Some Fundamental Groups in Arithmetic Geometry

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These lecture notes are more impressionistic than usual.

1 Deligne's conjectures: ℓ-adic theory

Theorem 1.1 (Deligne '87)**.** *If $X/\mathbb{C}$ is a smooth connected variety and a rank r is fixed, then there are finitely many rank r $\mathbb{Q}$-local systems which are direct factors of $\mathbb{Q}$-variations of polarizable pure Hodge structures of a given weight, definable over $\mathbb{Z}$.*

We are going to consider generalizations of this to positive characteristic.

Example 1.2. This is a generalization of a famous theorem of Faltings' 83, that there are finitely many isomorphism classes of abelian schemes on a variety of a given genus.

Deligne's recently (2012) proved an analogue for smooth projective varieties over a *finite* field. There are some new complications: you need to have a Cartier divisor D. The result says roughly that there are finitely many irreducible $\overline{\mathbb{Q}}_\ell$ Weil sheaves of rank r with ramification bounded by D, up to twist by Weil characters.

In characteristic zero, the proof is by a Lefschetz hyperplane theorem argument to reduce the theorem to a curve, since the theorem tells you that sufficiently "nice" curve $C \subset X$ has the property that $\pi_1(C) \twoheadrightarrow \pi_1(X)$. Thus, the study of representations of $\pi_1(X)$ reduces to that of the representations of $\pi_1(C)$>

Now this doesn't work in characteristic p. Drinfeld proved a kind of salvage for the *tame* part of the étale fundamental group. The proof was not too hard: the main ingredients are Bertini's Theorem and properties of tameness. This was subsequently enhanced by Kindler to obtain a "full package" Lefschetz Hyperplane Theorem for the tame part of the étale fundamental group.

In '14, Kerz-S. Saito recovered an even stronger version of Lefschetz hyperplane over perfect fields for the *abelian* part of the fundamental group. There are many technical hypotheses, but the punchline is that for a divisor D supported in $\overline{X} \setminus X$ and $\overline{Y} \subset \overline{X}$ in "good position" and of dimension at least 2, then

$$\pi_1(Y, D) \twoheadrightarrow \pi_1(X, D).$$

Question. Is there an analogue for the non-abelian fundamental group $\pi_1(X) \to \pi_1(X, D)$ generalizing $\pi_1^{\mathrm{ab}}(X, D)$? Then one could ask for a Lefschetz theorem $\pi_1(C, D_C) \twoheadrightarrow \pi_1(X, D)$ for a suitable curve C which would reflect Deligne's finiteness theorem.

2 Deligne's conjectures: crystalline theory

2.1 Crystals

Let X be a smooth geometrically connected variety over a perfect field k. Denote $W := W(k)$ the ring of Witt vectors and $K = \mathrm{Frac}(W)$ its field of functions.

One defines the crystalline sites X/W_n as PD("divided power" in French)-thickenings of coverings from Zariski opens. Then X/W is the limit.

- This defines a category of *crystals* $\mathrm{Crys}(X/W)$, i.e. sheaves of $O_{X/W}$-modules of finite presentation with transition maps which are isomorphisms.
- The $\mathbb{Q}$-linearization is the category of *crystals*. That means that the objects are the same, but the morphisms in the $\mathbb{Q}$-linearized category are vector spaces over K, i.e. we have tensored the Homs with $\mathbb{Q}$.
- The absolute Frobenius F acts on $\mathrm{Crys}(X/W)_{\mathbb{Q}}$. Berthelot-Ogus define the category of *convergent* isocrystals to be the largest full subcategory on which every object is F^∞-divisible.
- There's some categor y of *F-overconvergent* isocrystals which has a fully faithful embedding into F-convergent isocrystals.

There are many other seemingly technical variations and subcategories being mentioned ... we'll see if they come up.

2.2 Unipotent monodromy

The interest is in proving results on monodromy.

- Brieskorn proved that over $\mathbb{C}$, Gauss-Manin connections have "quasi-unipotent local monodromy" (i.e. unipotent after passing to a possibly ramified finite cover).
- Grothendieck proved an analogous result for lisse $\overline{\mathbb{Q}_\ell}$ sheaves over $\mathbb{F}_p$.
- Kedlaya proved a unipotent monodromy result for F-overconvergent crystals.

This suggests an analogy between $\mathbb{C}$ and $\mathbb{F}_q$.

- Irreducible F-overconvergent crystals with finite determinant should be analogous to irreducible lisse $\overline{\mathbb{Q}_\ell}$-sheaves with finite determinant.
- Upon bounding ramification at infinity, these two should in turn be analogous over $\mathbb{C}$ to irreducible $\mathbb{Q}$-variations of polarizable pure Hodge structures of pure weight, definable over $\mathbb{Z}$.

There are some results. One is a pretty interesting Cebotarev result:

Theorem 2.1 (Abe '13)**.** *If X is a quasiprojective smooth variety over $\mathbb{F}_q$ which is pure, then semisimple F-overconvergent isocrystals are determined by their eigenvalues at closed points.*

However, there doesn't seem to be a result for F-overconvergent isocrystals analogous to the Lefschetz stuff that was discussed in the first half.