Degenerations of line bundles on algebraic curves: new methods and results

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I decided to re-write my talk, so my talk will have very little to do with my abstract.

1 Historical Introduction

- In the 60s, Deligne Mumford introduced the compactification of the moduli space of stable curves $\overline{\mathcal{M}_{g,n}}$. To every $[X] \in \overline{\mathcal{M}_{g,n}}$ we can associate a *dual graph*, whose vertices are the irreducible components of X and whose edges are nodes.

  Pandharipande said, correctly, that we don't really need graph theory to study the moduli space of curves. However, I want to convince you that if we want to understand degenerations of Jacobians and line bundles, then you *do* need graph theory.

- Around the same time, Néron and Raynaud considered degenerations of Jacobians, and developed the *Néron-model* in this setting:

$$\begin{array}{ccccc} \mathcal{X}_K & \longrightarrow & \mathcal{X} & \longleftarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} R & \longleftarrow & s \end{array}$$

  The Néron model of the Jacobian of $\mathcal{X}_K$ has a group Φ_X. It was clear to Raynaud and his students that Φ_X is really an invariant of the dual graph G_X. We now understand this very well, thanks to people who studied compactified Jacobians. You can think of Néron models as a first approximation of compactificed Jacobians (they are not quite compact).

- In the 70s, Ode Seshodi ♠♠♠ TONY: [spelling???] gave a graph-theoretic description of Φ_X, as $\partial C_1(G_X, \mathbb{Z})/\partial\delta C_0(G_X, \mathbb{Z})$ (boundaries mod boundaries of coboundaries). This has the property that its cardinality is equal to the number of spanning trees of the graph. We mention this just to highlight that we are really venturing more deeply into graph theory.

- Since the 90s, algebraic graph theorists have studied the analogy between graphs and Riemann surfaces. They introduced the *Jacobian of a graph*:

$$\mathrm{Jac}(G) = \frac{\mathrm{Div}^0(G)}{\mathrm{Prim}(G)} \cong \Phi_G.$$

  They studied the Abel-Jacobi theory for graph, and investigated the possibility of a Torelli theorem for graphs. Simple examples of different graphs with the same Jacobian show that you can't such theorem.

2 Albanese varieties for tropical curves

2.1 Graphs

In 2008, I considered the following problem with Viriani. We wanted to extend the map $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$ (the moduli space of abelian varieties) to a compactification:

$$\begin{array}{ccc} \mathcal{M}_g & \hookrightarrow & \overline{\mathcal{M}_g} \\ \big\downarrow & & \big\downarrow \\ \mathcal{A}_g & \hookrightarrow & \overline{\mathcal{A}_g}^{\mathrm{mod}} \end{array}$$

This necessarily loses the injectivity. The compactification send X to the *compactified Jacobian* $\overline{P_{g-1}(X)}$ together with a polarization $\Theta(X)$.

We wanted to study the fibers of this map. We realized that the geometry of the pair $(\overline{P_{g-1}(X)}, \Theta(X))$ was completely ruled by the dual graph G_X.

We found that Kotami-Sumodo ♠♠♠ TONY: [spelling???] had introduced an Albanese variety associated to a graph: $\mathrm{Alb}(G) := H_1(G, \mathbb{R})/H_1(G, \mathbb{Z})$ with the polarization

$$(e, e') = \delta_{e,e'}.$$

Theorem 2.1 (Caporaso-Viriani)**.** *We have* $\mathrm{Alb}(G_1) \cong \mathrm{Alb}(G_2)$ *if and only if* $G_1^{(2)}$ *is "cyclically equivalent" to* $G_2^{(2)}$.

The "2-connectivization" $G^{(2)}$ means that you contract separating edges. The notion of "cyclically equivalent" is a well-known equivalence relation in graph theory.

This theorem allowed us to completely describe the fibers of the map $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}$, but that is a completely different story.

2.2 Tropical curves

Definition 2.2. A *tropical curve* $\Gamma = (G, \ell)$ is a graph together with a metric on the edges $\ell\colon E(G) \to \mathbb{R}_{>0}$.

We define the *Albanese variety* of a tropical curve Γ to be $Alb(\Gamma) = (H_1(G, \mathbb{R})/H_1(G, \mathbb{Z})$ with the polarization

$$(e, e') = \ell(e)\delta_{e,e'}.$$

Theorem 2.3 (Tropical Torelli)**.** *We have* $\mathrm{Alb}(\Gamma_1) \cong \mathrm{Alb}(\Gamma_2)$ *if and only if* $\Gamma_1^{(3)}$ *is cyclically equivalent to* $\Gamma_2^{(3)}$.

Here we need the "3-connectivization" which means you do the "2-connectivization" and then contract one of any pair of disconnecting edges.

3 Tropical moduli spaces

We wanted to work on a moduli space of tropical curves, but at the time no such thing had been discovered. Branetti, Melo, Viviani constructed a moduli space for tropical curves and tropical abelian varieties.

Mikhalkin and I contributed to define a compactification $\overline{\mathcal{M}_{g,n}}^{\mathrm{trop}}$. The moduli space of stable tropical curves has amazing analogies with the moduli space of stable curves.

Questions:

1. What is the meaning of the lengths? Can we associate to them something which is algebro-geometrically meaningful?

2. How can we explain the aforementioned amazing analogies?

If the lengths were *integers*, one would think instantly to the situation of a curve over a DVR, and think that the integers were labelling the singularities (which must be of type A_n). But why restrict ourselves to discrete valuation rings? We can consider *all* valuation rings, and allow the lengths to take real values.

Let R be a valuation ring. Given the data

$$\begin{array}{ccccc} \mathcal{X}_K & \longrightarrow & \mathcal{X} & \longleftarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \longrightarrow & \mathrm{Spec}\ R & \longleftarrow & s \end{array}$$

we want to associate a metric graph (G_X, ℓ). What is the length of an edge? Locally at e, $\mathcal{X} \to \mathrm{Spec}\ R$ has equation $xy = \rho_e$ for some $\rho_e \in K$. Since K has a valuation v, we can assign the length $v(\rho_e) \in \mathbb{R}$. So the data above gives an R-valued point of $\overline{\mathcal{M}_{g,n}}$

The theorem is that the Berkovich analytification of $\overline{\mathcal{M}_{g,n}}$ maps to $\overline{\mathcal{M}_{g,n}}^{\mathrm{trop}}$. What kind of stuff goes on in the compactification? If we have a trivial family at a node, then the local equation $xy = 0$ and we have to allow the length to be ∞.

To make the picture complete, we need to consider the Deligne-Mumford *stack* because the variety version is not toroidal, while the stack is. Then we can make a skeleton construction, and complete the picture to

$$\begin{array}{ccc} \overline{\mathcal{M}_{g,n}}^{\text{an}} & \xrightarrow{\qquad\qquad} & \overline{\mathcal{M}_{g,n}}^{\text{trop}} \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle \cong} & \\ & \Sigma(\overline{\mathcal{M}_{g,n}}) & \end{array}$$

Open question. Can one apply this to compactify the moduli space of *enriched curves*? (Enriched curves are curves plus a one-parameter deformation, modulo some equivalence relation.) I think that this would make a good project.