equal] \\ \bigoplus_{\widetilde{w}\in\widetilde{\underline{W}}}\mathrm{Frac}(\mathbb{S})[\widetilde{w}] \arrow[r, equal] & \bigoplus_{\widetilde{w}\in\widetilde{\underline{W}}}\mathrm{Frac}(\mathbb{S})[\widetilde{w}] \end{tikzcd}$$

By the equivariant formality of $\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}$ and $\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'}$ and Remark 6.2.1, (11.2.1) induces in turn an identification of the *non-equivariant* top Borel-Moore homology groups,

$$\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}) = \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'}). \tag{11.2.2}$$

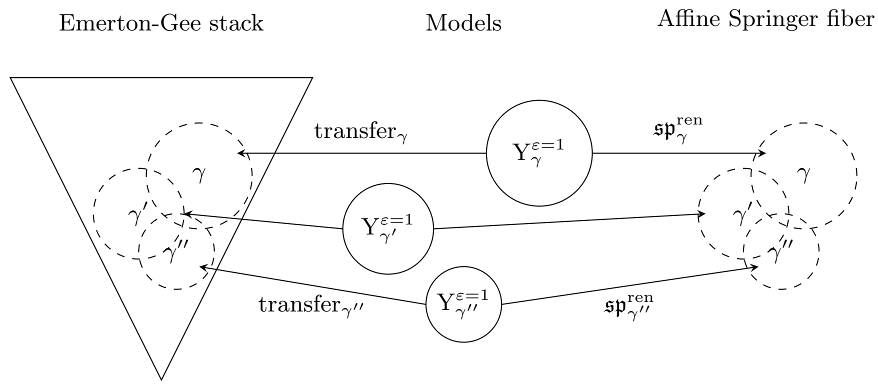

FIGURE 1. This cartoon (produced with the aid of ChatGPT after much coaxing) depicts the construction of Breuil-Mézard cycles. The triangle represents the p-dilated fundamental alcove $\underline{C}_0$, whose weights are the μ in the lowest-alcove presentation $R(w,\mu)$ for tame types; the choice of w and μ together is captured by $\gamma = \gamma(w,\mu)$. The models $\mathrm{Y}_\gamma^{\varepsilon=1}(\leq \lambda)$ provide charts for corresponding portions of $\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}}$; the sizes of the charts, depicted by the dashed circles, depend on the genericity of γ, which is measured by the distance from the walls. Cycles are initially produced on the affine Springer fiber via microlocal support, deformed to the models at $\varepsilon = 1$, and then transferred to the Emerton-Gee stack. Well-definedness of the cycles comes from the fact that transfer$_\gamma$ and $\mathfrak{sp}_\gamma^{\mathrm{ren}}$ effect the "same amount" of gluing.

Definition 11.2.1 (Renomalized specialization)**.** Suppose $\gamma = \gamma(w,\mu)$ where μ is h_λ-generic. Then we define the *renormalized specialization* map

$$\mathfrak{sp}_\gamma^{\mathrm{ren}} \colon \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq \lambda)) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$$

to be the composition of $\mathfrak{sp}_{\varepsilon\to 0}$ in the sense of Definition 4.2.4 followed by the monodromy-translation action of $t^\mu w$.

The significance of the renormalized specialization map is seen in the Proposition below, whose statement should be compared to Proposition 11.1.3, and interpreted as saying that "the maps $\mathfrak{sp}_\gamma^{\mathrm{ren}}$ implement the same combinatorial gluing relations among $\{\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mathrm{Y}_\gamma^{\varepsilon=1})\}_\gamma$ as the maps transfer$_\gamma$" (see Figure 1).

Lemma 11.2.2. *If γ is h_λ-generic, then the spaces $\underline{\mathrm{Y}}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda)$ are equivariantly formal for $\varepsilon_0 = 1$ or η.*

Proof. According to Lemma 3.7.2, the hypotheses imply that $\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq \lambda)$ has an affine paving and therefore that its cohomology is pure. □

Proposition 11.2.3. *Let $\gamma = \gamma(w,\mu)$ and $\gamma' = \gamma(w',\mu')$. Assume that γ is h_λ-generic and γ' is $h_{\lambda'}$-generic. Then the diagram*

$$\begin{tikzcd} \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma'}^{\varepsilon=1}(\leq \lambda')) \arrow[r, "\mathfrak{sp}_{\gamma'}^{\mathrm{ren}}"] & \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma'}^{\varepsilon=0}) \\ \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma}^{\varepsilon=1}(\leq \lambda)) \arrow[u, dashed, "\mathrm{tr}_{(w')^{-1}t^\delta w}"] \arrow[r, "\mathfrak{sp}_{\gamma}^{\mathrm{ren}}"] & \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma}^{\varepsilon=0}) \arrow[u, equal, "(11.2.2)"'] \end{tikzcd} \tag{11.2.3}$$

commutes, where $\delta := \mu - \mu'$ and the left vertical arrow is the partially defined map induced by (11.1.3).

Proof. By Lemma 11.2.2 and Remark 6.2.1, if γ is h_λ-generic then we have

$$\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda)) \xleftarrow{\sim} \mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\underline{\check{T}}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda)) \hookrightarrow \mathrm{Frac}(\mathbb{S}) \otimes_{\mathbb{S}} \mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\underline{\check{T}}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=\varepsilon_0}(\leq \lambda)) \tag{11.2.4}$$

for any $\varepsilon_0 \in \{0,1,\eta\}$. Therefore it suffices to check the commutativity of the diagram in rationalized $\underline{\check{T}}$-equivariant Borel-Moore homology. Then we may apply the equivariant localization theorem to compare the rationalized $\underline{\check{T}}$-equivariant homology version of (11.2.3) with the corresponding diagram for the rationalized equivariant homology version for the $\underline{\check{T}}$-fixed points. This embeds the corresponding diagram for

$\check{T}$-equivariant Borel-Moore homology

$$\begin{tikzcd} \mathrm{H}^{\mathrm{BM},\underline{\check{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma'}(\leq \lambda')) \arrow[r, "\mathfrak{sp}^{\mathrm{ren}}_{\gamma'}"] & \mathrm{H}^{\mathrm{BM},\underline{\check{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma'}) \\ \mathrm{H}^{\mathrm{BM},\underline{\check{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\leq \lambda)) \arrow[r, "\mathfrak{sp}_{\gamma}"] \arrow[u, dashed, "\mathrm{tr}_{(w')^{-1}t^{\delta}w}"] & \mathrm{H}^{\mathrm{BM},\underline{\check{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}) \arrow[u, equal, "(11.2.2)"'] \end{tikzcd} \tag{11.2.5}$$

into the similar diagram for $\underline{\check{T}}$-equivariant Borel-Moore homology of the $\underline{\check{T}}$-fixed points, tensored with Frac($\mathbb{S}$). Since the family $\underline{\mathrm{Y}}^{\varepsilon}_{\gamma}$ restricts to the constant family $\underline{\widetilde{W}} \times \mathrm{Spec}\, \mathbf{F}_p[\varepsilon]$ on $\underline{\check{T}}$-fixed points, the similar diagram on fixed points reads

$$\begin{tikzcd} \bigoplus_{\widetilde{w}\in\widetilde{W}} \mathbb{S}[\widetilde{w}] \arrow[r, hook, "t^{\mu'}w'\cdot"] & \bigoplus_{\widetilde{w}\in\widetilde{W}} \mathbb{S}[\widetilde{w}] \\ \bigoplus_{\widetilde{w}\in\widetilde{W}} \mathbb{S}[\widetilde{w}] \arrow[r, hook, "t^{\mu}w\cdot"] \arrow[u, dashed, "\mathrm{tr}_{(w')^{-1}t^{\delta}w}"] & \bigoplus_{\widetilde{w}\in\widetilde{W}} \mathbb{S}[\widetilde{w}] \arrow[u, equal] \end{tikzcd}$$

for which the commutativity is evident.

□

11.3. **Breuil-Mézard cycles on the model.** We now undertake the construction of Breuil-Mézard cycles. The plan is to define cycles on the affine Springer fibers $\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}$ using the microlocal support map (cf. Definition 8.3.1), then to deform them to the $\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}$, and finally to transfer them to the Emerton-Gee stack using the maps transfer$_\gamma$ from Theorem 11.1.1, as in the diagram below.

$$\mathrm{K}(\mathrm{Rep}^0(G_1T)) \xrightarrow{\mathrm{SS}} \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}) \xrightarrow{(\mathfrak{sp}_{\varepsilon\to 0})^{-1}} \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}) \xrightarrow{\mathrm{transfer}_\gamma} \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\lambda,\tau}_{\mathbf{F}_p}) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}})$$

Recall the definition of $\underline{\widetilde{W}}_1$ from §10.1.3.

Definition 11.3.1 (Cycles on the affine Springer fibers)**.** Let $\gamma = ts$ with $s \in \check{\underline{\mathfrak{t}}}(\overline{\mathbf{F}}_p)$ regular. For $u \in \underline{\widetilde{W}}_1$ and $\xi \in \underline{C}_0$, we define

$$\mathcal{Z}^{\varepsilon=0}_{\gamma}(u \bullet_p \xi) := t^{\xi+\rho}\mathrm{SS}[\widehat{L}_1(u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}). \tag{11.3.1}$$

For later use we record the effect of the monodromy action on the cycles just defined.

Lemma 11.3.2 (Monodromy action on the cycles)**.** *Let $\mu \in \rho + \underline{C}_0$ and $u \in \underline{\widetilde{W}}_1$. If $\nu \in \mathrm{X}^*(\underline{T})$ is sufficiently small so that $\mu - \rho + \underline{W}\nu \subset \underline{C}_0$, then*

$$wt^{-\mu} \cdot \mathcal{Z}^{\varepsilon=0}_{\gamma}(u \bullet_p (\mu - \rho + \nu)) = t^{-\mu} \cdot \mathcal{Z}^{\varepsilon=0}_{\gamma}(u \bullet_p (\mu - \rho + w\nu)) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}). \tag{11.3.2}$$

Proof. The LHS of (11.3.2) is

$$wt^{\nu} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)] = t^{w\nu}w \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)]$$

and the RHS of (11.3.2) is $t^{w\nu} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)]$. The two agree because $w \cdot [\widehat{L}_1(u \bullet_p 0)] = [\widehat{L}_1(u \bullet_p 0)]$ as explained in Example 7.6.1. □

Our next task is to transport the cycles to the model $\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}$. First we analyze when the cycle $\mathcal{Z}^{\varepsilon=0}_{\gamma}(u \bullet_p \xi)$ lies in the image of the renormalized specialization map $\mathfrak{sp}^{\mathrm{ren}}_{\gamma}$ on $\mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=1}_{\gamma}(\leq \lambda + \rho))$, and can therefore be deformed to $\varepsilon = 1$.

Definition 11.3.3 (Admissible tuples)**.** Let $u \in \underline{\widetilde{W}}_1, w \in \underline{W}$ and $\nu \in \mathrm{X}^*(\underline{T})$. We say that (u, w, ν) is *admissible* for $\lambda \in \mathrm{X}^*(\underline{T})^+$ if

$$(t^{-w^{-1}\nu})_{\mathrm{dom}} \leq w_0 t^{-\lambda-\rho} u.$$

Remark 11.3.4. Suppose μ is $(h_\lambda + 2h_\rho)$-generic. Then by [LHLM23b, Proposition 2.3.7], the Serre weight $F(\pi^{-1}(u) \bullet_p (\mu - \rho + \nu))$ appears as a Jordan-Hölder factor of $W(\lambda) \otimes R(w, \mu)$ if and only if (u, w, ν) is admissible for λ.

Lemma 11.3.5. *We have $t^{-\mu+\rho} \cdot \mathcal{Z}^{\varepsilon=0}_{\gamma}(u \bullet_p (\mu - \rho + w^{-1}\nu)) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma}(\leq \lambda + \rho))$ if and only if (u, w, ν) is admissible for λ.*

Proof. This follows from Lemma 9.4.2. □

If μ is $(h_\lambda + h_\rho)$-generic and $\gamma = \gamma(w, \mu)$, then

$$\mathfrak{sp}_\gamma^{\mathrm{ren}} \colon \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda + \rho)) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}) \tag{11.3.3}$$

is defined. By Lemmas 11.3.2 and 11.3.5, $w^{-1} t^{-\mu+\rho} \cdot \mathcal{Z}_\gamma^{\varepsilon=0}(u \bullet_p (\mu - \rho + \nu))$ lies in the image of (11.3.3) precisely when (u, w, ν) is admissible for λ.

We can now define the incarnation of Breuil-Mézard cycles on the models.

Definition 11.3.6 (Cycles on the model)**.** Assume $\mu \in \rho + \underline{C}_0$ is $(h_\lambda + h_\rho)$-generic and let $\gamma = \gamma(w, \mu)$. For (u, w, ν) which are admissible for λ, we define $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu))_\lambda \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda + \rho))$ to be the unique cycle satisfying

$$\mathfrak{sp}_\gamma^{\mathrm{ren}} \left(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu))_\lambda \right) = \mathcal{Z}_\gamma^{\varepsilon=0}(u \bullet_p (\mu - \rho + \nu)) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}).$$

(This notation reflects that in practice ν is "small" relative to μ, so we visualize the $\{\xi\}$ for which $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi)_\lambda$ is defined as forming a constellation of weights orbiting μ.)

For $\sigma = F(\pi^{-1} u \bullet_p (\mu - \rho + \nu))$, we denote

$$\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)_\lambda := \mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu))_\lambda.$$

Remark 11.3.7 (Independence of λ)**.** If $\lambda \le \lambda'$ and γ is $(h_{\lambda'} + h_\rho)$-generic, then γ is also $(h_\lambda + h_\rho)$-generic and the inclusion $\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda + \rho) \hookrightarrow \underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda' + \rho)$ sends $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu))_\lambda \mapsto \mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu))_{\lambda'}$. We therefore have a well-defined class

$$\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu)) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1})$$

as long as there exists λ such that μ is $(h_\lambda + h_\rho)$-generic and (u, w, ν) are admissible for λ. In this case we say that the Serre weight $\sigma = F(\pi^{-1} u \bullet_p (\mu - \rho + \nu))$ is *admissible* for (γ, λ), and we abbreviate $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma) := \mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu))$. With these definitions, the identity

$$\mathfrak{sp}_\gamma^{\mathrm{ren}} \left(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p (\mu - \rho + \nu)) \right) = \mathcal{Z}_\gamma^{\varepsilon=0}(u \bullet_p (\mu - \rho + \nu)) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$$

holds as long as all of its terms are defined.

We will next establish a certain "independence" of γ for the cycles $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$, in preparation for transferring these cycles to the Emerton-Gee stack.

Lemma 11.3.8 (Independence of γ)**.** *Let $\mu, w, \mu', w', \delta, \gamma, \gamma'$ be as in Proposition 11.2.3. Let $u \in \underline{\widetilde{W}}_1$ and $\xi \in \underline{C}_0$ such that $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi)$ and $\mathcal{Z}_{\gamma'}^{\varepsilon=1}(u \bullet_p \xi)$ are both defined. Then $\mathrm{tr}_{(w')^{-1} t^\delta w}(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi))$ is defined and*

$$\mathrm{tr}_{(w')^{-1} t^\delta w}(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi)) = \mathcal{Z}_{\gamma'}^{\varepsilon=1}(u \bullet_p \xi) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_{\gamma'}^{\varepsilon=1}). \tag{11.3.4}$$

Proof. Note that it is enough to check the equality in $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}$, since we have $\mathrm{Ch}_{\mathrm{top}} \hookrightarrow \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}$.

By definition, if $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi)$ is defined then $\mathfrak{sp}_\gamma^{\mathrm{ren}}(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi)) \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$ is defined and equals $t^{\xi+\rho} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)]$.

Similarly, if $\mathcal{Z}_{\gamma'}^{\varepsilon=1}(u \bullet_p \xi)$ is defined then $\mathfrak{sp}_{\gamma'}^{\mathrm{ren}}(\mathcal{Z}_{\gamma'}^{\varepsilon=1}(u \bullet_p \xi)) \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma'}^{\varepsilon=0})$ is defined and equals $t^{\xi+\rho} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)]$.

Taking λ and λ' so that $\sigma = F(\pi^{-1} u \bullet_p (\mu - \rho + \nu))$ is admissible for (γ, λ) and (γ', λ'), the partially defined map

$$\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda + \rho)) \xrightarrow{\mathrm{tr}_{(w')^{-1} t^\delta w}} \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma'}^{\varepsilon=1}(\le \lambda' + \rho))$$

includes $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi)$ in its domain, so $\mathrm{tr}_{(w')^{-1} t^\delta w}(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \xi))$ is defined in $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma'}^{\varepsilon=1}(\le \lambda' + \rho))$, and the equality (11.3.4) then follows from the previous paragraphs plus the commutativity of the diagram (11.2.3). □

11.4. **Breuil-Mézard relations on the model.** We will next verify a collection of relations on the models, which will later be seen to correspond to (11.0.1) under transfer$_\gamma$. In terms of Figure 1, we will show that the cycles $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ verify all the Breuil-Mézard relations which concern cycles contained in a single "chart" labeled by a single γ.

The most fundamental case is where $\lambda = 0$, corresponding to potentially Galois representations with minimal regular Hodge-Tate weights, which is handled by the Theorem below.

Theorem 11.4.1. *Suppose that $\mu \in \rho + \underline{C}_0$ is $2h_\rho$-generic. Let $\gamma = \gamma(w,\mu)$. Then we have*

$$\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\rho)] = \sum_\sigma [\overline{R(w,\mu)}\colon \sigma]\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}).$$

In this expression, we understand the summand to be 0 whenever $[\overline{R(w,\mu)}\colon \sigma] = 0$ (even when $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ is undefined – we are implicitly claiming that $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ is defined whenever $[R(w,\mu)\colon \sigma] \neq 0$).

Before giving the proof of Theorem 11.4.1 we record some representation-theoretic preliminaries. We will relate the multiplicities $[\overline{R(w,\mu)}\colon \sigma]$ to decomposition multiplicities for $\underline{G}_1\underline{T}$ using Jantzen's generic decomposition pattern (found in this generality in [GHS18, Proposition 10.1.2])[19]: if μ is $2h_\rho$-generic, then

$$\overline{R(w,\mu)} = \sum_{u\in \widetilde{\underline{W}}_1/\mathrm{X}^0(\underline{T})} \sum_{\nu \in \mathrm{X}^*(\underline{T})} [\widehat{Z}_1(\mu-\rho+p\rho)\colon \widehat{L}_1(u \bullet_p (\mu-\rho)+p\nu)] F(u \bullet_p (\mu - \rho + w\pi\nu)). \tag{11.4.1}$$

Let $m_{u,\nu}^\mu := [\widehat{Z}_1(\mu-\rho+p\rho)\colon \widehat{L}_1(u \bullet_p (\mu-\rho)+p\nu)]$. Note that by the translation principle, this is independent of μ as long as $\mu-\rho$ is regular in $\underline{C}_0$. In particular, we have

$$m_{u,\nu}^\mu = [\widehat{Z}_1(p\rho) : \widehat{L}_1(u \bullet_p 0 + p\nu)],$$

We also note for future use that

$$m_{\pi u,\pi\nu}^\mu = m_{u,\nu}^\mu, \tag{11.4.2}$$

since we have

$$\begin{aligned} m_{u,\nu}^\mu &= [\widehat{Z}_1(\mu-\rho+p\rho) : \widehat{L}_1(u \bullet_p (\mu-\rho)+p\nu)] \\ &= [\widehat{Z}_1(\pi(\mu-\rho)+p\rho) : \widehat{L}_1(\pi u \bullet_p \pi(\mu-\rho)+p\pi\nu)] \\ &= [\widehat{Z}_1(\mu-\rho+p\rho) : \widehat{L}_1(\pi u \bullet_p (\mu-\rho)+p\pi\nu)] = m_{\pi u,\pi\nu}^\mu, \end{aligned}$$

where on the third line we use the translation principle to replace $\pi(\mu-\rho)$ by $\mu-\rho$.

Lemma 11.4.2. *(1) If μ is m-generic for some $m \geq 2h_\rho$, then every σ such that $[\overline{R(w,\mu)}\colon \sigma] \neq 0$ is of the form $\sigma = F(\lambda)$ where $\lambda+\rho$ is $(m-h_\rho)$-generic.*

(2) If $\lambda+\rho$ is m-generic and $[\sigma(\tau)\colon F(\lambda)] \neq 0$ then $\sigma(\tau) = R(w,\mu)$ where μ is $(m-h_\rho)$-generic.

Proof. Part (1) follows from [LHLM23b, Proposition 2.3.7], and part (2) follows from [LHLM23b, Lemma 2.3.4]. ☐

Proof of Theorem 11.4.1. Since

$$\widetilde{\underline{W}} \bullet_p 0 = \bigcup_{u\in \widetilde{\underline{W}}_1/\mathrm{X}^0(\underline{T})} \bigcup_{\nu \in \mathrm{X}^*(\underline{T})} \{u \bullet_p 0 + p\nu\} \tag{11.4.3}$$

we have

$$\mathrm{SS}[\widehat{Z}_1(p\rho)] = \sum_{u\in \widetilde{\underline{W}}_1/\mathrm{X}^0(\underline{T})} \sum_{\nu \in \mathrm{X}^*(\underline{T})} m_{u,\nu}^\mu \, \mathrm{SS}[\widehat{L}_1(u \bullet_p 0 + p\nu)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.4}$$

By translation equivariance of SS, we have $\mathrm{SS}[\widehat{L}_1(u \bullet_p 0 + p\nu)] = t^\nu \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)]$, so we can rewrite (11.4.4) as

$$\mathrm{SS}[\widehat{Z}_1(p\rho)] = \sum_{u,\nu} m_{u,\nu}^\mu t^\nu \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.5}$$

[19] There is a discrepancy with the formula appearing in [GHS18, Proposition 10.1.2] because they use the dot action of the p-dilated extended affine Weyl group, while we use the p-dilated dot action of the extended affine Weyl group. Also note that their "$R(w,\mu+\rho)$" is our $R(w,\mu)$.

For later comparison, we reparametrize $u \mapsto \pi u$ and $\nu \mapsto \pi\nu$ so that inserting Theorem 9.4.1 into (11.4.5) gives

$$\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=0}(\rho)] = \sum_{u\in \widetilde{\underline{W}}_1/\mathrm{X}^0(\underline{T})} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m^\mu_{\pi u,\pi\nu} t^{\pi\nu} \cdot \mathrm{SS}[\widehat{L}_1(\pi u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.6}$$

Substituting the observation (11.4.2) that $m^\mu_{\pi u,\pi\nu} = m^\mu_{u,\nu}$ into (11.4.6) gives

$$\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=0}(\rho)] = \sum_{u\in \widetilde{\underline{W}}_1/\mathrm{X}^0(\underline{T})} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m^\mu_{u,\nu} t^{\pi\nu} \cdot \mathrm{SS}[\widehat{L}_1(\pi u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.7}$$

The genericity assumptions are such that Definition 11.3.6 says, using also (11.3.2),

$$\mathfrak{sp}_{\varepsilon\to 0}\mathcal{Z}_\gamma^{\varepsilon=1}(\pi u \bullet_p (\mu-\rho+w\pi\nu)) = t^{\pi\nu}\cdot \mathrm{SS}[\widehat{L}_1(\pi u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.8}$$

Using this we may rewrite (11.4.7) as

$$\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=0}(\rho)] = \sum_{u\in \widetilde{\underline{W}}_1/\mathrm{X}^0(\underline{T})} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m^\mu_{u,\nu} \mathfrak{sp}_{\varepsilon\to 0}\mathcal{Z}_\gamma^{\varepsilon=1}(\pi u \bullet_p (\mu-\rho+w\pi\nu)) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.9}$$

The assumptions imply that

$$\mathfrak{sp}_{\varepsilon\to 0}\colon \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq 2\rho)) \to \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}(\leq 2\rho))$$

is an isomorphism (cf. Lemma 4.2.3). Therefore each $\mathcal{Z}_\gamma^{\varepsilon=0}(\pi u \bullet_p (\mu-\rho+w\pi\nu))$ for which $m^\mu_{u,\nu}\neq 0$ lies in the domain where $\mathfrak{sp}_{\varepsilon\to 0}$ is an isomorphism, so we may apply $\mathfrak{sp}_{\varepsilon\to 0}^{-1}$ to (11.4.9) along with Proposition 4.2.1 (and also use that specializations in ε and p commute by Lemma 2.4.3) to find that[20]

$$\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\rho)] = \sum_{u\in \widetilde{\underline{W}}_1/p\mathrm{X}^0(\underline{T})} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m^\mu_{u,\nu} \mathcal{Z}_\gamma^{\varepsilon=1}(\pi u \bullet_p (\mu-\rho+w\pi\nu)). \tag{11.4.10}$$

Note that $\mathcal{Z}_\gamma^{\varepsilon=1}(\pi u \bullet_p (\mu-\rho+w\pi\nu)) = \mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ for $\sigma = F(u\bullet_p (\mu-\rho+w\pi\nu))$. Putting this into (11.4.10) completes the proof. $\square$

We will next show that our Breuil-Mézard cycles $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ satisfy the further relations expected from higher Hodge-Tate weights (the precise sense in which this is related to higher Hodge-Tate weights will be explained later in §11.6). We begin with a purely representation-theoretic lemma.

Lemma 11.4.3. *Assume $p\geq 2h_\rho$. Let $\lambda\in \mathrm{X}^*(\underline{T})^+$ be such that $\mu+\kappa\in\rho+\underline{C}_0$ is $2h_\rho$-generic for all weights κ of $W(\lambda)$. For $\nu\in\mathrm{X}^*(\underline{T})$, write $m_\kappa(\lambda)$ for the multiplicity of κ as weight of $W(\lambda)$. Then we have*

$$[\overline{W(\lambda)\otimes R(w,\mu)}] = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda)[\overline{R(w,\mu+\kappa)}] \in \mathrm{K}(\mathrm{Rep}_k(\underline{G}(\mathbf{F}_p))).$$

Proof. Given $\xi\in \mathrm{X}_1^*(\underline{T})$ such that $\xi+\kappa$ lies in the same alcove as ξ for all weights κ of $W(\lambda)$, [LLHLM20, Lemma 4.2.4] implies that we have

$$[W(\lambda)\otimes_k L(\xi)] = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda)[L(\xi+\kappa)] \in \mathrm{K}(\mathrm{Rep}_k(\underline{G}(\mathbf{F}_p))). \tag{11.4.11}$$

By (11.4.1) we have

$$[\overline{R(w,\mu)}] = \sum_{u\in\widetilde{\underline{W}}_1} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m_{u,\nu}[F(u\bullet_p(\mu-\rho+\nu))] \in \mathrm{K}(\mathrm{Rep}_k(\underline{G}(\mathbf{F}_p))),$$

where $m_{u,\nu} = [\widehat{Z}_1(\mu-\rho+p\rho) : \widehat{L}_1(u\bullet_p(\mu-\rho)+p\nu)]$ as before (recall that we saw it was independent of μ by the translation principle). Under our assumptions, $m_{u,\nu}\neq 0$ implies that $\xi := u\bullet_p(\mu-\rho+\nu)$ satisfies the condition needed to apply (11.4.11). We therefore have

$$[\overline{W(\lambda)\otimes R(w,\mu)}] = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda) \sum_{u\in\widetilde{\underline{W}}_1} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m_{u,\nu}[F(u\bullet_p(\mu-\rho+\nu)+\kappa)] \in \mathrm{K}(\mathrm{Rep}_k(\underline{G}(\mathbf{F}_p))). \tag{11.4.12}$$

[20]Here again, if $m^\mu_{u,\nu}=0$ then the corresponding summand is interpreted as 0, and we are implicitly asserting that $m^\mu_{u,\nu}=0$ if $\pi u\cdot(\mu-\rho+w\pi\nu)\notin \mathrm{X}_1^*(T)$ or if $\mathcal{Z}_\gamma^{\varepsilon=\eta}(u\bullet_p(\mu-\rho+w\pi\nu))$ is undefined.

Since the character of $W(\lambda)$ is invariant over W, we may rewrite

$$(11.4.12) = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda) \sum_{u\in \underline{\widetilde{W}}_1} \sum_{\nu\in \mathrm{X}^*(\underline{T})} m_{u,\nu}[F(u \bullet_p (\mu-\rho+\kappa+\nu))] \in \mathrm{K}(\mathrm{Rep}_k(\underline{G}(\mathbf{F}_p))). \tag{11.4.13}$$

Then we obtain the desired equality upon rearranging terms and applying (11.4.1).

□

The next theorem is the generalization of Theorem 11.4.1 that handles higher Hodge-Tate weights, although we will see that the proof is a reduction to Theorem 11.4.1.

Theorem 11.4.4 (Breuil-Mézard relations on the models)**.** *Let $\lambda \in \mathrm{X}^*(\underline{T})^+$. Suppose that μ is $(h_\lambda + 2h_\rho)$-generic. Let $\gamma = \gamma(w,\mu)$. Then we have*

$$\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\lambda+\rho)] = \sum_\sigma [W(\lambda)\otimes R(w,\mu) \colon \sigma]\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}). \tag{11.4.14}$$

Proof. By Lemma 4.2.3, the assumptions imply that

$$\mathfrak{sp}_{\varepsilon\to 0}\colon\ \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda+\rho)) \to \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}(\le \lambda+\rho))$$

is defined and is an isomorphism, so the identity can be checked after applying $\mathfrak{sp}_{\varepsilon\to 0}$. We first analyze what happens upon doing this to the LHS of (11.4.14). Applying Proposition 4.2.1 and then Theorem 9.2.1, we deduce that

$$\mathfrak{sp}_{\varepsilon\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\lambda+\rho)] = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda)\mathfrak{sp}_{p\to 0} t^\kappa \cdot [\underline{\mathrm{X}}_\gamma^{\varepsilon=0}(\rho)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.15}$$

Putting Theorem 11.4.1 into (11.4.15) yields

$$\mathfrak{sp}_{\varepsilon\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\lambda+\rho)] = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda) \sum_\sigma [\overline{R(w,\mu)} : \sigma] t^\kappa \cdot \mathcal{Z}_\gamma^{\varepsilon=0}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{11.4.16}$$

For $u \in \underline{\widetilde{W}}_1$ and $\nu \in \mathrm{X}^*(\underline{T})$, consider the contribution of $\sigma := F(u \bullet_p (\mu-\rho+w\pi\nu))$ on the RHS of (11.4.16). Set $\sigma' := F(u \bullet_p (\mu-\rho+\kappa+w\pi\nu))$. By construction we have $t^\kappa \cdot \mathcal{Z}_\gamma^{\varepsilon=0}(\sigma) = \mathcal{Z}_\gamma^{\varepsilon=0}(\sigma')$. Also we saw from the translation principle that $[\overline{R(w,\mu)} : \sigma] = [\overline{R(w,\mu+\kappa)} : \sigma']$, so we may rewrite (11.4.16) as

$$\mathfrak{sp}_{\varepsilon\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\lambda+\rho)] = \sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda) \sum_\sigma [R(w,\mu+\kappa) : \sigma']\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma'). \tag{11.4.17}$$

By Lemma 11.4.3, the coefficient of $\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma')$ in (11.4.17) is

$$\sum_{\kappa\in \mathrm{X}^*(\underline{T})} m_\kappa(\lambda)[\overline{R(w,\mu+\kappa)} : \sigma'] = [\overline{W(\lambda)\otimes R(w,\mu)} : \sigma'],$$

so that (11.4.17) agrees with

$$\sum_\sigma [\overline{W(\lambda)\otimes R(w,\mu)} \colon \sigma']\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma') \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}),$$

as desired.

□

11.5. **Breuil-Mézard cycles.** Here we will finally construct Breuil-Mézard cycles on the Emerton-Gee stack $\mathcal{X}^{\mathrm{EG}}$.

Composing Theorem 11.1.1 with the obvious embedding $\mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}_q}^{\lambda+\rho,\tau}) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}})$ gives a map

$$\mathrm{transfer}_\gamma\colon\ \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\le \lambda+\rho)) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}}).$$

We suppress the dependence of $\mathrm{transfer}_\gamma$ on λ, because it is independent of λ in the obvious sense whenever defined.

Lemma 11.5.1 (Independence of γ)**.** *For all γ such that $\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \kappa)$ is defined, the classes*

$$\mathrm{transfer}_\gamma(\mathcal{Z}_\gamma^{\varepsilon=1}(u \bullet_p \kappa)) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}})$$

coincide.

Proof. This follows from Proposition 11.1.3 and Lemma 11.3.8. $\square$

Definition 11.5.2 (Construction of Breuil-Mézard cycles)**.** Suppose that $\sigma = F(u \bullet_p \xi)$ occurs in a tame type $\tau = R(w, \mu)$ where μ is $2h_\rho$-generic (this is guaranteed whenever $\xi + \rho$ is $2h_\rho$-generic: up to adjusting u by an element in $\underline{\Omega}$, we can assume $u \in t^\rho \underline{\widetilde{W}}_{\mathrm{aff}}$ and we can then take $\nu = 0$ in Remark 11.3.4). Then the cycle $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1})$ is defined in Definition 11.3.6. Then we define

$$\mathcal{Z}^{\mathrm{EG}}(\sigma) := \mathrm{transfer}_\gamma(\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}}).$$

A priori this definition seems to depend on the choice of γ, but the independence of the choice of γ was established in Lemma 11.5.1.

11.6. **Proof of Theorem 11.0.1.** We now complete the proof of Theorem 11.0.1. Let $\lambda \in \mathrm{X}^*(\underline{T})^+$ and $\tau = \tau(w, \mu)$ be a lowest alcove presentation of a tame inertial parameter such that μ is $(2h_\rho + h_\lambda)$-generic.

Let $\gamma := \gamma(w, \mu)$. Then from Theorem 11.4.4 we have that

$$\mathfrak{sp}_{p \to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\lambda + \rho)] = \sum_\sigma n_\sigma(\lambda, \tau) \mathcal{Z}_\gamma^{\varepsilon=1}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}). \tag{11.6.1}$$

Consider applying $\mathrm{transfer}_\gamma$. Since $\mathrm{transfer}_\gamma(\mathfrak{sp}_{p \to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=0}(\lambda + \rho)]) = [\mathcal{X}^{\lambda+\rho,\tau}|_{\mathbf{F}_p}]$ by Theorem 11.1.1(1) and $\mathcal{Z}^{\mathrm{EG}}(\sigma) = \mathrm{transfer}_\gamma(\mathcal{Z}_\gamma^{\varepsilon=1})$ by definition, (11.6.1) becomes

$$[\mathcal{X}^{\mathrm{crys},\lambda+\rho,\tau}|_{\mathbf{F}_p}] = \sum_\sigma n_\sigma(\lambda, \tau) \mathcal{Z}^{\mathrm{EG}}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}}),$$

which is exactly what we wanted to show. $\square$

12. Complements on the Breuil-Mézard cycles

Here we establish some additional properties of the cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ produced in Theorem 11.0.1. In §12.1, we establish the second part of Theorem 1.3.1, asserting that if Conjecture 10.3.1 is true, then the "true" Breuil-Mézard cycles must agree with the cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ as soon as σ is sufficiently generic (which can be quantified effectively).

In §12.2, we show that the decomposition of Breuil-Mézard cycles into irreducible components is "the same" as the decomposition of the characteristic variety $\mathrm{SS}[\widehat{L}_1(\lambda)]$ into irreducible components. This fulfills Theorem 1.3.1(3).

In §12.3, we show that for sufficiently large p (which can again be quantified effectively), the cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ are effective. The proof is based on (!) the theory of quantum groups.

12.1. **Uniqueness.** First we investigate the uniqueness properties of the cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ constructed in the previous section. The main result of this subsection is:

Theorem 12.1.1. *If there are effective cycles $\mathcal{Z}(\sigma) \in \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}})$ satisfying Conjecture 10.3.1, then $\mathcal{Z}(\sigma)$ agrees with the $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ from Definition 11.5.2 whenever $\sigma = F(\lambda)$ such that λ is $6h_\rho$-generic.*

Remark 12.1.2. The proof of Theorem 12.1.1 only uses the following "bounded support" property of $\mathcal{Z}(\sigma)$: if $[\overline{R(w,\mu)} \colon \sigma] \neq 0$ for some $2h_\rho$-generic μ, then the support of $\mathcal{Z}(\sigma)$ is contained in $\mathcal{X}^{\mathrm{crys},\rho,\tau}|_{\mathbf{F}_p}$. This property is clearly implied by effectivity. Note that the $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ constructed by Theorem 11.0.1 have this bounded support property, but are not obviously effective.

12.1.1. *Reformulation in equivariant homology.* For each $\tau = \tau(w, \mu)$ and $\gamma = \gamma(w, \mu)$ such that μ is $2h_\rho$-generic, Theorem 11.1.1 gives an isomorphism

$$\mathrm{transfer}_\gamma^{-1} \colon \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}_p}^{\leq \rho, \tau}) \cong \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq \rho)),$$

which we also think as a partially defined (inverse) transfer map on $\mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}})$. Thus applying $\mathrm{transfer}_\gamma^{-1}$ to the cycles $\mathcal{Z}(\sigma)$ from the hypothesis of Theorem 12.1.1 gives a collection of cycles $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)^\dagger \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq \rho)$ for each $\sigma \in \mathrm{JH}(R(w, \mu))$. Recall the renormalized specialization map from Definition 11.2.1,

$$\mathfrak{sp}_\gamma^{\mathrm{ren}} \colon \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq \rho)) \cong \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}(\leq \rho)) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}).$$

Then we get cycles $\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma)^\dagger := \mathfrak{sp}_\gamma^{\mathrm{ren}}(\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)^\dagger) \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$. On the other hand, the Breuil-Mezard cycles $\mathcal{Z}_\gamma^{\varepsilon=1}(\sigma)$ we constructed in Definition 11.3.6 were characterized by the property (cf. also Definition 11.3.1)

$$\mathfrak{sp}_\gamma^{\mathrm{ren}}\left(\mathcal{Z}_\gamma^{\varepsilon=1}(F(\pi^{-1}u \bullet_p \xi))\right) = t^{\xi+\rho} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}).$$

Therefore Theorem 12.1.1 is equivalent to the statement

$$\mathcal{Z}_\gamma^{\varepsilon=0}(F(\pi^{-1}u \bullet_p \xi))^\dagger = t^{\xi+\rho} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)] \in \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0}). \tag{12.1.1}$$

Now recall that the spaces $\mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$ (resp. $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$, $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM},\underline{\check{T}}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=0})$) are canonically identified via (11.2.1) as we vary $\gamma = \gamma(w,\mu)$ over possible choices of (w,μ). It follows from Proposition 11.1.3 and Proposition 11.2.3 that under these identifications $\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma)^\dagger$ is independent of the choice of γ.

12.1.2. *Equivariant support bounds.* We establish some technical statements on the equivariant support of $\mathcal{Z}_\gamma^{\varepsilon=0}(F(\pi^{-1}u \bullet_p \xi))^\dagger$ for later use.

Lemma 12.1.3. *Assume all types containing $\sigma = F(\pi^{-1}u \bullet_p \xi)$ are $2h_\rho$-generic. Then $\mathrm{Loc}^{\underline{\check{T}}}(\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma)^\dagger_{\underline{\check{T}}})$ has equivariant support in $t^\xi(\underline{\widetilde{W}}_{\leq w_0 u})$.*

In particular, the lemma applies whenever $\xi + \rho$ is $3h_\rho$-generic.

Proof. The tame types $\sigma(\tau) = R(w,\mu)$ containing σ are exactly those such that for $\kappa := \mu - \rho - \xi$, the triple $(u, w, -\kappa)$ is admissible for 0 in the sense of Definition 11.3.3, i.e.,

$$(t^{w^{-1}\kappa})_{\mathrm{dom}} = (t^\kappa w)_{\mathrm{dom}} \leq w_0 t^{-\rho} u.$$

It follows from the definitions that any class in $\mathfrak{sp}_{\gamma(w,\mu)} \circ \mathrm{transfer}_{\gamma(w,\mu)}^{-1}\left(\mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}_p}^{\leq \rho,\tau})\right)$ has equivariant support in $t^\mu w \cdot \mathrm{Adm}(\rho)$. Hence (using the effectivity assumption) we learn that the equivariant support of $\mathcal{Z}_\gamma^{\varepsilon=0}(\sigma)^\dagger_{\underline{\check{T}}}$ belongs to

$$\bigcap_{\substack{\kappa \in \mathrm{X}^*(\underline{T}),\ w \in \underline{W}: \\ (t^\kappa w)_{\mathrm{dom}} = w_0 t^{-\rho} u}} t^{\xi+\rho} t^\kappa w \cdot \mathrm{Adm}(\rho).$$

But Lemma 12.1.4 below shows this intersection is exactly $t^{\xi+\rho}\underline{\widetilde{W}}_{\leq w_0 u}$. □

Lemma 12.1.4. *Let $u \in \underline{\widetilde{W}}_1$. Then we have*

$$\bigcap_{\substack{\kappa \in \mathrm{X}^*(\underline{T}),\ w \in \underline{W}: \\ (t^\kappa w)_{\mathrm{dom}} = w_0 t^{-\rho} u}} t^\kappa w\, \mathrm{Adm}(\rho) = \underline{\widetilde{W}}_{\leq w_0 u}.$$

Proof. Suppose $\widetilde{x}$ belong to the LHS. Since $(\underline{W} t^\kappa w)_{\mathrm{dom}} = (t^\kappa w)_{\mathrm{dom}}$, we see that

$$(w_0 t^{-\rho} u)^{-1} \sigma \widetilde{x} \in \mathrm{Adm}(\rho)$$

for all $\sigma \in \underline{W}$. In particular, for an appropriate choice of σ, we get

$$(w_0 t^{-\rho} u)^{-1} \sigma \widetilde{x} = (w_0 t^{-\rho} u)^{-1} w_0 \widetilde{y}$$

with $\widetilde{y} \in \underline{\widetilde{W}}^+$. But now [LHLM23b, Proposition 2.1.6 (4)] implies that $\widetilde{y} \leq u$, hence $\widetilde{x} = \sigma^{-1} w_0 \widetilde{y} \leq w_0 u$.

Conversely, if $\widetilde{x} \in \underline{\widetilde{W}}_{\leq w_0 u}$, then for a suitable $\sigma \in \underline{W}$ we have

$$(t^\kappa w)^{-1} \widetilde{x} = ((t^\kappa w)_{\mathrm{dom}})^{-1} \sigma \widetilde{x} \leq (w_0 t^{-\rho} u)^{-1} (w_0 u) = u^{-1} t^\rho u$$

since $(t^\kappa w)_{\mathrm{dom}} \leq w_0 t^{-\rho} u$, $\sigma \widetilde{x} \leq w_0 u$, and the second-to-last product is a reduced factorization by [LHLM23b, Lemma 2.1.4]. □

12.1.3. *A reconstruction algorithm.* The idea to prove (12.1.1) is that we should be able to reconstruct each side from the Breuil-Mézard relations. Here we explain how to carry out this reconstruction process.

Situation 12.1.5. Suppose we have a collection of cycles $Z(u,\xi) \in \mathrm{H}^{\mathrm{BM},\underline{\check{T}}}_{\mathrm{top}}(\underline{\mathrm{Y}}^{\varepsilon=0}_{\gamma})$ parametrized by a subset of $\underline{\widetilde{W}}_1^+ \times \underline{C}_0$ and an integer h such that:

(1) For any (w,μ) such that μ is $2h_\rho$-generic, we have for some $m^{\mu,w}_{u,\nu} \in \mathbb{S}$,
$$t^\mu w \cdot \mathrm{SS}[\widehat{Z}_1(p\rho)]_{\underline{\check{T}}} = \sum_{u,\nu} m^{\mu,w}_{u,\nu} Z(u, \mu - \rho + w\nu) \tag{12.1.2}$$
where
- The sum runs over $(u,\nu) \in \underline{\widetilde{W}}_1 \times \mathrm{X}^*(\underline{T})$ such that $(t^{-\nu})_{\mathrm{dom}} \leq w_0 t^{-\rho} u$, and each term in the sum is defined.
- When $(t^{-\nu})_{\mathrm{dom}} = t^\rho w_0 u$, we have $m^{\mu,w}_{u,\nu} = 1$.

(2) For every u and every h-generic ξ, $Z(u,\xi)$ is defined and has equivariant support in $t^{\xi+\rho} \cdot \underline{\widetilde{W}}_{\leq w_0 u}$.

Proposition 12.1.6. *In Situation 12.1.5, for any ξ which is $(h+3h_\rho)$-generic, $Z(u,\xi)$ is uniquely determined.*

Proposition 12.1.6 essentially follows from the recursive algorithm to compute generic Breuil-Mezard cycles in terms of Emerton-Gee stacks in [LHLM23b, §8.6.1]. For the convenience of the reader, we will adapt this algorithm to the more combinatorial setting of equivariant homology. This gets rid of the inputs from patching in *loc.cit.*, and our reformulation should be more practical for computer implementation.

By item (2), it is enough to compute the component of $\mathrm{Loc}^{\underline{\check{T}}}(Z(u,\xi))$ at each $t^\xi \widetilde{x} \in t^\xi \underline{\widetilde{W}}_{\leq w_0 u}$. If C is an equivariant cycle class and $\widetilde{z} \in \underline{\widetilde{W}}$, we will use the short hand $m_{\widetilde{z}}(C)$ for the component of $\mathrm{Loc}^{\underline{\check{T}}}(C)$ at $\widetilde{z}$.

Our algorithm will be based on recursion for the following notion of defect.

Definition 12.1.7. Let $\widetilde{z} = t^\xi \widetilde{x} \in \underline{\widetilde{W}}$ where $\widetilde{x} \leq w_0 t^{-\rho} u$. The *defect* of $\widetilde{z}$ with respect to $Z(u,\xi)$ is defined to be $\delta_{\widetilde{z}}(u,\xi) := \ell(u) - \ell(\widetilde{x}_{\mathrm{dom}}) \geq 0$.

We have the following key recursion relation, which expresses $m_{\widetilde{z}}(Z(u,\xi))$ in terms of the $m_{\widetilde{z}}$ for lower defect situations:

Lemma 12.1.8. *Suppose we are given $\widetilde{z} \in \underline{\widetilde{W}}$, $(u,\xi) \in \underline{\widetilde{W}}_1 \times \underline{C}_0$ such that $\widetilde{z} \in t^{\xi+\rho}\underline{\widetilde{W}}_{\leq w_0 u}$. Write*
$$\widetilde{z} = t^{\xi+\rho}\sigma w_0 \widetilde{x} \quad \textit{with} \quad \sigma \in \underline{W}, \quad \widetilde{x} \in \underline{\widetilde{W}}^+.$$
Assume $\widetilde{z}(0)$ is $(\max\{h,h_\rho\}+2h_\rho)$-generic.

Let $\kappa \in \mathrm{X}^*(\underline{T})$ and $w \in \underline{W}$ be such that
$$\sigma w_0 t^{-\rho} u = t^\kappa w$$
and set $\mu := \xi + \kappa - \rho$. Then
$$m_{\widetilde{z}}(t^\mu w \cdot \mathrm{SS}[\widehat{Z}_1(p\rho)]_{\underline{\check{T}}}) = m_{\widetilde{z}}(Z(u,\xi)) + \sum_{u',\xi'} m_{\widetilde{z}}(Z(u',\xi'))$$
where the sum runs over $(u',\xi') \in \underline{\widetilde{W}}_1 \times \underline{C}_0$ such that $\widetilde{z} \in t^{\xi'+\rho}\underline{\widetilde{W}}_{\leq w_0 u'}$ and $\delta_{\widetilde{z}}(u',\xi') < \delta_{\widetilde{z}}(u,\xi)$.

Proof. Note that
$$w^{-1}(-\mu + \widetilde{z}(0)) = (t^{\xi+\kappa}w)^{-1}\widetilde{z}(0) = (w_0 t^{-\rho}u)^{-1}(t^{\xi+\rho}\sigma)^{-1}t^{\xi+\rho}\sigma w_0 \widetilde{x}(0) = (w_0 t^{-\rho} u)^{-1} w_0 \widetilde{x}(0).$$
By [LHLM23b, Proposition 2.1.6], we have $(w_0 t^{-\rho} u)^{-1} w_0 \widetilde{x} \in \mathrm{Adm}(\rho)$, hence $\mu - \widetilde{z}(0)$ is h_ρ-small. It follows that μ is $2h_\rho$-generic. In particular, we have equation (12.1.2) for our choice of (w,μ).

We now observe:

(1) $Z(u,\xi)$ contributes to the right-hand side of (12.1.2) with coefficient $m_{u,-w^{-1}\kappa} = 1$. This is because $\xi = \mu - \rho - \kappa$ and
$$(t^\kappa w)_{\mathrm{dom}} = (t^{w^{-1}\kappa})_{\mathrm{dom}} = w_0 t^{-\rho} u.$$

(2) Any pair (u',ν') contributing to (12.1.2) for our choice (w,μ) has the property that $\xi' = \mu - \rho + w\nu'$ is h-generic. This is because $(t^{-\nu'})_{\mathrm{dom}} \leq w_0 t^{-\rho} u'$ belongs to $\underline{\widetilde{W}}_1$, hence $w\nu' \in -\underline{W}(t^{-\nu'}_{\mathrm{dom}})(0)$ is h_ρ-small.

The second item in particular implies that the equivariant support of any contributing $Z(u',\xi')$ belongs to $t^{\xi'+\rho}\widetilde{\underline{W}}_{\leq w_0 u'}$.

It remains to check $\delta_{\widetilde{z}}(u',\xi') < \delta_{\widetilde{z}}(u,\xi)$ when $(u',\xi') \neq (u,\xi)$. Assume $\widetilde{z}$ belongs to the equivariant support of some $Z(u',\xi')$ (thus $\xi' = \xi + \kappa + w\nu'$). Then we can write

$$\widetilde{z} = t^{\xi'+\rho}\widetilde{y}$$

with $\widetilde{y} \leq w_0 u'$. In particular, $\widetilde{y}_{\mathrm{dom}} \leq u'$, and $\delta_{\widetilde{z}}(u',\xi') = \ell(u') - \ell(\widetilde{y}_{\mathrm{dom}})$.

We have

$$t^{\xi}\sigma w_0 \widetilde{x} = t^{\xi+\kappa+w\nu'}\widetilde{y} = t^{\xi}t^{\kappa}wt^{\nu'}w^{-1}\widetilde{y} = t^{\xi}\sigma w_0 t^{-\rho}ut^{\nu'}w^{-1}(\widetilde{y})$$

and

$$(w_0 t^{-\rho}u)^{-1}w_0\widetilde{x} = t^{\nu'}w^{-1}\widetilde{y} = ((t^{-\nu'})_{\mathrm{dom}})^{-1}\sigma'\widetilde{y}_{\mathrm{dom}} \tag{12.1.3}$$

for some $\sigma' \in \underline{W}$.

Now by [LHLM23b, Lemma 2.1.4], we have

$$\ell((w_0 t^{-\rho}u)^{-1}w_0\widetilde{x}) = \ell((w_0 t^{\rho}u)^{-1}w_0) + \ell(\widetilde{x})$$

and

$$\ell(t^{\rho}) = \ell(u^{-1}t^{\rho}u) = \ell((w_0 t^{\rho}u)^{-1}w_0 u) = \ell((w_0 t^{\rho}u)^{-1}w_0) + \ell(u)$$

so that

$$\ell((w_0 t^{-\rho}u)^{-1}w_0\widetilde{x}_{\mathrm{dom}}) = \ell(t^{\rho}) - \delta_{\widetilde{z}}(u,\xi).$$

But this quantity also equals

$$\begin{aligned}\ell((t^{-\nu'}_{\mathrm{dom}})^{-1}\sigma'\widetilde{y}_{\mathrm{dom}}) &\leq \ell((t^{-\nu'})_{\mathrm{dom}}) + \ell(\sigma') + \ell(\widetilde{y}_{\mathrm{dom}}) \\ &\leq \ell(w_0 t^{-\rho}u') + \ell(w_0) + \ell(u') - \delta_{\widetilde{z}}(u',\xi') = \ell(t^{\rho}) - \delta_{\widetilde{z}}(u',\xi').\end{aligned}$$

We conclude that $\delta_{\widetilde{z}}(u',\xi') \leq \delta_{\widetilde{z}}(u,\xi)$. If equality occurs, then we must have

$$(t^{-\nu'})_{\mathrm{dom}} = t^{\rho}w_0 u', \quad \sigma' = w_0.$$

But applying [LHLM23b, Proposition 2.1.5], to the leftmost and rightmost factorization in equation (12.1.3) forces $u' = u$ and hence $(t^{-\nu'})_{\mathrm{dom}} = (t^{w^{-1}\kappa})_{\mathrm{dom}} = w_0 t^{-\rho}u$. But this implies

$$\nu' = (w_0 t^{-\rho}u)^{-1}(0) = -w^{-1}\kappa$$

so $\xi' = \xi + \kappa + w\nu' = \xi$.

∎

Corollary 12.1.9. *In Situation 12.1.5, the quantity $m_{\widetilde{z}}(Z(u,\xi))$ is uniquely determined whenever*

- *$\widetilde{z} \in \widetilde{\underline{W}}$ satisfies $\widetilde{z}(0)$ is $(\max\{h,h_\rho\}+2h_\rho)$-generic, and*
- *$(u,\xi) \in \widetilde{\underline{W}}_1 \times \underline{C}_0$ satisfies $\widetilde{z} \in t^{\xi+\rho}\widetilde{\underline{W}}_{\leq w_0 u}$.*

Proof. Lemma 12.1.8 gives a recursive formula for $m_{\widetilde{z}}(Z(u,\xi))$ with respect to the defect $\delta_{\widetilde{z}}(u,\xi) \geq 0$. ∎

Proof of Proposition 12.1.6. For any (u,ξ) such that $\xi+\rho$ is $(\max\{h,h_\rho\}+3h_\rho)$-generic, we know the equivariant support of $Z(u,\xi)$ is bounded by $t^{\xi+\rho}\widetilde{\underline{W}}_{\leq w_0 u}$. But if $\widetilde{z} \in t^{\xi+\rho}\widetilde{\underline{W}}_{\leq w_0 u}$, then $\xi+\rho-\widetilde{z}(0)$ is h_ρ-small, so $\widetilde{z}(0)$ is $(\max\{h,h_\rho\}+2h_\rho)$-generic. Thus Corollary 12.1.9 shows that each $m_{\widetilde{z}}(Z(u,\xi))$ are uniquely determined, hence so is $Z(u,\xi)$. ∎

12.1.4. *Proof of Theorem 12.1.1.* After the reformulation in §12.1.1, we need to check the equation

$$\mathcal{Z}^{\varepsilon=0}_{\gamma'}(F(\pi^{-1}u \bullet_p \xi))^{\dagger}_{\underline{\check{T}}} = t^{\xi+\rho}\cdot \mathrm{SS}[\widehat{L}_1(u\bullet_p 0)]_{\underline{\check{T}}}$$

whenever ξ is $6h_\rho$-generic.

We are in Situation 12.1.5 for the cycles $Z(u,\xi) := \mathcal{Z}^{\varepsilon=0}_{\gamma}(F(\pi^{-1}u\bullet_p\xi))^{\dagger}_{\underline{\check{T}}}$ with the choice of $h = 3h_\rho$ by Lemma 12.1.3. On the other hand, we are also in Situation 12.1.5 for the cycles $Z'(u,\xi) := t^{\xi+\rho}\cdot\mathrm{SS}[\widehat{L}_1(u\bullet_p 0)]_{\underline{\check{T}}}$ and the choice $h=0$, with the same structure constants $m^{\mu;w}_{u;\nu}$. Thus Proposition 12.1.6 applies and gives the desired equality.

12.2. **Decomposition of Breuil-Mézard cycles.** Let $\mathcal{X}^{\mathrm{EG}}$ be the Emerton-Gee stack for $G = \mathrm{GL}_n /\mathbf{Q}_q$. Emerton-Gee have explicitly described the irreducible components of $\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}$. They show that $\mathcal{X}^{\mathrm{EG}}_{\mathrm{red}}$ is equidimensional, and biject the (top-dimensional) irreducible components with Serre weights (these results can be found in [EG23, Theorem 1.2.1, Theorem 6.5.1, Errata]); for a Serre weight σ we denote the corresponding irreducible component by $\mathcal{C}_\sigma$.

It is an interesting problem to express $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ in terms of the irreducible component basis $[\mathcal{C}_\sigma]$ of $\mathrm{Ch}_{\mathrm{top}}(\mathcal{X}^{\mathrm{EG}})$. One expects that

$$\mathcal{Z}^{\mathrm{EG}}(\sigma) = [\mathcal{C}_\sigma] = \sum_{\sigma'} n_{\sigma\sigma'}[\mathcal{C}_{\sigma'}]$$

where the sum runs over σ' which are "less than" σ in some natural partial order, and $n_{\sigma\sigma'} = 1$. For $G = \mathrm{GL}_2 /\mathbf{Q}_q$, it turns out that $\mathcal{Z}^{\mathrm{EG}}(\sigma) = \mathcal{C}_\sigma$ except when σ is a Steinberg weight, which is not generic enough to be covered by our theory. However, numerical computations suggest that even for very generic σ, $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ can be reducible already for $n = 4$ [LHLM23b, Remark 1.5.11]. We hazard a guess that as n increases, the number of irreducible components of $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ is unbounded, even when quantified over very generic σ.

In general, if $X_1, \dots, X_m$ are the top-dimensional irreducible components of X (assumed to be equidimensional), then

$$\mathrm{Ch}_{\mathrm{top}}(X) \cong \bigoplus_{i=1}^{m} \mathbf{Q}[X_i].$$

Write $\mathrm{mult}(\alpha : [X_i])$ for the coefficient of $[X_i]$ in α. We say $\alpha \in \mathrm{Ch}_{\mathrm{top}}(X)$ is *effective* if each $\mathrm{mult}(\alpha : [X_i]) \geq 0$.

By Theorem 11.1.1(2) and the definition of $\mathcal{Z}^{\mathrm{EG}}(\sigma)$, the decomposition of $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ into the $[\mathcal{C}_{\sigma'}]$ is the same as the decomposition of the $\mathcal{Z}^{\varepsilon=1}_\gamma(\sigma)$ into the irreducible components $\mathrm{Y}^{\varepsilon=1}_\gamma(\widetilde{w})$. More precisely, if σ' and $\widetilde{w}$ are related as in Theorem 11.1.1(2), then we have

$$\mathrm{mult}(\mathcal{Z}^{\mathrm{EG}}(\sigma) \colon \mathcal{C}_{\sigma'}) = \mathrm{mult}\left(\mathcal{Z}^{\varepsilon=1}_\gamma(\sigma) \colon \mathrm{Y}^{\varepsilon=1}_\gamma(\widetilde{w})\right). \tag{12.2.1}$$

Proposition 12.2.1. *Let $\tau = \tau(w, \mu)$ be a lowest alcove presentation such that μ is $3h_\rho$-generic and $\mathcal{Z}^{\varepsilon=1}_\gamma(u \bullet_p \xi)$ is defined for $\gamma := \gamma(w, \mu)$. Then we have*

$$\mathrm{mult}\left(\mathcal{Z}^{\varepsilon=1}_\gamma(u \bullet_p \xi) : [\mathrm{Y}^{\varepsilon=1}_\gamma(\widetilde{w})]\right) = \mathrm{mult}\left(\mathrm{SS}[\widehat{L}_1(u \bullet_p 0)] : t^{\mu-\xi-\rho} w \cdot [\mathrm{Y}^{\varepsilon=0}_\gamma(\widetilde{w})]\right).$$

Proof. By definition we have

$$\mathfrak{sp}_{\varepsilon\to 0} \mathcal{Z}^{\varepsilon=1}_\gamma(u \bullet_p \xi) = w^{-1} t^{-\mu+\xi+\rho} \cdot \mathrm{SS}[\widehat{L}_1(u \bullet_p 0)],$$

and by Theorem 5.1.1 we have

$$\mathfrak{sp}_{\varepsilon\to 0}[\mathrm{Y}^{\varepsilon=1}_\gamma(\widetilde{w})] = [\mathrm{Y}^{\varepsilon=0}_\gamma(\widetilde{w})].$$

Comparing these gives the result. □

Proposition 12.2.1 and (12.2.1) equate the problem of finding the decomposition of Breuil-Mézard cycles (in the regime of generic σ) with the problem of finding the decomposition of $\mathrm{SS}[\widehat{L}_1(\lambda)]$ into irreducible components. This latter problem is an affine analogue of the famous representation-theoretic problem of calculating the characteristic variety of simple representations of $\mathfrak{g}_{\mathbf{C}}$, studied for example in [KS97, Wil15].

12.3. **Effectivity.** Next we prove that the cycles $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ are effective under the assumption that p is very large.

Theorem 12.3.1. *Assume that Lusztig's Conjecture holds for G at p. (This is satisfied for all p larger than an explicit bound, e.g., the quantity "$U(\widehat{w}_0)$" in* [Fie12, §1.3]*.) Then the $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ produced by Theorem 11.0.1 are effective.*

The proof shows a more general statement: under the hypothesis of Lusztig's Conjecture, the cycles $\mathcal{Z}^{\varepsilon=1}_\gamma(\sigma)$ (which were defined for any G with simply connected derived subgroup) are effective. For the proof we may and do return to the situation where G is split reductive over $\mathbf{F}_p$.

The assumption on Lusztig's conjecture is an artefact of our argument, which goes by comparison with the quantum group. The point is that our Breuil-Mézard cycles are constructed from the microlocal support of (derived) D-modules, so effectivity comes from knowing that the relevant complexes of D-modules actually lie in the heart. To control this, we need knowledge about the categorification of the map in Proposition 8.0.1. As that categorification occurs over $\mathbf{C}$, it can only be compared to representation theory over $\mathbf{C}$. The quantum group at a pth root of unity provides a $\mathbf{C}$-linear category of representations that resembles, at the level of Grothendieck groups, the representation theory of a Lie algebra in characteristic p, and enters the argument for that reason.

We comment in Remark 12.3.8 on how improvements in modular representation theory, which may be expected in the not-too-distant future, should prove the effectivity under the assumption that p exceeds an explicit linear bound on h_ρ, by a similar argument.

12.3.1. *Brief guide to quantum groups.* For the number theorist's convenience, we begin with a brief introduction to quantum groups. Let G be a split reductive group and $\mathfrak{g}_{\mathbf{C}}$ be its Lie algebra over $\mathbf{C}$. The quantum group associated to $\mathfrak{g}_{\mathbf{C}}$ is a Hopf algebra over $\mathbf{C}[v, v^{-1}]$, which is roughly a 1-parameter deformation (in the parameter v) of the universal enveloping algebra $\mathcal{U}\mathfrak{g}_{\mathbf{C}}$. For our purposes it is useful to think of this deformation as analogous to the "1-parameter deformation" $\mathcal{U}\mathfrak{g}_{\mathbf{Z}}$.

In fact there are two standard forms of a quantum group associated to $\mathfrak{g}_{\mathbf{C}}$, one called the *de Concini–Kac* form and denoted $\mathfrak{U}_v\mathfrak{g}$, and the other called *Lusztig's form* and denoted $\mathsf{U}_v\mathfrak{g}$ (we will only consider quantum groups over $\mathbf{C}$ and so suppress this subscript). These two forms are analogous to the two forms of enveloping algebras for Lie algebras over $\mathbf{Z}$: the universal enveloping algebra, and the hyperalgebra (i.e., the dual to the ring of functions on G). Either form of the quantum group can be "specialized" to any $q \in \mathbf{C}^\times$ by sending $v \mapsto q$; we denote these specializations by $\mathfrak{U}_q\mathfrak{g}$ and $\mathsf{U}_q\mathfrak{g}$. Just as the natural map from the universal enveloping algebra to the hyperalgebra is an isomorphism in characteristic zero, there is a natural map $\mathfrak{U}_q\mathfrak{g} \to \mathsf{U}_q\mathfrak{g}$, which is an isomorphism for generic q (i.e., when q is not a root of unity), and far from an isomorphism when q is a root of unity.

We continue to assume that p is larger than the Coxeter number of any simple factor of G, and also that $p > 3$. Let ζ be a primitive pth root of unity. Going forward, it will be more convenient for us to use the de Concini–Kac form, as its specialization to ζ, denoted $\mathfrak{U}_\zeta\mathfrak{g}$, is more analogous to $\mathcal{U}\mathfrak{g}_{\mathbf{F}_p}$. As the reader will see, our proof of Theorem 12.3.1 rests on the strength of the analogy between $\mathfrak{U}_\zeta\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}_{\mathbf{F}_p}$. (The number theorist might think of the analogy between number fields and function fields as a metaphor for this analogy.)

12.3.2. *Center of the quantum group.* We will now explain the structure of the center of the quantum group $\mathfrak{U}_\zeta\mathfrak{g}$, which should be compared to §7.2.

Under the assumptions on p, de Concini – Kac calculated $Z(\mathfrak{U}_\zeta\mathfrak{g})$. Reminiscent of (7.2.1), it has the form $Z(\mathfrak{U}_\zeta\mathfrak{g}) = {}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{HC}} \otimes_{{}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{HC}} \cap {}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{Fr}}} {}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{Fr}}$, where the subalgebra ${}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{HC}} \hookrightarrow Z(\mathfrak{U}_\zeta\mathfrak{g})$ is the *(quantum) Harish-Chandra center* and the subalgebra ${}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{Fr}}$ is the *(quantum) Frobenius center* (cf. [Tan16, p.48]). Hence a character of $Z(\mathfrak{U}_\zeta\mathfrak{g})$ is given by a compatible pair (λ, χ) where λ is a character of ${}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{HC}}$ and χ is a character of ${}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{Fr}}$. For such a compatible pair, we define:

$$\mathfrak{U}_\zeta\mathfrak{g}^\lambda := \mathfrak{U}_\zeta\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \lambda, \quad \mathfrak{U}_\zeta\mathfrak{g}_\chi := \mathfrak{U}_\zeta\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} \chi, \quad \mathfrak{U}_\zeta\mathfrak{g}_\chi^\lambda := \mathfrak{U}_\zeta\mathfrak{g} \otimes_{Z(\mathcal{U}\mathfrak{g})} (\lambda, \chi).$$

We also make the following definitions, recalling that "Rep" always means the category of finitely generated representations.

- Define $\mathrm{Rep}^\lambda(\mathfrak{U}_\zeta\mathfrak{g})$ to be the full subcategory of $\mathrm{Rep}(\mathfrak{U}_\zeta\mathfrak{g})$ where ${}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{HC}}$ acts with *generalized* eigenvalue λ.
- Define $\mathrm{Rep}_\chi(\mathfrak{U}_\zeta\mathfrak{g})$ to be the full subcategory of $\mathrm{Rep}(\mathfrak{U}_\zeta\mathfrak{g})$ where ${}^{\mathsf{q}}\mathfrak{Z}_{\mathrm{Fr}}$ acts with *generalized* eigenvalue χ.
- Define $\mathrm{Rep}^\lambda(\mathfrak{U}_\zeta\mathfrak{g}_\chi) := \mathrm{Rep}^\lambda(\mathcal{U}\mathfrak{g}) \cap \mathrm{Rep}(\mathfrak{U}_\zeta\mathfrak{g}_\chi)$, and $\mathrm{Rep}_\chi^\lambda(\mathfrak{U}_\zeta\mathfrak{g}) := \mathrm{Rep}^\lambda(\mathfrak{U}_\zeta\mathfrak{g}) \cap \mathrm{Rep}_\chi(\mathfrak{U}_\zeta\mathfrak{g})$, etc.

When T preserves λ and χ, we have the categories of "graded" representations $\mathrm{Rep}_\chi(\mathfrak{U}_\zeta\mathfrak{g}^\lambda, T)$, etc. defined analogously to §7.4 (see [Tan22, §7.2] for the definitions).

Example 12.3.2 (Quantum simple representations)**.** Take $\chi = 0$. The simple objects of $\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T)$ are in bijection with $\widetilde{W}$, where $\widetilde{w} \in \widetilde{W}$ corresponds to ${}^{\mathsf{q}}\widehat{L}(\widetilde{w} \bullet_p 0)$, the simple representation with highest weight $\widetilde{w} \bullet_p 0 \in X^*(T)$.

Example 12.3.3 (Quantum baby Verma modules)**.** Take $\chi = 0$ and $\lambda \in \widetilde{W} \bullet_p 0$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of B. There is an associated *baby Verma module*

$$\widehat{Z}_{\mathfrak{b}}(\lambda) := \mathfrak{U}_\zeta\mathfrak{g}_0 \otimes_{\mathfrak{U}_\zeta\mathfrak{b}} d\lambda$$

where $d\lambda$ has graded weight λ and the universal enveloping algebras are equipped with their natural gradings.

12.3.3. *Comparison with modular representation theory.* Take χ and λ to be the trivial characters. In this case, the quantum analogue of BMR localization[21] [Tan22, §1.9] is an equivalence

$$
{}^{\mathfrak{q}}\gamma_0^0 \colon D^b(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T)) \cong D^b(\mathrm{Coh}^{T_{\mathbf{C}}}_{\mathcal{B}_{\mathbf{C}}}(\widetilde{\mathcal{N}}_{\mathbf{C}})) \tag{12.3.1}
$$

which is exact with respect to the usual t-structure on the LHS and the exotic t-structure on the RHS. It induces an isomorphism

$$
\mathrm{K}(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T)) \xrightarrow{\sim} \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}_{\mathcal{B}_{\mathbf{C}}}(\widetilde{\mathcal{N}}_{\mathbf{C}})) \cong \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}}))
$$

which is equivariant for the action of $\mathrm{Rep}(T_{\mathbf{C}})$, and sends

$$
[{}^{\mathfrak{q}}\widehat{Z}_{w_0\mathfrak{b}}(p\rho)] \mapsto t^{-\rho} \cdot [\mathcal{O}_{\mathfrak{b}}]. \tag{12.3.2}
$$

In Lemma 7.7.3 we produced an isomorphism $\mathrm{K}(\mathrm{Coh}^{T_{\mathbf{F}_p}}(\mathcal{B}_{\mathbf{F}_p})) \cong \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}}))$. Combining this with (7.5.2) and (12.3.1) gives a chain of isomorphisms

$$
\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \cong \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{F}_p}}(\mathcal{B}_{\mathbf{F}_p})) \cong \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}})) \cong \mathrm{K}(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T)). \tag{12.3.3}
$$

Lemma 12.3.4. *Let $B_{\mathbf{Z}} < G_{\mathbf{Z}}$ be a Borel subgroup defined over $\mathbf{Z}$. Then the composite isomorphism* (12.3.3) *sends the class of the baby Verma $[\widehat{Z}_{w_0\mathfrak{b}}(\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ to the class of the baby Verma $[{}^{\mathfrak{q}}\widehat{Z}_{w_0\mathfrak{b}}(\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T))$.*

Proof. We saw in Lemma 7.7.3 that the middle isomorphism $\mathrm{K}(\mathrm{Coh}^{T_{\mathbf{F}_p}}(\mathcal{B}_{\mathbf{F}_p})) \cong \mathrm{K}(\mathrm{Coh}^{T_{\mathbf{C}}}(\mathcal{B}_{\mathbf{C}}))$ sends the skyscraper $[\mathcal{O}_{wB_{\mathbf{F}_p}}] \mapsto [\mathcal{O}_{wB_{\mathbf{C}}}]$, and is equivariant for the tensoring action of $\mathrm{Rep}(T)$.

The result then follows from the fact that $\widehat{Z}_{w_0\mathfrak{b}}(\lambda)$ and ${}^{\mathfrak{q}}\widehat{Z}_{w_0\mathfrak{b}}(\lambda)$ localize to the skyscraper sheaf at the mod p and complex fibers of $B_{\mathbf{Z}}$ (with the same grading), respectively, cf. Example A.1.1. □

In general, the isomorphism of (12.3.3) does *not* send the class of the simple module $[\widehat{L}(\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ to the class of the simple module $[{}^{\mathfrak{q}}\widehat{L}(\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T))$ in general. This is related to the *failure* of *Lusztig's Conjecture* [Lus80], a conjectural formula for the characters of simple representations of G in characteristic p. Lusztig originally conjectured that this character formula would hold as soon as $p > 2h-2$ where h is the Coxeter number of G. The character formula was proven to hold for all sufficiently large p (building on a long program by many people) by Andersen-Jantzen-Soergel [AJS94]. The bound on p was made effective by Fiebig [Fie12]. On the other hand, Williamson showed in [Wil17] that there are counterexamples to the character formula for p super-polynomially large in h. The significance of Lusztig's Conjecture to us comes from the following Proposition.

Proposition 12.3.5. *If Lusztig's character formula holds for G at p, then the composite map in* (12.3.3) *sends $[\widehat{L}(\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T))$ to $[{}^{\mathfrak{q}}\widehat{L}(\lambda)] \in \mathrm{K}(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T))$.*

Proof. Since these K-groups are free as $\mathbf{Z}[X^*(T)]$-modules (acting by change of grading), it suffices to check the equality after tensoring with $\mathbf{Q}(X^*(T))$. The classes of the baby Vermas (with respect to any chosen $\mathfrak{b}$) generate in

$$
\mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)) \otimes_{\mathbf{Z}[X^*(T)]} \mathbf{Q}(X^*(T)) \cong \mathrm{K}(\mathrm{Rep}_0(\mathfrak{U}_\zeta\mathfrak{g}^0, T)) \otimes_{\mathbf{Z}[X^*(T)]} \mathbf{Q}(X^*(T)) \tag{12.3.4}
$$

since the characters of the simple modules $\widehat{L}(\lambda)$ for $\lambda \in X_1^*(T)$ may be written in terms of the characters of $w \cdot [\widehat{Z}_{\mathfrak{b}}(p\rho)]$ after passing to the fraction field $\mathbf{Q}(X_*(\check{T}))$. (As an aside, we remark that this may also be seen from the coherent realization using Theorem 7.5.1 and equivariant localization in K-theory.)

Therefore we have unique $(n_{\widetilde{w}\widetilde{w}'} \in \mathbf{N})_{\widetilde{w},\widetilde{w}'}$ such that

$$
[\widehat{Z}_{\mathfrak{b}}(\widetilde{w} \bullet_p 0)] = \sum_{\widetilde{w}' \in \widetilde{W}} n_{\widetilde{w}\widetilde{w}'} [\widehat{L}(\widetilde{w}' \bullet_p 0)] \in \mathrm{K}(\mathrm{Rep}_0(\mathcal{U}\mathfrak{g}^0, T)).
$$

Define $({}^{\mathfrak{q}}n_{\widetilde{w}\widetilde{w}'} \in \mathbf{N})_{\widetilde{w}\widetilde{w}'}$ analogously for the quantum group. Lusztig's Conjecture says that the characters of $\widehat{L}(\widetilde{w} \bullet_p 0)$ and ${}^{\mathfrak{q}}\widehat{L}(\widetilde{w} \bullet_p 0)$ coincide for all $\widetilde{w} \in \widetilde{W}$ (cf. [Jan03, H.12]). Since we have seen in Lemma 12.3.4 that the composite map (12.3.3) takes $[\widehat{Z}_{\mathfrak{b}}(\widetilde{w} \bullet_p 0)] \mapsto [{}^{\mathfrak{q}}\widehat{Z}_{\mathfrak{b}}(\widetilde{w} \bullet_p 0)]$, we must therefore have $(n_{\widetilde{w}\widetilde{w}'})_{\widetilde{w},\widetilde{w}'} =$

[21]The accreditation of this result seems tricky. It was claimed the main theorem of [BK08], but there appear to be serious gaps in the proof there – see [Tan14, Remark 5.4].

$({}^{\natural}n_{\widetilde{w}\widetilde{w}'})_{\widetilde{w},\widetilde{w}'}$, and they are uni-triangular with respect to the partial order $\uparrow$, hence invertible in $\mathbf{Q}(X^*(T))$. Therefore the composite map (12.3.3) must send $[\widehat{L}(\widetilde{w}' \bullet_p 0)] \mapsto [{}^{\natural}\widehat{L}(\widetilde{w}' \bullet_p 0)]$. □

12.3.4. *Effectivity of Breuil-Mézard cycles.* The key input to effectivity is the following t-exactness property from [BBAMY23].

Proposition 12.3.6 ([BBAMY23, Theorem 6.3.5])**.** *The functor $\mathcal{D}^b(\mathrm{Coh}_{\mathcal{B}}^{\check{T}}(\widetilde{\mathcal{N}})) \to \mathcal{D}_{\psi,\check{\mathbf{I}}}$ from (8.1.1) is t-exact for the perverse t-structure on the target and the exotic t-structure on source.*

Corollary 12.3.7. *(1) The singular support of ${}^{\natural}\gamma_0^0({}^{\natural}\widehat{L}(\lambda))$ is effective.*

(2) If Lusztig's Conjecture holds for G at p, then the class $\mathrm{SS}[\widehat{L}(\lambda)] \in \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\underline{\mathrm{Y}}_{\gamma}^{\varepsilon=0})$ is effective.

Proof. (1) This follows from Proposition 12.3.6 and the effectivity of the singular support of perverse sheaves.

(2) Combine Proposition 12.3.5 with (1). □

Proof of Theorem 12.3.1. By Corollary 12.3.7(2), the hypothesis implies that $\mathrm{SS}[\widehat{L}(\lambda)]$ is effective. (Note that if we were willing to impose a small genericity assumption, we could simply conclude by invoking Proposition 12.2.1 at this point.) By definition, $\mathcal{Z}^{\mathrm{EG}}(\sigma)$ is obtained from $\mathrm{SS}[\widehat{L}(\lambda)]$ by the composition of (a) applying $w^{-1}t^{-\mu}$, (b) applying $(\mathfrak{sp}_{\varepsilon\to 0})^{-1}$, and (c) applying $\mathfrak{sp}_{\varepsilon\to 1}$. Then observe that all of these operations preserve effectivity: (b) by Theorem 5.1.1(1) and (c) by Lemma 2.4.1. □

Remark 12.3.8. The above argument is in some sense a hack to circumvent the issue that "mod p Geometric Langlands" is not currently available. There should be an analogue of (8.1.1) for $G/\overline{\mathbf{F}}_p$ on the LHS, and mod p sheaves on the RHS. The main missing input is an analogue of Bezrukavnikov's equivalence in characteristic p, but this has seen significant recent progress by Bezrukavnikov–Riche et al. We are therefore optimistic that we will soon be able to apply this same t-exactness argument to show the effectivity $\mathcal{Z}_{\gamma}^{\varepsilon=0}(\sigma)$ without the assumption that p is extremely large.

Part 4. Appendices

Appendix A. The microlocal support of baby Vermas by R. Bezrukavnikov, P. Boixeda Alvarez, T. Feng, B. Le Hung

We will prove Theorem 9.4.1 by showing that $\mathrm{SS}[\widehat{Z}_1(\rho)]$ satisfies the conditions of the Recognition Principle 9.3.1. Conditions (2) and (3) are not difficult to verify by direct computation. However, the "eigenclass" condition is less accessible; indeed, at first glance it seems to require calculating the equivariant fundamental class, which was the difficulty in the first place. However, it admits a more conceptual interpretation in terms of the interplay between the two actions of $\widetilde{W}$ restricted to $X^*(T)$. The key point is then that under BMR localization, baby Vermas are localized to *skyscraper sheaves* in the coherent realization. This suggests that they are "Hecke eigensheaves" in some sense, and making this precise leads to the "eigenclass" condition; this was in fact the origin of the name.

A.1. **Eigenclass condition.** We remind that we regard $T < B < G$, etc. as being over $\mathbf{F}_p$ in this part. The G-action on our fixed Borel $B < G$ induces an isomorphism $G/B \xrightarrow{\sim} \mathcal{B}$. The fixed points for the T-action on $\mathcal{B}$ are identified with W, with $w \in W$ corresponding to $wB \in \mathcal{B}$. We let $\mathcal{O}_{wB}$ be the skyscraper sheaf at wB viewed as a point in the zero section of $\widetilde{\mathcal{N}} := T^*(\mathcal{B})$. Since wB is T-fixed, $\mathcal{O}_{wB}$ carries a native T-equivariant structure, which we use to view it as an object of $\mathrm{Coh}_{\mathcal{B}}^{T}(\widetilde{\mathcal{N}})$.

Example A.1.1 (BMR localization of baby Verma)**.** As explained in Example 7.5.4, $\widehat{Z}_{\mathfrak{b}}(2\rho)$ localizes to the class of the skyscraper sheaf $[\mathcal{O}_B] \in \mathrm{Coh}_{\mathcal{B}}^{T}(\widetilde{\mathcal{N}})$. We will calculate the BMR localization of $[\widehat{Z}_{w_0\mathfrak{b}}(p\rho)]$. To this end, observe by comparing characters that we have

$$[\widehat{Z}_{\mathfrak{b}}(2\rho)] = [\widehat{Z}_{w_0\mathfrak{b}}(2p\rho)] \in \mathrm{K}(\mathrm{Coh}_{\mathcal{B}}^{T}(\widetilde{\mathcal{N}}))$$

where $w_0 \in W$ is the longest Weyl element. Hence writing $\widehat{Z}_{\mathfrak{b}}(p\rho) = t^{-\rho} \cdot_p \widehat{Z}_{\mathfrak{b}}(2p\rho)$ and using the equivariance of SS for the $(\widetilde{W}, \cdot_p)$-action, we find that

$$\mathrm{SS}[\widehat{Z}_{w_0\mathfrak{b}}(p\rho)] = t^{-\rho} \cdot [\mathcal{O}_{\mathfrak{b}}] \in \mathrm{K}(\mathrm{Coh}_{\mathcal{B}}^{T}(\widetilde{\mathcal{N}})). \tag{A.1.1}$$

Recall from §7.4.2 that for $\lambda \in X^*(T)$ we defined the representation $\widehat{Z}_1(\lambda) \in \mathrm{Rep}^0(G_1T)$ to correspond to $\widehat{Z}_{w_0\mathfrak{b}}(\lambda)$. Hence we may rewrite (A.1.1) as

$$\mathrm{SS}[\widehat{Z}_1(p\rho)] = t^{-\rho} \cdot [\mathcal{O}_{\mathfrak{b}}] \in \mathrm{K}(\mathrm{Coh}^T_{\mathcal{B}}(\widetilde{\mathcal{N}})). \tag{A.1.2}$$

Recall that $X^*(T)$ acts on $\mathrm{K}(\mathrm{Coh}^T_{\mathcal{B}}(\widetilde{\mathcal{N}})) \cong \mathrm{K}(\mathrm{Coh}^T(\mathcal{B}))$ in two ways through the embedding $X^*(T) \hookrightarrow \widetilde{W}$.

(1) For the $\cdot$-action, $\lambda \in X^*(T)$ acts by tensoring with the T-equivariant line bundle $\mathcal{O}\langle\lambda\rangle$, which is the pullback of the T-equivariant line bundle on a point corresponding to the character λ of T.
(2) Fir the $\bullet$-action, $\lambda \in X^*(T)$ acts by tensoring with the G-equivariant line bundle $\mathcal{O}(\lambda) = G \times^B \mathbf{A}^1$ on $\mathcal{B}$, where B acts on $\mathbf{A}^1$ through the character λ.

Moreover, it is evident that the two actions of $X^*(T)$ commute with each other.

Lemma A.1.2. *For any $\mu \in X^*(T)$, on the object $t^\mu \cdot \mathcal{O}_{wB} \in \mathrm{Coh}^T(\mathcal{B})$, we have*

$$t^{w\lambda} \cdot (t^\mu \cdot [\mathcal{O}_{wB}]) = (t^\mu \cdot [\mathcal{O}_{wB}]) \bullet t^\lambda \in \mathrm{K}(\mathrm{Coh}^T(\mathcal{B})).$$

Proof. We immediately reduce to the case $\mu = 0$ since the two actions of $X^*(T)$ commute. By definition, $t^\lambda \cdot \mathcal{O}_{wB} = \mathcal{O}\langle\lambda\rangle \otimes \mathcal{O}_{wB}$ is $\mathcal{O}_{wB}$ equipped with T-equivariant structure placing it in graded degree λ.

For $g \in T$, we have $gwB = w(w^{-1}gwB) \in \mathcal{B}$. Therefore, the left translation of $g \in T$ on $\mathcal{O}_{wB} \bullet t^\lambda = \mathcal{O}_{wB} \otimes \mathcal{O}(-\lambda)$ acts as multiplication by $\lambda(w^{-1}gw) = (w\lambda)(g)$. Hence $\mathcal{O}_{wB} \otimes \mathcal{O}(-\lambda)$ is $\mathcal{O}_{wB}$ in graded degree $w\lambda$ for the left translation action of t. $\square$

Proposition A.1.3. *The translation action of $X_*(\check{T})$ on $\mathrm{SS}[\widehat{Z}_1(p\rho)]$ agrees with the affine Springer action of $X_*(\check{T})$ on $\mathrm{SS}[\widehat{Z}_1(p\rho)]$. In particular, the equivariant support of $\mathrm{SS}[\widehat{Z}_1 p\rho)]_{\check{T}}$ is concentrated on the translation elements $X_*(\check{T}) \subset \widetilde{W}$.*

Proof. By (A.1.2) and Lemma A.1.2 we see that

$$t^\lambda \cdot \mathrm{SS}[\widehat{Z}_1(p\rho)] = \mathrm{SS}[\widehat{Z}_1(p\rho)] \bullet t^\lambda \in \mathrm{K}(\mathrm{Coh}^T(\mathcal{B}))$$

for all $\lambda \in X^*(T)$. This gives the first statement of the Proposition. This implies that the two actions of $X_*(\check{T})$ – one restricted from $(\widetilde{W}, \cdot)$ and the other restricted from $(\widetilde{W}, \bullet)$ – also agree after lifting to $\check{T}$-equivariant Borel–Moore homology, so that

$$t^{-\nu} \cdot \mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}} \bullet t^\nu = \mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}}.$$

By the description of the two $\widetilde{W}$-actions in $\check{T}$-equivariant Borel-Moore homology (cf. §6.6), this implies that if $\widetilde{w}$ belongs to the equivariant support of $\mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}}$ then so does $t^\nu \widetilde{w} t^{-\nu} = t^{\nu - w\nu}\widetilde{w}$ for any $\nu \in X_*(\check{T})$, where w is the projection of $\widetilde{w}$ to W. But if $w \neq 1$, the set $t^{\nu - w\nu}\widetilde{w}$ is unbounded, a contradiction; this concludes the proof. $\square$

A.2. **Support bound and normalization conditions.**

Proposition A.2.1. *Assume that $p > 2h_\rho$. Then $\mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}}$ has equivariant support in $\mathrm{Adm}(\rho) \subset \widetilde{W}$.*

Proof. By [GHS18, Lemma 10.1.5], the simple constituents of $\widehat{Z}_1(p\rho)$ consists of $\widehat{L}_1(p\nu + \widetilde{w} \bullet_p 0)$ where

$$\sigma t^{-\nu} \uparrow w_0 t^{-\rho} \widetilde{w}$$

for all $\sigma \in W$. This is equivalent to $\sigma t^{-\nu} \uparrow w_0 t^{-\rho}\widetilde{w}$ for the choice of σ such that $\sigma t^{-\nu}$ is dominant. In turn, this is equivalent to $\sigma t^{-\nu} \leq w_0 t^{-\rho}\widetilde{w}$ for this choice. By Lemma 9.4.3, it suffices to show that $t^\nu \widetilde{W}_{\leq w_0\widetilde{w}} \subset \mathrm{Adm}(\rho)$ for such $\widetilde{w}, \nu, \sigma$.

Let w be the projection of $\widetilde{w}$ to W. Then we have the reduced factorization

$$t^{w^{-1}\rho} = (w_0 t^{-\rho}\widetilde{w})^{-1} w_0 \widetilde{w}.$$

We have

$$(\sigma t^{-\nu})^{-1} \leq (w_0 t^{-\rho}\widetilde{w})^{-1}, \quad \sigma \leq w_0$$

hence for any $u \leq w_0\widetilde{w}$,

$$t^\nu u = (\sigma t^{-\nu})^{-1}\sigma u \leq (w_0 t^{-\rho}\widetilde{w})^{-1} w_0 \widetilde{w} = t^{w^{-1}\rho}$$

belongs to $\mathrm{Adm}(\rho)$. This shows the desired inclusion $t^\nu \widetilde{W}_{\leq w_0\widetilde{w}} \subset \mathrm{Adm}(\rho)$. $\square$

Proposition A.2.2. *The coefficient of* $\mathrm{Loc}^{\check{T}}\left(\mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}}\right)$ *at* $[t^\rho]$ *is* $1/\beta \in \mathrm{Frac}(\mathbb{S})$.

Proof. From the proof of Proposition A.2.1, the only factor $\widehat{L}_1(p\nu+\widetilde{w}\bullet_p 0)$ of $\widehat{Z}_1(p\rho)$ whose microlocal support can contribute to the coefficient of $[t^\rho]$ is the one with $\widetilde{w}=1$ and $w_0t^{-\nu}=w_0t^{-\rho}$, i.e., $\widehat{L}_1(p\nu+\widetilde{w}\bullet_p 0)=\widehat{L}_1(p\rho)$. Since $\widehat{L}_1(p\rho)$ occurs in $\widehat{Z}_1(p\rho)$ with multiplicity one, and $\mathrm{SS}[\widehat{L}_1(p\rho)]_{\check{T}}=t^\rho[\check{G}/\check{B}]_{\check{T}}$ has coefficient $1/\beta$ by Example 6.4.2, we are done. ☐

Proof of Theorem 9.4.1. The equality

$$\mathrm{Loc}^{\check{T}}\left(\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]_{\check{T}}\right)=\frac{1}{\beta}\sum_{w\in W}\mathrm{sgn}(w)[t^{w\rho}]$$

comes from (9.3.1). As observed in Lemma 9.3.5, this shows that $\mathfrak{sp}_{p\to 0}[\mathrm{X}_\gamma^{\varepsilon=0}(\rho)]_{\check{T}}$ satisfies the properties of Proposition 9.3.1.

Propositions A.1.3, A.2.1 and A.2.2 show that $\mathrm{SS}[\widehat{Z}_1(p\rho)]_{\check{T}}$ also satisfy the characterizing properties in Proposition 9.3.1. Hence they coincide. ☐

Appendix B. The homological model theorem by B. Le Hung

In this appendix, we explain some ideas developed in [LH] (which deals with the much more complicated setup of non-generic tame types) to give a sketch of proof of Theorem 11.1.1. As we will cite several lengthy formulas from [LHLM23b], we will align our conventions with this reference. Especially, our affine flag varieties are written as right cosets, and we work with the upper triangular Borel. We can convert between the two conventions by transposing all statements involving matrices.

We recall the basic setup. We wish to study Galois representations of G_K where $K=\mathbf{Q}_{p^f}=\mathbf{Q}_q$. All spaces we study will be over the ring of integer $\mathcal{O}$ of a sufficiently large finite extension $E/\mathbf{Q}_p$, with residue field $\mathbf{F}$. Unlike [LHLM23b], we will suppress subscripts $\mathcal{O}$ for the spaces over $\mathcal{O}$. Set $\mathcal{J}=\mathrm{Hom}(K,\overline{\mathbf{Q}}_p)$, which is identified with $\mathbf{Z}/f$ using arithmetic Frobenius according to the conventions of [LHLM23b, §1.9.2].

B.1. Generalities.

B.1.1. *Loop groups.* For a $\mathcal{O}$-algebra R, we denote by $R[v]^{\wedge_{v(v+p)}}$ the completion of $R[v]$ with respect to $v(v+p)$. Consider the functors on Noetherian test rings R

$$\begin{aligned}
L\mathcal{G}(R)&=\mathrm{GL}_n(R[v]^{\wedge_{v(v+p)}}[\frac{1}{v(v+p)}])\\
L\mathcal{M}(R)&=\mathrm{M}_n(R[v]^{\wedge_{v(v+p)}}[\frac{1}{v(v+p)}])\\
L^+\mathcal{G}(R)&=\{A\in\mathrm{GL}_n(R[v]^{\wedge_{v(v+p)}}),\ A\text{ is upper triangular modulo }v\}\\
L^+\mathcal{M}(R)&=\{A\in\mathrm{M}_n(R[v]^{\wedge_{v(v+p)}}),\ A\text{ is upper triangular modulo }v\}
\end{aligned}$$

and for integers a,b

$$L\mathcal{G}^{[a,b]}(R)=\{g\in L\mathcal{G}(R)\,|\,g\in(v+p)^aL^+\mathcal{M}(R)\text{ and }(v+p)^bg^{-1}\in L^+\mathcal{M}(R)\}$$

For $\lambda=(\lambda_1,\cdots\lambda_n)\in X_*(T)^+$ with $\lambda_i\in[0,h]$, we also define $L^{\leq\lambda}\mathcal{G}(R)\subset L^{[0,h]}\mathcal{G}(R)$ to be the subset of A such that all its $k\times k$ minors belong to $(v+p)^{\lambda_{n-k+1}+\cdots+\lambda_n}R[v]^{\wedge_{v(v+p)}}$, and its determinant is in $(v+p)^{\lambda_1+\cdots\lambda_n}(R[v]^{\wedge_{v(v+p)}})^\times$.

Remark B.1.1. Compared to [LHLM23b], the difference is that we complete $R[v]$ with respect to $v(v+p)$ instead of $(v+p)$. This makes no difference for p-adically complete test rings, but makes a difference when we work over rings where p is invertible.

For each $N\geq 0$ we have a reduction mod v^n maps

$$r_N: L^+\mathcal{M}(R)\twoheadrightarrow\mathcal{M}_N(R)\hookrightarrow\mathrm{M}_n(R[v]/v^N)$$

where $\mathcal{M}_N(R)$ is the subset which is upper triangular mod v. Define $L_N^+\mathcal{G}(R)$, $L_N^+\mathcal{M}(R)$ to be the pre-image of 1 in $L^+\mathcal{G}(R)$, and of 0 in $L^+\mathcal{M}(R)$, respectively. The choice of coordinate v gives a functorial R-linear splitting $R[v]/v^N \to R[v]^{\wedge_{v(v+p)}}$, and hence gives a section $s_N : \mathcal{M}_N \to L^+\mathcal{M}$. We warn the reader that the restriction of r_N to $L^+\mathcal{G}(R)$ does not surject onto $\mathcal{M}_N \cap \mathrm{GL}_n(R[v]/v^N)$ in general. Finally, we note that $v \mapsto v^p$ gives a well-defined map $\mathcal{M}_N \to \mathcal{M}_{pN}$.

B.1.2. *Twisted affine Grassmannians.* Let $\mathrm{Gr}_{\mathcal{G}} = L^+\mathcal{G}\backslash L\mathcal{G}$. It is an ind-scheme over $\mathcal{O}$, with the following properties:

- The generic fiber $(\mathrm{Gr}_{\mathcal{G}})_E \cong (\mathrm{Gr}_{\mathrm{GL}_n})_E \times (\mathrm{Fl}_{\mathrm{GL}_n})_E$ is the product of affine Grassmannian for loop variable $v+p$ and the affine flag variety for the loop variable v (over E).
- The special fiber $(\mathrm{Gr}_{\mathcal{G}})_{\mathbf{F}} \cong (\mathrm{Fl}_{\mathrm{GL}_n})_{\mathbf{F}}$, the affine flag variety (for the upper triangular Iwahori).

For each $a, b \in \mathbf{Z}$, $\mathrm{Gr}_{\mathcal{G}}^{[a,b]} = L^+\mathcal{G}\backslash L^{[a,b]}\mathcal{G}$ is a finite type closed subscheme of $\mathrm{Gr}_{\mathcal{G}}$, and under the isomorphism $(\mathrm{Gr}_{\mathcal{G}})_E \cong (\mathrm{Gr}_{\mathrm{GL}_n})_E \times (\mathrm{Fl}_{\mathrm{GL}_n})_E$ its projection to second factor is the base point of $(\mathrm{Fl}_{\mathrm{GL}_n})_E$.

For $N > 0$, we also have $\mathrm{Gr}_{\mathcal{G},N} = L^+_N\mathcal{G}\backslash L\mathcal{G}$. Then $\mathrm{Gr}_{\mathcal{G},N} \to \mathrm{Gr}_{\mathcal{G}}$ is a torsor for the group $L^+\mathcal{G}/L^+_N\mathcal{G}$.

B.1.3. *Twisted φ action.* Suppose R is p-adically complete. Then $R[v]^{\wedge_{v(v+p)}} = R[\![v+p]\!] = R[\![v]\!]$, so Frobenius $\varphi : R[v] \to R[v]$ sending $v \mapsto v^p$ extends to $R[v]^{\wedge_{v(v+p)}}$. Given $(s, \mu) \in W^{\mathcal{J}} \times X^*(T)^{\mathcal{J}}$, we define the (s,μ)-twisted φ-conjugation action of $(L^+\mathcal{G})^{\mathcal{J},\wedge_p}$ on $(L\mathcal{G})^{\mathcal{J},\wedge_p}$ by

$$(I^{(j)})_j \cdot (A^{(j)})_j = I^{(j)} A^{(j)} \big(\operatorname{Ad}(s_j^{-1} v^{\mu_j})\big(\varphi(I^{(j-1)})^{-1}\big)\big)$$

We warn the reader that this is different from the definition in [LHLM23b, §5.2], in that we do not incorporate the shift by η in *loc.cit.*

An important feature of the twisted action is that it can often be straightened over p-power torsion bases:

Proposition B.1.2. *Suppose $N \geq \frac{p+h+M-1}{p-1}$, and $(s,\mu) \in W^{\mathcal{J}} \times X_1(T)^{\mathcal{J}}$. Then over $\mathcal{O}/p^M$, the (s,μ)-twisted φ-conjugation action and left translation action by $(L^+_N\mathcal{G})^{\mathcal{J}}$ have the same orbits, and hence*

$$[(L^{[0,h]}\mathcal{G})^{\mathcal{J}}/_{\varphi,(s,\mu)}(L^+_N\mathcal{G})^{\mathcal{J}}]_{\mathcal{O}/p^M} \cong (\mathrm{Gr}^{[0,h]}_{\mathcal{G},N})^{\mathcal{J}}_{\mathcal{O}/p^M}$$

Proof. Exactly as in the proof of [LHLM23b, Lemma 5.2.2], this follows once for $A \in L\mathcal{G}^{[0,h]}(R)$, the operation

$$X \mapsto A \operatorname{Ad}(s_j^{-1} v^{\mu_j})(\varphi(X)) A^{-1}$$

defines a topologically nilpotent endomorphism on $\operatorname{Lie} L^+_N\mathcal{G} = v^N \mathrm{M}_n(R[\![v]\!])$. The result now follows from the fact that $A \operatorname{Ad}(s_j^{-1} v^{\mu_j})(\varphi(X)) A^{-1}$ belongs to $\frac{v^{pN-p+1}}{(v+p)^h}\mathrm{M}_n(R[\![v]\!]) \in v^{pN-p+1-h-M+1}\mathrm{M}_n(R[\![v]\!])$ when R is an $\mathcal{O}/p^M$-algebra. ∎

If we let $\tau = \tau(s,\mu)$ be the inertial type corresponding to (s,μ), the quotient stack

$$Y^{[0,h],\tau} = [(L^{[0,h]}\mathcal{G})^{\mathcal{J},\wedge_p}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$$

is exactly the p-adic formal stack of Breuil-Kisin modules of height $\leq h$ with descent data of type τ, cf [LHLM23b, Proposition 5.2.1]. If $\lambda \in (X_*(T)^+)^{\mathcal{J}}$, we also have the stack $Y^{\leq\lambda,\tau} := [\prod_{j\in\mathcal{J}}(L^{\leq\lambda_j}\mathcal{G})^{\wedge_p}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$. Note however that the object with the same name in [LHLM23b, §5.3] is the p-saturation of the reduced part of our $Y^{\leq\lambda,\tau}$.

B.2. **Monodromy condition.** Fix $h \geq 0$. Let $\tau = \tau(s,\mu)$ be a tame inertial type, with μ at least m-generic with $m \geq 2h$. Set $\mathcal{J}' = \mathrm{Hom}(\mathbf{Q}_{q^r}, \overline{\mathbf{Q}}_p) \cong \mathbf{Z}/fr$ where r is the order of s. The data (s,μ) gives rises to various quantities $s'_{\mathrm{or},j'}, \mathbf{a}'^{(j')}$ as explicated in [LHLM23b, Example 2.4.1] (noting that the role of $\mu+\eta$ in *loc.cit.* is played by μ here). As in the main text, the precise formulas for these quantities are not important for us.

Let R be a p-adically complete $\mathcal{O}$-algebra. Let $A = (A^{(j)}) \in (L^{[0,h]}\mathcal{G}(R))^{\mathcal{J}}$, which we inflate to a tuple in $(L^{[0,h]}\mathcal{G}(R))^{\mathcal{J}'}$ by demanding $A^{(j')}$ to only depend on j' mod f. Set $A^{*,(j')} = (v+p)^h (A^{(j')})^{-1} \in L^+\mathcal{M}(R)$. Define the expression $\mathcal{N}_\infty^{(j')}(A)$ as

$$\begin{aligned}
&\mathcal{N}_0^{(j')} + \frac{p}{(p+v^p)^h}\mathcal{N}_1^{(j')} + \frac{p^2}{(p+v^p)^h(p+v^{p^2})^{h-1}}\mathcal{N}_2^{(j')} + \cdots \\
=&\mathcal{N}_0^{(j')} + \frac{p}{(p+v^p)^h} A^{(j')} \operatorname{Ad}(s_{j'}^{-1} v^{\mu_{j'}})\big(\varphi(\mathcal{N}_0^{(j'-1)})\big) A^{*,(j')} + \frac{p}{(p+v^p)^h(p+v^{p^2})^{h-1}} A^{(j')} \operatorname{Ad}(s_{j'}^{-1} v^{\mu_{j'}})\big(\varphi(\mathcal{N}_1^{(j'-1)})\big) A^{*,(j')} + \cdots \\
=&\mathcal{N}_0^{(j')} + A^{(j')} \operatorname{Ad}(s_{j'}^{-1} v^{\mu_{j'}})\Big(\sum_{i=1}^{\infty} \frac{p^i}{\prod_{k=1}^{i}(p+v^{p^k})^h}\varphi(\mathcal{N}_{i-1}^{(j'-1)})\Big) A^{*,(j')}
\end{aligned}$$

where the $\mathcal{N}_i^{(j')}$ are defined recursively by

$$\mathcal{N}_0^{(j')} = (e' v \frac{d}{dv} A^{(j')} + [\mathrm{Diag}((s'_{\mathrm{or},j'})^{-1}(\mathbf{a'}^{(j')})), A^{(j')}]) A^{*,(j')}$$
$$\mathcal{N}_i^{(j')} = A^{(j')} \mathrm{Ad}(s_{j'}^{-1} v^{\mu_{j'}}) \Big(\varphi(\mathcal{N}_{i-1}^{(j'-1)}) \Big) A^{*,(j')}$$

Remark B.2.1. It follows from the proof of [LHLM23b, Equation (7.8)] that

$$\mathcal{N}_i^{(j')} \in v^{1+m\frac{p^i-1}{p-1}} \mathrm{Mat}_n(R[\![v+p]\!])$$

The *monodromy condition* for type τ is the condition

$$\mathcal{N}_{A,\infty}^{(j')} \in (v+p)^{h-1} L^+\mathcal{M}(R[\frac{1}{p}]) \tag{B.2.1}$$

or, more precisely, the condition that

$$(\frac{d}{dv})^t|_{v=-p} \big(v^{-\delta_{k>\ell}} N_{A,\infty}^{(j')} \big)_{k\ell} = 0 \tag{B.2.2}$$

for $0 \le t \le h-2$, $1 \le k, \ell \le n$. The formula defining $N_{A,\infty}^{(j')}$ only makes sense in $R[\frac{1}{p}][\![v]\!]$, however the quantity in (B.2.2) belongs to R, so that condition (B.2.1) makes sense already over R. Note also the condition for j' is the same as the condition for $(j'+f)$. Hence, we can define $L\mathcal{G}^{[0,h],\nabla_{\tau,\infty}}$ to be the closed subfunctor of $(L^{[0,h]}\mathcal{G})^{\mathcal{J},\wedge_p}$ cut out by equation (B.2.1). We also define $L\mathcal{G}^{\le\lambda,\nabla_{\tau,\infty}}$ with its obvious meaning. All these objects are stable under the twisted φ conjugation action of type τ.

The crucial feature of this condition is the following:

Proposition B.2.2. *(*[LHLM23b, Proposition 7.2.3]*) Let $\tau = \tau(s,\mu)$ and $\mathcal{X}^{[0,h]\lambda,\tau} = \bigcup \mathcal{X}^{\le\lambda,\tau}$ be the potentially crystalline Emerton-Gee stack of tame type τ and Hodge-Tate weights belong to $[0,h]$. Assume μ is $(h+1)$-generic. Then $\mathcal{X}^{[0,h],\tau}$ is isomorphic to the p-saturation of the generic fiber of $[L\mathcal{G}^{[0,h],\nabla_{\tau,\infty}}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$. If λ is a collection of Hodge-Tate weights belonging to $[0,h]$, under this isomorphism $\mathcal{X}^{\le\lambda,\tau}$ identifies with the p-saturation of the generic fiber of $[L\mathcal{G}^{\le\lambda,\nabla_{\tau,\infty}}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$.*

B.2.1. *Approximating monodromy.* The monodromy condition (B.2.1) only makes sense after p-adic completion, since the formula for $\mathcal{N}_\infty^{(j')}(A)$ has infinitely many terms. Define the truncation

$$\mathcal{N}_N^{(j')}(A) = \mathcal{N}_0^{(j')} + A^{(j')} \mathrm{Ad}(s_{j'}^{-1} v^{\mu_{j'}}) \Big(\sum_{i=1}^{\infty} tr_N \big(\frac{p^i}{\prod_{k=1}^i (p+v^{p^k})^h} \big) s_N r_N \big(\varphi(r_N(\mathcal{N}_{i-1}^{(j'-1)})) \big) \Big) A^{*,(j')}$$

where the term $tr_N\big(\frac{p^i}{\prod_{k=1}^i (p+v^{p^k})^h}\big)$ is the polynomial

$$\frac{1}{p^{(h-1)i}} s_N r_N \Big(\prod_{k=1}^{i} \big(1 - \frac{v^{p^k}}{p} + \frac{v^{2p^k}}{p^2} + \cdots\big) \Big) \in R[\frac{1}{p}][v]$$

Define the N-th truncated monodromy condition as

$$\mathcal{N}_N^{(j')}(A) \in (v+p)^{h-1} L^+\mathcal{M}(R[\frac{1}{p}]) \tag{B.2.3}$$

which again is understood as the vanishing of appropriate derivatives evaluated at $v = -p$.

By Remark B.2.1, the sum defining $\mathcal{N}_N^{(j')}(A)$ has only finitely many terms, and furthermore, the structure of the recursion shows that

$$\mathcal{N}_N^{(j')}(A) = \mathcal{N}_0^{(j')} + A^{(j')} f_N(r_N(A)) A^{*,(j')}$$

where f_N is a regular function $\mathcal{M}_N^{\mathcal{J}} = (L^+\mathcal{M}/L_N^+\mathcal{M})_E^{\mathcal{J}} \to (L^+\mathcal{M})_E^{\mathcal{J}}$. This implies that the N-th truncated monodromy condition can be decompleted to give subfunctor $L\mathcal{G}^{[0,h],\nabla_{\tau,N}}, L\mathcal{G}^{\le\lambda,\nabla_{\tau,N}}$ of $(L^{[0,h]}\mathcal{G})^{\mathcal{J}}$. These subfunctors are invariant under the left translation action by $L_N^+\mathcal{G}$, thus induce closed subschemes $\mathrm{Gr}_{\mathcal{G},N}^{[0,h],\nabla_{\tau,N}}, \mathrm{Gr}_{\mathcal{G},N}^{\le\lambda,\nabla_{\tau,N}} \subset \mathrm{Gr}_{\mathcal{G},N}$.

Remark B.2.3. The $N = 0$th truncation $(\mathrm{Gr}_{\mathcal{G},0}^{\le\lambda,\nabla_{\tau,0}})_{\mathbf{F}}$ has the same underlying reduced scheme as the deformed affine Springer fiber $\underline{Y}_\gamma^{\varepsilon=1}(\le\lambda)$ for the element $\gamma = \gamma(s,\mu) = t(w^{-1}\mu)$ attached to the type $\tau = \tau(s,\mu)$, up to transposing the situation.

We have the following estimate:

Proposition B.2.4. *Let R be a p-adically complete $\mathcal{O}$-algebra and suppose we are given $f : \mathrm{Spf} R \to (L^{[0,h]}\mathcal{G})^{\mathcal{J}}$, corresponding to $A = (A^{(j)})$. Let $I_{\nabla_{\tau,\infty}}$, $I_{\nabla_{\tau,N}}$ be the pull back along f of the ideals defining the subfunctors $L\mathcal{G}^{\nabla_{\tau,\infty}}$,$L\mathcal{G}^{\nabla_{\tau,N}}$. Then*

(1) For all $N \geq 0$

$$(I_{\nabla_{\tau,N}}, p^{m-2h+3}) = (I_{\nabla_{\tau,\infty}}, p^{m-2h+3})$$

(2) Given an integer M, for any sufficiently large N

$$(I_{\nabla_{\tau,N}}, p^{M}) = (I_{\nabla_{\tau,\infty}}, p^{M})$$

Proof. The first item follows immediately from (the proof of) [LHLM23b, Proposition 7.1.10], noting that maximal power of v dividing $s_N r_N\big(\varphi(r_N(\mathcal{N}_{i-1}^{(j'-1)}))$ is at least that of $\varphi(\mathcal{N}_{i-1}^{(j'-1)})$.

For the second item, observe:

- $s_N r_N\big(\varphi(r_N(\mathcal{N}_{i-1}^{(j'-1)})) - \varphi(\mathcal{N}_{i-1}^{(j'-1)}\big)$ belongs to $v^{\max\{N, 1+m\frac{p^i-1}{p-1}\}}\,\mathrm{Mat}_n(R[\![v+p]\!])$.
- $(\frac{d}{dv})^s|_{v=-p}\Big(tr_N\big(\frac{p^i}{\prod_{k=1}^i(p+v^{p^k})^h}\big) - \frac{p^i}{\prod_{k=1}^i(p+v^{p^k})^h}\Big) \in p^{(1-\frac{1}{p})N-s-(h-1)i}\mathbf{Z}_p$.

A crude estimate then shows that $(\frac{d}{dv})^s|_{v=-p}\big(\mathcal{N}_\infty^{(j')}(A) - \mathcal{N}_N^{(j')}(A)\big) \in p^{\min_i\{\max\{(1-\frac{1}{p})N, 1+m\frac{p^i-1}{p-1}\}-p-(h-1)(i+1)\}}R$ for $s \leq h-2$, which immediately gives what we want. □

We can also control the generic fiber:

Lemma B.2.5. *Suppose we are given a morphism $f_N : (L^+\mathcal{M}/L_N^+\mathcal{M})_E^{\mathcal{J}} = (\mathcal{M}_N)_E^{\mathcal{J}} \to (L^+\mathcal{M})_E^{\mathcal{J}}$. Let $\mathcal{X} \subset (L^{[0,h]}\mathcal{G})_E^{\mathcal{J}}$ be the (left $L_N^+\mathcal{G}$-invariant) subfunctor of $(L\mathcal{G})_E^{\mathcal{J}}$ cut out by the condition*

$$v\frac{d}{dv}(A)A^{-1} + Af_N r_N(A)A^{-1} \in \frac{1}{v+p}(L^+\mathcal{M})_E^{\mathcal{J}}$$

Recall there is a natural isomorphism $\iota : (L\mathcal{G})_E^{\mathcal{J}} \cong (L\,\mathrm{GL}_n)_E^{\mathcal{J}} \times (L\,\mathrm{GL}_n)_E^{\mathcal{J}}$ where the first factor has loop variable $v+p$ and the second has loop variable v. Then $\mathcal{X} \cong (\coprod_{\lambda\in([0,h]^n)^{\mathcal{J}}}(L^+\mathrm{GL}_n)^{\mathcal{J}}(v+p)^\lambda \mathrm{GL}_n^{\mathcal{J}})_E \times \mathcal{I}_E^{\mathcal{J}}$, where $\mathcal{I} \in L\,\mathrm{GL}_n$ is the standard upper triangular Iwahori group scheme of $L\,\mathrm{GL}_n$ in loop variable v.

In particular $\dim(L_N^+\mathcal{G})^{\mathcal{J}}\backslash\mathcal{X} \leq (\dim(L^+\mathcal{G}/L_N^+\mathcal{G}) + \dim(\mathrm{GL}_n/B))\#\mathcal{J}$.

Proof. Let R be an E-algebra. Write $\iota(f_N r_N(A)) = (B, C) \in M_n(R[\![v+p]\!]) \times M_n(R[\![v]\!])$. The differential equation

$$(v\frac{d}{dv})X + XB = 0$$

has a unique solution $X(B)$ in $GL_n(R[\![v+p]\!])$ which is congruent to 1 mod $(v+p)$, and the assignment $B \mapsto X(B)$ is a regular function on $\mathrm{M}_n(E[\![v+p]\!])$.

The map $A = \iota(B, C) \mapsto (AX(B), C)$ thus gives an isomorphism from $\mathcal{X}(R)$ to the space of pairs (Y, Z) such that

- $Y \in M_n(R[\![v+p]\!])$ such that $(v+p)^h Y^{-1} \in M_n(R[\![v+p]\!])^{\mathcal{J}}$ and
$$\frac{dY}{dv}Y^{-1} \in \frac{1}{v+p}M_n(R[\![v+p]\!])^{\mathcal{J}}$$
- $Z \in M_n(R[\![v]\!])^{\mathcal{J}}$, and is upper triangular mod v.

The result now follows from the well-known fact that the space of regular singular lattices on the trivial bundle with trivial connection on the punctured disk in characteristic 0 is isomorphic to the space of filtrations on the fiber at 0 of the trivial bundle. □

B.2.2. *Comparing limit cycles.* By Proposition B.2.4, for each $N \geq 0$, the spaces $L\mathcal{G}_{\mathbf{F}}^{\leq\lambda,\nabla_{\tau,N}}$ all coincide and are equal to $L\mathcal{G}_{\mathbf{F}}^{\leq\lambda,\nabla_{\tau,\infty}}$, and this common space is invariant under both the left action and the (s,μ)-twisted φ-conjugation action of $(L^+\mathcal{G})_{\mathbf{F}}^{\mathcal{J}}$. By Proposition B.1.2 these two actions have the same orbits when restricted the subgroup $(L_1^+\mathcal{G})_{\mathbf{F}}^{\mathcal{J}}$. The common quotient stack by either restricted action is then exactly the $B_{\mathbf{F}}^{\mathcal{J}}$-torsor over the deformed affine Springer fiber $(\mathrm{Gr}_{\mathcal{G},0}^{\leq\lambda,\nabla_{\tau,0}})_{\mathbf{F}}$ obtained by pulling back along $(\mathrm{Gr}_{\mathcal{G},1})^{\mathcal{J}} \to (\mathrm{Gr}_{\mathcal{G}})^{\mathcal{J}}$. In particular, it has dimension $\leq \dim(\mathrm{GL}_n^{\mathcal{J}})$, with equality achieved when λ is regular.

For each $N \geq 0$, define $\widetilde{\mathcal{Z}}_N^{\leq\lambda}$ to be the special fiber of the closure of the generic fiber of $\mathrm{Gr}_{\mathcal{G},N}^{\leq\lambda,\nabla_{\tau,N}}$, which is in fact descends (or ascends if $N=0$) to a closed subscheme of $(\mathrm{Gr}_{\mathcal{G},1})_{\mathbf{F}}^{\mathcal{J}}$ of dimension $\leq \dim(\mathrm{GL}_n^{\mathcal{J}})$, with equality if λ is regular. At the level of cycle classes, it is even invariant under left translation by $(L^+\mathcal{G})^{\mathcal{J}}$, and hence descends to a cycle class $[\mathcal{Z}_N^{\leq\lambda,\tau}] \in \mathrm{Ch}_{\mathrm{top}}((\mathrm{Gr}_{\mathcal{G},0}^{\leq\lambda,\nabla_{\tau,0}})_{\mathbf{F}}) = \mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq\lambda))$.

We also denote by $\mathcal{Z}_\infty^{\leq\lambda}$ the special fiber of the closure of the generic fiber of $[L\mathcal{G}^{\leq\lambda,\nabla_{\tau,\infty}}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$. This also induces a cycle class $[\mathcal{Z}_\infty^{\leq\lambda,\tau}] \in \mathrm{Ch}_{\mathrm{top}}((\mathrm{Gr}_{\mathcal{G},0}^{\leq\lambda,\nabla_{\tau,0}})_{\mathbf{F}})$ by ascending along the natural $B^{\mathcal{J}}$-torsor and then descending by left translation under $B^{\mathcal{J}}$. We have the following key invariance property

Theorem B.2.6. *The cycle classes $[\mathcal{Z}_N^{\leq\lambda,\tau}]$ are independent of $N \in \mathbf{N}\cup\{\infty\}$.*

Proof. Fix $N \geq 0$, and consider the following family $\mathrm{Gr}_{\mathcal{G},N}^{\leq\lambda,\nabla_{\tau,N},\varepsilon} \to \mathbf{A}^1$: For an $\mathcal{O}$-algebra R, its R-points consists of $(A,\varepsilon) \in (\mathrm{Gr}_{\mathcal{G},N}^{\leq\lambda})^{\mathcal{J}}(R) \times \mathbf{A}^1(R)$ such that

$$\varepsilon\mathcal{N}_0^{(j')}(A) + (1-\varepsilon)\mathcal{N}_N^{(j')}(A) \in (v+p)^{h-1}L^+\mathcal{M}(R[\frac{1}{p}])$$

By Proposition B.2.4, the special fiber of this family is the constant family over $\mathbf{A}^1_{\mathbf{F}}$ with fiber the natural $(L\mathcal{G}/L_N^+\mathcal{G})^{\mathcal{J}}$-cover of $(\mathrm{Gr}_{\mathcal{G},0}^{\leq\lambda,\nabla_{\tau,0}})_{\mathbf{F}}$. On the other hand, by Lemma B.2.5, after inverting p, this family is isomorphic to a constant family over $\mathbf{A}^1_E$ of unions of flag varieties. By Lemma 2.4.3, these two facts imply

$$[\mathcal{Z}_N^{\leq\lambda,\tau}] = [\mathcal{Z}_0^{\leq\lambda,\tau}].$$

To finish, it suffices to show that for sufficiently large N

$$[\mathcal{Z}_N^{\leq\lambda,\tau}] = [\mathcal{Z}_\infty^{\leq\lambda,\tau}].$$

This is a consequence of the fact that $[L\mathcal{G}^{\leq\lambda,\nabla_{\tau,\infty}}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$ is a topologically finite type p-adic formal stack with regular generic fiber. In particular, the p^∞-torsion of its structure sheaf is bounded.

The boundedness and Proposition B.2.4 implies that for all sufficiently large N, $\widetilde{\mathcal{Z}}_N^{\leq\lambda,\tau}$ embeds into $\widetilde{\mathcal{Z}}_{N,\infty}$, the pullback of $\mathcal{Z}_\infty^{\leq\lambda,\tau}$ to $(\mathrm{Gr}_{\mathcal{G},N})^{\mathcal{J}}$. On the other hand we can find a finite type smooth affine cover $U \to [L\mathcal{G}^{\leq\lambda,\nabla_{\tau,\infty}}/_{\varphi,(s,\mu)}(L^+\mathcal{G})^{\mathcal{J},\wedge_p}]$ such that the map can be lifted to a map $U \to (L\mathcal{G})^{\mathcal{J},\wedge_p}$. Using such a cover and Proposition B.2.4, an elementary argument shows that for any finite flat $\mathcal{O}$-algebra Λ and a point in $U(\Lambda)$ with image $x \in L\mathcal{G}^{\mathcal{J}}(\Lambda)$, we can find a point in $L\mathcal{G}^{\leq\lambda,\nabla_{\tau,N}}(\Lambda)$ lifting x mod p. This implies that in fact $\widetilde{\mathcal{Z}}_N^{\leq\lambda,\tau} = \widetilde{Z}_{N,\infty}$, and the proof is finished. ☐

B.3. **Proof of Theorem 11.1.1.** It follows from Proposition B.2.2 and the discussion on the previous sectioc that we have a natural inclusion

$$\iota : \mathrm{Ch}_{\mathrm{top}}(\mathcal{X}_{\mathbf{F}}^{\leq\lambda+\rho,\tau}) \hookrightarrow \mathrm{Ch}_{\mathrm{top}}((\mathrm{Gr}_{\mathcal{G},0}^{\leq\lambda+\rho,\nabla_{\tau,0}})_{\mathbf{F}})$$

and the group on the right identifies with $\mathrm{Ch}_{\mathrm{top}}(\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq\lambda+\rho))$ by Remark B.2.3. The first part of Theorem 11.1.1 now follows from Theorem B.2.6 and an easy induction on λ' (note that if λ' is irregular, the equality we need to check reduces $0=0$).

Once we have proven the first part, we learn that the image $\iota\,\mathrm{Ch}_{\mathrm{top}}(X_{\mathbf{F}}^{\leq\lambda+\rho,\tau})$ contains classes of irreducible components occurring in $\mathfrak{sp}_{p\to 0}[\underline{\mathrm{X}}_\gamma^{\varepsilon=1}(\lambda+\rho)]$. But Corollary 9.4.4 shows that all top dimensional component of $\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\leq\lambda+\rho)$ occurs this way, hence ι is an isomorphism, and we can define $\mathrm{transfer}_\gamma = \iota^{-1}$.

Finally, we prove second part of Theorem 11.1.1. Let $\widetilde{w} = \widetilde{v}^{-1}w_0\widetilde{u} \in \mathrm{Adm}^{\mathrm{reg}}(\lambda+\rho)$ and set $\sigma = F(\pi^{-1}(\widetilde{u}) \bullet_p (t^\mu w\widetilde{v}^{-1}(0)-\rho))$. Then from what we have already proven, $\iota^{-1}\left([\underline{\mathrm{Y}}_\gamma^{\varepsilon=1}(\widetilde{w})]\right)$ must correspond to some irreducible component $\mathcal{C}_{\sigma'}$ of $\mathcal{X}_{\mathrm{red}}^{\mathrm{EG}}$. Then [LHLM23b, Lemma 7.4.6], which forces $\sigma'=\sigma$ by looking at restrictions to inertia of a generic Galois representation occurring in $\mathcal{C}_{\sigma'}$ (cf the proof of [LHLM23b, Theorem 7.4.2]).

Remark B.3.1. As alluded to in the main text, Theorem 11.1.1 actually holds under the weaker hypothesis that μ is $h_{\lambda+\rho}$-generic. The sole source for the stronger hypothesis in the above proof is the first part Proposition B.2.4 cited from [LHLM23b, Proposition 7.1.10] which controls how much p-divisibility there is in the error terms of the monodromy operator. In fact, by working with the slightly modified monodromy condition which keep track of the data of the residue $\frac{\mathcal{N}_0(A)(0)}{p^{h-1}}$ integrally, one would obtain the sought-after improved version of Proposition B.2.4. The details will occur in [LH].

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