Diamonds for the Perplexed

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for a talk by Dennis Gaitsgory

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These are notes for a 20-minute discussion session by Dennis Gaitsgory on the basics of diamonds and their application to the moduli stack of bundles on the Fargues-Fontaine curve.

1 Diamonds for the perplexed

The "diamondification" $Y \rightsquigarrow Y^\diamond$ is a functor

$$\text{Presheaves on } \mathrm{Perf}_{\mathrm{Spa}\, E} \to \text{Presheaves on } \mathrm{Perf}_{\overline{\mathbb{F}}_q}.$$

What is $Y^\diamond$? Its functor of points is

$$Y^\diamond(R) = \begin{Bmatrix} R^\# = \text{ untilt of } R/\mathbb{Q}_p \\ \mathrm{Spa}(R^\#) \to Y/\, \mathrm{Spa}\, E \end{Bmatrix}.$$

Suppose Y is representable, i.e. is given by a perfectoid space over $\mathrm{Spa}\, E$. In this case, we claim that $Y^\diamond$ is also representable, namely by $Y^\flat$.

Proof. The R-points are untilts $R^\#$ of R over $\mathrm{Spa}\, E$. There is a theorem due to Scholze that perfectoid spaces over a perfectoid Y are equivalent to those over $Y^\flat$.

$$\begin{array}{ccc} R^\# & \longleftrightarrow & R \\ \downarrow & & \downarrow \\ Y & & Y^\flat \end{array}$$

□

Definition 1.1. A *diamond* $\mathcal{Z}$ is a presheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ such that there exists a map $Y \to \mathcal{Z}$ such that Y is representable and the morphism is representable and quasi-profinite covering.

The point is not a diamond.

Lemma 1.2. *Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$. Then*

$$(Y_{S,E})^\diamond = S \times (\mathrm{Spa}\, E)^\diamond$$

Proof. Assume $S = \mathrm{Spa}(R, R^+)$. Let's check the functor of points on $\mathrm{Spa}\, B \in \mathrm{Perf}_{\mathbb{F}_p}$. For the left side, we have by definition

$$Y_{S,E}^\diamond(B) = \left\{ \begin{matrix} B^\# / \mathrm{Spa}(E) \\ \iota\colon (B^\#)^\flat \cong B \\ \mathrm{Spa}\, B^\# \to Y_{S,E} \end{matrix} \right\}.$$

At the level of rings, a map $\mathrm{Spa}\, B^\# \to Y_{S,E}$ is the same as $W_E(R^+) \to B^{\#+}$ satisfying an invertibility condition, which by the adjunction is the same as $R^+ \to B^{\#+\flat} = B^+$.

On the right side, we have by definition

$$S \times (\mathrm{Spa}\, E)^\diamond(B) = \left\{ \begin{matrix} B^\# / \mathrm{Spa}(E) \\ \iota\colon (B^\#)^\flat \cong B \\ R^+ \to B^+ \end{matrix} \right\}.$$

These look more or less the same. The only thing left is to carefully track the invertibility condition that comes packaged in with the map $\mathrm{Spa}\, B^\# \to Y_{S,E}$. The condition is that the map $W_E(R^+) \to B^{\#+}$ must not kill ϖ or $[\varpi^\flat]$. But ϖ is a unit in E, hence also in $B^\#$, so that is built into the the fact that $B^\#$ is over $\mathrm{Spa}(E)$, and the analogous condition for $\varpi^\flat$ comes from similar reasoning with respect to the isomorphism $\iota\colon (B^\#)^\flat \cong B$.

□

2 The stack Bun_G

2.1 The classical version

Bun_G is a presheaf on $\mathrm{Perf}_{\mathbb{F}_q}$, with $\mathrm{Bun}_G(S)$ is the groupoid of G-bundles on Y_S equivariant with respect to Frob_S.

There is a map $\mathrm{Gr}_G^{B_{\mathrm{dR}}} \to \mathrm{Bun}_G \times (\mathrm{Spa}\, E)^\diamond$. This is supposed to be smooth after taking some kind of Bott-Samuelson resolution of the affine Grassmannian.

In the usual world of algebraic geometry, there is a Hecke stack Hecke which parametrizes

$$S \mapsto \left\{ (\mathcal{E}, \mathcal{E}', x, \iota)\colon \begin{matrix} x\colon S \to X \\ \mathcal{E}, \mathcal{E}' = G\text{-bundles}/S \times X \\ \iota\colon \mathcal{E} \cong \mathcal{E}'|_{S \times X - \Gamma_x} \end{matrix} \right\}.$$

We have maps

$$\begin{array}{ccccc} & & \mathrm{Hecke} & & \\ & \swarrow^{\mathcal{E}} & \downarrow{\scriptstyle x} & \searrow^{\mathcal{E}'} & \\ \mathrm{Bun}_G & & X & & \mathrm{Bun}_G \end{array}$$

2.2 The diamond version

We denote a mirror curve $X' = \operatorname{Spa} E^{\diamond} / \operatorname{Frob}$. What does this mean?

$$X'(S) = \left\{ \begin{matrix} S^{\#}/E \\ \iota\colon (S^{\#})^{\flat} \cong S \end{matrix} \right\} / \sim$$

where the equivalence relation is for the action of Frobenius on ι.

Again we have maps

$$\begin{array}{ccccc} & & \mathrm{Hecke} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{Bun}_G & & X' & & \mathrm{Bun}_G \end{array}$$

where Hecke has the functor of points

$$\mathrm{Hecke}(S) = \left\{ \begin{matrix} \mathcal{E}, \mathcal{E}' = G\text{-bundles} / Y_S, \mathrm{Frob}_S\text{-equivariant} \\ \iota\colon (S^{\#})^{\flat} \cong S \leftrightarrow S^{\#} \xrightarrow{\theta} Y_S \\ \mathcal{E} \cong \mathcal{E}' \text{ away from } \theta \text{ and Frobenius translates} \end{matrix} \right\} / \sim$$

The datum is up to Frobenius, hence the map to X'.