# Discussion session on the Fargues-Fontaine Curve

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April 5, 2016

These are notes from an impromptu discussion session to elaborate upon / clarify aspects of the Fargues-Fontaine curve. Dennis Gaitsgory recalled the basic setup of the Fargues-Fontaine curve and posed some questions, and then Peter Scholze discussed the answers and some complements.

## 1 Basic setup of the Fargues-Fontaine curve

We have $A_{\inf} = W(\mathcal{O}^\flat_{\mathbb{C}_p})$. The tilt is

$$B^\flat = \varprojlim_{\Phi} B/pB = \varprojlim_{b \mapsto b^p} B.$$

The universal property of Witt vectors is that if $R$ is perfect and $B$ is $p$-adically complete, then we have

$$\mathrm{Hom}(W(R), B) = \mathrm{Hom}(R, B^\flat).$$

In other words, formation of Witt vectors is left adjoint to tilting. The unit of the adjunction is

$$\theta \colon A_{\inf} \to \mathcal{O}_{\mathbb{C}_p}.$$

There are two possible generators of $\ker\theta$.

1. $p - [p^\flat]$.
2. $\frac{1-[\epsilon]}{1-[\epsilon^{\flat}]}$ where $\epsilon = (1, \zeta_p, \zeta_p^2, \ldots) \in \mathcal{O}^\flat_{\mathbb{C}_p}$. (This is like a Gauss sum.)

We have maps

$$\begin{array}{ccc} \mathcal{O}^\flat_{\mathbb{C}_p} = A_{\inf}/(p) & & \\ \uparrow & & \\ A_{\inf} & \xrightarrow{\theta} & \mathcal{O}_{\mathbb{C}_p} \\ \downarrow & & \\ W(\overline{\mathbb{F}}_p) & & \end{array}$$

We consider $Y := \text{Spec } A_{\text{inf}} - (\text{Spec } \mathcal{O}_{\mathbb{C}_p}^{\flat} \cup \text{Spec } W(\overline{\mathbb{F}}_p))$. What points can we write down in here? Take $a \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}^{\flat}} - 0$ and consider the ideal $(p - [a])$. (For example we could take $a = p^{\flat}$.)

**Lemma 1.1.** *We have* $(A_{\text{inf}}/(p - [a]))^{\flat} = \mathcal{O}_{\mathbb{C}_p}^{\flat}$.

This is a preview of the fact we'll see later that the closed points of the the Fargues-Fontaine curve correspond to "untilts".

*Proof.* Let's try going directly to the definition:

$$(A_{\text{inf}}/(p - [a]))^{\flat} = \varprojlim_{\Phi} (A_{\text{inf}}/(p, p - [a]))^{\flat} = \varprojlim_{\Phi} \mathcal{O}_{\mathbb{C}_p}^{\flat}/a.$$

Why is this the same as $\mathcal{O}_{\mathbb{C}_p}^{\flat}$? We have a (non-canonical) isomorphism

$$\mathcal{O}_{\mathbb{C}_p}^{\flat} = \overline{\mathbb{F}_q[[T]]}^{\wedge}$$

(by which we mean the normalization in the algebraic closure of its fraction field, completed). We can arrange so that $a = T$. Then the result follows from inspection. □

## 2 Questions

**Question 1.** Is it true that all closed primes in this thing are of this form?

**Question 2.** How are these primes parametrized via Lubin-Tate theory?

**Question 3.** What is the map $Y \to (0, \infty)$.

**Question 4.** Why do we have $\pi_1(X) = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$?

## 3 Answers

### 3.1 Question 1

Not quite: you also have the 0 ideal, but that's the only exception. We're not going to discuss why.

### 3.2 Question 2

Take $a \in 1 + \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}^{\flat}}$, which we can regard as a group under multiplication. (The Lubin-Tate group for $\mathbb{Q}_p$ is just the multiplicative group, which is why we only have to consider this basic object.)

**Proposition 3.1.** *All prime ideals as in Question 1 are of the form* $\frac{1-[a]}{1-[a^{1/p}]}$ *where* $a \in (1 + \mathfrak{m} \setminus \{1\})/\mathbb{Z}_p*$.

*Remark* 3.2. We know by Question 1 that this ideal is of the form $p - [b]$ for some (not unique) $b$, but it is difficult to express thi $b$ in terms of $a$; the relation would be a horrible formula.

The proof of this proposition is by approximation on the characteristic $p$ side.

### 3.3 Question 3

*Notation.* Unless otherwise noted, we abbreviate $\mathrm{Spa}(R) := \mathrm{Spa}(R, R^0)$.

Consider $\mathrm{Spa}\,\mathbb{F}_p((t)) \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,\mathbb{F}_p((u))$.

**Lemma 3.3.** *Let K be a complete nonarchimedean field of characteristic p. Then*

$$\mathrm{Spa}\,\mathbb{F}_p((t)) \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,K$$

*is the punctured open disk. The result is that*

$$\mathrm{Spa}\,\mathbb{F}_p((t)) \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,K = \mathbb{D}_K^* = \text{“}\{x \mid 0 < |x| < 1\}\text{”}.$$

*The quotation marks mean that this is true at the level of K-points, and this is universally true with respect to all fields.*

*Remark* 3.4. We are used to thinking of $\mathrm{Spa}\,\mathbb{F}_p((t))$ as a punctured open disk, but it has only one point so this doesn't quite make sense. However, once we base change to a complete non-archimedean field it does make sense.

*Proof.* You first compute a fibered product at the level of rings of integral elements:

$$\mathrm{Spa}\,\mathbb{F}_p[[t]] \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,\mathcal{O}_K.$$

This is really the fiber product at the level of formal schemes; anyways the result is

$$\mathrm{Spa}\,\mathbb{F}_p[[t]] \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,\mathcal{O}_K = \mathrm{Spa}\,\mathcal{O}_K[[t]].$$

In here we have $\mathrm{Spa}\,\mathbb{F}_p((t)) \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,K$, which is open and corresponds to $\{t\varpi \neq 0\}$ where $\varpi \in K$ is a pseudo-uniformizer.

$$\begin{array}{ccc}
\mathrm{Spa}\,\mathbb{F}_p[[t]] \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,\mathcal{O}_K & = & \mathrm{Spa}\,\mathcal{O}_K[[t]] \\
\uparrow & & \uparrow \\
\mathrm{Spa}\,\mathbb{F}_p((t)) \times_{\mathrm{Spa}\,\mathbb{F}_p} \mathrm{Spa}\,K & = & \{t\varpi \neq 0\}
\end{array}$$

Note the similarity with the situation with $\mathbb{A}_{\mathrm{inf}}$.

*Guideline.* $O_K[[t]]$ is an analogue of $A_{\inf}$ in equal characteristic.

(Indeed, $O_K[[t]]$ is the starting point for the construction Fargues-Fontaine curve in equal characteristic, just as $A_{\inf}$ is the starting point in mixed characteristic.)

Let's consider imposing the conditions one by one. First, the open subset $\{\varpi \neq 0\}$ is the generic fiber (in the sense of Bertholot) of $\operatorname{Spa} O_K[[t]] \to \operatorname{Spa} O_K$, and that turns out to be $\mathbb{D}_K$: the open unit disk. (Why? Consider mapping out of $O_K[[t]]$: a homomorphism over $O_K$ is determined by the image of $t$, and the only restriction is that you have to send $t$ to something topologically nilpotent since it is itself topologically nilpotent.)

Then adding in the condition $t \neq 0$ is the punctured disk $\mathbb{D}_K^*$. □

Let us emphasize again: *if $\mathbb{Q}_p$ is replaced by $\mathbb{F}_p((t))$ and $K$ is algebraically closed then the equal characteristic version of $A_{\inf}$ is $O_K[[t]]$.*

*Remark* 3.5. In constructing the curve we should have started with a complete algebraically closed field of characteristic $p$, instead of 0. (In the notation of Colmez's talk, we should have started with $C^\flat$ rather than $C$.) Starting with char 0 gives a *pointed curve* because there is a distinguished choice of untilt.

## 3.4 Question 4

Finally, what is the map $\operatorname{Spa}(A_{\inf}) \setminus \{p[p^\flat]\} \to (0, \infty)$?

In equal characteristic, you have

$$\operatorname{Spa}(O_K[[t]]) \setminus \{t\varpi = 0\} \to (0, \infty).$$

We can understand this from the picture of the punctured disk. We know that

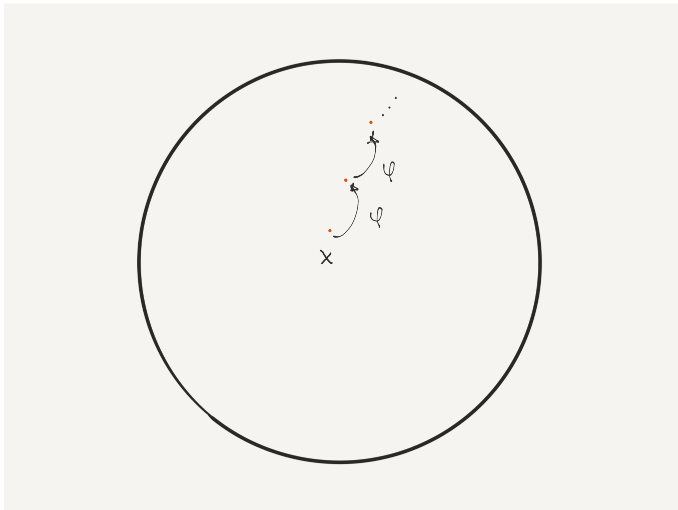
$$\operatorname{Spa}(O_K[[t]]) = \operatorname{Spa} \mathbb{F}_p((t)) \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} K = \mathbb{D}_K^*.$$

The map is a normalization of the "radius". For $x \in \mathbb{D}_K^*(K)$, you have a radius function on the punctured disk which is

$$\kappa(x) = \log_{|\varpi(x)|} |t(x)|.$$

(The "value" of a valuation is not really well-defined, since valuations are only considered up to isomorphism. However, the ratio between two values *is* well-defined.)

There is an action of $\varphi$ on $O_K[[t]]$ via $\varphi$ on $O_K$ and $t \mapsto t$. (To remember this, think to the mixed characteristic case, where $t$ is replaced by $p$. Of course there can be no nontrivial action on $p$.) This induces an action on $\mathbb{D}_K^*$. In terms of its effect on $\kappa$, it decreases $\kappa$ hence *increases* the "radius" $p^{-\kappa(x)}$.




*Warning* 3.6. As is obvious from the definition, this is *not* an action over $K$. It is a "geometric" rather than "arithmetic" Frobenius.

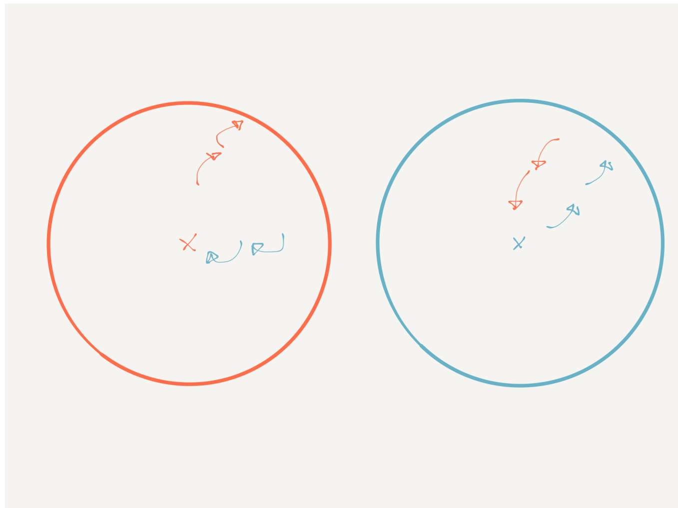
The action of $\varphi$ on $Y^{\mathrm{ad}} = \mathbb{D}_K^*$ is totally discontinuous and proper. Don't worry about the precise meaning of "proper"; suffice to say that it satisfies the properties that one would want to take a quotient.

*Definition* 3.7. The *adic Fargues-Fontaine curve* is $X^{\mathrm{ad}} = Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$. Via $\kappa$ this is fibered over $(0, \infty)/p^{\mathbb{Z}} = S^1$.

Here's a confusing thing. We have

$$\operatorname{Spa} \mathbb{F}_p((t)) \times_{\operatorname{Spa} \mathbb{F}_p} \operatorname{Spa} \mathbb{F}_p((u)) = \mathbb{D}^*_{\mathbb{F}_p((u))}.$$

Switching factors, we can apply the same reasoning to view this as $\mathbb{D}^*_{\mathbb{F}_p((t))}$. But these two disks have somehow "opposite" coordinates. In particular, if Frobenius is expanding in one picture then it is contracting in the other picture.



The space $Y$ can be compactified by adding two points. For each one the result is easy to understand, but not so for both at once.

In mixed characteristic the story is the same except that $p, [p^\flat]$ replace $t, u$.

## 4 The scheme-theoretic Fargues-Fontaine curve

### 4.1 Line bundles on $X^{\mathrm{ad}}$

By a *$\varphi$-equivariant vector bundle* on $Y^{\mathrm{ad}}$ we mean a vector bundle on $Y^{\mathrm{ad}}$ equipped with an action of $\varphi$ over $Y^{\mathrm{ad}}$. For an integer $d \in \mathbb{Z}$, we can form the equivariant line bundle

$$(\mathcal{O}_{Y^{\mathrm{ad}}}, \varphi_d = t^{-d}\varphi).$$

(In mixed characteristic this would be $p^{-d}$ instead.) This descends to the line bundle $\mathcal{O}(d)$ on $X^{\mathrm{ad}}$.

We basically declare $\mathcal{O}(1)$ to be ample. The justification comes from a Theorem of Kedlaya (or Hartl in equal characteristic) that twisting by high enough powers kills cohomology (although that was not the original motivation of Fargues-Fontaine). We can define

$$P := \bigoplus_{d \geq 0} H^0(X^{\mathrm{ad}}, \mathcal{O}(d)).$$

This is a graded $\mathbb{F}_p((t))$-algebra, but it's huge. Letting $P_d := H^0(X^{\mathrm{ad}}, \mathcal{O}(d))$, the fundamental exact sequence reads

$$0 \to \mathbb{F}_p((t)) \to P_1 \to K \to 0.$$

where the map $P_1 \to K$ is evaluation at one fixed point. But $K$ is huge over $\mathbb{F}_p((t))$; for instance it's infinite-dimensional. (There are similar sequences for $d > 1$, which we'll see later.)

*Definition* 4.1. The *schematic Fargues-Fontaine curve* is $X := \operatorname{Proj} P$.

**Theorem 4.2** (GAGA)**.** *There is a morphism of locally ringed topological spaces $X^{\mathrm{ad}} \to X$, and pullback induces an equivalence of categories*

$$\operatorname{Bun}(X) \xrightarrow{\sim} \operatorname{Bun}(X^{\mathrm{ad}}).$$

## 4.2 The mixed characteristic case

Now we replace $\mathbb{F}_p((t))$ by $\mathbb{Q}_p$. Start with a field $C$ which is complete and algebraically closed of char $p$. (In the notation of Colmez's talk, this is $C^\flat$.)

We would like to take "$\operatorname{Spa}\mathbb{Q}_p \times \operatorname{Spa} C$".

> *"Fact":* If $R$ is perfect, then one should have
>
> $$\text{“}\operatorname{Spa}\mathbb{Z}_p \times \operatorname{Spa} R\text{”} := \operatorname{Spa} W(R).$$
>
> (The point is that there is no real object to take the fiber product over; for this reason people sometimes write the base as $\mathbb{F}_1$.)

Therefore

$$\text{“}\operatorname{Spa}\mathbb{Z}_p \times \operatorname{Spa} \mathcal{O}_C\text{”} = \operatorname{Spa} W(\mathcal{O}_C) = \operatorname{Spa} A_{\mathrm{inf}}$$

and also

$$\text{“}\operatorname{Spa}\mathbb{Q}_p \times \operatorname{Spa} C\text{”} = \{p[\varpi] \neq 0\}$$

for $\varpi \in C$ a pseudo-uniformizer.

As before, we have a Frobenius $\varphi$ acting on $Y^{\mathrm{ad}}$ and a map

$$\kappa\colon Y^{\mathrm{ad}} \to (0, \infty).$$

**Lemma 4.3.** *The closed non-zero prime ideals of $Y^{\mathrm{ad}}$ are in bijection with the set of untilts, which is*

$$\left\{ (C^\#, \iota) \mid \begin{matrix} C^\# = \text{ complete, algebraically closed extension of } \mathbb{Q}_p \\ \iota\colon (C^\#)^\flat \cong C \end{matrix} \right\}.$$

*Proof.* We give one direction. Starting with $C^\#$ and an isomorphism $\iota\colon C \cong (C^\#)^\flat$, we can form

$$\ker\left(\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_{C^\#}\right).$$

This is a closed prime ideal. □

This is why fixing a point on $Y$ is fixing an untilt. Alternatively, one can think of it as "giving a $\mathbb{Q}_p$ structure on $C^\flat$." In these terms, the action of $\varphi$ on $Y$ is through its action on $\iota$.

*Remark* 4.4. In equal characteristic, untilts are just maps

$$\mathbb{F}_p((t)) \to K.$$

For $t \neq 0$, the map $t \mapsto a$ corresponds to the prime ideal $(t - a)$ on the curve.

Again we get line bundles $\mathcal{O}(d)$, and we define

$$P := \bigoplus_{d \geq 0} H^0(X^{\mathrm{ad}}, \mathcal{O}(d))$$

where $P_d = (B^+_{\mathrm{cris}})^{\varphi = p^d}$.

### 4.3 $p$-adic period rings

What's the connection to Colmez's talk? Fixing $C^\#$, we get a closed point $\infty \in Y^{\mathrm{ad}}$ with residue field $C^\#$ and hence also $\infty \in X^{\mathrm{ad}}$. Since the adic curve maps to the scheme-theoretic curve (a general fact), we also get $\infty \in X$.

In these terms, the $p$-adic period rings can be described as

- $B^+_{\mathrm{dR}}(C^\#) = \widehat{\mathcal{O}}_{X,\infty}$.
- $B_e = H^0(X - \infty, \mathcal{O}_X) = B^{\varphi=1}_{\mathrm{cris}}$.
- $B_{\mathrm{dR}} = \mathrm{Frac}(B^+_{\mathrm{dR}})$.

There is an element $t = \log[\epsilon] \in (B^+_{\mathrm{cris}})^{\varphi=p} = P_1$. The fundamental short exact sequence is

$$0 \to \mathbb{Q}_p t^d \to P_d \to B^+_{\mathrm{dR}} / \mathrm{Fil}^d \to 0.$$

This is like recording the first $d$ steps of the power series expansion at $\infty$.

For projective space, one would get finite-dimensional vector spaces over the base field. We have here a *mix* between $\mathbb{Q}_p$ and $C^\#$-vector spaces; this type of object is called a "Banach-Colmez space".

Dividing by $t^d$ and taking the colimit over $d$ gives the fundamental exact sequence

$$0 \to \mathbb{Q}_p \to B_e \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$