

Diamonds for the Perplexed

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for a talk by Dennis Gaitsgory

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These are notes for a 20-minute discussion session by Dennis Gaitsgory on the basics of diamonds and their application to the moduli stack of bundles on the Fargues-Fontaine curve.

1 Diamonds for the perplexed

The “diamondification” $Y \rightsquigarrow Y^\diamond$ is a functor

$$\text{Presheaves on } \text{Perf}_{\text{Spa } E} \rightarrow \text{Presheaves on } \text{Perf}_{\overline{\mathbb{F}}_q} .$$

What is Y^\diamond ? Its functor of points is

$$Y^\diamond(R) = \left\{ \begin{array}{l} R^\# = \text{untilt of } R/\mathbb{Q}_p \\ \text{Spa}(R^\#) \rightarrow Y/\text{Spa } E \end{array} \right\} .$$

Suppose Y is representable, i.e. is given by a perfectoid space over $\text{Spa } E$. In this case, we claim that Y^\diamond is also representable, namely by Y^\flat .

Proof. The R -points are untilts $R^\#$ of R over $\text{Spa } E$. There is a theorem due to Scholze that perfectoid spaces over a perfectoid Y are equivalent to those over Y^\flat .

$$\begin{array}{ccc} R^\# & \longleftrightarrow & R \\ \downarrow & & \downarrow \\ Y & & Y^\flat \end{array}$$

□

Definition 1.1. A *diamond* \mathcal{Z} is a presheaf on $\text{Perf}_{\overline{\mathbb{F}}_q}$ such that there exists a map $Y \rightarrow \mathcal{Z}$ such that Y is representable and the morphism is representable and quasi-profinite covering.

The point is not a diamond.

Lemma 1.2. *Let $S \in \text{Perf}_{\mathbb{F}_q}$. Then*

$$(Y_{S,E})^\diamond = S \times (\text{Spa } E)^\diamond$$

Proof. Assume $S = \text{Spa}(R, R^+)$. Let's check the functor of points on $\text{Spa } B \in \text{Perf}_{\mathbb{F}_p}$. For the left side, we have by definition

$$Y_{S,E}^\diamond(B) = \left\{ \begin{array}{l} B^\# / \text{Spa}(E) \\ \iota: (B^\#)^b \cong B \\ \text{Spa } B^\# \rightarrow Y_{S,E} \end{array} \right\}.$$

At the level of rings, a map $\text{Spa } B^\# \rightarrow Y_{S,E}$ is the same as $W_E(R^+) \rightarrow B^{\#\dagger}$ satisfying an invertibility condition, which by the adjunction is the same as $R^+ \rightarrow B^{\#\dagger} = B^+$.

On the right side, we have by definition

$$S \times (\text{Spa } E)^\diamond(B) = \left\{ \begin{array}{l} B^\# / \text{Spa}(E) \\ \iota: (B^\#)^b \cong B \\ R^+ \rightarrow B^+ \end{array} \right\}.$$

These look more or less the same. The only thing left is to carefully track the invertibility condition that comes packaged in with the map $\text{Spa } B^\# \rightarrow Y_{S,E}$. The condition is that the map $W_E(R^+) \rightarrow B^{\#\dagger}$ must not kill ϖ or $[\varpi^b]$. But ϖ is a unit in E , hence also in $B^\#$, so that is built into the fact that $B^\#$ is over $\text{Spa}(E)$, and the analogous condition for ϖ^b comes from similar reasoning with respect to the isomorphism $\iota: (B^\#)^b \cong B$.

□

2 The stack Bun_G

2.1 The classical version

Bun_G is a presheaf on $\text{Perf}_{\mathbb{F}_q}$, with $\text{Bun}_G(S)$ is the groupoid of G -bundles on Y_S equivariant with respect to Frob_S .

There is a map $\text{Gr}_G^{\text{BdR}} \rightarrow \text{Bun}_G \times (\text{Spa } E)^\diamond$. This is supposed to be smooth after taking some kind of Bott-Samuelson resolution of the affine Grassmannian.

In the usual world of algebraic geometry, there is a Hecke stack Hecke which parametrizes

$$S \mapsto \left\{ \begin{array}{l} x: S \rightarrow X \\ (\mathcal{E}, \mathcal{E}', x, \iota): \mathcal{E}, \mathcal{E}' = G\text{-bundles}/S \times X \\ \iota: \mathcal{E} \cong \mathcal{E}'|_{S \times X - \Gamma_x} \end{array} \right\}.$$

We have maps

$$\begin{array}{ccc} & \text{Hecke} & \\ \varepsilon \swarrow & \downarrow x & \searrow \varepsilon' \\ \text{Bun}_G & X & \text{Bun}_G \end{array}$$

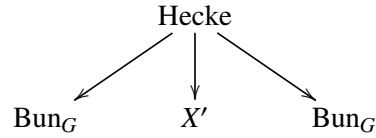
2.2 The diamond version

We denote a mirror curve $X' = \text{Spa } E^\circ / \text{Frob}$. What does this mean?

$$X'(S) = \left\{ \begin{array}{l} S^\# / E \\ \iota: (S^\#)^b \cong S \end{array} \right\} / \sim$$

where the equivalence relation is for the action of Frobenius on ι .

Again we have maps



where Hecke has the functor of points

$$\text{Hecke}(S) = \left\{ \begin{array}{l} \mathcal{E}, \mathcal{E}' = G\text{-bundles} / Y_S, \text{Frob}_S\text{-equivariant} \\ \iota: (S^\#)^b \cong S \leftrightarrow S^\# \xrightarrow{\theta} Y_S \\ \mathcal{E} \cong \mathcal{E}' \text{ away from } \theta \text{ and Frobenius translates} \end{array} \right\} / \sim$$

The datum is up to Frobenius, hence the map to X' .