LOCAL LANGLANDS FUNCTORIALITY FOR YU'S SUPERCUSPIDALS I: KALETHA'S PARAMETRIZATION

SEAN COTNER AND TONY FENG

Abstract. This is the first of two papers dedicated to the explicit computation of the Fargues–Scholze correspondence. In this first part, we construct (inspired by, and extending, Kaletha's explicit Local Langlands Correspondence for non-singular supercuspidal representations) semisimple inertial L-parameters for all cuspidal representations arising from Yu's constructions, with arbitrary coefficients. We then establish a characterization of this correspondence in terms of certain functoriality properties. This characterization will be used in the sequel paper in order to compute the Fargues–Scholze correspondence after restriction to inertia.

Contents

1. Introduction

This paper is the first in a two-part series which computes the Fargues–Scholze correspondence [FS21] on inertia for supercuspidal representations arising from Yu's construction.

1.1. **Informal preview of results.** Let p be a prime and let G be a reductive group over a nonarchimedean local field F of residue characteristic p.

1.1.1. *The Local Langlands Correspondence.* The Local Langlands Correspondence (LLC) predicts a relationship between the smooth irreducible representations of $G(F)$ and the Galois representations (more generally, L-parameters) into the L-group ${}^LG$. Several rather distinct approaches to the LLC have now emerged, at least in characteristic 0. In this paper, we focus on the following ones.

(1) The "classical" approach stemming from the seminal work of Laumon–Rapoport–Stuhler [LRS93], Harris–Taylor [HT01], and Henniart [Hen00] on the Local Langlands Correspondence for GL_n. This has been extended to certain other families of groups, as we shall review in §1.6.
(2) The "explicit" approach developed by Kaletha [Kal19], [Kal21b]. This has earlier roots in the depth 0 correspondence of DeBacker–Reeder [DR09], and draws inspiration from the character formulas of Adler–Spice and DeBacker–Spice [AS09, DS18].
(3) The "geometric" approach of Fargues–Scholze [FS21], which is based on adapting the Geometric Langlands program and Lafforgue's work [Laf18] to the Fargues–Fontaine curve. Other geometric approaches due to Genestier–Lafforgue [GL18] and Zhu [Zhu25] will also be reviewed in §1.6.

Each of these approaches has its own advantages and disadvantages. The classical approach furnishes a complete Local Langlands Correspondence, but only for specific families of groups, and it is difficult to make it explicit. Kaletha's correspondence is explicit and applies uniformly to quite general groups, but only for specific families of representations, and it carries technical assumptions (e.g., that p not be too small relative to G). The Fargues–Scholze correspondence is completely general and uniform in G, and it is part of a broader categorical framework, but it is difficult to compute explicitly, and its fundamental expected properties (such as surjectivity and finiteness of fibers) remain open in general. Therefore, it is of interest and utility to reconcile these approaches. In this paper and its sequel [CF26b], we will compare (2) and (3).

1.1.2. *Our results.* One of the main results [CF26b, Theorem 10.1.1] of this two-part series *computes the Fargues–Scholze correspondence on inertia for cuspidal representations arising from Yu's construction*, assuming a certain refinement of [Fen24] which is expected in a forthcoming revision (see §1.1.3). Moreover, for non-singular cuspidal representations, we calculate the restriction of the Fargues–Scholze parameter to an explicit finite index subgroup. The answer is compatible with Kaletha's explicit Local Langlands parametrization (when the latter is applicable).

In particular, when G splits after a tamely ramified extension and p does not divide the order of the absolute Weyl group of G, work of Fintzen [Fin21b], [Fin22], building on work of Kim [Kim07], shows that Yu's construction [Yu01] exhausts all cuspidal representations, so our results determine the Fargues–Scholze parameters on inertia for all irreducible smooth representations of $G(F)$. In this case, we are also able to deduce important qualitative consequences for the Fargues–Scholze correspondence, including:

- [CF26b, Theorem 10.1.1(2)] A characterization of the representations with supercuspidal L-parameter.
- [CF26b, Theorem 10.2.1] Depth preservation, and finiteness of the fibers.
- [CF26b, Theorem 10.3.1] Surjectivity onto inertial L-parameters.

There has been considerable recent progress towards some of these results in [DL26], [Ete23], [Fu26], [GHSBP24], and [BPHT25] under various hypotheses on F and G, which we will recall in more detail in the introduction to [CF26b]. One motivation for the "soft" results bulleted above is that they are among the inputs needed in the strategy of Hansen–Mann [HM26] to prove the categorical local Langlands conjecture of [FS21]. A depth 0 version of the categorical conjecture has already been established by other means by Zhu [Zhu25] for his version of categorical local Langlands, and he has compared the resulting Local Langlands Correspondence to DeBacker–Reeder in [Zhu25, §5.3.4] when G is unramified of adjoint type.

Remark 1.1.1. There has been much recent work on the compatibility of (3) with the semisimplification of (1). In [FS21, Theorem I.9.6(ix)], Fargues–Scholze proved compatibility for $G = \mathrm{GL}_n$. Most recently, Daniels–van Hoften–Kim–Zhang [DvHKZ26, Theorem II] (building on [FS21], [HKW22], [BMHN24], [Ham25], [Han26], and [Pen26]) establish this compatibility over arbitrary p-adic local fields for most inner forms of classical groups. This does not imply our results, even for the groups to which it applies, because the compatibility of (1) and (2) was only previously known in special cases [OT21], [Tok23], [Oi26]. We note that our results imply inertial compatibility of (1) and (2) in the cases in which (1) and (3) are known to be equivalent.

1.1.3. *Our methods.* Our methods are completely different from prior approaches. In particular, they are purely local and representation-theoretic, and uniform in the group. The starting point is the theory of *modular functoriality* as developed in [Fen24], building on ideas of Treumann–Venkatesh [TV16], which gives concrete constructions of functorial (in the sense of Langlands) descents of $\overline{\mathbf{F}}_\ell$-representations in certain very special situations.

The basic idea is to use modular functoriality to probe the Fargues–Scholze parameters modulo arbitrarily large prime numbers. In particular, although the applications described above are to representation theory with characteristic 0 coefficients, *the proofs make essential use of miracles occurring in positive characteristic.*

The precise statement of modular functoriality is formulated in [CF26b, Hypothesis 4.1.4]. At the time of writing, [CF26b, Hypothesis 4.1.4] is only written in [Fen24] when ℓ is large enough (with explicit bounds) relative to G. *A forthcoming revision to* [Fen24] *is expected to prove* [CF26b, Hypothesis 4.1.4] *for all* $\ell \neq p$. Some of our main results can be proven unconditionally from the version of [CF26b, Hypothesis 4.1.4] which is currently available in the literature, and we indicate where this is the case, but the full strength of our results requires the hypothesis for every prime $\ell \neq p$.

1.2. **The role of this paper.** Let us highlight some of the main outcomes of this paper. First, as noted above, our strategy requires passage through positive characteristic coefficients, even for the ultimate characteristic 0 applications. Moreover, to obtain finiteness of L-packets, we need to (partially) compute the Fargues–Scholze L-parameters of arbitrary cuspidal representations. Therefore, we define (following the construction of Kaletha for non-singular supercuspidal representations with characteristic 0 coefficients) explicit semisimplified *inertial* L-parameters for cuspidal representations with arbitrary coefficients.[1] For "non-singular" cuspidal representations, we are able to define full L-parameters, which coincide with those constructed by Kaletha.

One of the difficulties in comparing different approaches to the LLC is that, beyond the case of GL_n, there is no known list of properties characterizing it. One of our main results gives a partial characterization of the explicit (inertial) L-parameters above, using known properties of ρ^{FS} and two forms of functoriality described below. This will be described in more detail shortly in §1.3. Both of these forms of functoriality will be verified in [CF26b] for the Fargues–Scholze correspondence.

Given a cuspidal representation π, we define "descents" of π to cuspidal representations of unramified twisted Levi subgroups H of G, under some hypotheses on H. These descents should be thought of as adjoint to the "functorial transfer" corresponding to a natural L-homomorphism ${}^L j_{H,G} \colon {}^L H \to {}^L G$. Similarly, for a cyclic ℓ-extension E/F we define (under some hypotheses) a cuspidal representation π_E of $G(E)$, which should be regarded as the "base change lifting" in the sense of Langlands. A major subtlety in both cases is that the sign characters of [FKS23] show up in nontrivial ways in these constructions. This will be sketched further in §1.4.

1.3. **Kaletha's parametrization.** Throughout the rest of this introduction, we assume that G splits after a tamely ramified extension of F and that p does not divide the order of the absolute Weyl group of G. For an algebraically closed field k of characteristic $\neq p$, let $\Pi_k(G)_{\mathrm{cusp}}$ denote the set of irreducible cuspidal smooth k-representations of $G(F)$ up to isomorphism, and let $\Phi_k^{\mathrm{ss}}(G)$ denote the set of semisimple L-parameters $W_F \to {}^L G(k)$ up to $\widehat{G}(k)$-conjugation.

In [Kal21b], when k is of characteristic 0, Kaletha singled out a class $\Pi_k(G)_{\mathrm{cusp,ns}}$ of irreducible supercuspidal smooth k-representations which he called *non-singular*, and he defined a map of sets

$$\rho^{\mathrm{Kal}} \colon \Pi_k(G)_{\mathrm{cusp,ns}} \to \Phi_k^{\mathrm{ss}}(G)$$

with the property that $\rho^{\mathrm{Kal}}(\pi)$ is always a *supercuspidal* L-parameter, i.e., there is no proper parabolic subgroup of $\widehat{G}$ normalized by $\rho^{\mathrm{Kal}}(\pi)$. Roughly, the recipe is as follows: from the representation π, one can extract a ($G(F)$-conjugacy class of) pair(s) (T,θ) consisting of an elliptic maximal F-torus $T \subset G$ and a character $\theta \colon T(F) \to k^\times$. From this, one defines

$$\rho^{\mathrm{Kal}}(\pi) = \left(W_F \xrightarrow{{}^L\theta} {}^L T(k) \xrightarrow{{}^L j_{T,G}} {}^L G(k) \right), \tag{1.3.1}$$

where ${}^L\theta \colon W_F \to {}^L T(k)$ is the L-parameter arising from the LLC for tori and ${}^L j_{T,G} \colon {}^L T \to {}^L G$ is a certain L-embedding. One of the innovations of [Kal21b], developed from [FKS23], is the definition of ${}^L j_{T,G}$, which is highly non-canonical.[2] Supporting evidence for this definition of $\rho^{\mathrm{Kal}}(\pi)$ comes from the Harish–Chandra character formulas and from stability and the endoscopic character identities proved for certain endoscopic elements in [FKS23, §4.3, Theorem 4.4.4] at least under additional technical hypotheses on F.

The first goal of this paper is to extend Kaletha's definition in two directions, as we explain in §7: first, one can allow k to be of positive characteristic with few changes. Second, for arbitrary cuspidal π, one can still extract a pair (T,θ) and attempt to use (1.3.1) to define a semisimple L-parameter. However, if π is not non-singular, then this cannot be the "correct" semisimple L-parameter in general because it has finite image mod center, while the correct L-parameter should not (in characteristic 0); moreover, the pair (T,θ) is not unique, so this definition depends on a choice. We will show that both objections disappear for the

[1] A technical point is that our definitions are only made on the level of Yu data defining a cuspidal representation. There is more than one Yu datum associated to a given representation, and we do not know that our definitions are independent of this choice because we have not proven the analogue of [HM08, Theorem 6.6], [Kal19, Corollary 3.5.5] with positive characteristic coefficients. For the applications, this distinction will be irrelevant, so we will elide this in the remainder of the introduction.

[2] In particular, despite our notation, ${}^L j_{T,G}$ depends on π and not just T!

inertial (semisimple) L-parameter

$$\rho_I^{\mathrm{Kal}}(\pi) = \left(I_F \xrightarrow{{}^L\theta} {}^L T(k) \xrightarrow{{}^L j_{T,G}} {}^L G(k) \right).$$

We propose $\rho_I^{\mathrm{Kal}}(\pi)$ as a construction of the "true" semisimple inertial L-parameter associated to π. The following theorem, which will be stated far more precisely in Theorem 10.2.1, gives some justification for believing this.

Theorem 1.3.1 (Informal; see Theorem 10.2.1)**.** *Let k be an algebraically closed field of characteristic $\neq p$. Suppose that for every finite tamely ramified extension E/F and every twisted Levi E-subgroup $H \subset G_E$, we are given a map of sets*

$$\rho = \rho_{E,H} \colon \Pi_k(H)_{\mathrm{cusp}} \to \Phi_k^{\mathrm{ss}}(H)$$

satisfying the following conditions:

(1) Agreement with class field theory in the case of tori.
(2) Compatibility with formation of central characters.
(3) Compatibility with homomorphisms which are isomorphisms up to center.
(4) A weak form of the existence of base change liftings along certain cyclic extensions of prime degree $\neq p$ for a certain class of cuspidal representations.
(5) Existence of descent to unramified twisted Levi subgroups.

Then

(A) $\rho(\pi)|_{I_F} \sim \rho_I^{\mathrm{Kal}}(\pi)$ for all $\pi \in \Pi_k(G)_{\mathrm{cusp}}$.
(B) If π is non-singular, then there is an explicit integer N (depending on π) such that

$$\rho(\pi)|_{W_{F_N}} \sim \rho^{\mathrm{Kal}}(\pi)|_{W_{F_N}}. \tag{1.3.2}$$

Moreover, $\rho(\pi)$ is irreducible if and only if $\rho^{\mathrm{Kal}}(\pi)$ is irreducible.

Remark 1.3.2. The integer N in (1.3.2) depends on the torus T associated to π. If T is maximally unramified (i.e., T contains a maximal unramified F-torus), then one can take $N = 1$. In Example 10.2.6, we study the case that $T \cap G_{\mathrm{der}}$ is totally ramified[3], and we show that if G is an unramified form of GL_r then one can take $N = r$, while if G is an unramified form of SO_{2r+1}, Sp_{2r}, or SO_{2r} then one can take $N \leq 2^{\lceil \log_2(2r) \rceil}$ to be a power of 2. In Example 10.2.8, we give a brief indication of how to establish concrete bounds on N when G is an unramified form of an exceptional group, if ρ is compatible with Weil restriction and products; in all cases, one can show $N \leq 36$. We will show in [CF26b, Theorem 10.4.3] that if $G_{F^{\mathrm{unr}}} \cong \mathrm{SL}_r$, then one can take $N = 1$.

Remark 1.3.3. In [CF26b], we will apply Theorem 1.3.1 to $\rho = \rho^{\mathrm{FS}}$. Properties (1), (2), and (3) follow from [FS21, Theorem I.9.6(i), (iii), (v)], and (5) uses the existing [Fen24, Theorem 1.3.1]. For (4), we assume [CF26b, Hypothesis 4.1.4] for cyclic base change at all primes $\ell \neq p$. As mentioned, this hypothesis is known in many cases, and it is expected in general in a forthcoming revision of [Fen24]. Theorem 10.2.1 gives a sharper statement with more flexible hypotheses and explicit bounds.

The very rough idea of the proof of Theorem 1.3.1 is to use (5) to reduce to the case that $T \cap G_{\mathrm{der}}$ is totally ramified, then use (4) (which applies after this reduction) to pass to a field extension E/F such that T_E is elliptic and maximally unramified. The a priori surprising feature of Theorem 1.3.1 is that assumption (4) will only concern *wild inertia*, so this naive argument only determines $\rho(\pi)|_{P_F}$ a priori. However, once one passes to the case that $T \cap G_{\mathrm{der}}$ is totally ramified, the situation becomes extremely rigid, to the point that the L-parameter $\rho(\pi)$ is almost completely determined by its restriction to P_F.

1.4. **Functoriality.** To make hypotheses (4) and (5) in Theorem 1.3.1 more precise, we must specify the sense in which they hold for ρ^{Kal} and ρ_I^{Kal}. We start with (5).

[3] Using assumption (5) of Theorem 10.2.1, one can reduce to this case by replacing G by an unramified twisted Levi. However, the bounds become more complicated because these twisted Levis are products (up to center) of several smaller rank groups, and the bound one obtains is roughly the least common multiple of the bounds associated to such factors.

1.4.1. *Unramified twisted Levis.* Recall that a closed F-subgroup $H \subset G$ is an *unramified twisted Levi* if H becomes a Levi factor of a parabolic subgroup of G after passing to an unramified extension of F. For such a subgroup H, there is a canonical L-embedding ${}^L j_{H,G} \colon {}^L H \to {}^L G$, whose definition we recall in Definition 4.5.1.

Theorem 1.4.1 (Theorem 9.3.3)**.** *Let π be an irreducible cuspidal k-representation of $G(F)$, let (T, θ) be a torus-character pair associated to π as above, and let $H \subset G$ be an unramified twisted Levi F-subgroup containing T. Then there is an explicit finite set of pairs $\{(L, \pi_L)\}$, where L is a Levi F-subgroup of H and π_L is a cuspidal irreducible smooth k-representation of $L(F)$ such that*

$$\rho_I^{\mathrm{Kal}}(\pi) \sim {}^L j_{L,G} \circ \rho_I^{\mathrm{Kal}}(\pi_L).$$

If π is non-singular, then $L = H$ for every such pair, π_H is non-singular, and

$$\rho^{\mathrm{Kal}}(\pi) \sim {}^L j_{H,G} \circ \rho^{\mathrm{Kal}}(\pi_H).$$

The existence of *some* pair (L, π_L) as in Theorem 1.4.1 has limited content, at least when G is quasi-split, since [Kal21b] already shows that every irreducible L-parameter is in the image of ρ^{Kal}; the content of the theorem is that π_L is constructed explicitly. The fact that this is subtle can be seen from Remark 9.4.2, which shows that if π is non-singular and (T, θ) is the torus-character pair associated to π, then although the pair (T_H, θ_H) associated to π_H satisfies $T_H = T$, the character θ_H need not be $N_G(T)(F)$-conjugate to θ. What is true is that θ_H is conjugate to $\theta\epsilon$ for some sign character $\epsilon \colon T(F) \to k^\times$, but ϵ is generally nontrivial and depends on π; we will discuss this in §1.5. Beyond dealing with sign issues, the main content in the construction of Theorem 1.4.1 is to carefully track the semi-rational Lusztig series associated to the special fibers of Bruhat–Tits group schemes, as introduced in [Cot26, §2.8].

Assumption (5) in Theorem 1.3.1 is a slight variant of the statement that Theorem 1.4.1 holds with ρ (resp. $\rho|_{I_F}$) in place of ρ^{Kal} (resp. ρ_I^{Kal}) for a *single* pair (L, π_L).

1.4.2. *Cyclic field extensions.* Let E/F be a cyclic extension of degree $\ell \neq p$. If π is an irreducible cuspidal k-representation of $G(F)$, then we do not have a general recipe for producing a representation π_E of $G(E)$ such that $\rho^{\mathrm{Kal}}(\pi_E) \sim \rho^{\mathrm{Kal}}(\pi)|_{W_E}$. Even for depth 0 representations, this appears to be complicated: see, for example, the treatment of ramified $\mathrm{U}(2,1)$ in [AL10, §7]. However, there are two situations in which matters simplify considerably.

Theorem 1.4.2 (Proposition 8.2.3)**.** *Let π be an irreducible cuspidal k-representation of $G(F)$, let $\ell \neq p$ be a prime, and let E/F be the unramified extension of degree ℓ. If ℓ is sufficiently large, then there is an explicit irreducible cuspidal k-representation π_ℓ of $G(E)$ such that*

$$\rho_I^{\mathrm{Kal}}(\pi_\ell) \sim \rho_I^{\mathrm{Kal}}(\pi).$$

If π is non-singular, then π_ℓ is non-singular and

$$\rho^{\mathrm{Kal}}(\pi_\ell) \sim \rho^{\mathrm{Kal}}(\pi)|_{W_E}.$$

Once again, the significance of Theorem 1.4.2 is that the representation π_ℓ is explicit. The construction of π_ℓ in Theorem 1.4.2 involves the Glauberman correspondence [Gla68], which, roughly speaking, provides base change lifts for finite reductive groups under field extensions of large prime degree.[4] Plugging this into Yu's construction leads to the existence of π_ℓ. The sign issues described above are relatively minor in this case by Lemma 5.2.1 and Lemma 5.2.4.

Next, consider the case that π is *toral*, i.e., that the pair (T, θ) associated to π is toral in the sense of [CO25, Definition 3.7].[5] Any toral irreducible cuspidal k-representation is non-singular.

Theorem 1.4.3 (Proposition 8.3.1)**.** *Let π be a toral irreducible cuspidal k-representation of $G(F)$, let $\ell \neq p$ be a prime, let (T, θ) be a torus-character pair associated to π, and let E/F be a cyclic extension of prime degree $\ell \neq p$ such that T_E is elliptic. There is an explicit irreducible cuspidal k-representation π_E of $G(E)$ such that if either ℓ is odd, or $\ell = 2$ and $k = \overline{\mathbf{F}}_2$, we have*

$$\rho^{\mathrm{Kal}}(\pi_E) \sim \rho^{\mathrm{Kal}}(\pi)|_{W_E}.$$

[4] It is likely true that the "correct" base change lifting is given by Shintani descent. As explained in [Cot26, Remark 4.3.1], Shintani descent and the Glauberman correspondence differ by a twist, which when reduced modulo ℓ corresponds to the ℓ-Frobenius endomorphism. Thus our construction also involves a certain twist.

[5] The word *toral* has two competing definitions in the literature; our definition is the less restrictive one.

The same sign issues described above for unramified twisted Levis arise in the setting of Theorem 1.4.3, particularly when E/F is ramified. If $\ell = 2$, then one can still define a candidate base change lifting π_E for arbitrary k, but the sign issues are even more severe; this is the reason for the restriction $k = \overline{\mathbf{F}}_2$.

Assumption (4) of Theorem 1.3.1 is the statement that, in the setting of Theorem 1.4.3, one has

$$\rho(\pi_E)|_{P_F} \sim \rho(\pi)|_{P_F}.$$

We emphasize that this is a fairly weak hypothesis: it only concerns toral representations, and it only requires functoriality after restriction to wild inertia. Although Theorem 1.4.3 shows that π_E is an actual base change lifting of π from F to E, adding this as a further assumption to Theorem 1.3.1 does not significantly improve (1.3.2). In any case, for the Fargues–Scholze correspondence we can only prove that π_E is a base change lifting of π in characteristic ℓ.

1.5. **Sign characters.** One important theme of the explicit Local Langlands Correspondence is the necessity of certain subtle sign characters, as alluded to above and treated definitively in [FKS23]. For simplicity, we will assume in the following discussion that π is non-singular, but the issues are essentially the same in general.

In the proof, we see sign characters arise in two independent ways. First, if $T \subset H \subset G$ are as in Theorem 1.4.1, then there is a naive way to construct a candidate π_H using Yu's construction [Yu01], which we temporarily call π'_H, such that (T, θ) is a torus-character pair associated to π'_H. From π'_H, one obtains an L-embedding ${}^L j_{T,H} \colon {}^L T \to {}^L H$ as above. However, in general,

$$ {}^L j_{T,G} \circ {}^L \theta \not\sim {}^L j_{H,G} \circ {}^L j_{T,H} \circ {}^L \theta, \tag{1.5.1}$$

so π'_H cannot be the correct functorial descent.[6] This discrepancy gives rise to one sign character.

The second source of sign characters is (as explained in [FKS23]) that Yu's construction should itself be twisted by a sign character which depends on the data involved in the definition. This suggests that the naive construction of π'_H should also be twisted by a sign character; finding the "correct" twist, however, requires a careful analysis of [FKS23]. The precise sign characters that intervene appear slightly ad hoc; they will be partially explained by the Tate cohomology of the Weil representation in [CF26b], where they will ultimately arise from the sign characters in [Gér77].

1.6. **Context and related work.** In this subsection we discuss further context on the different approaches to the Local Langlands correspondence, and related work on their compatibility.

1.6.1. *The classical approach.* After the Local Langlands Correspondence was established for GL_n [LRS93, HT01, Hen00], it was extended to more general groups in work of many people.

Over p-adic fields (in characteristic 0), twisted endoscopy was used to extend the correspondence to quasi-split symplectic and special orthogonal groups by Arthur [Art13], to quasi-split unitary groups by Mok [Mok15], and to their inner forms of unitary groups by Kaletha–Mínguez–Shin–White [KMSW14]. Mœglin–Renard [MR18, §3.6] establish the tempered correspondence for pure inner forms of quasi-split special orthogonal and unitary groups. Applying the Langlands classification to these tempered correspondences, including those for Levi subgroups, gives the correspondence for all irreducible representations.[7]

The theta correspondence gives another construction: for even orthogonal groups, this was used by Chen–Zou [CZ21], and one can deduce a correspondence for special orthogonal groups up to outer automorphism. Gan–Takeda and Gan–Tantono [GT11, GT14] also deduce the correspondence for GSp_4 and its non-split inner form; Gan–Takeda and Choiy treat Sp_4 and its non-split inner form [GT10, Cho17]. For G_2, Gan–Savin [GS23] use an exceptional theta correspondence, while Aubert–Xu [AX22, Theorem 10.2.1] give an explicit construction when $p \neq 2, 3$.

Other classical constructions include Hiraga–Saito's work on inner forms of SL_n [HS12], completed over arbitrary nonarchimedean local fields by Aubert–Baum–Plymen–Solleveld [ABPS16]. They use restriction from inner forms of GL_n, whose correspondence follows from the Jacquet–Langlands correspondence. For split odd special orthogonal groups, Jiang–Soudry [JS03] established the correspondence for generic supercuspidal representations using local converse theorems and descent.

[6] Note that this does not contradict [Kal21a, Proposition 6.9], which looks similar to the negation of this statement but whose notation implicitly refers to different L-embeddings than those appearing in (1.5.1).

[7] For even special orthogonal groups, these endoscopic classifications retain the usual ambiguity under the outer automorphism induced by the full orthogonal group.

Over local fields of positive characteristic, the method of close fields has been used to construct a Local Langlands correspondence in some situations: Ganapathy [Gan15] treats GSp_4 when $p > 2$, Ganapathy–Varma [GV17] treat split classical groups under bounds on p, and [ABPS16] treats inner forms of SL_n without such bounds. Gan–Lomelí [GL18, §7] give a different construction of parameter maps for quasi-split classical groups using globalization and Lafforgue's work, under some hypotheses.

In a different direction, for coefficients of characteristic $\ell \neq p$, Vignéras [Vig01] constructs the semisimple correspondence for $\mathrm{GL}_n(F)$ over an arbitrary nonarchimedean local field F. Her proof uses congruences between automorphic representations following Khare [Kha01].

1.6.2. *The geometric approach.* The "geometric" approach to the Local Langlands Correspondence itself divides into several different approaches.

The earliest is that of of Genestier–Lafforgue [GL18], which constructs a semisimplified Local Langlands correspondence over function fields. It uses the global Langlands correspondence of V. Lafforgue [Laf18] plus local-global compatibility. Later, Li-Huerta [LH23] has shown that the Genestier–Lafforgue and Fargues–Scholze correspondences agree in this case.

Zhu [Zhu25] has developed his own approach to the categorical Local Langlands Conjecture. While this does not yet have a "spectral action" (hence no construction of L-parameters attached to higher depth representations), Zhu is able to construct a fully faithful embedding of depth zero blocks into the spectral category. The recent work [GHI$^+$26] compares the relevant categories of sheaves in the approaches of Fargues–Scholze and Zhu. For $G = \mathrm{GL}_n$, this had been done earlier by Ben-Zvi–Chen–Helm–Nadler [BZCHN24].

Building on Zhu's work, Helm–Solleveld–Xu [HSX26, Theorem 1.2] construct fully faithful functors from Bernstein blocks with generic supercuspidal support to ind-coherent sheaves on parameter stacks for a class of quasi-split groups over p-adic fields that includes symplectic, special orthogonal, and unitary groups, and G_2. This in turn has applications to the Fargues–Scholze correspondence through the work of Hansen–Mann [HM26].

1.7. **Outline of the paper.** We now give some details on the various sections appearing in this work.

In §2, we detail the various conventions on notations and definitions which we will use.

In §3, we recall the sign characters defined in [FKS23] which will be used throughout. In §4, we recall the definition of sets of χ-data and their associated L-embeddings from [LS87] (mainly following [Kal21a]), and we define the particular L-embeddings which will be of use to us. In §5, we establish certain compatibilities between the sign characters of §3 and the L-embeddings of §4 under base change and descent to unramified twisted Levi subgroups, and in particular we show how to modify (1.5.1) to a true statement. Since several of the proofs involve rather long and complicated calculations, they are mostly relegated to Appendix A.

In §6, we recall the twisted Yu construction from [FKS23], which involves modifying the classical Yu construction [Yu01] with a certain sign character. With an eye towards applications to functoriality, we carefully analyze the behavior of this construction under base change and passage to unramified twisted Levis.

In §7, we describe our extension of Kaletha's parametrization in detail, and we prove some basic compatibility results for it. In §8, we describe the explicit base change liftings required for Theorems 1.4.2 and 1.4.3, and we prove those theorems. The proofs mainly involve applying the results of §5. In §9, we describe the explicit descent to unramified twisted Levi subgroups used in Theorem 1.4.1, and we prove that theorem. In §9.4, we give an explicit example showing that the sign issues can be quite subtle.

Finally, in §10, we prove Theorem 1.3.1, establishing a partial characterization of the LLC in terms of Theorems 1.4.1 and 1.4.3.

1.8. **AI methodology.** After an initial draft of this paper was written, AI tools (specifically GPT 5.6 and Claude Fable 5) were used for proofreading, revision, and the creation of all tables and figures. Finally, GPT 6 was used for literature search in order to give appropriate attributions in the introduction. AI tools were not used in any other way.

1.9. **Acknowledgements.** We thank Jeff Adler, Adèle Bourgeois, Charlotte Chan, Stephen DeBacker, Jessica Fintzen, Michael Harris, Alex Hazeltine, Alex Ivanov, Josh Lansky, Monica Nevins, Sian Nie, Peter Scholze, David Schwein, Jack Sempliner, Loren Spice, Jay Taylor, and Cheng-Chiang Tsai for helpful conversations.

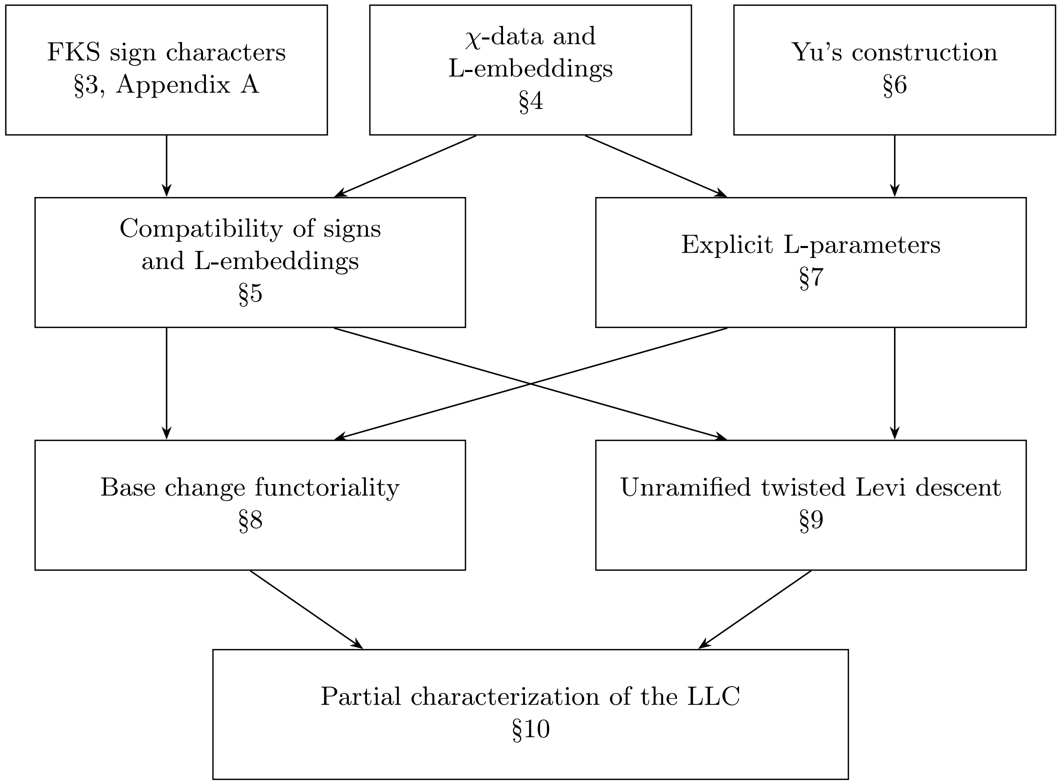

FIGURE 1.7.1. Leitfaden of the paper.

S.C. acknowledges support from the National Science Foundation under Award No. 2402231 and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 950326).

T.F. was supported by the NSF (grants DMS-2302520 and DMS-2441922), the Simons Foundation, and the Alfred P. Sloan Foundation.

2. NOTATION AND CONVENTIONS

2.1. **Local fields.** Let F be a local field with ring of integers $\mathcal{O}_F$ and residue field $\mathbf{F}_q$ of characteristic $p > 0$. We fix a separable closure $\overline{F}/F$. A separable algebraic extension of F is always understood as a subfield of $\overline{F}$. For any finite separable field extension E/F and integer $n \geq 1$, we denote by E_n the unramified subextension of $\overline{F}/E$ of degree n. We write $F^{\mathrm{unr}} \subset \overline{F}$ for the maximal unramified extension of F. Unless another field is specified, ϖ denotes a uniformizer of F.

We write W_F for the Weil group of F, $I_F \triangleleft W_F$ for the inertia subgroup, and $P_F \triangleleft W_F$ for the wild inertia subgroup.

We write Fr for geometric Frobenius in W_F/I_F, and also for a chosen lift to W_F.

2.2. **Reductive groups.** We denote by G a connected reductive group over F. (Our convention is that reductive groups over fields are not necessarily assumed to be connected.) All representations and characters of $G(F)$ are understood to be smooth unless otherwise stated. If $C \subset G$ is a closed subscheme, then we denote by $Z_G(C)$ (resp. $N_G(C)$) the scheme-theoretic centralizer (resp. normalizer) of C in G.

If H is a group scheme of finite type over a field k, then we use H° to denote the identity component of H. The notation H^0, which is sometimes used instead of H°, will be reserved for Yu's construction. We write $\pi_0(H) = H/H^\circ$ for the group scheme of connected components and H_{red} for the underlying reduced subscheme. Similarly, if $\mathcal{O}$ is a discrete valuation ring and $\mathcal{G}$ is a smooth separated $\mathcal{O}$-group scheme, then we will use $\mathcal{G}^\circ$ to denote the (open) relative identity component of $\mathcal{G}$ over $\mathcal{O}$, a smooth finite type $\mathcal{O}$-group scheme with connected fibers.

For a connected reductive group H, we write $Z(H)$ for its center, H_{der} for its derived subgroup, $H_{\mathrm{ad}} = H/Z(H)$ for its adjoint quotient, and $H_{\mathrm{ab}} = H/H_{\mathrm{der}}$ for its abelianization. We write $\mathrm{rk}\, H$ for its absolute rank. For semisimple H, $\pi_1(H)$ denotes its algebraic fundamental group. We use $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathrm{Lie}^*(G)$ for the linear dual of $\mathrm{Lie}(G)$.

If T is an F-torus, then $T(F)_{\mathrm{b}}$ denotes the maximal bounded subgroup of $T(F)$.

We make compatible choices of square roots of q in $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and $\overline{\mathbf{F}}_\ell$. We use ind for ordinary induction in finite or finite-index settings and c-Ind for compact induction.

2.3. **Dual groups.** The Langlands dual group $\widehat{G}$ is regarded over $\mathbf{Z}[1/p]$, though we will often base change to a field of characteristic $\neq p$ without changing the notation.

We write ${}^L G = \widehat{G} \rtimes W_F$ for the L-group of G, with the usual pinning-preserving action of W_F on $\widehat{G}$. If ρ_1, ρ_2 are (inertial) L-parameters valued in ${}^L G(k)$, then $\rho_1 \sim \rho_2$ means that ρ_1 and ρ_2 are $\widehat{G}(k)$-conjugate. We reserve *equality* of L-parameters for actual equality of chosen representatives by homomorphisms.

2.4. **Bruhat–Tits buildings.** We let $\mathcal{B}(G)$ denote the extended Bruhat–Tits building of G over F. For a point $x \in \mathcal{B}(G)$, we let $[x] = [x]_G$ denote the corresponding point of $\mathcal{B}(G_{\mathrm{der}})$. Let $\mathcal{G}_x$ (resp. $\mathcal{G}_{[x]}$) denote the smooth separated $\mathcal{O}_F$-group scheme with generic fiber G and $\mathcal{G}_x(\mathcal{O}_F) = G(F)_x$ (resp. $\mathcal{G}_{[x]}(\mathcal{O}_F) = G(F)_{[x]}$), the stabilizer of x (resp. $[x]$) in $G(F)$, as in [KP23, Remark 8.3.4]. Note that $\mathcal{G}_x$ is of finite type, but $\mathcal{G}_{[x]}$ might not be. Let $\overline{G}_x$ (resp. $\overline{G}_{[x]}$) denote the quotient of the special fiber of $\mathcal{G}_x$ (resp. $\mathcal{G}_{[x]}$) by the unipotent radical of its identity component.

For $r \geq 0$, the notations $G(F)_{x,r}$ and $G(F)_{x,r+}$ refer to the Moy–Prasad filtration at x; analogous notation is used for the Lie algebra. We implicitly use the normalized valuation v of F in this definition; however, if E/F is a finite separable extension then $G(E)_{x,r}$ and $G(E)_{x,r+}$ will be defined using the unique extension of v to E. Thus $G(F)_{x,r} = G(E)_{x,r} \cap G(F)$.

2.5. **Coefficients.** We use k for coefficient rings, which need not be fields. Our coefficient prime ℓ is always distinct from p. We write $\overline{\mathbf{Z}}_\ell$ for the valuation ring of $\overline{\mathbf{Q}}_\ell$, with residue field $\overline{\mathbf{F}}_\ell$.

For a k-algebra k' and a k-module or representation V, we write $V_{k'} = V \otimes_k k'$. Subscripts on group schemes and homomorphisms likewise indicate base change.

3. The FKS sign character

There is a notorious sign error in [Yu01], noticed by Loren Spice and arising from a misprint in [Gér77, Theorem 2.4(b)]; see the introduction to [Fin21a]. This has led to substantial "sign adjustments" in the literature, the culmination of which is [FKS23]. The latter paper introduces a complicated sign character $\epsilon = \epsilon_G$ and shows (among other things) that if Yu's construction is adjusted by ϵ then the errant results in [Yu01] become valid.

The character ϵ is the product of five individually complicated sign characters, called $\epsilon_{\sharp,x}$, $\epsilon_{\flat,0}$, ϵ_f, $\epsilon_{\flat,1}$, and $\epsilon_{\flat,2}$ in [FKS23]. The character $\epsilon_{\sharp,x}$ was introduced in [DS18, §4.3]; the character $\epsilon_{\flat,0}$ first appeared in [Kal21a, Proposition 5.27]; and the character ϵ_f is the quotient of two sign characters which appear in [Kal19, Definition 4.7.3].

Write $\mathbf{F} = \mathbf{F}_q$ for the residue field of F. Whenever a coefficient ring $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{Z}}_\ell, \overline{\mathbf{F}}_\ell\}$ is fixed in the following constructions, we also fix once and for all an additive character $\psi = \psi_k \colon F \to k^\times$ which is trivial on the maximal ideal $\mathfrak{p}_F \subset \mathcal{O}_F$ and whose restriction to $\mathcal{O}_F$ is the inflation of a nontrivial character $\psi^0 \colon \mathbf{F}_q \to k^\times$ of the form $\mathbf{F}_q \xrightarrow{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}} \mathbf{F}_p \xrightarrow{\overline{\psi}} k^\times$. The characters $\psi_{\overline{\mathbf{Q}}_\ell}$ and $\psi_{\overline{\mathbf{F}}_\ell}$ are chosen to be compatible with $\psi_{\overline{\mathbf{Z}}_\ell}$ by extension of scalars and reduction, respectively.

Assume that $p \neq 2$ and that G is a connected reductive F-group of adjoint type, and let M be a tamely ramified twisted Levi F-subgroup of G. Fix an embedding $\mathcal{B}(M) \subset \mathcal{B}(G)$ of enlarged Bruhat–Tits buildings, and let $x \in \mathcal{B}(M)$. Let $\mathcal{M}_x$ be the smooth affine $\mathcal{O}_F$-group scheme with $\mathcal{M}_x(\mathcal{O}_F) = M(F)_x$, and let $\overline{M}_x$ be the quotient of its special fiber by the unipotent radical of its identity component. If $T \subset G$ is a tamely ramified maximal torus, let $\overline{T}$ denote the analogous quotient of the special fiber of the finite-type Néron model of T. Then [FKS23, Theorem 3.4] exhibits a sign character

$$\epsilon_x^{G/M} \colon \overline{M}_x(\mathbf{F}) \longrightarrow \{\pm 1\},$$

whose definition we now recall.

3.1. **Preliminary notions.** For an F-torus $T \subset M$, let $\Phi(G_{\overline{F}}, T_{\overline{F}})$ (resp. $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})$) denote the set of nonzero weights (which we will call roots, even if T is not maximal) for the action of $T_{\overline{F}}$ on $\mathfrak{g}_{\overline{F}}$ (resp. $(\mathfrak{g}/\mathfrak{m})_{\overline{F}}$). Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$ and $\Sigma = \Gamma \times \{\pm 1\}$, so Γ and Σ act on $\Phi(G_{\overline{F}}, T_{\overline{F}})$ and $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})$.

Definition 3.1.1. A root $\alpha \in \Phi(G_{\overline{F}}, T_{\overline{F}})$ is said to be *symmetric* if $-\alpha$ lies in the Γ-orbit of α; otherwise α is said to be *asymmetric*. For each absolute root $\alpha \in \Phi(G_{\overline{F}}, T_{\overline{F}})$, let F_α be the finite extension of F such that $\mathrm{Gal}(\overline{F}/F_\alpha)$ is the stabilizer of α under the Γ-action, and let $F_{\pm\alpha}$ be the subextension of F_α such that $\mathrm{Gal}(\overline{F}/F_{\pm\alpha})$ is the (setwise) stabilizer of the set $\{\pm\alpha\}$ under the Γ-action.

A symmetric root α is *unramified* if the quadratic extension $F_\alpha/F_{\pm\alpha}$ is unramified; otherwise α is *ramified*.

Below we indicate the corresponding subsets of $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})$ with self-explanatory subscripts, e.g., $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,unram}}$ denotes the subset of symmetric unramified roots in $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})$.

Let $\mathbf{F}_\alpha$ (resp. $\mathbf{F}_{\pm\alpha}$) denote the residue field of F_α (resp. $F_{\pm\alpha}$). If α is unramified, so that $\mathbf{F}_\alpha \neq \mathbf{F}_{\pm\alpha}$, then let $\mathbf{F}_\alpha^1$ denote the kernel of the norm map $\mathrm{Nm}_{\mathbf{F}_\alpha/\mathbf{F}_{\pm\alpha}} \colon \mathbf{F}_\alpha^\times \to \mathbf{F}_{\pm\alpha}^\times$. Let $\mathrm{sgn}_{\mathbf{F}_\alpha^\times}$ (resp. $\mathrm{sgn}_{\mathbf{F}_\alpha^1}$) denote the unique nontrivial character $\mathbf{F}_\alpha^\times \to \{\pm 1\}$ (resp. $\mathbf{F}_\alpha^1 \to \{\pm 1\}$). Let $\mathrm{ord}_x(\alpha) = \mathrm{ord}_x(\alpha)_F$ denote the set of real numbers t such that $\mathfrak{g}_\alpha(F_\alpha)_{x,t+} \subsetneq \mathfrak{g}_\alpha(F_\alpha)_{x,t}$.

If $M \subset G$ is a twisted Levi F-subgroup, then we will write M_{sc} to denote the preimage of $M \cap G_{\mathrm{der}}$ in the universal cover $G_{\mathrm{sc}} \to G_{\mathrm{der}}$.

3.2. **Definition of the sign character.** Fix a *good* element[8] $X \in \mathrm{Lie}^*(M_{\mathrm{sc,ab}})(F) \subset \mathrm{Lie}^*(M_{\mathrm{sc}})(F)$, i.e., an element such that there exists $r \in \mathbf{R}$ such that

$$\mathrm{ord}(\langle X, H_\alpha \rangle) = -r \quad \text{ for all } \alpha \in \Phi((G/M)_{\overline{F}}, S_{\overline{F}}),$$

where S ranges over all tamely ramified maximal F-tori in M and $H_\alpha = \mathrm{d}\alpha^\vee(1)$.[9] For a tamely ramified maximal F-torus $T \subset M$ such that $x \in \mathcal{B}(T)$, the character $\epsilon_x^{G/M}|_{\overline{T}(\mathbf{F})}$ is characterized as a product

$$\epsilon_x^{G/M}|_{\overline{T}(\mathbf{F})} = \epsilon_{\sharp,x}^{G/M} \cdot \epsilon_{\flat,0}^{G/M} \cdot \epsilon_f^{G/M} \cdot \epsilon_{\flat,1}^{G/M} \cdot \epsilon_{\flat,2}^{G/M} \tag{3.2.1}$$

where each term on the right hand side is a character of $\overline{T}(\mathbf{F})$. Remarkably, [FKS23, Theorem 3.4 and Corollary 3.6] shows that there is a unique character $\epsilon_x^{G/M} \colon \overline{M}_x(\mathbf{F}) \to \{\pm 1\}$ whose restriction to $\overline{T}(\mathbf{F})$ is given by (3.2.1) for every tamely ramified maximal F-torus $T \subset M$ with $x \in \mathcal{B}(T)$.

We now define the sign characters appearing in (3.2.1). Fix a tamely ramified maximal F-torus $T \subset M$ such that $x \in \mathcal{B}(T)$, and fix $\gamma \in \overline{T}(\mathbf{F})$. Define first

$$\epsilon_{\sharp,x}^{G/M}(\gamma) = \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{asym}}/\Sigma \\ r/2 \in \mathrm{ord}_x(\alpha)}} \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,unram}}/\Gamma \\ r/2 \in \mathrm{ord}_x(\alpha)}} \mathrm{sgn}_{\mathbf{F}_\alpha^1}(\alpha(\gamma)). \tag{3.2.2}$$

Let Z_M denote the maximal central torus of M, and for $\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})$ let $\alpha_M = \alpha|_{Z_M}$. Let $e(\alpha/\alpha_M) = e(\alpha/\alpha_M)_F$ denote the ramification degree of F_α/F_{α_M}, where F_{α_M} is the finite extension of F corresponding to α_M. We extend the terminology of Definition 3.1.1 to the nonzero weights of Z_M: the field $F_{\pm\alpha_M}$ is the fixed field of the setwise stabilizer of $\{\pm\alpha_M\}$ in Γ, and α_M is symmetric ramified precisely when $-\alpha_M \in \Gamma \cdot \alpha_M$ and $F_{\alpha_M}/F_{\pm\alpha_M}$ is ramified. Define

$$\epsilon_{\flat,0}^{G/M}(\gamma) = \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{asym}}/\Sigma \\ \alpha_M \in \Phi((G/M)_{\overline{F}}, (Z_M)_{\overline{F}})_{\mathrm{sym,ram}} \\ 2 \nmid e(\alpha/\alpha_M)}} \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,unram}}/\Gamma \\ \alpha_M \in \Phi((G/M)_{\overline{F}}, (Z_M)_{\overline{F}})_{\mathrm{sym,ram}} \\ 2 \nmid e(\alpha/\alpha_M)}} \mathrm{sgn}_{\mathbf{F}_\alpha^1}(\alpha(\gamma)). \tag{3.2.3}$$

For a symmetric ramified root $\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})$, choose a nonzero element $X_\alpha \in \mathfrak{g}_\alpha(F_\alpha)$, and choose $\tau \in \Gamma$ such that $\tau \cdot \alpha = -\alpha$. It is observed in [Kal15, §3.1] that

$$c_\alpha = \frac{[X_\alpha, \tau X_\alpha]}{H_\alpha} \tag{3.2.4}$$

[8]Here we follow the definitions in [FKS23], which differ from those of [Yu01]; see [FKS23, Remark 4.1.3] for a discussion of the difference.

[9]Generic elements (the definition of which we will recall later) are good, so such elements will arise in the setting of Yu's construction.

lies in $F_{\pm\alpha}^\times$. Let

$$\kappa_\alpha \colon F_{\pm\alpha}^\times/\operatorname{Nm}_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times) \cong \{\pm 1\}$$

denote the unique isomorphism. Replacing X_α by aX_α, with $a \in F_\alpha^\times$, multiplies (3.2.4) by $a\tau(a) = \operatorname{Nm}_{F_\alpha/F_{\pm\alpha}}(a)$. Thus $\kappa_\alpha(c_\alpha)$ is independent of the choice of X_α; we denote it by $f_{(G,T)}(\alpha)$. Moreover, $f_{(G,T)}(\alpha)$ depends only on the $\operatorname{Gal}(\overline{F}/F)$-orbit of α. If $\mathfrak{p}_\alpha$ denotes the maximal ideal of $\mathcal{O}_{F_\alpha}$, then we define

$$\epsilon_f^{G/M}(\gamma) = \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,ram}}/\Gamma \\ \alpha(\gamma) \in -1+\mathfrak{p}_\alpha}} f_{(G,T)}(\alpha). \tag{3.2.5}$$

For any $\alpha \in \Phi(G_{\overline{F}}, T_{\overline{F}})$, let $\ell_{p'}(\alpha^\vee) = \ell_{G,p'}(\alpha^\vee) \in \{1,2,3\}$ denote the prime-to-p part of the normalized square length of $\alpha^\vee$, as in [FKS23, Definition 5.4.1]. Let e_α denote the ramification degree of F_α/F. We define then

$$\epsilon_{\flat,1}^{G/M}(\gamma) = \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,ram}}/\Gamma \\ \alpha(\gamma) \in -1+\mathfrak{p}_\alpha}} (-1)^{[\mathbf{F}_\alpha:\mathbf{F}]+1} \operatorname{sgn}_{\mathbf{F}_\alpha^\times}(e_\alpha \ell_{p'}(\alpha^\vee)). \tag{3.2.6}$$

By [FKS23, Lemma 5.6.5], the integer $e(\alpha/\alpha_M)$ is odd for every symmetric ramified root α, so the exponent in (3.2.7) is an integer. Finally, define

$$\epsilon_{\flat,2}^{G/M}(\gamma) = \prod_{\substack{\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,ram}}/\Gamma \\ \alpha(\gamma) \in -1+\mathfrak{p}_\alpha}} \operatorname{sgn}_{\mathbf{F}_\alpha^\times}(-1)^{(e(\alpha/\alpha_M)-1)/2}. \tag{3.2.7}$$

We have now defined $\epsilon_x^{G/M}$ as a character of $\overline{M}_x(\mathbf{F})$; inflation along $M(F)_x \to \overline{M}_x(\mathbf{F})$ gives a character of $M(F)_x$. For a general connected reductive G, set $G_{\mathrm{ad}} = G/Z(G)$ and let M_{ad} be the image of M in G_{ad}. If $[x]_G$ denotes the image of x in $\mathcal{B}(G_{\mathrm{der}}) = \mathcal{B}(G_{\mathrm{ad}})$, then we define (cf. [FKS23, Lemma 4.1.2 and Definition 4.1.10]):

$$\epsilon_x^{G/M} \colon M(F)_{[x]_G} \longrightarrow M_{\mathrm{ad}}(F)_{[x]_G} \xrightarrow{\epsilon_x^{G_{\mathrm{ad}}/M_{\mathrm{ad}}}} \{\pm 1\}.$$

We emphasize that, despite the notation, the character $\epsilon_x^{G/M}$ depends on the good element X, although it only does so through r.

3.2.1. *Unramified twisted Levis.* We record here a couple of basic properties concerning the behavior of $\epsilon_x^{G/M}$ under passage to unramified twisted Levi subgroups of G; this will be studied further in §5.1.

Lemma 3.2.1. *Let $H \subset G$ be an unramified twisted Levi, and let $T \subset H$ be an F-subtorus containing Z_H. Every symmetric ramified root in $\Phi(G_{\overline{F}}, T_{\overline{F}})$ lies in $\Phi(H_{\overline{F}}, T_{\overline{F}})$.*

Proof. Choose a finite unramified extension E/F for which H_E is a Levi factor of an E-parabolic subgroup $P \subset G_E$, and let U be the unipotent radical of P. Observe that

$$\Phi(G_{\overline{F}}, T_{\overline{F}}) - \Phi(H_{\overline{F}}, T_{\overline{F}}) = \Phi(U_{\overline{F}}, T_{\overline{F}}) \sqcup -\Phi(U_{\overline{F}}, T_{\overline{F}}), \tag{3.2.8}$$

where $\operatorname{Gal}(\overline{F}/E)$ preserves the two summands. Thus the $\operatorname{Gal}(\overline{F}/E)$-orbit of a weight α in (3.2.8) cannot contain $-\alpha$, which implies that $F_\alpha/F_{\pm\alpha}$ is unramified. □

Kaletha has informed us that in general there is no character of $M(F)_{[x]_G}$ extending the characters $\epsilon_{\sharp,x}^{G/M}$ associated to the various tamely ramified maximal F-tori T of M such that $x \in \mathcal{B}(T)$. The following lemma shows, among other things, that this is not the case if M is an unramified twisted Levi of G.

Lemma 3.2.2. *Let $M \subset G$ be a tamely ramified twisted Levi, and let $H \subset G$ be an unramified twisted Levi such that $M \cap H$ contains a maximal F-torus of G. Assume $x \in \mathcal{B}(M \cap H)$. There exists a unique character $\epsilon_{\sharp,x}^{G/M,H}$ of $(M \cap H)(F)_{[x]_G}$ such that for each tamely ramified maximal F-torus T of $M \cap H$ such that $x \in \mathcal{B}(T)$, the restriction of $\epsilon_{\sharp,x}^{G/M,H}$ to $\overline{T}_{[x]}(\mathbf{F}_q)$ is equal to $\epsilon_{\sharp,x}^{G/M} \cdot \epsilon_{\sharp,x}^{H/M\cap H}$. If*

$$V = \mathfrak{g}(F)_{x,\frac{r}{2}} / \left(\mathfrak{g}(F)_{x,\frac{r}{2}+} + \mathfrak{h}(F)_{x,\frac{r}{2}} + \mathfrak{m}(F)_{x,\frac{r}{2}}\right),$$

and there exists a parabolic F-subgroup $P \subset G$ with Levi factor H, and V_+ denotes the image of $\mathfrak{p}(F)_{x,\frac{r}{2}}$ in V, then

$$\epsilon_{\sharp,x}^{G/M,H} = \operatorname{sgn}_{\mathbf{F}_q^\times} \circ \det(-|V_+).$$

Proof. Uniqueness follows from [FKS23, Lemma 3.5], so we need only prove existence. We may and do assume that G is of adjoint type, in which case $[x]_G = x$. We will apply [FKS23, Proposition 5.1.13]. Let $Z_{M\cap H}$ be the maximal central F-subtorus of $M \cap H$. Following the notation of *loc. cit.*, define

$$\mathfrak{S} = \{\alpha \in \Phi((G/(M\cup H))_{\overline{F}}, (Z_{M\cap H})_{\overline{F}})\colon r/2 \in \mathrm{ord}_x(\alpha)\}/I_F,$$

Note that $\mathfrak{S}$ inherits a Σ-action from the Σ-action on $\Phi(G_{\overline{F}}, (Z_{M\cap H})_{\overline{F}})$. By Lemma 3.2.1, every root $\alpha \in \Phi((G/(M\cup H))_{\overline{F}}, (Z_{M\cap H})_{\overline{F}})$ is either asymmetric or symmetric unramified; such an α is asymmetric if and only if its image $\mathcal{O}_\alpha$ in $\mathfrak{S}$ is asymmetric in the sense that $\mathcal{O}_{-\alpha} \not\subset \Gamma\cdot\mathcal{O}_\alpha$. Observe the $(M\cap H)(F)_x$-equivariant decomposition

$$V_{\overline{\mathbf{F}}_q} = \bigoplus_{\mathcal{O}\in\mathfrak{S}} V_{\mathcal{O}},$$

where $V_{\mathcal{O}}$ is the sum of the weight spaces for the action of $Z_{M\cap H}(F)$ on $V_{\overline{\mathbf{F}}_q}$ corresponding to weights in $\mathcal{O}$. Given $\mathcal{O}\in\mathfrak{S}$, let $\mathbf{F}_{\mathcal{O}}$ be the residue field of the fixed field of the stabilizer of $\mathcal{O}$ in Γ, and define a character $\chi_{\mathcal{O}}\colon (M\cap H)(F)_x \to \mathbf{F}_{\mathcal{O}}^\times$ by

$$\chi_{\mathcal{O}}(\gamma) = \det(\gamma, V_{\mathcal{O}}).$$

Define $\mathbf{F}_{\pm\mathcal{O}}$ similarly, and let $\mathbf{F}^1_{\mathcal{O}} = \ker(\mathrm{Nm}_{\mathbf{F}_{\mathcal{O}}/\mathbf{F}_{\pm\mathcal{O}}}\colon \mathbf{F}^\times_{\mathcal{O}} \to \mathbf{F}^\times_{\pm\mathcal{O}})$. Let $L_{\mathcal{O}}\colon \mathbf{F}^1_{\mathcal{O}} \to \mathbf{F}^\times_{\mathcal{O}}/\mathbf{F}^\times_{\pm\mathcal{O}}$ be the isomorphism which is inverse to the natural isomorphism $t\mapsto t\sigma(t)^{-1}$, where $\sigma\in\mathrm{Gal}(\mathbf{F}_{\mathcal{O}}/\mathbf{F}_{\pm\mathcal{O}})$ is the nontrivial element.

By [FKS23, Proposition 5.1.13], there is a character $\epsilon_{\mathfrak{S}}\colon (M\cap H)(F)_x \to \mathbf{F}^\times/\mathbf{F}^{\times 2}$ which satisfies

$$\epsilon_{\mathfrak{S}}(\gamma) = \prod_{\mathcal{O}\in\mathfrak{S}_{\mathrm{asym}}/\Sigma} \mathrm{Nm}_{\mathbf{F}_{\mathcal{O}}/\mathbf{F}}(\chi_{\mathcal{O}}(\gamma)) \cdot \prod_{\mathcal{O}\in\mathfrak{S}_{\mathrm{sym}}/\Gamma} \mathrm{Nm}_{\mathbf{F}_{\mathcal{O}}/\mathbf{F}}(L_{\mathcal{O}}(\chi_{\mathcal{O}}(\gamma))) \pmod{\mathbf{F}^{\times 2}}$$

for all $\gamma\in(M\cap H)(F)_x$. If $T\subset M\cap H$ is a tamely ramified maximal F-torus such that $x\in\mathcal{B}(T)$, then applying [FKS23, Remark 5.1.12] to the Σ-equivariant surjection

$$\{\alpha\in\Phi((G/(M\cup H))_{\overline{F}}, T_{\overline{F}})\colon r/2\in\mathrm{ord}_x(\alpha)\} \to \mathfrak{S}$$

and noting again Lemma 3.2.1, we see that the character $\epsilon_{\mathfrak{S}}$ restricts to $T(F)_x$ as $\epsilon^{G/M}_{\sharp,x}\epsilon^{H/M\cap H}_{\sharp,x}$, as desired.

Now suppose that there exists a parabolic F-subgroup $P\subset G$ with Levi factor H, and let V_+ be as in the lemma statement. Fix a tamely ramified maximal F-torus T of $M\cap H$ such that $x\in\mathcal{B}(T)$. Let Φ denote the set of roots $\alpha\in\Phi((G/(M\cup H))_{\overline{F}}, T_{\overline{F}})$ such that $r/2\in\mathrm{ord}_x(\alpha)$. We can decompose

$$V = \bigoplus_{\substack{\omega\in\Phi/\Gamma \\ r/2\in\mathrm{ord}_x(\omega)}} V_\omega,$$

where $\mathrm{ord}_x(\omega) \coloneqq \mathrm{ord}_x(\alpha)$ for any $\alpha\in\omega$, and

$$V_\omega = \left(\bigoplus_{\alpha\in\omega/I_F} (V_{\overline{\mathbf{F}}_q})_\alpha\right)\cap V.$$

The subspace V_+ of V corresponds to a Γ-stable subset $\Psi\subset\Phi$ such that $\Phi = \Psi\sqcup -\Psi$. We have therefore

$$\det(t|V_+) = \prod_{\alpha\in\Psi/I_F}\alpha(t).$$

Since Ψ is Γ-stable, every element of Φ is asymmetric. Fix an orbit $\omega\in\Phi/\Gamma$, and let $\alpha\in\omega$; without loss of generality, we may assume $\omega\subset\Psi$. Then for $t\in\overline{T}_{[x]}(\mathbf{F}_q)$ we have

$$\begin{aligned}\mathrm{sgn}_{\mathbf{F}_q^\times}(\det(t|V_+\cap(V_\omega\oplus V_{-\omega}))) &= \prod_{\alpha\in\omega/I_F}\alpha(t)^{(q-1)/2} \\ &= \mathrm{Nm}_{\mathbf{F}_\alpha/\mathbf{F}_q}(\alpha(t)^{(q-1)/2}) \\ &= \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(t)),\end{aligned}$$

as desired. ∎

3.3. **Sign characters associated to a twisted Levi tower.** Suppose we are given a sequence $G^0 \subset G^1 \subset \cdots \subset G^d = G$ of twisted Levi F-subgroups each of which splits after a tamely ramified extension of F. Fix embeddings $\mathcal{B}(G^0) \subset \mathcal{B}(G^1) \subset \cdots \subset \mathcal{B}(G^d)$ and suppose $x \in \mathcal{B}(G^0)$.[10] For every i, let $[x]_{G^{i+1}}$ be the image of x in $\mathcal{B}(G^{i+1}_{\mathrm{der}})$. Note

$$G^0(F)_{[x]_G} \subset G^i(F)_{[x]_{G^{i+1}}},$$

so the restriction in (3.3.1) is defined. If we are given good (relative to G^{i+1}) elements $X_i^* \in \mathrm{Lie}^*(G^i_{\mathrm{sc,ab}})(F)$ of depth $-r_i$ for $0 \le i \le d-1$, then we define a character $\epsilon = \epsilon_G = \epsilon_G(\vec{G}, x, \vec{r})$ on $G^0(F)_{[x]_G}$ by

$$\epsilon = \prod_{i=0}^{d-1} \epsilon_x^{G^{i+1}/G^i}|_{G^0(F)_{[x]_G}}. \tag{3.3.1}$$

Similarly, if $T \subset G^0$ is a tamely ramified maximal F-torus such that $x \in \mathcal{B}(T)$, then the formulas (3.2.2)–(3.2.7) define characters of $T(F)_{[x]_G}$. We may therefore define a character $\epsilon_{\sharp,x} = \epsilon_{G,\sharp,x} = \epsilon_{\vec{G},\sharp,x}$ of $T(F)_{[x]_G}$ by

$$\epsilon_{\sharp,x} = \prod_{i=0}^{d-1} \epsilon_{\sharp,x}^{G^{i+1}/G^i}|_{T(F)_{[x]_G}}. \tag{3.3.2}$$

We define characters $\epsilon_{\flat,0}$, ϵ_f, $\epsilon_{\flat,1}$, and $\epsilon_{\flat,2}$ of $T(F)_{[x]_G}$ completely analogously. Define also

$$\epsilon_\flat \coloneqq \epsilon_{G,\flat} \coloneqq \epsilon_{\flat,0}\epsilon_{\flat,1}\epsilon_{\flat,2}. \tag{3.3.3}$$

Finally, if $H \subset G$ is an unramified twisted Levi F-subgroup such that $G^0 \cap H$ contains a maximal F-torus of G, then define

$$\epsilon_{\sharp,x}^{\vec{G},H} = \prod_{i=0}^{d-1} \epsilon_{\sharp,x}^{G^{i+1}/G^i, H\cap G^{i+1}}|_{(G^0\cap H)(F)_{[x]_G}} \tag{3.3.4}$$

with notation as in Lemma 3.2.2.

4. χ-data and L-embeddings

As mentioned in the introduction, the explicit (inertial) L-parameter associated to a supercuspidal $\overline{\mathbf{Q}}_\ell$-representation $\pi = \pi(\Psi)$ arising from a Yu datum Ψ is obtained by first constructing a pair (T, θ) consisting of a tamely ramified maximal F-torus $T \subset G$ and a character $\theta\colon T(F) \to \overline{\mathbf{Q}}_\ell^\times$, and then defining $\rho^{\mathrm{Kal}}(\Psi)$ as the composition of ${}^L\theta\colon W_F \to {}^LT(\overline{\mathbf{Q}}_\ell)$ and an L-embedding ${}^Lj_{T,G}\colon {}^LT \to {}^LG$. When T is unramified, the L-embedding ${}^Lj_{T,G}$ is canonical, but in general it is not, and another innovation of [FKS23] is to select a "good" choice of ${}^Lj_{T,G}$ (depending on both T and θ).

We begin by recalling a number of definitions which originate from [LS87], though we will largely follow [Kal19], [Kal21a], and [Kal21b]. Fix a tamely ramified F-torus $S \subset G$. Let $\Phi(G_{\overline{F}}, S_{\overline{F}})$ denote the set of nonzero weights for the action of $S_{\overline{F}}$ on $\mathfrak{g}_{\overline{F}}$. We follow the notation of §3.1: let $\Gamma = \mathrm{Gal}(\overline{F}/F)$ and $\Sigma = \Gamma \times \{\pm 1\}$, and for each nonzero weight $\alpha \in \Phi(G_{\overline{F}}, S_{\overline{F}})$, let F_α be the fixed field of the stabilizer of α in Γ, and let $F_{\pm\alpha} \subset F_\alpha$ be the fixed field of the (setwise) stabilizer of $\{\pm\alpha\}$. Let

$$\kappa_\alpha\colon F_{\pm\alpha}^\times / \mathrm{Nm}_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times) \to \{\pm 1\} \tag{4.0.1}$$

denote the unique injection, which is an isomorphism if and only if α is symmetric. Throughout this section, let k be a ring among $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and $\overline{\mathbf{F}}_\ell$, where $\ell \neq p$ is a prime number. We choose once and for all a square root of q in k, and we will assume that the choices for $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and $\overline{\mathbf{F}}_\ell$ are compatible in the obvious sense.

4.1. **χ-data and a-data.** In [LS87], the following definition is introduced.

Definition 4.1.1. A *set of χ-data* $(\chi_\alpha)_{\alpha\in\Phi(G_{\overline{F}}, S_{\overline{F}})}$ for S with coefficients in k is a collection of continuous homomorphisms $\chi_\alpha\colon F_\alpha^\times \to k^\times$ satisfying the following properties for all $\alpha \in \Phi(G_{\overline{F}}, S_{\overline{F}})$.

(1) $\chi_{-\alpha} = \chi_\alpha^{-1}$.
(2) $\chi_{\gamma(\alpha)} = \chi_\alpha \circ \gamma^{-1}$ for all $\gamma \in \Gamma$.
(3) $\chi_\alpha|_{F_{\pm\alpha}^\times} = \kappa_\alpha$ if α is symmetric.

In some cases, there are canonical choices of χ-data.

[10] This property does not depend on the choice of embedding $\mathcal{B}(G^0) \subset \mathcal{B}(G)$.

Definition 4.1.2. If $(\chi_\alpha)_\alpha$ is a set of χ-data such that $\chi_\alpha = 1$ for all asymmetric α and χ_α is the unramified character $F_\alpha^\times \to k^\times$ sending a uniformizer to -1 for all symmetric unramified roots α, then we will call (χ_α) *minimally ramified*. Given a set $\chi = (\chi_\alpha)_\alpha$, we will let $\min\chi = (\min\chi_\alpha)_\alpha$ denote the unique minimally ramified set of χ-data such that $\min\chi_\alpha = \chi_\alpha$ for all symmetric ramified α.

If all roots for $S_{\overline{F}}$ are either asymmetric or symmetric unramified, then we will refer to the unique minimally ramified set of χ-data $(\chi_\alpha)_\alpha$ as the *canonical set of χ-data*.

Example 4.1.3. If S is unramified, or more generally if S becomes maximally unramified after a tame extension of *odd* ramification degree, then every root in $\Phi(G_{\overline{F}}, S_{\overline{F}})$ is either asymmetric or symmetric unramified and thus $\Phi(G_{\overline{F}}, S_{\overline{F}})$ has a canonical set of χ-data.

As we will soon recall, if $M \subset G$ is a tamely ramified twisted Levi with maximal central torus Z, then a set of χ-data (χ_α) for Z can be used to induce a $\widehat{G}(k)$-conjugacy class of L-embeddings ${}^L j_\chi \colon {}^L M \to {}^L G$.

One of the key technical points in [Kal19] (resp. [Kal21b] and [FKS23]) is the definition of a set of χ-data (χ'_α) (resp. (χ''_α)) associated to a non-singular torus-character pair (T, θ). When T is unramified, this set of χ-data is the canonical set described above, but in general it depends on a number of subtle choices.

We now recall the notion of sets of a-data, introduced in [LS87].

Definition 4.1.4 ([Kal19, §4.6])**.** If $M \subset G$ is a tamely ramified twisted Levi F-subgroup and $T \subset M$ is a tamely ramified maximal F-torus, then a *set of a-data* $\{a_\alpha\}$ for $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})$ is an assignment to each $\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})$ of an element $a_\alpha \in F_\alpha^\times$ such that $a_{\gamma\alpha} = \gamma(a_\alpha)$ and $a_{-\alpha} = -a_\alpha$ for all $\gamma \in \Gamma$ and all α.

4.2. **The set of χ-data associated to a quadruple.** In this subsection, we assume $p \neq 2$. Let $\Psi = (\vec{G}, x, \vec{r})$ be a triple as at the end of §3.3, and for each $0 \le i \le d-1$ suppose that we are given a good element $X_i^* \in \mathrm{Lie}^*(G^i_{\mathrm{sc,ab}})(F)_{-r_i}$. Fix a maximally unramified maximal F-torus $T \subset G^0$. Recall the character $\psi^0 \colon \mathbf{F}_q \to k^\times$ fixed in §3. For each $0 \le i \le d-1$ and each $\alpha \in \Phi((G^{i+1}/G^i)_{\overline{F}}, T_{\overline{F}})$, define $a_\alpha \in F_\alpha^\times$ by $a_\alpha = \langle X_i^*, H_\alpha \rangle$. As observed in [FKS23, Notation 4.2.2], for each i the collection of a_α constitutes a set of a-data for $\Phi((G^{i+1}/G^i)_{\overline{F}}, T_{\overline{F}})$, and $a_\alpha \in F_{\alpha_i}^\times$.

We now use this set of a-data to define a set of χ-data (χ''_α). By convention, set $G^{-1} = Z^{-1} = T$. Given an integer $-1 \le i \le d-1$ and $\alpha \in \Phi(G^{i+1}_{\overline{F}}, T_{\overline{F}}) - \Phi(G^i_{\overline{F}}, T_{\overline{F}})$, let α_i denote the restriction of α to the maximal central torus $Z^i_{\overline{F}}$ of $G^i_{\overline{F}}$. Define $\chi''_{\alpha_i} = \chi''_{G,\alpha_i} \colon F_{\alpha_i}^\times \to k^\times$ as in [FKS23, Notation 4.2.2], as follows.[11]

(1) If α_i is asymmetric, then $\chi''_{\alpha_i} = 1$.
(2) If α_i is symmetric unramified, then χ''_{α_i} is the unique unramified quadratic character.
(3) If α_i is symmetric ramified, then χ''_{α_i} is the unique tamely ramified character whose restriction to $\mathcal{O}_{F_{\alpha_i}}^\times$ is the inflation of the quadratic character of the residue field $\mathbf{F}_{\alpha_i}^\times$, whose restriction to $F_{\pm\alpha_i}^\times$ is the κ_{α_i} from (4.0.1), and which has the property that
$$\chi''_{\alpha_i}(\ell_{G,p'}(\alpha^\vee) a_\alpha) = (-1)^{f_{\alpha_i}+1} \mathfrak{G}_{\mathbf{F}_{\alpha_i}}(\psi^0), \tag{4.2.1}$$
where $\ell_{G,p'}(\alpha^\vee)$ is the prime-to-p part of the normalized squared length of $\alpha^\vee$ (so $\ell_{G,p'}(\alpha^\vee) \in \{1,2,3\}$)[12], the positive integer f_{α_i} is the degree of the residue field extension $\mathbf{F}_{\alpha_i}/\mathbf{F}_q$ and for any finite field extension $\mathbf{F}'/\mathbf{F}_q$ we define the normalized Gauss sum[13]
$$\mathfrak{G}_{\mathbf{F}'}(\psi^0) = |\mathbf{F}'|^{-1/2} \sum_{x \in \mathbf{F}'^\times} \mathrm{sgn}_{\mathbf{F}'^\times}(x) \psi^0(\mathrm{tr}_{\mathbf{F}'/\mathbf{F}_q}(x)).$$

Observe that χ''_{α_i} only depends on α_i by [FKS23, Notation 4.2.2]. Finally, define $\chi''_\alpha = \chi''_{G,\alpha} = \chi''_{\alpha_i} \circ \mathrm{Nm}_{F_\alpha/F_{\alpha_i}}$. We will call $\chi''_G = \chi''_{(\vec{G},x,\vec{r},\vec{X},T)} = (\chi''_\alpha)$ the *set of χ-data associated to* $(\vec{G}, x, \vec{r}, \vec{X}, T)$. We note that if T becomes maximally unramified in G after an extension of odd ramification degree, then χ''_G is the canonical set of χ-data associated to T.

[11] Another similar description of a set of χ-data, also denoted by χ'', is described in the arXiv version of [Kal21b, §3.5]; Kaletha has informed us that these are likely different, and that the definition in [FKS23] is the "correct" one.

[12] Since we have assumed $p \neq 2$ throughout this section, the subscript p' is only relevant if $p = 3$ and G has a simple factor of type G_2. The set of χ-data introduced in [FKS23, §4.2] uses $\ell_G(\alpha^\vee)$ in place of $\ell_{G,p'}(\alpha^\vee)$, but that section also assumes that p does not divide the order of any bond in the absolute Dynkin diagram for G, which rules out the G_2 case above.

[13] Recall that we have fixed a square root of q.

4.3. **The L-embedding associated to a set of χ-data.** We now drop the assumption $p \neq 2$ from the previous section. As we have mentioned, from a set of χ-data (χ_α) for the maximal central torus Z of a tamely ramified twisted Levi $M \subset G$, we obtain a $\widehat{G}(k)$-conjugacy class ${}^L j_\chi \colon {}^L M \to {}^L G$ of L-embeddings. We recall this construction now, following [Kal21a, §§3.3, 3.4, 6.1], but we will restrict the generality for simplicity. The content below amounts essentially to unraveling the constructions in [Kal21a]; because our use of sets of χ-data will be limited to a few rather concrete settings, we choose not to recall the notion of gauges or the notations t_p, $s_{p/q}$, etc.

4.3.1. *The quasi-split case.* Suppose first that G and M are quasi-split, and let (T_M, B_M) and (T_G, B_G) be F-rational Borel pairs in M and G, respectively. Choose $g \in G(\overline{F})$ such that $(T_M)_{\overline{F}} = g(T_G)_{\overline{F}}g^{-1}$ and $(B_M)_{\overline{F}} \subset g(B_G)_{\overline{F}}g^{-1}$ and such that the system of simple absolute roots $\Delta((B_M)_{\overline{F}}, (T_M)_{\overline{F}})$ for (B_M, T_M) is contained in the system of simple absolute roots $g\Delta((B_G)_{\overline{F}}, (T_G)_{\overline{F}})g^{-1}$ for $(g(B_G)_{\overline{F}}g^{-1}, g(T_G)_{\overline{F}}g^{-1})$. Given $\gamma \in \Gamma$, define $\gamma_{M,G}$ to be the image of $g^{-1}\gamma(g)$ in the Weyl group $\Omega(G_{\overline{F}}, (T_G)_{\overline{F}})$.

Choose a Γ-stable pinning $(\widehat{B}, \widehat{T}, \{X_{\alpha^\vee}\}_{\alpha \in \Delta(G_{\overline{F}}, (T_G)_{\overline{F}})})$ for $\widehat{G}$, so we obtain a standard Levi k-subgroup scheme $\widehat{M} \subset \widehat{G}$ containing $\widehat{T}$ with set of simple roots $\Delta(\widehat{M}, \widehat{T})$ dual to

$$g^{-1}\Delta((B_M)_{\overline{F}}, (T_M)_{\overline{F}})g \subset \Delta(G_{\overline{F}}, (T_G)_{\overline{F}}) = \Delta(\widehat{G}, \widehat{T})^\vee.$$

Given $\gamma \in \Gamma$, let γ_G denote the pinned automorphism of $\widehat{G}$ induced by γ, and also use $\gamma_{M,G}$ to denote the element of $\Omega(\widehat{G}, \widehat{T})$ corresponding to the above element $\gamma_{M,G}$ under the canonical isomorphism $\Omega(\widehat{G}, \widehat{T}) \cong \Omega(G_{\overline{F}}, (T_G)_{\overline{F}})$. Observe that $\Delta(\widehat{M}, \widehat{T}) \subset \Delta(\widehat{G}, \widehat{T})$ is stable under the automorphism $\gamma_{M,G} \rtimes \gamma_G \in \Omega(\widehat{G}, \widehat{T}) \rtimes \mathrm{Aut}(\widehat{G}, \widehat{B}, \widehat{T}, \{X_{\alpha^\vee}\})$, but $\Delta(\widehat{G}, \widehat{T})$ is not stable under $\gamma_{M,G} \rtimes \gamma_G$ in general.

Let $n \colon \Omega(\widehat{G}, \widehat{T}) \to N_{\widehat{G}}(\widehat{T})(k)$ denote the Tits section of the projection map $N_{\widehat{G}}(\widehat{T})(k) \to \Omega(\widehat{G}, \widehat{T})$, given on simple reflections s_α by defining $n(s_\alpha) = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$, where $x_{\pm\alpha}$ are the root subgroup homomorphisms determined by the chosen pinning, and defined in general by $n(w) = n(s_1) \cdots n(s_r)$ if $w = s_1 \cdots s_r$ is any reduced expression of w into simple reflections.

Let $A \subset \Phi := \Phi((G/M)_{\overline{F}}, (T_M)_{\overline{F}})/\Omega(M_{\overline{F}}, (T_M)_{\overline{F}})$ be a chosen system of representatives for the Σ-orbits. We now define a function $Q = Q_\chi \colon W_F \to Z_{\widehat{M}}(k)$, where $Z_{\widehat{M}}$ denotes the maximal central torus of $\widehat{M}$, using the given set of χ-data $(\chi_\alpha)_\alpha$. We will define Q_χ as a product

$$Q_\chi = s \cdot \prod_{\alpha \in A} Q_{\chi,\alpha},$$

so it suffices to define s and each $Q_{\chi,\alpha}$. Observe that there is a natural map

$$\Phi \to X_*(Z_{\widehat{M}}),$$

given by $\mathcal{O} \mapsto \sum_{\alpha \in \mathcal{O}} \widehat{\alpha}^\vee$. If $\alpha \in \mathcal{O}$, then we will abusively use $\widehat{\alpha}^\vee$ to denote the image of $\mathcal{O}$ under this map. We will also abuse notation by using α to refer to an element of A.

For each $\alpha \in A$, let W_α be the stabilizer of α in W_F and let $W_{\pm\alpha}$ be the (setwise) stabilizer of the set $\{\pm\alpha\}$. Fix a system of representatives $w_1, \ldots, w_m \in W_F$ for the quotient $W_{\pm\alpha}\backslash W_F$, and let $v_0 \in W_\alpha$ be arbitrary. If α is symmetric, let $v_1 \in W_{\pm\alpha} - W_\alpha$ be arbitrary. Define a function $P \colon \Phi \to \{\pm 1\}$ by

$$P(\beta) = \begin{cases} 1 & \text{if } \beta = w_i^{-1}\alpha \text{ for some } \alpha \in A \text{ and } 1 \leq i \leq m, \\ -1 & \text{otherwise.} \end{cases}$$

Let $\Phi^+ \subset \Phi$ denote (the image in Φ of) the positive system of roots corresponding to $g(B_G)_{\overline{F}}g^{-1}$. Using P, for each $w \in W_F$ we define the set

$$\begin{aligned} M(w) = \{\alpha \in \Phi \colon \alpha \in \Phi^+, w^{-1}\alpha \notin \Phi^+, P(\alpha) = P(w^{-1}\alpha) = 1\} \\ \sqcup \{\alpha \in \Phi \colon \alpha \in \Phi^+, w^{-1}\alpha \in \Phi^+, P(\alpha) = -1, P(w^{-1}\alpha) = 1\}. \end{aligned}$$

Using this, for each $w \in W_F$ we define

$$s(w) = \prod_{\alpha \in M(w)} \widehat{\alpha}^\vee(-1) \in Z_{\widehat{M}}(k).$$

Now fix $\alpha \in A$. For $w \in W_F$ and $1 \le i \le m$, define $u_i(w) \in W_{\pm\alpha}$ by $w_i \cdot w = u_i(w) \cdot w_j$ for some (uniquely determined) j. Similarly, for $u \in W_{\pm\alpha}$ and $i \in \{0,1\}$, define $v_i(u) \in W_\alpha$ by $v_i \cdot u = v_i(u) \cdot v_j$ for some (uniquely determined) $j \in \{0,1\}$. We define

$$Q_{\chi,\alpha}(w) = \prod_{i=1}^{m} (w_i^{-1}\widehat{\alpha}^\vee)(\chi_\alpha(\mathrm{rec}_{F_\alpha}(v_0(u_i(w))))), \tag{4.3.1}$$

where $\mathrm{rec}_{F_\alpha}\colon W_{F_\alpha} \to F_\alpha^\times$ is the reciprocity map from class field theory (normalized to send a lift of *geometric* Frobenius to a uniformizer of $F_\alpha^\times$). We then define ${}^L j_\chi\colon {}^L M \to {}^L G$ by

$${}^L j_\chi(h,w) = (hQ(w)n(\gamma_{M,G}), w), \tag{4.3.2}$$

where $\gamma \in \Gamma$ is the image of $w \in W_F$. It is proven in [Kal21a, Corollary 6.7] that ${}^L j_\chi$ is an L-embedding. This completes the construction in the case that G and M are quasi-split.

4.3.2. *The general case.* In the general case, let G_0 be a quasi-split inner form of G. By [Kal21a, Lemma 6.4], there exists an inner twist $\xi\colon G_{0,\overline{F}} \to G_{\overline{F}}$ with the property that $\xi^{-1}(M_{\overline{F}})$ is the base change to $\overline{F}$ of a quasi-split twisted Levi $M_0 \subset G_0$, and the restricted map $\xi\colon M_{0,\overline{F}} \to M_{\overline{F}}$ is an inner twist. The inner twist ξ therefore induces compatible isomorphisms ${}^L M \cong {}^L M_0$ and ${}^L G \cong {}^L G_0$. If χ_0 is the set of χ-data for the maximal central torus Z_0 of M_0 induced by χ, then we define ${}^L j_\chi = {}^L j_{\chi_0}$ via the above isomorphisms.

4.4. **Dual L-homomorphisms.** In the remainder of this subsection, fix a tamely ramified maximal F-torus $T \subset G$, and retain the notation of the previous subsection. The following remark partially explains the otherwise somewhat mystifying formulas occurring in the definitions of L-embeddings given above.

Remark 4.4.1 (Explicit form of Shapiro's lemma)**.** Let Γ be a locally profinite group, let $\Gamma_0 \subset \Gamma$ be an open subgroup of finite index, and let M be a locally profinite group equipped with a continuous Γ_0-action. The continuous noncommutative Shapiro lemma gives a natural bijection $\mathrm{H}^1(\Gamma_0, M) \cong \mathrm{H}^1(\Gamma, \mathrm{ind}_{\Gamma_0}^{\Gamma} M)$. This bijection can be described concretely on the level of cocycles as follows.

First, recall that $\mathrm{ind}_{\Gamma_0}^{\Gamma} M$ can be described as the space of Γ_0-equivariant functions $f\colon \Gamma \to M$, with Γ-action given by $(\gamma \cdot f)(\gamma_0) = f(\gamma_0\gamma)$. Choose a set of representatives $\eta_1, \dots, \eta_n \in \Gamma$ for $\Gamma_0\backslash\Gamma$, and observe that an element $f \in \mathrm{ind}_{\Gamma_0}^{\Gamma} M$ is determined by the elements $f(\eta_1), \dots, f(\eta_n)$. For $\gamma \in \Gamma$, we define $\eta_i(\gamma) \in \Gamma_0$ by the equation

$$\eta_i \cdot \gamma = \eta_i(\gamma) \cdot \eta_j,$$

where j is uniquely determined. For 1-cocycles, the map in Shapiro's lemma can be described on cocycles as sending a 1-cocycle $c_0\colon \Gamma_0 \to M$ to the 1-cocycle $c\colon \Gamma \to \mathrm{ind}_{\Gamma_0}^{\Gamma} M$ given by

$$c(\gamma)(\eta_i) = c_0(\eta_i(\gamma)).$$

It is easy to check that c is a 1-cocycle and that it is independent of the choice of representatives $\eta_1, \dots, \eta_n$ up to a coboundary. Moreover, the inverse bijection sends a 1-cocycle c to the 1-cocycle $\Gamma_0 \to M$, $\gamma \mapsto c(\gamma)(1)$.

Now we prove a technical lemma which will be necessary later. For each $\alpha \in \Phi(G_{\overline{F}}, T_{\overline{F}})$, define an F-torus J_α as in [Kal21a, §3.1] as follows: if α is asymmetric, let $J_\alpha = \mathrm{Res}_{F_\alpha/F}\,\mathbf{G}_m$; if α is symmetric, let

$$J_\alpha = \mathrm{Res}_{F_{\pm\alpha}/F}\mathrm{Res}^1_{F_\alpha/F_{\pm\alpha}}\mathbf{G}_m = \ker\big(\mathrm{Res}_{F_\alpha/F}\,\mathbf{G}_m \xrightarrow{\mathrm{Nm}_{F_\alpha/F_{\pm\alpha}}} \mathrm{Res}_{F_{\pm\alpha}/F}\,\mathbf{G}_m\big).$$

Observe that the root α is an F_α-homomorphism $T_{F_\alpha} \to \mathbf{G}_m$, which corresponds to an F-homomorphism $\alpha\colon T \to J_\alpha$. In turn, this is dual to an L-homomorphism ${}^L\alpha\colon {}^L J_\alpha \to {}^L T$. Let $\widehat{\alpha}^\vee\colon \mathbf{G}_m \to \widehat{T}$ denote the cocharacter corresponding to α, and let $\widehat{\alpha}\colon \widehat{J}_\alpha \to \widehat{T}$ be the restriction of ${}^L\alpha$ to dual groups.

Recall that the dual group of $\mathrm{Res}_{F_\alpha/F}\,\mathbf{G}_m$ can be described (functorially) as the space of functions $W_{F_\alpha}\backslash W_F \to \mathbf{G}_m$. If α is symmetric, then $\widehat{J}_\alpha$ can be described as the space of functions $W_{F_\alpha}\backslash W_F \to \mathbf{G}_m$ modulo pointwise multiplication by functions $W_{F_{\pm\alpha}}\backslash W_F \to \mathbf{G}_m$.

Recall that k is a ring among $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and $\overline{\mathbf{F}}_\ell$, where $\ell \neq p$. For each α, let $\xi_\alpha\colon F_\alpha^\times \to k^\times$ be a character. If α is asymmetric, identify $J_\alpha(F)$ with $F_\alpha^\times$ and set $\xi_\alpha^1 = \xi_\alpha$. If α is symmetric, assume that ξ_α factors through the surjection $F_\alpha^\times \twoheadrightarrow F_\alpha^1$ given by $t \longmapsto t\,v_1(t)^{-1}$, where $F_\alpha^1 := \ker(\mathrm{Nm}_{F_\alpha/F_{\pm\alpha}}\colon F_\alpha^\times \to F_{\pm\alpha}^\times)$, and let

$$\xi_\alpha^1\colon J_\alpha(F) = F_\alpha^1 \to k^\times$$

be the factored character. Let $\widehat{\xi}^1_\alpha \colon W_F \to \widehat{J}_\alpha(k)$ denote its dual cocycle. Let $\widehat{\xi}_{\alpha,0} \colon W_{F_\alpha} \to k^\times$ be the character corresponding to ξ_α as a character of $\mathbf{G}_m(F_\alpha)$, and let $\widehat{\xi}_\alpha \colon W_F \to (\operatorname{Res}_{F_\alpha/F}\mathbf{G}_m)^\wedge(k)$ be the cocycle corresponding to ξ_α as a character of $(\operatorname{Res}_{F_\alpha/F}\mathbf{G}_m)(F)$.

Lemma 4.4.2. *We have*

$$[\widehat{\alpha} \circ \widehat{\xi}^1_\alpha] = \left[w \longmapsto \prod_{i=1}^m (w_i^{-1}\widehat{\alpha}^\vee)(\widehat{\xi}_{\alpha,0}(v_0(u_i(w)))) \right] \quad \text{in } \mathrm{H}^1(W_F, \widehat{T}(k)). \tag{4.4.1}$$

Proof. Suppose first that α is asymmetric, so $F_\alpha = F_{\pm\alpha}$. The dual L-homomorphism $\widehat{\alpha} \colon \widehat{J}_\alpha \to \widehat{T}$ is given by

$$\widehat{\alpha}(f) = \prod_{i=1}^m (w_i^{-1}\widehat{\alpha}^\vee)(f(v_0 w_i)), \tag{4.4.2}$$

where $f \colon W_{F_\alpha}\backslash W_F \to \mathbf{G}_m$ as above. By Remark 4.4.1, for $w \in W_F$ we have

$$\widehat{\xi}_\alpha(w)(v_0 w_i) = \widehat{\xi}_{\alpha,0}(v_0(u_i(w))). \tag{4.4.3}$$

Combining (4.4.2) and (4.4.3) yields (4.4.1).

Now suppose that α is symmetric. Note that $\widehat{\alpha} \colon \widehat{J}_\alpha \to \widehat{T}$ is given by

$$\widehat{\alpha}(f^1) = \prod_{i=1}^m (w_i^{-1}\widehat{\alpha}^\vee)(f^1(v_0 w_i) \cdot f^1(v_1 w_i)^{-1}), \tag{4.4.4}$$

where we regard f^1 as a function $W_{F_\alpha}\backslash W_F \to \mathbf{G}_m$, up to multiplication by a function $W_{F_{\pm\alpha}}\backslash W_F \to \mathbf{G}_m$. By Remark 4.4.1, the dual L-parameter $\widehat{\xi}_\alpha \colon W_F \to (\operatorname{Res}_{F_\alpha/F}\mathbf{G}_m)^\wedge(k)$ is given for $w \in W_F$ by

$$\widehat{\xi}_\alpha(w)(v_j w_i) = \widehat{\xi}_{\alpha,0}(v_j(u_i(w))). \tag{4.4.5}$$

The injective homomorphism $\widehat{J}_\alpha \to (\operatorname{Res}_{F_\alpha/F}\mathbf{G}_m)^\wedge$ which is dual to $t \mapsto t \cdot v_1(t)^{-1}$ is given by sending the equivalence class of a function $f^1 \colon W_{F_\alpha}\backslash W_F \to \mathbf{G}_m$ to the function $f \colon W_{F_\alpha}\backslash W_F \to \mathbf{G}_m$ given by

$$f(v_j w_i) = f^1(v_j w_i) \cdot f^1(v_{1-j} w_i)^{-1}$$

for all j and i. For each $w \in W_F$, represent the class $\widehat{\xi}^1_\alpha(w) \in \widehat{J}_\alpha(k)$ by the unique function normalized by $\widehat{\xi}^1_\alpha(w)(v_1 w_i) = 1$ for every i. With this normalization,

$$\widehat{\xi}^1_\alpha(w)(v_0 w_i) = \widehat{\xi}_{\alpha,0}(v_0(u_i(w))) \quad \text{and} \quad \widehat{\xi}^1_\alpha(w)(v_1 w_i) = 1 \tag{4.4.6}$$

for all i. Combining (4.4.4), (4.4.5), and (4.4.6), we obtain (4.4.1). $\square$

4.5. **Associated L-embeddings.** We have now associated explicit L-embeddings to twisted Levi F-subgroups of G in two important settings.

Definition 4.5.1. If $H \subset G$ is an unramified twisted Levi with maximal central torus Z and χ is the canonical set of χ-data for $\Phi(G_{\overline{F}}, Z_{\overline{F}})$ (which exists by Lemma 3.2.1), then we let ${}^L j_{H,G} \colon {}^L H \to {}^L G$ denote the L-embedding ${}^L j_{H,G} = {}^L j_\chi$.

Similarly, if $p \neq 2$ and $(\vec{G}, x, \vec{r}, \vec{X})$ is a tuple for G as in §4.2 and $T \subset G^0$ is a maximally unramified maximal F-torus and χ'' is the set of χ-data associated to $(\vec{G}, x, \vec{r}, \vec{X}, T)$ in §4.2, let ${}^L j_{T,G} \colon {}^L T \to {}^L G$ denote the L-embedding ${}^L j_{T,G} = {}^L j_{\chi''}$. When these two notations overlap, it will be clear from context which is meant.

We summarize some basic properties of ${}^L j_{H,G}$ in the following lemma.

Lemma 4.5.2. *Let $H \subset G$ be an unramified twisted Levi F-subgroup. The L-embedding ${}^L j_{H,G}$ enjoys the following properties.*

(1) If H is the Levi of an F-rational parabolic subgroup of G, then ${}^L j_{H,G}$ agrees with the canonical dual embedding $(t, w) \mapsto (t, w)$.

(2) If $\pi\colon G' \to G$ is an F-homomorphism which induces an isomorphism on adjoint groups, and we let $H' = \pi^{-1}(H)$, then the diagram

$$\begin{tikzcd} {}^L H \arrow[r, "{}^L j_{H,G}"] \arrow[d] & {}^L G \arrow[d, "\pi"] \\ {}^L H' \arrow[r, "{}^L j_{H',G'}"] & {}^L G' \end{tikzcd}$$

commutes.

(3) If $G \cong G_1 \times G_2$, so $H \cong H_1 \times H_2$, then ${}^L j_{H,G}$ is the product of ${}^L j_{H_1,G_1}$ and ${}^L j_{H_2,G_2}$.

Proof. (1) We may and do assume that G is quasi-split by [Kal21a, Lemma 6.4], since passing to an inner twist which preserves H also preserves the property that H is the Levi of a parabolic F-subgroup of G. In this case, there is a Borel F-subgroup $B \subset G$ such that $B \cap H$ is a Borel F-subgroup. In the construction of §4.3.1, we may choose $B_G = B$, $B_H = B \cap H$, and $T_G = T_H = T$ a maximal F-torus in B. We therefore have $\gamma_{H,G} = 1$ for all $\gamma \in \Gamma$, and it suffices to show $Q = 1$ by (4.3.2). Note that we may choose representatives w_i as in §4.3.1 so that $w_i^{-1}\alpha$ is positive (relative to B) whenever α is positive. With this choice, the set $M(w)$ is empty and thus $s = 1$. Moreover, every element $\alpha \in \Phi((G/H)_{\overline{F}}, T_{\overline{F}})$ is asymmetric, so the set of χ-data is trivial and thus $Q_{\chi,\alpha} = 1$ for all α by (4.3.1).

(2) First, it is clear that H' is an unramified twisted Levi in G', so the diagram makes sense. We may again assume that G is quasi-split, hence G' is also quasi-split. In this case, it suffices to observe that the definitions of s, n, and Q in §4.3.1 are all evidently compatible with π, since the sets of χ-data defining ${}^L j_{H,G}$ and ${}^L j_{H',G'}$ are the same.

(3) This is clear from the definitions. $\square$

5. Compatibility of sign characters and L-embeddings

It will be important to understand the behavior of the L-embedding ${}^L j_{T,G}$ associated to a quadruple $(\vec{G}, x, \vec{r}, \vec{X})$ as in Definition 4.5.1 when G is replaced by an unramified twisted Levi subgroup H such that $x \in \mathcal{B}(H)$. For such an H, we obtain a quadruple $(\vec{H}, x, \vec{r}, \vec{X}_H)$, where $H^i = G^i \cap H$ for all i and $X^*_{H,i} = X^*_i|_{\mathrm{Lie}(H^i)}$, and from this we obtain an L-embedding ${}^L j_{T,H}\colon {}^L T \to {}^L H$. There is a canonical L-embedding ${}^L j_{H,G}\colon {}^L H \to {}^L G$, and we want to compare ${}^L j_{T,G}$ and ${}^L j_{H,G} \circ {}^L j_{T,H}$. It is not true that

$${}^L j_{T,G} \sim {}^L j_{H,G} \circ {}^L j_{T,H}$$

in general; see Example 9.4.1. The technical issues are twofold: first, the definition of ${}^L j_{T,G}$ (resp. ${}^L j_{T,H}$) involves a product over the set of roots $\Phi(G_{\overline{F}}, T_{\overline{F}})$ (resp. $\Phi(H_{\overline{F}}, T_{\overline{F}})$), and these sets are of course different. Second, the definitions both involve the numbers $\ell_{G,p'}(\alpha^\vee)$ (resp. $\ell_{H,p'}(\alpha^\vee)$), and these differ even when $\alpha \in \Phi(H_{\overline{F}}, T_{\overline{F}})$.

However, we will prove in Proposition 5.1.5 that

$${}^L j_{T,G} \circ {}^L\theta \sim {}^L j_{H,G} \circ {}^L j_{T,H} \circ {}^L(\theta \epsilon_G \epsilon_H \epsilon_{\sharp,x,G} \epsilon_{\sharp,x,H}),$$

where the ϵ are the sign characters from [FKS23] recalled above, whenever T is elliptic and $\theta\colon T(F) \to k^\times$ is a character. We will similarly analyze base change, i.e., the behavior of ${}^L j_{T,G}$ under passage from F to a cyclic tamely ramified extension E/F. Since the calculations become rather involved, the proofs are mostly relegated to Appendix A.

We retain the notation of Sections 3 and 4. Fix a tuple $(\vec{G}, x, \vec{r}, \vec{X})$ as in §4.2, and let k be a ring among $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and $\overline{\mathbf{F}}_\ell$.

5.1. **Unramified twisted Levis.** We first deal with the case of descent to unramified twisted Levis. We assume in this section that $p \neq 2$.

Lemma 5.1.1. *Let H and M be tamely ramified twisted Levi F-subgroups of G such that H is an unramified twisted Levi and $H \cap M$ contains a maximal F-torus T of G*[14] *such that $x \in \mathcal{B}(T)$. Then*

$$\epsilon_f^{G/M} = \epsilon_f^{H/(M\cap H)}$$

as characters of $T(F)_{[x]}$.

[14] Note that this automatically implies that $H \cap M$ is a twisted Levi F-subgroup of H.

Proof. Observe that if $\alpha \in \Phi((G/M)_{\overline{F}}, T_{\overline{F}})$ is symmetric ramified then $\alpha \in \Phi((H/(M \cap H))_{\overline{F}}, T_{\overline{F}})$ by Lemma 3.2.1, and it is clear from the definitions that $f_{(G,T)}(\alpha) = f_{(H,T)}(\alpha)$. □

Throughout this section, let $H \subset G$ be an unramified twisted Levi F-subgroup such that $H \cap G^0$ contains a maximal F-torus of G, and let $H^i = H \cap G^i$ for all $0 \leq i \leq d$. Fix a maximally unramified maximal F-torus $T \subset H^0$, and let χ''_G (resp. χ''_H) be the set of χ-data for $\Phi(G_{\overline{F}}, T_{\overline{F}})$ (resp. $\Phi(H_{\overline{F}}, T_{\overline{F}})$) associated to the tuple $(\vec{G}, x, \vec{r}, \vec{X})$ where X_i^* is as in §4.2.[15] Below we will use the characters defined in §3.3.

For the following statements, we recall that for a character $\eta\colon T(F)_{\mathrm{b}} \to k^\times$, there is an associated inertial L-parameter $I_F \to {}^LT(k)$; we will denote this homomorphism by ${}^L\eta|_{I_F}$.[16] If T is not anisotropic, then the notation ${}^L\eta|_{I_F}$ is slightly abusive, because there is no canonical choice of L-parameter ${}^L\eta\colon W_F \to {}^LT(k)$ of which ${}^L\eta|_{I_F}$ is the restriction.

Lemma 5.1.2. *Let $\theta\colon T(F) \to k^\times$ be a character. Recall from Definition 4.1.2 that $\min \chi''_G$ denotes the minimally ramified χ-data associated to χ''_G. We have*

$$ {}^Lj_{\chi''_G} \circ {}^L\theta|_{I_F} \sim {}^Lj_{\min \chi''_G} \circ {}^L(\theta \cdot \epsilon_{G,\flat,0})|_{I_F}. $$

If moreover T is elliptic in G[17]*, then we have*

$$ {}^Lj_{\chi''_G} \circ {}^L\theta \sim {}^Lj_{\min \chi''_G} \circ {}^L(\theta \cdot \epsilon_{G,\flat,0}). $$

The completely analogous claims hold for H.

Proof. The proof is given in Appendix A.1. □

Let $\chi''_{H,G}$ denote the unique minimally ramified set of χ-data for $\Phi(G_{\overline{F}}, T_{\overline{F}})$ such that for all $\alpha \in \Phi(H_{\overline{F}}, T_{\overline{F}})$ we have $\chi''_{H,G,\alpha} = \min \chi''_{H,\alpha}$; this determines $\chi''_{H,G}$ by Lemma 3.2.1.

Lemma 5.1.3. *If $\theta\colon T(F) \to k^\times$ is a character, then*

$$ {}^Lj_{\chi''_{H,G}} \circ {}^L\theta|_{I_F} \sim {}^Lj_{\min \chi''_G} \circ {}^L(\theta \cdot \epsilon_{G,\flat,1}\epsilon_{H,\flat,1}\epsilon_{G,\flat,2}\epsilon_{H,\flat,2})|_{I_F}. $$

If moreover T is elliptic in G, then

$$ {}^Lj_{\chi''_{H,G}} \circ {}^L\theta \sim {}^Lj_{\min \chi''_G} \circ {}^L(\theta \cdot \epsilon_{G,\flat,1}\epsilon_{H,\flat,1}\epsilon_{G,\flat,2}\epsilon_{H,\flat,2}). $$

Proof. The proof is given in Appendix A.2. □

Let χ_H be the canonical (minimally ramified) set of χ-data for $\Phi((G/H)_{\overline{F}}, (Z_H)_{\overline{F}})$, and let $\chi_{H,G}$ be the unique set of χ-data for $\Phi(G_{\overline{F}}, T_{\overline{F}})$ whose restriction to $\Phi((G/H)_{\overline{F}}, T_{\overline{F}})$ is the inflation of χ_H in the sense of [Kal21a, Definition 5.16] and whose restriction to $\Phi(H_{\overline{F}}, T_{\overline{F}})$ is equal to the restriction of $\chi''_{H,G}$.

Lemma 5.1.4. *We have ${}^Lj_{\chi''_{H,G}}|_{\widehat{T} \rtimes I_F} \sim {}^Lj_{\chi_{H,G}}|_{\widehat{T} \rtimes I_F}$. If T is elliptic in G, then ${}^Lj_{\chi''_{H,G}} \sim {}^Lj_{\chi_{H,G}}$.*

Proof. The two sets of χ-data agree on $\Phi(H_{\overline{F}}, T_{\overline{F}})$, so fix a Σ-orbit represented by $\alpha \in \Phi((G/H)_{\overline{F}}, T_{\overline{F}})$ and let α_H be its restriction to Z_H. Set

$$ \zeta_\alpha \coloneqq \chi_{H,G,\alpha}(\chi''_{H,G,\alpha})^{-1}. $$

If α is asymmetric, set $\zeta^1_\alpha = \zeta_\alpha$ on $J_\alpha(F) = F^\times_\alpha$; if α is symmetric, then ζ_α is trivial on $F^\times_{\pm\alpha}$, so it factors through $F^\times_\alpha \to F^1_\alpha = J_\alpha(F)$, and we denote the factored character by ζ^1_α. Put $\eta_\alpha = \zeta^1_\alpha \circ \alpha$, so the product $\prod_{\alpha \in \Phi((G/H)_{\overline{F}}, T_{\overline{F}})/\Sigma} \eta_\alpha$ is the character ζ_T of $T(F)$ from [Kal21a, §3.2].

By Lemma 3.2.1, neither α nor α_H is symmetric ramified. This implies that the restriction of ζ_T to $T(F)_{[x]}$ is trivial (as in the proof of [Kal21a, Proposition 5.27]). Lemma 4.4.2, applied to ζ_α, identifies

$$ [Q_{\chi_{H,G},\overline{\alpha}} Q^{-1}_{\chi''_{H,G},\overline{\alpha}}] = [\widehat{\eta}_\alpha] \quad \text{in } \mathrm{H}^1(W_F, \widehat{T}(k)). $$

---

[15] Note that the restrictions of the good elements X_i^* to $\mathrm{Lie}(H^i)$ are automatically good relative to H^{i+1}, since the roots of H^{i+1}/H^i relative to T form a subset of the roots of G^{i+1}/G^i.

[16] This amounts to the claim that if $\eta_1, \eta_2\colon T(F) \to k^\times$ are two characters extending η, then ${}^L\eta_1|_{I_F} \sim {}^L\eta_2|_{I_F}$. From the construction of the Local Langlands Correspondence for tori, this reduces to the case $T = \mathbf{G}_m$, in which case it follows from the fact that the isomorphism $W_F^{\mathrm{ab}} \cong F^\times$ from class field theory sends I_F to $\mathcal{O}_F^\times = \mathbf{G}_m(F)_{\mathrm{b}}$.

[17] Observe that the following displayed equation would not make sense if T were not assumed elliptic, because then $T(F) \neq T(F)_{[x]}$, so $\epsilon_{G,\flat,0}$ would not be a character of $T(F)$. Completely similar remarks apply to the subsequent results.

The right hand class restricts trivially to I_F by local Langlands for tori, so the same is true of the left hand class. By the definitions (4.3.2) and (4.3.1), this implies ${}^L j_{\chi''_{H,G}}|_{\widehat{T}\rtimes I_F} \sim {}^L j_{\chi_{H,G}}|_{\widehat{T}\rtimes I_F}$. If T is elliptic, then the same argument applies without restricting to I_F. $\square$

Proposition 5.1.5. *Let $\theta\colon T(F) \to k^\times$ be a character. Then, with $\epsilon_{G,\sharp,x}$ and $\epsilon_{H,\sharp,x}$ defined as in* (3.3.2), *we have*

$$ {}^L j_{\chi''_G} \circ {}^L\theta|_{I_F} \sim {}^L j_{H,G} \circ {}^L j_{\chi''_H} \circ {}^L(\theta \cdot \epsilon_G \epsilon_H \epsilon_{G,\sharp,x} \epsilon_{H,\sharp,x})|_{I_F}. $$

If moreover T is elliptic in G, then

$$ {}^L j_{\chi''_G} \circ {}^L\theta \sim {}^L j_{H,G} \circ {}^L j_{\chi''_H} \circ {}^L(\theta \cdot \epsilon_G \epsilon_H \epsilon_{G,\sharp,x} \epsilon_{H,\sharp,x}). $$

Proof. By definition, the restriction of $\chi_{H,G}$ to $\Phi((G/H)_{\overline{F}}, T_{\overline{F}})$ is equal to the inflation (in the sense of [Kal21a, Definition 5.16]) of χ_H, the canonical set of χ-data for $\Phi((G/H)_{\overline{F}}, (Z_H)_{\overline{F}})$. The restriction of $\chi_{H,G}$ to $\Phi(H_{\overline{F}}, T_{\overline{F}})$ is by definition equal to $\min \chi''_H$. Thus by [Kal21a, Proposition 6.9], we have

$$ {}^L j_{\chi_{H,G}} \sim {}^L j_{H,G} \circ {}^L j_{\min \chi''_H}. $$

By Lemmas 5.1.2, 5.1.3, and 5.1.4, we have

$$ \begin{aligned} {}^L j_{\chi''_G} \circ {}^L\theta|_{I_F} &\sim {}^L j_{\min \chi''_G} \circ {}^L(\theta \cdot \epsilon_{G,\flat,0})|_{I_F} \\ &\sim {}^L j_{\chi''_{H,G}} \circ {}^L(\theta \cdot \epsilon_{G,\flat,0}\epsilon_{G,\flat,1}\epsilon_{H,\flat,1}\epsilon_{G,\flat,2}\epsilon_{H,\flat,2})|_{I_F} \\ &\sim {}^L j_{\chi_{H,G}} \circ {}^L(\theta \cdot \epsilon_{G,\flat,0}\epsilon_{G,\flat,1}\epsilon_{H,\flat,1}\epsilon_{G,\flat,2}\epsilon_{H,\flat,2})|_{I_F} \\ &\sim {}^L j_{H,G} \circ {}^L j_{\min \chi''_H} \circ {}^L(\theta \cdot \epsilon_{G,\flat,0}\epsilon_{G,\flat,1}\epsilon_{H,\flat,1}\epsilon_{G,\flat,2}\epsilon_{H,\flat,2})|_{I_F} \\ &\sim {}^L j_{H,G} \circ {}^L j_{\chi''_H} \circ {}^L(\theta \cdot \epsilon_{G,\flat}\epsilon_{H,\flat})|_{I_F}, \end{aligned} $$

where $\epsilon_{G,\flat}$ and $\epsilon_{H,\flat}$ are as in (3.3.3). By (3.2.1), (3.3.3), and Lemma 5.1.1, we obtain

$$ \epsilon_{G,\flat}\epsilon_{H,\flat} = \epsilon_G \epsilon_H \epsilon_{G,\sharp,x} \epsilon_{H,\sharp,x}, $$

as required. The calculation in the case that T is elliptic is identical. $\square$

5.2. **Base change.** Now we move on to the case of base change functoriality. In this section, we continue to assume $p \neq 2$. For any finite extension E/F and any weight α, write $\mathbf{F}_{E,\alpha}$ and $\mathbf{F}_{E,\pm\alpha}$ for the residue fields of E_α and $E_{\pm\alpha}$, respectively.

Lemma 5.2.1. *Let $T \subset M$ be a tamely ramified maximal F-torus such that $x \in \mathcal{B}(T)$, and let E/F be a finite unramified extension of odd degree. Then*

$$ \epsilon_x^{G/M} \circ \mathrm{Nm}_{E/F}|_{T(E)_{[x]}} = \epsilon_x^{G_E/M_E}|_{T(E)_{[x]}}. $$

Proof. The proof is given in Appendix A.4. $\square$

Lemma 5.2.2. *Let $H \subset G$ be an unramified twisted Levi F-subgroup, and let E/F be a finite extension. Recall from §2 that E_2/E denotes the unramified extension of degree* 2. *Then*

$$ {}^L j_{H_E,G_E}|_{\widehat{H}\rtimes W_{E_2}} \sim {}^L j_{H,G}|_{\widehat{H}\rtimes W_{E_2}}. $$

Proof. By twisting, we may assume that G is quasi-split. In this case, observe that each χ_α is unramified of order at most 2 and the maps s and n are essentially independent of F, so the result follows from the definitions (4.3.1) and (4.3.2). $\square$

Remark 5.2.3. The restriction to W_{E_2} in Lemma 5.2.2 may look strange, but it is necessary in general. For instance, let $G = \mathrm{PGL}_2$ and let $T \subset G$ be an unramified elliptic maximal F-torus. If $E = F_2$, the representatives furnished by the construction satisfy ${}^L j_{T_E,G_E}(1,w) = (1,w)$ for every $w \in W_E$, since T_E is split. On the other hand, if $w \in W_F$ has image Fr^2 in W_F/I_F, then

$$ {}^L j_{T,G}(1,w) = (-1,w). $$

We expect a more natural-looking compatibility if one uses the c-group in place of the L-group, but the difference will not ultimately be relevant to this paper.

Let E/F be a finite separable, tamely ramified field extension. Choose a maximally unramified maximal F-torus $T \subset G^0$, and let χ''_F (resp. χ''_E) denote the set of χ-data for $\Phi(G_{\overline{F}}, T_{\overline{F}})$ associated to the tuple $(\vec{G}, x, \vec{r}, \vec{X})$ (resp. $(\vec{G}_E, x, \vec{r}, \vec{X})$). Let ${}^L j_F \colon {}^L T \to {}^L G$ (resp. ${}^L j_E \colon {}^L(T_E) \to {}^L(G_E)$) denote the L-embedding associated to χ''_F (resp. χ''_E) by the mechanism of §4. Note that since the characters associated to χ''_F and χ''_E are valued in μ_4, we may regard ${}^L j_F$ and ${}^L j_E$ as homomorphisms of $\overline{\mathbf{Z}}[1/p]$-group schemes. In particular, we may consider the base change k-homomorphisms $({}^L j_F)_k$ and $({}^L j_E)_k$ for our chosen coefficient ring k.

Put $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$, and let $\psi_E^0 \colon \mathbf{F}_E \to k^\times$ be the induced character of the residue field; this is the character relative to which we define the L-embedding ${}^L j_E$.

Lemma 5.2.4. *Suppose that E/F is a cyclic extension of prime degree ℓ.*

(1) If $\ell = 2$, then $({}^L j_E)_{\overline{\mathbf{F}}_2} \sim ({}^L j_F)_{\overline{\mathbf{F}}_2}|_{{}^L(T_E)}$. If moreover E/F is unramified, then ${}^L j_E|_{\widehat{T} \rtimes I_F} \sim {}^L j_F|_{\widehat{T} \rtimes I_F}$.
(2) If ℓ is odd and E/F is unramified, then ${}^L j_E \sim {}^L j_F|_{{}^L(T_E)}$.

Proof. We may and do assume that G is quasi-split. Note that the characters associated to χ''_F and χ''_E are μ_4-valued, so the products Q associated to both by (4.3.1) are trivial after reduction modulo 2; the factor s is trivial modulo 2 because it is a product of values $\widehat{\alpha}^\vee(-1)$. Finally, the Tits section $n \colon \Omega(\widehat{G}, \widehat{T}) \to N_{\widehat{G}}(\widehat{T})(\mathbf{F}_2)$ is a homomorphism. Formula (4.3.2) therefore proves the first claim of (1).

For the second claim of (1), choose the systems of representatives used in (4.3.1) compatibly with the decomposition of every $\mathrm{Gal}(\overline{F}/F)$-orbit into $\mathrm{Gal}(\overline{F}/E)$-orbits. Note that for $w \in I_F$, the Weyl element $\gamma_{T,G}$, its Tits lift $n(\gamma_{T,G})$, and the factor $s(w)$ in (4.3.2) are the same over F and over E. Thus norm functoriality in local class field theory reduces us to the claim

$$\chi''_{F,\alpha} \circ \mathrm{Nm}_{E_\alpha/F_\alpha}|_{\mathcal{O}_{E_\alpha}^\times} = \chi''_{E,\alpha}|_{\mathcal{O}_{E_\alpha}^\times}$$

for every absolute root α. In fact, by definition of $\chi''_{F,\alpha}$ and $\chi''_{E,\alpha}$, it is enough to show

$$\chi''_{F,\alpha_i} \circ \mathrm{Nm}_{E_{\alpha_i}/F_{\alpha_i}}|_{\mathcal{O}_{E_{\alpha_i}}^\times} = \chi''_{E,\alpha_i}|_{\mathcal{O}_{E_{\alpha_i}}^\times} \tag{5.2.1}$$

if $\alpha \in \Phi(G^{i+1}_{\overline{F}}, T_{\overline{F}}) - \Phi(G^i_{\overline{F}}, T_{\overline{F}})$.

If α_i is either asymmetric or symmetric unramified over F, then (5.2.1) is clear from the definitions (since both sides are trivial), so we may and do assume that α_i is symmetric ramified over F. Since E/F is unramified, it follows that α_i is symmetric ramified over E, and thus both sides of (5.2.1) are the unique nontrivial sign character of $\mathcal{O}_{E_{\alpha_i}}^\times$, and in particular they are equal.

For (2), arguing as above reduces us to proving

$$\chi''_{F,\alpha_i} \circ \mathrm{Nm}_{E_{\alpha_i}/F_{\alpha_i}} = \chi''_{E,\alpha_i}. \tag{5.2.2}$$

for every absolute root α. Unramified base change of odd degree preserves whether α_i is asymmetric, symmetric unramified, or symmetric ramified. If it is asymmetric, both characters in (5.2.2) are trivial. If it is symmetric unramified, both are the unramified quadratic character: the degree $[E_{\alpha_i} : F_{\alpha_i}]$ is odd, so pullback by the norm preserves the value -1 on a uniformizer.

It remains to treat the symmetric ramified case. The pullback $\chi''_{F,\alpha_i} \circ \mathrm{Nm}_{E_{\alpha_i}/F_{\alpha_i}}$ has the required restriction to $\mathcal{O}_{E_{\alpha_i}}^\times$, because the quadratic character of $\mathcal{O}_{E_{\alpha_i}}^\times$ is compatible with norms; it has the required restriction to $E_{\pm\alpha_i}^\times$ by norm functoriality for the quadratic characters κ_{α_i}. Thus it remains only to check the final condition in (4.2.1).

Put $f_{F,\alpha_i} = [\mathbf{F}_{\alpha_i} : \mathbf{F}]$ and $f_{E,\alpha_i} = [\mathbf{F}_{E,\alpha_i} : \mathbf{F}_E]$, and let $m = [E_{\alpha_i} : F_{\alpha_i}]$. The Hasse–Davenport relation gives the identity of Gauss sums

$$-\mathfrak{G}_{\mathbf{F}_{E,\alpha_i}}(\psi_E^0) = \left(-\mathfrak{G}_{\mathbf{F}_{\alpha_i}}(\psi^0)\right)^m.$$

Since m is odd, we have $f_{F,\alpha_i} \equiv f_{E,\alpha_i} \pmod 2$ and thus (4.2.1) gives

$$\begin{aligned}(\chi''_{F,\alpha_i} \circ \mathrm{Nm}_{E_{\alpha_i}/F_{\alpha_i}})(\ell_{G,p'}(\alpha^\vee)a_\alpha) &= \chi''_{F,\alpha_i}(\ell_{G,p'}(\alpha^\vee)a_\alpha)^m \\ &= \left((-1)^{f_{F,\alpha_i}+1}\mathfrak{G}_{\mathbf{F}_{\alpha_i}}(\psi^0)\right)^m \\ &= (-1)^{f_{E,\alpha_i}+1}\mathfrak{G}_{\mathbf{F}_{E,\alpha_i}}(\psi_E^0).\end{aligned}$$

This is precisely the remaining defining condition, so (5.2.2) follows. $\square$

We now deal with the analogue of Lemma 5.2.4 for ramified extensions of odd degree. Suppose that $G^0 = T$, so the image of the point x in $\mathcal{B}((G^0_{\mathrm{der}})_E)$ is a vertex. Let ϵ_F (resp. ϵ_E) denote the sign character associated to the tuple $(\vec{G}, x, \vec{r}, \vec{X})$ (resp. $(\vec{G}_E, x, \vec{r}, \vec{X}_E)$) as in §3.3.

Proposition 5.2.5. *Let E/F be a cyclic extension of prime degree ℓ. If $\ell \neq 2$ and $\theta\colon T(F)_{[x]} \to k^\times$ is a character and $\theta_E := \theta \circ \mathrm{Nm}_{E/F}|_{T(E)_{[x]}}$, then*

$$ {}^L j_E \circ {}^L(\theta_E \cdot \epsilon_E \epsilon_{E,\sharp,x} \cdot (\epsilon_F \epsilon_{F,\sharp,x}) \circ \mathrm{Nm}_{E/F})|_{I_E} \sim {}^L j_F \circ {}^L\theta|_{I_E}. $$

If moreover T_E is elliptic, then

$$ {}^L j_E \circ {}^L(\theta_E \cdot \epsilon_E \cdot \epsilon_F \circ \mathrm{Nm}_{E/F}) \sim {}^L j_F \circ {}^L\theta|_{W_E}. $$

The same assertions hold if $\ell = 2$ and $k = \overline{\mathbf{F}}_2$.

Proof. If $\ell = 2$ and $k = \overline{\mathbf{F}}_2$, then the result follows from Lemma 5.2.4(1) and the fact that ϵ_E and ϵ_F are both trivial; thus one may assume $\ell \neq 2$. If E/F is unramified, then the result follows from Lemma 5.2.1 and Lemma 5.2.4(2); thus one may assume that E/F is totally ramified. In this case, the proof is given in Appendix A.5. ☐

5.3. **Composition of canonical L-embeddings.** Now drop the assumption $p \neq 2$. In this subsection, let $M \subset H \subset G$ be unramified twisted Levi F-subgroups. Let Z_M and Z_H denote the maximal central F-tori of M and H, respectively. Recall that we have defined a canonical L-embedding ${}^L j_{M,H}\colon {}^L M \to {}^L H$, and similarly for ${}^L j_{H,G}$ and ${}^L j_{M,G}$ in Definition 4.5.1. For clarity, let $\chi_{M,H}$, $\chi_{H,G}$, and $\chi_{M,G}$ denote the associated sets of χ-data. We also define a set of χ-data

$$ \chi_{M,H,G} = (\chi_{M,H,G,\alpha})_{\alpha \in \Phi(G_{\overline{F}}, (Z_M)_{\overline{F}})} $$

as follows: for $\alpha \in \Phi(G_{\overline{F}}, (Z_M)_{\overline{F}})$, we set

$$ \chi_{M,H,G,\alpha} = \begin{cases} \chi_{M,H,\alpha} & \text{if } \alpha \in \Phi(H_{\overline{F}}, (Z_M)_{\overline{F}}), \\ \chi_{H,G,\alpha_H} \circ \mathrm{Nm}_{F_\alpha/F_{\alpha_H}} & \text{if } \alpha \in \Phi((G/H)_{\overline{F}}, (Z_M)_{\overline{F}}), \end{cases} $$

where α_H denotes the restriction of α to Z_H.

The following two results will not be used in this paper; they will be used in [CF26b] to accommodate inductive arguments.

Lemma 5.3.1. *We have ${}^L j_{\chi_{M,G}}|_{\widehat{M} \rtimes I_F} \sim {}^L j_{\chi_{M,H,G}}|_{\widehat{M} \rtimes I_F}$. If $Z(M)/Z(G)$ is anisotropic, then ${}^L j_{\chi_{M,G}} \sim {}^L j_{\chi_{M,H,G}}$.*

Proof. The proof of Lemma 5.1.4 adapts nearly verbatim to this setting, so we omit it. ☐

The following proposition is a technical extension (in view of Lemma 5.3.1) of [Kal21a, Proposition 6.9] to the above choices of χ-data.[18]

Proposition 5.3.2. *We have ${}^L j_{M,G}|_{\widehat{M} \rtimes I_F} \sim {}^L j_{H,G} \circ {}^L j_{M,H}|_{\widehat{M} \rtimes I_F}$. If $Z(M)/Z(G)$ is anisotropic, then ${}^L j_{M,G} \sim {}^L j_{H,G} \circ {}^L j_{M,H}$.*

Proof. The proof is nearly identical to the proof of [Kal21a, Proposition 6.9], but we recall the outline for convenience of the reader. We will only deal with the case that $Z(M)/Z(G)$ is anisotropic, the general case being completely similar. By Lemma 5.3.1, we have ${}^L j_{M,G} \sim {}^L j_{\chi_{M,H,G}}$. Applying [Kal21a, Lemma 6.4] twice, we may and do assume that M, H, and G are quasi-split. Let $Q_{H,G}$, $Q_{M,H}$, and $Q_{M,H,G}$ be the functions as defined in (4.3.1) using $\chi_{H,G}$, $\chi_{M,H}$, and $\chi_{M,H,G}$, respectively. Unraveling the construction in §4.3 (following the proof of [Kal21a, Proposition 6.9]), one reduces to showing that the 1-cocycle $w \mapsto Q_{M,H}(w) Q_{H,G}(w) Q_{M,H,G}(w)^{-1}$ is a 1-coboundary.

Let $T \subset M$ be a maximal F-torus. If $\Sigma = \mathrm{Gal}(\overline{F}/F) \times \{\pm 1\}$ as in §4.3, then $Q_{M,H,G}$ is defined as a product over $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})/(\Sigma \ltimes \Omega(M_{\overline{F}}, T_{\overline{F}}))$, and $Q_{M,H}$ is the subproduct indexed only over the orbits in $\Phi((H/M)_{\overline{F}}, T_{\overline{F}})/(\Sigma \ltimes \Omega(M_{\overline{F}}, T_{\overline{F}}))$. Thus $Q_{M,H} Q_{M,H,G}^{-1}$ is equal, in the notation of [Kal21a, §6], to the 1-cochain from [Kal21a, Definition 3.17] for the torus M_{ab} and the set $\Phi((G/H)_{\overline{F}}, (M_{\mathrm{ab}})_{\overline{F}})$, and the set

[18] Note that, despite appearances, [Kal21a, Proposition 6.9] if M is a torus then only applies because of Lemma 5.3.1; the sets of χ-data implicitly appearing in [Kal21a, Proposition 6.9] are $\chi_{M,H}$, $\chi_{H,G}$, and $\chi_{M,H,G}$, not $\chi_{M,G}$.

of χ-data inflated (in the sense of [Kal21a, Definition 5.16]) from $\chi_{H,G}$ as above. Also, $Q_{H,G}$ is equal to the 1-cochain from [Kal21a, Definition 3.17] for the torus H_{ab}, the set $\Phi((G/H)_{\overline{F}}, (H_{\mathrm{ab}})_{\overline{F}})$, and the set of χ-data $\chi_{H,G}$. Thus [Kal21a, Proposition 5.23(2)] applied to the surjection $M_{\mathrm{ab}} \to H_{\mathrm{ab}}$ shows that $Q_{H,G}$ is cohomologous to $Q_{M,H}Q_{M,H,G}^{-1}$, as desired. □

6. Yu's construction

In this section we recall Yu's construction of cuspidal representations from Yu data [Yu01], following the exposition (but not the notation) of [Fin21a]. We use the "twisted Yu construction" from [FKS23], which incorporates a sign character derived from those recalled in §3 and is better suited for functoriality. With an eye toward later applications, we work with $\overline{\mathbf{Q}}_\ell$-, $\overline{\mathbf{F}}_\ell$-, and $\overline{\mathbf{Z}}_\ell$-coefficients, and we record the behavior of the construction under tamely ramified extensions of F and restriction to twisted Levi subgroups of G. Our definitions will be slightly broader than the standard ones because of these concerns.

6.1. **Yu data.** Assume $p \neq 2$, let $\ell \neq p$ be another prime, let k be a ring among $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and $\overline{\mathbf{F}}_\ell$, and use the fixed additive character $\psi\colon F \to k^\times$ introduced at the beginning of §3. As in §2, $\mathcal{B}(G)$ denotes the enlarged Bruhat–Tits building of G over F. In this paper, a k-valued *Yu datum* for G is a 5-tuple $\Psi = ((G^i)_{0\le i\le d}, x, (r_i)_{0\le i\le d}, \tau, (\phi_i)_{0\le i\le d})$, where

(a) $G^0 \subset G^1 \subset \cdots \subset G^d = G$ is a sequence of twisted Levi F-subgroups[19] of G which split over a tamely ramified extension of F,
(b) x is a point of $\mathcal{B}(G^0) \subset \mathcal{B}(G)$,
(c) $(r_i)_{0\le i\le d}$ is a sequence of real numbers such that $0 < r_0 < r_1 < \cdots < r_{d-1} \le r_d$ if $d > 0$, and $0 \le r_0$ if $d = 0$,
(d) τ is a k-representation of $G^0(F)_{[x]}$ (where $G^0(F)_{[x]} \subset G^0(F)$ is the stabilizer of the image $[x]$ of x in the reduced Bruhat–Tits building of G/F) which is trivial on $G^0(F)_{x,0+}$ and whose underlying k-module is finite free,
(e) $(\phi_i)_{0\le i\le d}$ is a sequence of characters $\phi_i\colon G^i(F) \to k^\times$. For $0 \le i \le d-1$, we assume that ϕ_i is of depth r_i.
By convention we set $r_{-1} \coloneqq 0$ if $d = 0$. If $r_{d-1} < r_d$, then ϕ_d is of depth r_d; if $r_{d-1} = r_d$, then ϕ_d is trivial.

The constituents of this 5-tuple are moreover required to satisfy the following properties:

(1) $Z(G^0)/Z(G)$ is anisotropic,
(2) $[x]$ is a vertex in $\mathcal{B}(G^0_{\mathrm{der}})$,
(3) $\tau \otimes_k \mathrm{Frac}(k)$ is a cuspidal representation of $G^0(F)_{[x]}/G^0(F)_{x,0+}$ in the sense of [Cot26, §2.10]. (See also [Cot26, Proposition 2.1.4].)
(4) ϕ_i is G^{i+1}-generic of depth r_i for $0 \le i \le d-1$: this means that ϕ_i is of depth r_i and under the Moy–Prasad isomorphism $G^i(F)_{x,r_i}/G^i(F)_{x,r_i+} \cong \mathfrak{g}^i(F)_{x,r_i}/\mathfrak{g}^i(F)_{x,r_i+}$, there is an element $X_i^* \in \mathrm{Lie}^*(G^i)^{G^i}(F)_{-r_i}$ which is G^{i+1}-generic in the sense of [Yu01, §8] (see [FKS23, Remark 4.1.3] for a slight correction) such that $\phi_i|_{G^i(F)_{x,r_i}}$ is induced by X_i^* and the fixed additive character ψ.[20]

If $\tau \otimes_k \mathrm{Frac}(k)$ is an irreducible representation of $G^0(F)_{[x]}/G^0(F)_{x,0+}$, then we will call Ψ *irreducible*. If $G^i \neq G^{i+1}$ for all $0 \le i < d$, then we will call Ψ *irredundant*.

Remark 6.1.1. Typically, one only considers Yu data for which k is an algebraically closed field of characteristic $\neq p$ and Ψ is irreducible. Our ultimate interest is in representations arising from irreducible Yu data over $\overline{\mathbf{Q}}_\ell$, but for the study of L-parameters we will need to understand the modular reductions of $\overline{\mathbf{Z}}_\ell$-lattices in these; such modular reductions may a priori arise from Yu's construction applied to Yu data which are not irreducible.

[19] In most references, it is further assumed that $G^i \neq G^{i+1}$ for all $0 \le i \le d-1$. However, relaxing this requirement does not significantly affect the theory, and for the purposes of functoriality with respect to unramified twisted Levi subgroups of G, our convention is more convenient. This generalization (with slightly different conditions on the r_i and ϕ_i) was introduced in [Yu01, §15], where it was used to analyze the irreducible supercuspidal representations of products.

[20] Note that if $G^i = G^{i+1}$, then every character of $G^i(F)$ which is induced by an element of $\mathrm{Lie}^*(G^i)^{G^i}$ is G^{i+1}-generic. (This can fail; it is essentially Hypothesis C($\vec{G}$) from [HM08, §2.6].

From a Yu datum Ψ, [Yu01] constructs a compact-mod-center open subgroup $K \subset G(F)$, a k-representation ρ of K, and then [FKS23] defines a sign character $\epsilon\colon K \to k^\times$ and sets

$$\pi(\Psi) := \text{c-Ind}_K^{G(F)}(\epsilon \otimes \rho).$$

We will now recall the definitions of K, ρ, and ϵ.

6.2. **The inducing subgroup and extended characters.** From a k-valued Yu datum Ψ, one constructs a smooth representation $\pi = \text{c-Ind}_K^{G(F)}(\epsilon\otimes\rho)$ as follows. First, we define a compact-mod-center open subgroup $K \subset G(F)$ by

$$K = K(\vec{G}, x, \vec{r}) = G^0(F)_{[x]} G^1(F)_{x,\frac{r_0}{2}} \cdots G^d(F)_{x,\frac{r_{d-1}}{2}}. \tag{6.2.1}$$

See Figure 6.2.1 for an illustration of K.

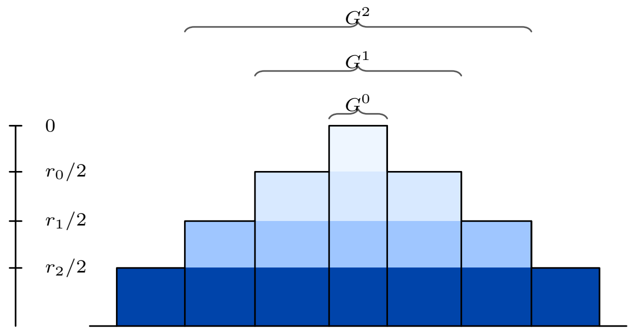

FIGURE 6.2.1. Illustration of the subgroup K from (6.2.1). The vertical axis measures depth, and the horizontal axis records the roots present in G^i.

Next, we recall the definition of the representation ρ as above. We note that K acts on τ by requiring that $G^1(F)_{x,\frac{r_0}{2}} \cdots G^d(F)_{x,\frac{r_{d-1}}{2}}$ acts trivially. We will define $\rho = \tau \otimes \kappa \otimes \phi_d$, where κ is a representation of K which we will define below. Roughly speaking, we have

$$\kappa = \bigotimes_{i=0}^{d-1} V_{\widehat{\phi}_i},$$

where each $V_{\widehat{\phi}_i}$ is constructed using a Weil–Heisenberg representation whose central character is derived from the character ϕ_i.[21]

Fix a maximal F-torus $T' \subset G^0$ which splits over a tamely ramified Galois extension E'/F and satisfies $x \in \mathcal{A}(T'_{E'})$. As in [Yu01, §4], for $0 \le i \le d-1$ let $\widehat{\phi}_i$ be the unique $k^\times$-valued character of $G^0(F)_{[x]} G^i(F)_{x,0} G(F)_{x,\frac{r_i}{2}+}$ satisfying the following two properties:

- $\widehat{\phi}_i|_{G^0(F)_{[x]}G^i(F)_{x,0}} = \phi_i|_{G^0(F)_{[x]}G^i(F)_{x,0}}$.
- $\widehat{\phi}_i|_{G(F)_{x,\frac{r_i}{2}+}}$ factors through

$$\begin{aligned} G(F)_{x,\frac{r_i}{2}+}/G(F)_{x,r_i+} &\cong \mathfrak{g}(F)_{x,\frac{r_i}{2}+}/\mathfrak{g}(F)_{x,r_i+} \\ &= (\mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F))_{x,\frac{r_i}{2}+}/(\mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F))_{x,r_i+} \\ &\to \mathfrak{g}^i(F)_{x,\frac{r_i}{2}+}/\mathfrak{g}^i(F)_{x,r_i+} \\ &\cong G^i(F)_{x,\frac{r_i}{2}+}/G^i(F)_{x,r_i+}, \end{aligned}$$

  where $\mathfrak{n}^i$ is defined as $\mathfrak{g} \cap \bigoplus_{\alpha \in \Phi(G,T'_{E'}) - \Phi(G^i,T'_{E'})} (\mathfrak{g}_{E'})_\alpha$, and the map $\mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F) \to \mathfrak{g}^i(F)$ is the first projection.

Observe that the induced map $\widehat{\phi}_i\colon G^i(F)_{x,\frac{r_i}{2}+} \to k^\times$ is equal to $\phi_i|_{G^i(F)_{x,\frac{r_i}{2}+}}$, so the two bullet points are compatible, hence this construction is well-defined. By convention, we set $\widehat{\phi}_d = \phi_d$.

[21]This notation is not standard; for example, [HM08] use ϕ_i' to denote $V_{\widehat{\phi}_i}$, while [Fin21a] uses (ω_i, V_{ω_i}).

6.3. **The Weil–Heisenberg factors.** Let $\widetilde{\mathbf{R}} = \mathbf{R} \cup \{r+ \colon r \in \mathbf{R}\}$, ordered such that for $r < s \in \widetilde{\mathbf{R}}$ we have $r < r+ < s$. For $\widetilde{r} \geq \widetilde{r}' \geq \frac{\widetilde{r}}{2} > 0$ in $\widetilde{\mathbf{R}}$ and $1 \leq i \leq d$, let

$$\begin{aligned} G^i(F)_{x,\widetilde{r},\widetilde{r}'} = G(F) & \\ & \cap \langle T'(E')_{x,\widetilde{r}}, U_\alpha(E')_{x,\widetilde{r}}, U_\beta(E')_{x,\widetilde{r}'} : \\ & \quad \alpha \in \Phi(G^{i-1}_{E'}, T'_{E'}), \\ & \quad \beta \in \Phi((G^i/G^{i-1})_{E'}, T'_{E'}) \rangle. \end{aligned}$$

In particular, define $J_i = J_i(\vec{G}, x, \vec{r}) = G^i(F)_{x, r_{i-1}, \frac{r_{i-1}}{2}}$ and $J_i^+ = G^i(F)_{x, r_{i-1}, \frac{r_{i-1}}{2}+}$. Observe that

$$K = G^0(F)_{[x]} J_1 \cdots J_d.$$

Figure 6.3.1 illustrates J_i; we learned this picture from Stephen DeBacker.

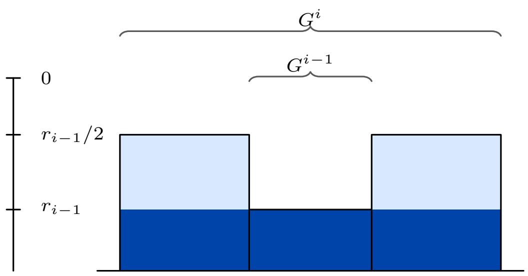

FIGURE 6.3.1. Illustration of $J_i = G^i(F)_{x, r_{i-1}, r_{i-1}/2}$. The vertical axis measures depth, and the horizontal axis records the roots present in G^i. The roots present in G^{i-1} enter at depth r_{i-1}, while the remaining roots enter at depth $r_{i-1}/2$.

symbol	range	role
$\widehat{\phi}_i$	$0 \leq i \leq d-1$	extension of ϕ_i used to define the $(i+1)$st Heisenberg datum
J_i, J_i^+	$1 \leq i \leq d$	compact subgroups attached to the jump from G^{i-1} to G^i
V_i	$1 \leq i \leq d$	symplectic quotient J_i/J_i^+, with central character from $\widehat{\phi}_{i-1}$
B_i	$1 \leq i \leq d$	$\mathbf{F}_q$-valued symplectic form on V_i induced by X^*_{i-1}
$V_{\widehat{\phi}_i}$	$0 \leq i \leq d-1$	Weil–Heisenberg representation attached to J_{i+1}

TABLE 1. Table of objects and indices appearing in Yu's construction.

Recall that $\overline{\psi} \colon \mathbf{F}_p \to k^\times$ is the nontrivial additive character fixed in §3. It is proven in [Yu01, Proposition 11.4] that if we let $V_i = V_i(\vec{G}, x, \vec{r}) = J_i/J_i^+$ for $1 \leq i \leq d$ and equip V_i with the pairing $\langle \cdot, \cdot \rangle_i$ defined by $\langle a, b \rangle_i = \overline{\psi}^{-1}(\widehat{\phi}_{i-1}(aba^{-1}b^{-1}))$, then $(V_i, \langle \cdot, \cdot \rangle_i)$ is a symplectic $\mathbf{F}_p$-vector space.

Lemma 6.3.1. *For all $1 \leq i \leq d$, the Moy–Prasad isomorphism*

$$G^i(F)_{x, \frac{r_{i-1}}{2}} / G^i(F)_{x, \frac{r_{i-1}}{2}+} \cong \mathfrak{g}^i(F)_{x, \frac{r_{i-1}}{2}} / \mathfrak{g}^i(F)_{x, \frac{r_{i-1}}{2}+}$$

restricts to an isomorphism

$$V_i \cong \mathfrak{g}^i(F)_{x, \frac{r_{i-1}}{2}} / (\mathfrak{g}^i(F)_{x, \frac{r_{i-1}}{2}+} + \mathfrak{g}^{i-1}(F)_{x, \frac{r_{i-1}}{2}}). \tag{6.3.1}$$

Under this isomorphism, for all $v, w \in \mathfrak{g}^i(F)_{x, \frac{r_{i-1}}{2}}$ we have

$$\langle \overline{v}, \overline{w} \rangle_i = \mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p} \overline{X^*_{i-1}([v, w])}, \tag{6.3.2}$$

where the bar denotes reduction modulo $\mathfrak{p}_F$ of the element $X^*_{i-1}([v, w]) \in \mathcal{O}_F$.

Proof. The identity (6.3.1) is standard and is observed in the proof of [Yu01, Lemma 11.1]. The identity

$$\overline{\psi}(\langle \overline{v}, \overline{w}\rangle_i) = \psi(X^*_{i-1}([v,w]))$$

is also observed in the proof of [Yu01, Lemma 11.1], and (6.3.2) follows from the fact that $\overline{\psi}$ is injective and $\psi^0 = \overline{\psi} \circ \mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}$. □

Let $B_i = B_i(\vec{G}, x, \vec{r})\colon V_i \times V_i \to \mathbf{F}_q$ denote the symplectic form which is transported under the Moy–Prasad isomorphism to the symplectic form

$$(\overline{v}, \overline{w}) \mapsto \overline{X^*_{i-1}([v,w])}.$$

Our interest in this symplectic form (as opposed to $\langle \cdot, \cdot \rangle_i$, which is $\mathbf{F}_p$-valued) comes from the following lemma. Recall that F_r denotes the unramified extension of F of degree r.

Lemma 6.3.2. *The maps $f_{i,r}\colon \overline{G}^0_{[x]}(\mathbf{F}_{q^r}) \to \mathrm{Sp}(V_i(\vec{G}_{F_r}), B_i(\vec{G}_{F_r}))$ (for $r \in \mathbf{Z}_{\geq 1}$) induced by the actions of $G^0(F_r)_{[x]}$ on $J_i(\vec{G}_{F_r})$ and $J_i^+(\vec{G}_{F_r})$ are algebraic, i.e., there is an $\mathbf{F}_q$-homomorphism $\overline{G}^0_{[x]} \to \mathbf{Sp}(V_i, B_i)$ which restricts to $f_{i,r}$ on $\mathbf{F}_{q^r}$-points for all r.*

Proof. Put

$$L_i = \mathfrak{g}(F)_{x, r_{i-1}/2}, \qquad L_i^+ = \mathfrak{g}(F)_{x, r_{i-1}/2+},$$

considered as vector groups over $\mathcal{O}_F$, and let $\mathcal{G}^0_{[x]}$ be the smooth separated $\mathcal{O}_F$-model of G^0 with $\mathcal{G}^0_{[x]}(\mathcal{O}_{F^{\mathrm{unr}}}) = G^0(F^{\mathrm{unr}})_{[x]}$. The relative identity component of $\mathcal{G}^0_{[x]}$ is affine and open, and its cosets form an affine open cover whose $\mathcal{O}_{F^{\mathrm{unr}}}$-points cover $\mathcal{G}^0_{[x]}(\mathcal{O}_{F^{\mathrm{unr}}})$. Applying [KP23, Corollary 2.10.10] to each coset, we see that the maps $G^0(F^{\mathrm{unr}})_{[x]} \to \mathrm{Aut}(L_i \otimes \mathcal{O}_{F^{\mathrm{unr}}})$ and $G^0(F^{\mathrm{unr}})_{[x]} \to \mathrm{Aut}(L_i^+ \otimes \mathcal{O}_{F^{\mathrm{unr}}})$ induced by the adjoint action extend uniquely to homomorphisms of $\mathcal{O}_{F^{\mathrm{unr}}}$-group schemes $(\mathcal{G}^0_{[x]})_{\mathcal{O}_{F^{\mathrm{unr}}}} \to \mathrm{GL}(L_i \otimes \mathcal{O}_{F^{\mathrm{unr}}})$, and similarly for L_i^+. These maps are Galois-invariant, and hence they are defined over $\mathcal{O}_F$ by descent. Since $\varpi L_i \subset L_i^+$, these maps induce an algebraic action of $(\mathcal{G}^0_{[x]})_{\mathbf{F}_q}$ on L_i/L_i^+. Thus the action factors through $\overline{G}^0_{[x]}$. The inverse image of the unipotent radical of $(\mathcal{G}^0_{[x]})^\circ_{\mathbf{F}_q}$ is $G^0(F^{\mathrm{unr}})_{x,0+}$, which acts trivially on L_i/L_i^+ by the Moy–Prasad commutator relation. Since $G^0 \subset G^{i-1} \subset G^i$, equation (6.3.1) induces an algebraic action of $\overline{G}^0_{[x]}$ on V_i. Finally, the G^{i-1}-invariance of X^*_{i-1} shows that the action preserves B_i. This gives the required algebraic homomorphism, and unramified base change gives the asserted maps on all $\mathbf{F}_{q^r}$-points. □

By [Yu01, Proposition 11.4] again, for $1 \leq i \leq d$ there is a canonical special isomorphism

$$j_i = j_i(\vec{G}, x, \vec{r}, \vec{\phi})\colon J_i/\big(J_i^+ \cap \ker \widehat{\phi}_{i-1}\big) \to V_i^\sharp, \tag{6.3.3}$$

where $V_i^\sharp$ is the Heisenberg group corresponding to $(V_i, \frac{1}{2}\langle \cdot, \cdot \rangle)$. In other words, $V_i^\sharp$ is the group with underlying set $V_i \times \mathbf{F}_p$ and group law $(v,a)\cdot(w,b) = (v+w, a+b+\frac{1}{2}\langle v, w\rangle_i)$.

For $0 \leq i \leq d-1$, let $V_{\widehat{\phi}_i}$ denote the Heisenberg representation of $J_{i+1}/(J_{i+1}^+ \cap \ker \widehat{\phi}_i)$ with central character the character of $J_{i+1}^+/(J_{i+1}^+ \cap \ker \widehat{\phi}_i)$ induced by $\widehat{\phi}_i|_{J_{i+1}^+}$. By (6.3.3) applied with index $i+1$, the group $J_{i+1}/(J_{i+1}^+ \cap \ker \widehat{\phi}_i)$ is identified with the Heisenberg group $V_{i+1}^\sharp$ attached to the symplectic space $V_{i+1} = J_{i+1}/J_{i+1}^+$. When k is a field, $V_{\widehat{\phi}_i}$ is the unique irreducible k-representation of $V_{i+1}^\sharp$ with central character $\widehat{\phi}_i$. For $k = \overline{\mathbf{Z}}_\ell$, take a $V_{i+1}^\sharp$-stable lattice in the corresponding Heisenberg representation over $\overline{\mathbf{Q}}_\ell$. Existence and uniqueness up to isomorphism follow from [Ser77, Part III, no. 15.5, Proposition 43] and the fact that $V_{i+1}^\sharp$ is a p-group and $\ell \neq p$.

The representation κ is defined to have underlying space $\bigotimes_{i=0}^{d-1} V_{\widehat{\phi}_i}$; we will define the action of K separately on each $V_{\widehat{\phi}_i}$. We let J_{i+1} act on $V_{\widehat{\phi}_i}$ through its quotient $V_{i+1}^\sharp$. If $j \neq i$, let J_{i+1} act on $V_{\widehat{\phi}_j}$ through $\widehat{\phi}_j|_{J_{i+1}}$. Moreover, $G^0(F)_{[x]}$ acts on $V_{\widehat{\phi}_i}$ through $\phi_i \cdot W_i$, where W_i is the representation

$$G^0(F)_{[x]}/G^0(F)_{x,0+} \to \mathrm{Sp}(V_{i+1}) \to \mathrm{GL}(V_{\widehat{\phi}_i}),$$

where $\mathrm{Sp}(V_{i+1}) \to \mathrm{GL}(V_{\widehat{\phi}_i})$ is the Weil representation, as defined in [Gér77, Theorem 2.4]. (See also [Fin22, Lemma 2.5] for an explanation of the existence of these representations over $\overline{\mathbf{Z}}_\ell$, and by extension $\overline{\mathbf{F}}_\ell$.)

6.4. **Compatibility with tame base change and restriction to twisted Levis.**

Ansatz 6.4.1. In the functoriality statements below, let $H \subset G$ be a twisted Levi F-subgroup containing a tamely ramified elliptic maximal F-torus in common with G^0, assume $x \in \mathcal{B}(H)$, and let E/F be a finite tamely ramified Galois extension. We set $H^i = G^i \cap H$ for all i and write $\vec{H} = (H^i)_{0 \le i \le d}$. If ϕ_i is a character of $G^i(F)$, we write $\phi_{H,i} = \phi_i|_{H^i(F)}$.

Lemma 6.4.2. *Assume we are in the setting of Ansatz 6.4.1.*

(1) $K(\vec{G}, x, \vec{r}) \cap H(F) = K(\vec{H}, x, \vec{r})$.
(2) $K(\vec{G}_E, x_E, \vec{r})^{\mathrm{Gal}(E/F)} = K(\vec{G}, x, \vec{r})$.

Proof. For (2), we induct on d, the case $d = 0$ being clear. If $d > 0$, then there is a short exact sequence

$$1 \to G^d(E)_{x,\frac{r_{d-1}}{2}} \to K(\vec{G}_E, x_E, \vec{r}) \to K((G^i_E)_{0 \le i < d}, x_E, (r_i)_{0 \le i < d})/G^{d-1}(E)_{x,\frac{r_{d-1}}{2}} \to 1. \tag{6.4.1}$$

There is also a tautological short exact sequence

$$1 \to G^{d-1}(E)_{x,\frac{r_{d-1}}{2}} \to K((G^i_E)_{0 \le i < d}, x_E, (r_i)_{0 \le i < d}) \to K((G^i_E)_{0 \le i < d}, x_E, (r_i)_{0 \le i < d})/G^{d-1}(E)_{x,\frac{r_{d-1}}{2}} \to 1. \tag{6.4.2}$$

Taking $\mathrm{Gal}(E/F)$-fixed points in (6.4.1) and (6.4.2), applying [Yu01, Proposition 2.2], and using the fact that the Moy–Prasad isomorphism restricts well with respect to finite field extensions of F (which follows easily from the proof of [KP23, Theorem 13.5.1]), we obtain

$$1 \to G^d(F)_{x,\frac{r_{d-1}}{2}} \to K(\vec{G}_E, x_E, \vec{r})^{\mathrm{Gal}(E/F)} \to K((G^i_E)_{0 \le i < d}, x_E, (r_i)_{0 \le i < d})^{\mathrm{Gal}(E/F)}/G^{d-1}(F)_{x,\frac{r_{d-1}}{2}} \to 1. \tag{6.4.3}$$

By induction, we have

$$K((G^i_E)_{0 \le i < d}, x_E, (r_i)_{0 \le i < d})^{\mathrm{Gal}(E/F)} = K((G^i)_{0 \le i < d}, x, (r_i)_{0 \le i < d}). \tag{6.4.4}$$

Since clearly $K(\vec{G}, x, \vec{r}) \subset K(\vec{G}_E, x_E, \vec{r})^{\mathrm{Gal}(E/F)}$, it follows from (6.4.3) and (6.4.4) that this inclusion is an equality, as desired.

For (1), let Z denote the maximal central torus of H. By (2), we may pass to a finite tamely ramified extension of F to assume that there is an element $z \in Z(F)$ of finite order prime to p such that $Z_G(z) = H$. Then the claim is that

$$K(\vec{G}, x, \vec{r})^z = K(\vec{H}, x, \vec{r}). \tag{6.4.5}$$

Since z is of order prime to p, the set $\mathrm{H}^1(z, G^i(F)_{x,\frac{r_{i-1}}{2}})$ is a singleton. Moreover, we have $(G^i(F)_{x,\frac{r_{i-1}}{2}})^z = H^i(F)_{x,\frac{r_{i-1}}{2}}$ for all i and $(G^0(F)_{[x]})^z = H^0(F)_{[x]}$, so the same argument as in the previous paragraph allows us to conclude inductively that (6.4.5) holds. □

To study the behavior of the $\widehat{\phi}_i$ under base change to a tamely ramified Galois extension E/F, we need to extend each ϕ_i from $G^i(F)$ to a $\mathrm{Gal}(E/F)$-invariant character of $G^i(E)$; this is not possible in complete generality, and it will lead to some small restrictions on p.

Lemma 6.4.3. *Assume we are in the setting of Ansatz 6.4.1.*

(1) $\widehat{\phi}_{H,i} = \widehat{\phi}_i|_{H^0(F)_{[x]}H^i(F)_{x,0}H(F)_{x,\frac{r_i}{2}+}}$ *for all* $0 \le i \le d-1$.
(2) If ϕ_i *extends to a character* $\phi_{E,i}$ *of* $G^i(E)$ *which is generic of depth* r_i*, then* $\widehat{\phi}_i = \widehat{\phi}_{E,i}|_{G^0(F)_{[x]}G^i(F)_{x,0}G(F)_{x,\frac{r_i}{2}+}}$ *for all* $0 \le i \le d-1$.

Proof. We begin with (1). It is clear that $\widehat{\phi}_i|_{H^0(F)_{[x]}H^i(F)_{x,0}H(F)_{x,\frac{r_i}{2}+}}$ satisfies the first defining property in §6.2. Compatibility of the Moy–Prasad isomorphism with restriction to twisted Levi subgroups shows that the map

$$H(F)_{x,\frac{r_i}{2}+}/H(F)_{x,r_i+} \to H^i(F)_{x,\frac{r_i}{2}+}/H^i(F)_{x,r_i+}$$

defined there is the restriction of the corresponding map for G. Since $\phi_{H,i}$ is the restriction of ϕ_i, it is therefore clear that the second defining property in §6.2 holds as well. The argument for (2) is completely similar, using compatibility of the Moy–Prasad isomorphism with tame extensions, as in [Yu01, Corollary 2.4]. □

Lemma 6.4.4. *With notation and assumptions as in Ansatz 6.4.1, we have*

(1) $J_i(\vec{G}, x, \vec{r}) \cap H(F) = J_i(\vec{H}, x, \vec{r})$ *and similarly for* J_i^+*,*
(2) $J_i(\vec{G}_E, x_E, \vec{r})^{\mathrm{Gal}(E/F)} = J_i(\vec{G}, x, \vec{r})$ *and similarly for* J_i^+*.*

Proof. The proof is completely analogous to that of Lemma 6.4.2. □

Lemma 6.4.5. *Assume we are in the setting of Ansatz 6.4.1.*

(1) Write $\vec{\phi}_H = (\phi_{H,j})_j$*, and let* $\iota_{H,i} \colon V_i(\vec{H})^\sharp \hookrightarrow V_i(\vec{G})^\sharp$ *be the canonical embedding. Then*

$$\iota_{H,i} \circ j_i(\vec{H}, x, \vec{r}, \vec{\phi}_H) = j_i(\vec{G}, x, \vec{r}, \vec{\phi})\big|_{J_i(\vec{H},x,\vec{r})/(J_i^+(\vec{H},x,\vec{r}) \cap \ker \widehat{\phi}_{H,i-1})}.$$

(2) If $\vec{\phi}_E = (\phi_{E,j})_j$ *extends* $\vec{\phi}$*, with each* $\phi_{E,j}$ *being* G_E^{j+1}*-generic of depth* r_j *for* $0 \le j < d$*, let* $\iota_{E,i} \colon V_i(\vec{G})^\sharp \hookrightarrow V_i(\vec{G}_E)^\sharp$ *be the canonical embedding. Then*

$$\iota_{E,i} \circ j_i(\vec{G}, x, \vec{r}, \vec{\phi}) = j_i(\vec{G}_E, x_E, \vec{r}, \vec{\phi}_E)\big|_{J_i(\vec{G},x,\vec{r})/(J_i^+(\vec{G},x,\vec{r}) \cap \ker \widehat{\phi}_{i-1})}.$$

Proof. We briefly recall the construction of j_i as in the proof of [Yu01, Proposition 11.4]. First suppose that G^i is split, let $S \subset G^i$ be a split maximal F-torus, and choose a system of positive roots $\Phi^+ \subset \Phi(G^i, S)$. Let $J_i(+)$ (resp. $J_i(-)$) denote the subgroup of J_i generated by $U_\alpha(F)_{x, \frac{r_{i-1}}{2}}$ for $\alpha \in \Phi^+$ (resp. $\alpha \in -\Phi^+$) not lying in $\Phi(G^{i-1}, S)$. If $N_i = J_i^+ \cap \ker \widehat{\phi}_{i-1}$, then the maps $\pi_+ \colon J_i(+)N_i/N_i \to J_i/J_i^+$ and $\pi_- \colon J_i(-)N_i/N_i \to J_i/J_i^+$ are injective, and they form a complete polarization of J_i/J_i^+. By [Yu01, Lemma 10.1], there is a canonical isomorphism $j_i \colon J_i/N_i \to V_i^\sharp$ given by

$$j_i(w_+ w_- c) = (\pi_+(w_+) + \pi_-(w_-), c + \frac{1}{2}\langle \pi_+(w_+), \pi_-(w_-)\rangle)$$

for $w_+ \in J_i(+)N_i/N_i$, $w_- \in J_i(-)N_i/N_i$, and $c \in J_i^+/N_i$, identified with an element of $\mathbf{F}_p$ via $\overline{\psi}^{-1} \circ \widehat{\phi}_{i-1}$. It is shown in the proof of [Yu01, Proposition 11.4] that j_i is independent of the choices of S and Φ^+. If G^i is not necessarily split, let $T \subset G^i$ be a tamely ramified maximal F-torus, let F'/F be a finite tamely ramified extension such that $T_{F'}$ is F'-split, and let j_i denote the restriction to J_i/N_i of the isomorphism $J_i(F')/N_i(F') \to (J_i(F')/J_i^+(F'))^\sharp$ constructed above. As noted in the proof of [Yu01, Proposition 11.4] (see also the discussion preceding [HM08, Definition 3.15]), this gives rise to a special isomorphism over F by [Yu01, Lemma 10.3] which is independent of the choice of F'. From this construction, both points are evident. □

6.5. **The FKS twist and compact induction.** Finally, for a Yu datum Ψ, we define the character ϵ of K to be trivial on $G^i(F)_{x, \frac{r_{i-1}}{2}}$ for $1 \le i \le d$, and to restrict to the character ϵ on $G^0(F)_{[x]}$ defined in (3.3.1) using the twisted Levi sequence (G^i) and the generic elements X_i^* in the definition.

Finally, we define

$$\pi = \pi(\Psi) := \text{c-Ind}_K^{G(F)}(\epsilon \otimes \tau \otimes \bigotimes_{i=0}^{d-1} V_{\widehat{\phi}_i} \otimes \phi_d). \tag{6.5.1}$$

For $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$, [Fin22, Theorem 3.1] and exactness of compact induction show that $\pi(\Psi)$ has finite length; it is irreducible and cuspidal when $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$ and Ψ is irreducible. If moreover p does not divide the order of the absolute Weyl group of G, then [Fin22, Theorem 4.1] shows that every irreducible cuspidal smooth k-representation of $G(F)$ arises from this construction.

6.6. **Equivalence and normalization of Yu data.** It is natural to ask when $\pi(\Psi) \cong \pi(\Psi')$ for two different Yu data Ψ, Ψ'. This leads to the notion of *G-equivalence* on irredundant Yu data, an equivalence relation introduced by Hakim–Murnaghan in [HM08, Definition 6.1], whose definition we will shortly recall.

6.6.1. *Removing redundant stages.* Recall from §6.1 that a Yu datum is irredundant if $G^i \ne G^{i+1}$ for all $0 \le i < d$. First, we show that one can reduce to irredundant Yu data.

Lemma 6.6.1. *Let*

$$\Psi = ((G^i)_{0 \le i \le d}, x, (r_i)_{0 \le i \le d}, \tau, (\phi_i)_{0 \le i \le d})$$

be a Yu datum for G*. Suppose that* $G^m = G^{m+1}$ *for some* $0 \le m < d$*, and let*

$$\Psi' = ((G'^i)_{0 \le i \le d-1}, x, (r'_i)_{0 \le i \le d-1}, \tau, (\phi'_i)_{0 \le i \le d-1})$$

be the Yu datum defined by

(1) $G'^i = G^i$ *for* $0 \le i < m$ *and* $G'^i = G^{i+1}$ *for* $m \le i \le d-1$,
(2) $r'_i = r_i$ *for* $0 \le i < m$ *and* $r'_i = r_{i+1}$ *for* $m \le i \le d-1$,
(3) $\phi'_i = \phi_i$ *for* $0 \le i < m$ *and* $\phi'_m = \phi_m\phi_{m+1}$ *and* $\phi'_i = \phi_{i+1}$ *for* $m < i \le d-1$.

Then $\pi(\Psi) \cong \pi(\Psi')$. *In particular, there is an irredundant Yu datum* Ψ_0 *such that* $\pi(\Psi) \cong \pi(\Psi_0)$.

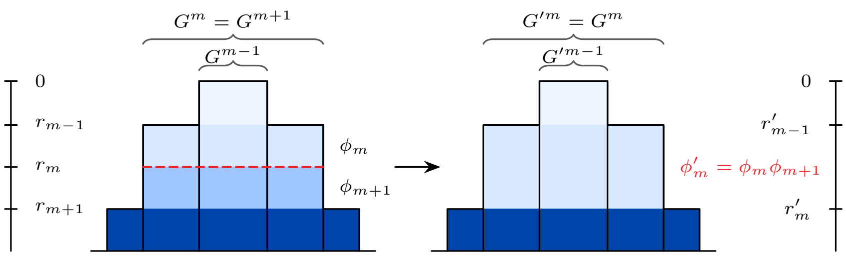

FIGURE 6.6.1. Collapsing a redundant step in a Yu datum. The vertical axis records character depth, and the horizontal axis records the roots present in the corresponding twisted Levi.

Proof. The final claim is immediate from the first by induction, so it suffices to prove the first claim. The fact that Ψ' is a Yu datum is trivial. One checks that $K(\Psi) = K(\Psi')$; moreover, $\epsilon_\Psi = \epsilon_{\Psi'}$, since $\epsilon_x^{G^{m+1}/G^m}$ is trivial. It is just as straightforward to check that the inducing representations $\rho(\Psi)$ and $\rho(\Psi')$ are isomorphic, so we leave this to the reader. ☐

6.6.2. *G-equivalence.* We aim to explicitly describe the equivalence relation on Yu data induced by isomorphism of the corresponding representations.

Definition 6.6.2. Let $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ and $\Psi' = ((G'^i), x', (r'_i), \tau', (\phi'_i))$ be irredundant Yu data over k. We say that Ψ and Ψ' are related by an *elementary transformation* if $(G^i) = (G'^i)$, $x = x'$, $(r_i) = (r'_i)$, $\tau \cong \tau'$, and $(\phi_i) = (\phi'_i)$. We say that Ψ and Ψ' are *G-conjugate* if there exists some $g \in G(F)$ such that every term of Ψ' is obtained from the corresponding term of Ψ by g-conjugation, where we interpret $g \cdot (r_i) = (r_i)$. We say that Ψ' is a *refactorization* of Ψ if $G^i = G'^i$ for all i and $[x] = [x']$, and the following conditions on the characters $\chi_i \colon G^i(F) \to k^\times$ defined by $\chi_i(g) = \prod_{j=i}^d \phi_j(g)\phi'_j(g)^{-1}$ hold:

(1) If $\phi_d = 1$, then $\phi'_d = 1$.
(2) The character χ_i is of depth at most r_{i-1} for all $0 \le i \le d$, where $r_{-1} = 0$.
(3) $\tau' = \tau \otimes \chi_0$.

Finally, we say that Ψ and Ψ' are *G-equivalent* if they are equivalent under the equivalence relation generated by elementary transformations, G-conjugacy, and refactorization.

By [Kal19, Corollary 3.5.5] (which builds on [HM08, Theorem 6.6]), one knows that the supercuspidal representations of $G(F)$ over $k = \overline{\mathbf{Q}}_\ell$ constructed from irreducible Yu data Ψ and Ψ' are isomorphic if and only if Ψ and Ψ' are G-equivalent. We expect that the same claim holds for $k = \overline{\mathbf{F}}_\ell$, but we have not checked it. However, we will use one direction of this result, proven in [HM08, Proposition 4.24].

Proposition 6.6.3. *Let* Ψ *and* Ψ' *be irredundant Yu data for* G *over a ring* k *among* $\overline{\mathbf{Q}}_\ell$, $\overline{\mathbf{Z}}_\ell$, *and* $\overline{\mathbf{F}}_\ell$. *If* Ψ *and* Ψ' *are G-equivalent, then* $\pi(\Psi) \cong \pi(\Psi')$.

Proof. The proof of [HM08, Proposition 4.24] is largely self-contained, and it extends nearly verbatim to our setting once Hypothesis C($\vec{G}$) from [HM08, §4.3] is arranged to hold by passing to a z-extension as in [Kal19, Lemmas 3.5.2–3.5.4]. ☐

6.6.3. *Normalized Yu data.* We introduce the following definition of "normalized" characters, whose nomenclature was chosen for consistency with the notion of normalized Yu data in [Kal19, Definition 3.7.1].

Definition 6.6.4. Let H be a connected reductive F-group and let k be a commutative ring. A smooth character $\phi\colon H(F) \to k^\times$ is *normalized* if it is trivial on the image of $H_{\mathrm{sc}}(F) \to H(F)$, where H_{sc} is the simply connected cover of H_{der}.

With this terminology, [Kal19, Definition 3.7.1] says that a Yu datum Ψ is *normalized* if each ϕ_i is normalized as a character of $G^i(F)$. By [Kal19, Lemma 3.7.2][22], if p does not divide $|\pi_1(G_{\mathrm{der}})|$, then every irredundant Yu datum is G-equivalent to a normalized Yu datum.

The following lemma will both facilitate the study of modular reductions and allow us to prove finiteness of L-packets later.

Lemma 6.6.5. *If Ψ is an irredundant Yu datum over k, then there is a Yu datum Ψ' which is a refactorization of Ψ such that each ϕ'_i is of finite order. Moreover, for each i there is some integer N_i depending only on G^i, x, and r_i such that ϕ'_i is of order at most N_i. If Ψ is normalized, then Ψ' can be chosen to be normalized.*

Proof. Recall the subgroup $G^i(F)^1 \subset G^i(F)$ defined by

$$G^i(F)^1 = \{g \in G^i(F) \colon |\chi(g)| = 1 \text{ for all } \chi \colon G^i \to \mathbf{G}_m\}.$$

Note that $G^i(F)/G^i(F)^1$ is a free abelian group, so the map $G^i(F)/G^i(F)_{\mathrm{der}} \to G^i(F)/G^i(F)^1$ of abelian groups admits a section, and thus

$$G^i(F)/G^i(F)_{\mathrm{der}} \cong G^i(F)^1/G^i(F)_{\mathrm{der}} \times G^i(F)/G^i(F)^1. \tag{6.6.1}$$

For each i, let ψ_i be the pullback, under projection to the second factor in (6.6.1), of the restriction of ϕ_i to that factor. Thus ψ_i is trivial on $G^i(F)^1$. Put $\phi'_i = \phi_i\psi_i^{-1}$. Then ϕ'_i factors through the compact group $G^i(F)^1/G^i(F)_{\mathrm{der}}$ and, since it is trivial on $G^i(F)_{x,r_i+}$, its order divides

$$N_i := \left|G^i(F)^1/\big(G^i(F)_{\mathrm{der}}G^i(F)_{x,r_i+}\big)\right|.$$

Put $\chi_i = \prod_{j=i}^d \psi_j|_{G^i(F)}$ and $\tau' = \tau \otimes \chi_0$. If $j \geq i$, then $G^i(F)^1 \subset G^j(F)^1$, so χ_i is trivial on $G^i(F)^1$ and hence has depth at most $0 \leq r_{i-1}$. Moreover,

$$\prod_{j=i}^d \phi_j\phi_j'^{-1} = \chi_i,$$

while $\phi_d = 1$ implies $\phi'_d = 1$. Thus $\Psi' = ((G^i), x, (r_i), \tau', (\phi'_i))$ is a refactorization of Ψ. Finally, the image of $G^i_{\mathrm{sc}}(F)$ lies in $G^i(F)^1$, so each ϕ'_i has the same restriction as ϕ_i to this image; therefore Ψ' is normalized whenever Ψ is. ☐

7. Explicit construction of L-parameters

In this section, we recall Kaletha's definition of a Local Langlands Correspondence for non-singular tame supercuspidal representations. Moreover, we extend Kaletha's candidate to positive characteristic coefficients and partially to singular tame cuspidal representations. The key results established in this section are certain compatibilities of Kaletha's construction with Langlands functoriality, which will be needed later for comparison to the Fargues–Scholze correspondence.

Throughout this section, assume that $p \neq 2$. Recall from §2 that F_n/F denotes the unramified extension of degree n. The notation ${}^L j_{T,G}$ for the L-embedding of a torus T associated to a Yu datum is defined in Definition 4.5.1.

7.1. **Almost equivalence of torus-character pairs.** Recall from §2 the notation $\mathcal{G}_{[x]}$ and $\overline{G}_{[x]}$ associated to a point $x \in \mathcal{B}(G)$. Let F^{unr} denote the maximal unramified extension of F. If $T \subset G$ is a maximally unramified maximal torus and a point $x \in \mathcal{B}(G)$ is given which lies in $\mathcal{B}(T)$, then we let $\overline{T}$ denote the image in $\overline{G}_{[x]}$ of the special fiber of the schematic closure of T in $\mathcal{G}_{[x]}$, so $\overline{T}(\mathbf{F}_q) = T(F)_{[x]}/T(F)_{x,0+}$. Note in particular that depth 0 characters of $T(F)_{[x]}$ are equivalent to characters of $\overline{T}(\mathbf{F}_q)$.

Definition 7.1.1. Let $T, T' \subset G$ be two tamely ramified maximal F-tori such that $x \in \mathcal{B}(T) \cap \mathcal{B}(T')$, let k be a field among $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$, and let $\theta \colon T(F)_{[x]} \to k^\times$ and $\theta' \colon T'(F)_{[x]} \to k^\times$ be characters. Since $Z(G)(F)$ acts trivially on $\mathcal{B}(G_{\mathrm{der}})$, it is contained in both $T(F)_{[x]}$ and $T'(F)_{[x]}$. We will say that (T, θ) and (T', θ') are *almost equivalent* if the following conditions hold:

[22] The errata on Kaletha's website notes that Section 3.7 (among others) of [Kal19] does not, but should, require that p is a good prime for G and that p does not divide the order of $\pi_0(Z(G))$. However, this error only arises because the proof of [Kal19, Lemma 3.6.8] uses a misstated form of [Yu01, Lemma 8.1] in its proof. The proof of [Kal19, Lemma 3.7.2] does not use [Kal19, Lemma 3.6.8], and it is still correct. Moreover, the proof applies without change over $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$.

(1) $\theta|_{Z(G)(F)} = \theta'|_{Z(G)(F)}$.
(2) There exists a positive integer n such that the pairs

$$\left(T_{F_n}, \theta \circ \mathrm{Nm}_{F_n/F}|_{T(F_n)_{[x]}}\right) \quad \text{and} \quad \left(T'_{F_n}, \theta' \circ \mathrm{Nm}_{F_n/F}|_{T'(F_n)_{[x]}}\right)$$

are $G(F_n)$-conjugate.

The notion of almost equivalence should be considered as a weak analogue for p-adic groups of the equivalence relation on tori arising from rational Lusztig series, as explained in [Cot26, §2.8]. The terminology is justified by Lemma 7.1.3.

Proposition 7.1.2. *Let $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Z}}_\ell, \overline{\mathbf{Q}}_\ell\}$, let E/F be a finite extension of degree n, and let $\phi\colon G(F) \to k^\times$ be a normalized character (cf. Definition 6.6.4). There is a canonical normalized character $\phi_E\colon G(E) \to k^\times$ such that*

$$\phi_E|_{T(E)} = \phi|_{T(F)} \circ \mathrm{Nm}_{E/F}$$

for every maximal F-torus $T \subset G$. Moreover, $\phi_E|_{G(F)} = \phi^n$, and ϕ_E is $\mathrm{Gal}(E/F)$-stable when E/F is Galois.

Proof. Choose a z-extension

$$1 \longrightarrow Z \longrightarrow \widetilde{G} \xrightarrow{q} G \longrightarrow 1$$

with Z induced and $\widetilde{G}_{\mathrm{der}}$ simply connected, so that $q|_{\widetilde{G}_{\mathrm{der}}}$ is the universal covering map onto G_{der}. Put $D = \widetilde{G}/\widetilde{G}_{\mathrm{der}}$, write $\pi_D\colon \widetilde{G} \to D$ for the quotient, and let $i\colon Z \to D$ be the induced map. For $L \in \{F, E\}$, Hilbert 90 and Kneser's theorem [KP23, Theorem 10.6.4] give

$$A_L := G(L)/q(\widetilde{G}_{\mathrm{der}}(L)) \xrightarrow{\sim} D(L)/i(Z(L)).$$

Since ϕ is normalized, the pullback $q^*\phi$ descends to a character $\phi_D\colon D(F) \to k^\times$ which is trivial on $i(Z(F))$. The composite

$$\widetilde{G}(E) \xrightarrow{\pi_D} D(E) \xrightarrow{\mathrm{Nm}_{E/F}} D(F) \xrightarrow{\phi_D} k^\times$$

is trivial on $Z(E)$ by functoriality of the norm, and hence descends to a character ϕ_E of $G(E)$. It is normalized, and it is Galois-stable when E/F is Galois. If $g \in G(F)$ and $\widetilde{g} \in \widetilde{G}(F)$ lifts it, then

$$\phi_E(g) = \phi_D\big(\mathrm{Nm}_{E/F}(\pi_D(\widetilde{g}))\big) = \phi_D(\pi_D(\widetilde{g})^n) = \phi(g)^n.$$

Finally, if $T \subset G$ is a maximal F-torus and $\widetilde{T} = q^{-1}(T)$, then $\widetilde{T}(E) \to T(E)$ is surjective. Lifting $t \in T(E)$ to $\widetilde{t} \in \widetilde{T}(E)$ and using functoriality of norms gives

$$\phi_E(t) = \phi_D\big(\mathrm{Nm}_{E/F}(\pi_D(\widetilde{t}))\big) = \phi\big(\mathrm{Nm}_{E/F}(t)\big),$$

which proves the proposition. □

Lemma 7.1.3. *Let $T, T' \subset G$ be maximally unramified maximal F-tori such that $x \in \mathcal{B}(T) \cap \mathcal{B}(T')$, let k be a field among $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$, and let $\theta_0\colon T(F)_{[x]} \to k^\times$ and $\theta'_0\colon T'(F)_{[x]} \to k^\times$ be characters of depth 0. Let $\phi\colon G(F) \to k^\times$ be a normalized character, and let $\theta = \theta_0 \cdot \phi|_{T(F)_{[x]}}$ and $\theta' = \theta'_0 \cdot \phi|_{T'(F)_{[x]}}$. If $\mathcal{E}(\overline{G}_{[x]}, [\overline{T}, \theta_0]) = \mathcal{E}(\overline{G}_{[x]}, [\overline{T}', \theta'_0])$, with notation as in* [Cot26, Definition 2.8.2]*, then the pairs (T, θ) and (T', θ') are almost equivalent.*

Proof. We first reduce to the case $\phi = 1$. By geometric conjugacy of $(\overline{T}, \theta_0)$ and $(\overline{T}', \theta'_0)$, there is a finite unramified extension E/F and an element of $G(E)$ conjugating $(T_E, \theta_0 \circ \mathrm{Nm}_{E/F})$ to $(T', \theta'_0 \circ \mathrm{Nm}_{E/F})$. To show that the same element also conjugates $(T, \theta \circ \mathrm{Nm}_{E/F})$ to $(T', \theta' \circ \mathrm{Nm}_{E/F})$, it suffices to construct a character ϕ_E of $G(E)$ whose restrictions to $T(E)$ and $T'(E)$ are the corresponding norm pullbacks of ϕ. Such a character is provided by Proposition 7.1.2, so we have made the desired reduction.

Let n be a positive integer such that the pairs $(\overline{T}_{\mathbf{F}_{q^n}}, \theta \circ \mathrm{Nm}_{\mathbf{F}_{q^n}/\mathbf{F}_q})$ and $(\overline{T}'_{\mathbf{F}_{q^n}}, \theta' \circ \mathrm{Nm}_{\mathbf{F}_{q^n}/\mathbf{F}_q})$ are $\overline{G}_{[x]}(\mathbf{F}_{q^n})$-conjugate, and let $E = F_n$. Let $\overline{g} \in \overline{G}_{[x]}(\mathbf{F}_{q^n})$ be an element such that

$$\overline{g}(\overline{T}_{\mathbf{F}_{q^n}}, \theta \circ \mathrm{Nm}_{\mathbf{F}_{q^n}/\mathbf{F}_q})\overline{g}^{-1} = (\overline{T}'_{\mathbf{F}_{q^n}}, \theta' \circ \mathrm{Nm}_{\mathbf{F}_{q^n}/\mathbf{F}_q}).$$

Let S (resp. S') denote the maximal unramified F-subtorus of T (resp. T'), and let $\mathcal{S}$ (resp. $\mathcal{S}'$) denote the $\mathcal{O}_F$-torus with generic fiber S (resp. S'). Note that there are canonical closed embeddings $\mathcal{S} \to \mathcal{G}_{[x]}$ and $\mathcal{S}' \to \mathcal{G}_{[x]}$ by [KP23, Axiom 4.1.20].

Let
$$\pi_x\colon (\mathcal{G}_{[x]})_{\mathbf{F}_{q^n}} \longrightarrow (\overline{G}_{[x]})_{\mathbf{F}_{q^n}}$$
be the quotient by the unipotent radical of $(\mathcal{G}_{[x]})^{\circ}_{\mathbf{F}_{q^n}}$. By Lang's theorem, π_x is surjective on $\mathbf{F}_{q^n}$-points. Thus $\overline{g}$ lifts to $\mathcal{G}_{[x]}(\mathbf{F}_{q^n})$, and then, by smoothness of $\mathcal{G}_{[x]}$ and completeness of $\mathcal{O}_E$, to some $g \in \mathcal{G}_{[x]}(\mathcal{O}_E)$. The images of $\mathcal{S}_{\mathbf{F}_{q^n}}$ and $\mathcal{S}'_{\mathbf{F}_{q^n}}$ under π_x are $\overline{T}^{\circ}_{\mathbf{F}_{q^n}}$ and $\overline{T}'^{\circ}_{\mathbf{F}_{q^n}}$, respectively. Hence the tori $g\mathcal{S}g^{-1}$ and $\mathcal{S}'$ have the same image under π_x, so they are conjugate by $\ker \pi_x$. After passing to a translate of g by an element of $\mathcal{G}_{[x]}(\mathcal{O}_E)$ whose special fiber lies in $(\ker \pi_x)(\mathbf{F}_{q^n})$, we may assume that $g\mathcal{S}g^{-1}$ and $\mathcal{S}'$ have the same special fiber. By deformation theory for tori [SGA3$_{\mathrm{II}}$, Exp. IX, Corollaire 7.3], there is some
$$h \in \ker\big(\mathcal{G}^{\circ}_{[x]}(\mathcal{O}_E) \to \overline{G}^{\circ}_{[x]}(\mathbf{F}_{q^n})\big) = G(E)_{x,0+}$$
such that $hgSg^{-1}h^{-1} = S'$. Since T and T' are maximally unramified, we have $T = Z_G(S)$ and $T' = Z_G(S')$, so also $hgTg^{-1}h^{-1} = T'$. Since h has trivial image in $\overline{G}^{\circ}_{[x]}(\mathbf{F}_{q^n})$ and θ, θ' have depth 0, we deduce
$$hg(T_E, \theta \circ \mathrm{Nm}_{E/F}\,|_{T(E)_{[x]}})g^{-1}h^{-1} = (T'_E, \theta' \circ \mathrm{Nm}_{E/F}\,|_{T'(E)_{[x]}}),$$
as desired. □

7.2. **Associated torus-character pairs.** In [Kal21b], Kaletha introduced his Local Langlands Correspondence for non-singular tame supercuspidal $\overline{\mathbf{Q}}_\ell$-representations of $G(F)$. In this section, we recall this correspondence and partially extend it to general tame cuspidal representations.

Let $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ be an irreducible Yu datum for G over a field k which is either $\overline{\mathbf{Q}}_\ell$ or $\overline{\mathbf{F}}_\ell$. Let $\overline{Z}$ be the center of $\overline{G}^0_{[x]}$, and recall that $\overline{G}^0_{[x]}$ is a paraductive $\mathbf{F}_q$-group scheme in the sense of [Cot26, Definition 2.1.1].

7.2.1. *Associated depth* 0 *torus-character pair.* By [Cot26, Proposition 2.6.2], there is a maximal $\mathbf{F}_q$-torus $\overline{T}^{\circ} \subset (\overline{G}^0_{[x]})^{\circ}$ and a character $\theta_0\colon \overline{T}(\mathbf{F}_q) \to k^{\times}$, where $\overline{T} = \overline{T}^{\circ} \cdot \overline{Z}$, such that τ lies in the semi-rational Lusztig series $\mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}, \theta_0])$ (with notation as in [Cot26, Definition 2.8.2]). The pair $(\overline{T}, \theta_0)$ is not unique in general; however, the Lusztig series $\mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}, \theta_0])$ is unique by [Cot26, Lemma 2.8.4]. Moreover, if there is such a pair with θ_0 non-singular, then the pair $(\overline{T}, \theta_0)$ is unique up to $\overline{G}^0_{[x]}(\mathbf{F}_q)$-conjugacy by [Cot26, Lemma 2.9.2].

7.2.2. *Associated p-adic torus-character pair.* Choose an $\mathcal{O}_F$-torus $\mathcal{T}_0$ of $\mathcal{G}^0_{[x]}$ whose special fiber $\overline{\mathcal{T}}_0$ has image $\overline{T}^{\circ}$ in $\overline{G}^0_{[x]}$ (which exists by [Bor91, Proposition 11.14(1)] and deformation theory for tori [SGA3$_{\mathrm{II}}$, Exposé IX, Théorème 3.6, Théorème 7.1]), and let T_0 be the generic fiber of $\mathcal{T}_0$, an F-torus of G^0. If $T = Z_{G^0}(T_0)$, then T is a maximally unramified maximal F-torus of G^0 and thus also a maximal F-torus of G, and $x \in \mathcal{B}(T)$. Define a character $\theta\colon T(F)_{[x]} \to k^{\times}$ by
$$\theta = \theta_0 \cdot \prod_{i=0}^{d} \phi_i|_{T(F)_{[x]}}.$$
If τ is non-singular, then it follows that $\overline{T}^{\circ}$ is elliptic and thus T is also elliptic since $Z(G^0)/Z(G)$ is anisotropic.

Definition 7.2.1. Define $\mathcal{T}(\Psi)$ as the set of $G(F)$-conjugacy classes $[(T, \theta)]$ of pairs $(T, \theta\colon T(F)_{[x]} \to k^{\times})$ constructed above. It is clear that $\mathcal{T}(\Psi)$ only depends on the G-equivalence class of Ψ, and it is unaffected by the procedure of Lemma 6.6.1.

Let $\mathcal{T}_0(\Psi) \subset \mathcal{T}(\Psi)$ denote the set of $[(T, \theta)]$ constructed as above such that $\tau \in \mathcal{E}_0(\overline{G}, [\overline{T}, \theta])$, with notation as in [Cot26, Definition 2.8.2].

Note that if $k = \overline{\mathbf{Q}}_\ell$ and τ is cuspidal, then for every $[(T, \theta)] \in \mathcal{T}_0(\Psi)$, the torus T is elliptic in G. The set $\mathcal{T}_0(\Psi)$ will be important for studying the Fargues–Scholze correspondence in [CF26b] because it retains direct contact with Deligne–Lusztig induction, but the following lemma will eventually show that it is irrelevant for the study of Kaletha's correspondence.

Lemma 7.2.2. *Assume that Ψ is normalized. If $[(T, \theta)]$ and $[(T', \theta')]$ lie in $\mathcal{T}(\Psi)$, then the pairs (T, θ) and (T', θ') are almost equivalent in the sense of Definition 7.1.1. The central character of $\pi(\Psi)$ is equal to $\theta|_{Z(G)(F)}$.*

Proof. Put $\phi = \prod_{i=0}^d \phi_i|_{G^0(F)}$. The inclusions $G^0_{\mathrm{der}} \to G^i_{\mathrm{der}}$ lift to the simply connected covers, so the hypothesis that Ψ is normalized implies that ϕ is normalized. The fact that (T,θ) and (T',θ') are almost equivalent is immediate from Lemma 7.1.3. The final claim is clear from the fact that the inducing representation $\epsilon \otimes \tau \otimes \kappa \otimes \phi_d$ in Yu's construction has central character $\theta|_{Z(G)(F)}$ (since $\epsilon|_{Z(G)(F)} = 1$). □

Lemma 7.2.3. *Assume that Ψ is normalized. If $[(T,\theta)]$ and $[(T',\theta')]$ lie in $\mathcal{T}(\Psi)$, then*

$$ {}^L j_{T,G} \circ {}^L(\theta|_{T(F)_{\mathrm{b}}})|_{I_F} \sim {}^L j_{T',G} \circ {}^L(\theta'|_{T'(F)_{\mathrm{b}}})|_{I_F}. $$

Proof. By Lemma 7.2.2, there are an unramified extension E/F and $g \in G^0(E)$ such that $g(T_E, \theta \circ \mathrm{Nm}_{E/F})g^{-1} = (T'_E, \theta' \circ \mathrm{Nm}_{E/F})$. Let $c_g\colon T_E \xrightarrow{\sim} T'_E$ be the conjugation map, and let ${}^L c_g\colon {}^L T'_E \xrightarrow{\sim} {}^L T_E$ be the associated L-isomorphism. It is visible from (4.3.2) that

$$ {}^L j_{T'_E, G_E} \sim {}^L j_{T_E, G_E} \circ {}^L c_g. \tag{7.2.1} $$

Let

$$ \theta_E \coloneqq \theta \circ \mathrm{Nm}_{E/F}|_{T(E)_{[x]}}, \qquad \theta'_E \coloneqq \theta' \circ \mathrm{Nm}_{E/F}|_{T'(E)_{[x]}}. $$

The choice of g implies that $c_g^* \theta'_E = \theta_E$, so naturality of the inertial Local Langlands Correspondence for tori and (7.2.1) yield

$$ {}^L j_{T'_E, G_E} \circ {}^L(\theta'_E|_{T'(E)_{\mathrm{b}}})|_{I_E} \sim {}^L j_{T_E, G_E} \circ {}^L(\theta_E|_{T(E)_{\mathrm{b}}})|_{I_E}. $$

Thus we conclude by norm functoriality for the inertial Local Langlands Correspondence for tori and Lemma 5.2.4(1)–(2) (applied to a filtration of E/F by extensions of prime degree). □

Lemma 7.2.4. *If $T \subset G^0$ is a tamely ramified maximal F-torus such that $x \in \mathcal{B}(T)$, then $\phi_i|_{T(F)}$ is of depth r_i for all $0 \le i < d$, and $\phi_d|_{T(F)}$ is of depth r_d if $\phi_d \ne 1$.*

Proof. Fix $i < d$. By genericity, the character induced by ϕ_i on $G^i(F)_{x,r_i}/G^i(F)_{x,r_i+}$ is represented, under the Moy–Prasad isomorphism and the fixed additive character ψ, by the class of some $X_i^* \in \mathrm{Lie}^*(G^i)^{G^i}(F)_{x,-r_i}$. Choose $Y \in \mathfrak{g}^i(F)_{x,r_i}$ whose pairing with X_i^* lies in $\mathcal{O}_F^\times$; this exists since X_i^* has nontrivial image in $\mathrm{Lie}^*(G^i)(F)_{x,-r_i:-r_i+}$. Let E/F be a tame extension splitting T, so there is a direct sum decomposition

$$ \mathfrak{g}^i(E)_{x,r_i} = \mathfrak{t}(E)_{r_i} \oplus \bigoplus_\alpha \mathfrak{g}^i_\alpha(E)_{x,r_i}. $$

Since X_i^* is G^i-invariant, it annihilates every root space, each of which lies in $[\mathfrak{g}^i, \mathfrak{g}^i]$. Thus the projection of Y to $\mathfrak{t}(E)_{r_i}$ has the same nonzero pairing with X_i^*. This projection lies in $\mathfrak{t}(F)_{r_i}$, so $\phi_i|_{T(F)_{r_i}}$ is nontrivial. Thus $\phi_i|_{T(F)}$ has depth r_i.

Now assume $\phi_d \ne 1$, so ϕ_d is of depth d by assumption. Choose a z-extension $\pi\colon \widetilde{G} \to G$, and note that by [Kal19, Lemma 3.5.3], the map $\widetilde{G}(F)_{x,r_d} \to G(F)_{x,r_d}$ is surjective; thus the induced character $\widetilde{\phi}_d$ of $\widetilde{G}(F)$ is of depth r_d. Now [Kal19, Lemma 3.5.2] shows that $\widetilde{\phi}_d$ is represented by an element $X_i^* \in \mathrm{Lie}^*(\widetilde{G})^{\widetilde{G}}(F)_{x,-r_d}$, so the same argument as above shows that if $\widetilde{T} = \pi^{-1}(T)$ then $\widetilde{\phi}_d|_{\widetilde{T}(F)}$ is of depth r_d. Since $\widetilde{T}(F)_{r_d} \to T(F)_{r_d}$ is surjective by [Kal19, Lemma 3.5.3] again, we conclude. □

7.2.3. *Non-singular characters.* We adapt the following definition from [Kal21b, Definition 3.1.1].[23]

Definition 7.2.5. Let k be a field of characteristic not equal to p. Let $T \subset G$ be a maximally unramified maximal F-torus, fix a point $x \in \mathcal{B}(T) \subset \mathcal{B}(G)$, let E_0/F be a finite tamely ramified extension splitting T, and let F'/F be the maximal unramified subextension, so F' splits the maximal unramified F-subtorus $S \subset T$. Let $\mathbf{F}_{F'}$ be the residue field of F', and let $\overline{S}^\circ$ denote the special fiber of the relative identity component of the lft Néron model of S, viewed as a maximal torus of $\overline{G}^\circ_{[x]}$. Fix a subgroup $T^\dagger \subset T(F)$ containing $T(F)_{\mathrm{b}}$ and a character $\theta\colon T^\dagger \to k^\times$. Assume

$$ \theta \circ \mathrm{Nm}_{E_0/F} \circ \alpha^\vee|_{1+\mathfrak{p}_{E_0}} = 1 \text{ for every coroot } \alpha^\vee \in \Phi^\vee(G_{E_0}, T_{E_0}). \tag{7.2.2} $$

We say that

(1) θ is $\mathbf{F}$-*non-singular* if $(\theta \circ \mathrm{Nm}_{F'/F} \circ \alpha^\vee_{\mathrm{res}})|_{\mathcal{O}_{F'}^\times}$ is nontrivial
for each $\alpha_{\mathrm{res}} \in \Phi((\overline{G}^\circ_{[x]})_{\mathbf{F}_{F'}}, \overline{S}^\circ_{\mathbf{F}_{F'}}) \subset \Phi(G_{F'}, S_{F'})$.

[23] In fact, for reasons explained below, our definition differs slightly from [Kal21b, Definition 3.1.1].

(2) θ is *F-non-singular* if $(\theta \circ \mathrm{Nm}_{F'/F} \circ \alpha_{\mathrm{res}}^\vee)|_{\mathcal{O}_{F'}^\times}$ is nontrivial for each $\alpha_{\mathrm{res}} \in \Phi(G_{F'}, S_{F'})$.

We now generalize Definition 7.2.5 beyond the case of maximally unramified tori as in [Kal21b, Definition 3.4.1], for which we must introduce some notation. Let k be a field of characteristic different from p, fix a tamely ramified maximal F-torus $T \subset G$, a subgroup $T^\dagger$ with $T(F)_{\mathrm{b}} \subset T^\dagger \subset T(F)$, and a character $\theta\colon T^\dagger \to k^\times$. Define the depth of θ to be the depth of $\theta|_{T(F)_{\mathrm{b}}}$. Let E/F be the splitting field of T. For each positive real number r, we define

$$\Phi_r = \Phi_r(T,\theta) = \{\alpha \in \Phi(G_E, T_E) \colon \theta(\mathrm{Nm}_{E/F}(\alpha^\vee(E_r^\times))) = 1\}$$

where $E_r^\times \coloneqq 1 + \mathfrak{p}_E^{\lceil er \rceil}$ for e the ramification degree of E/F. Put $E_{r+}^\times \coloneqq \bigcup_{s>r} E_s^\times$. Let $\Phi_{r+} = \bigcap_{s>r} \Phi_s$ and say that $r \in \mathbf{R}_{>0}$ is a *break* in the filtration $\Phi_\bullet(T,\theta)$ if $\Phi_r \neq \Phi_{r+}$. There are finitely many (possibly 0) breaks, which we denote $r_{d-1} > r_{d-2} > \ldots > r_0 > 0$. We define by convention $r_{-1} \coloneqq 0$ and $r_d \coloneqq \mathrm{depth}(\theta)$. If $d > 0$, then this gives a sequence $r_d \geq r_{d-1} > \cdots > r_0 > r_{-1}$, while if $d = 0$ (no breaks in Φ_r) then $r_0 \geq r_{-1} = 0$.

For $d \geq i \geq 0$, we define $G^i = G^i(T,\theta) \subset G$ to be the connected reductive subgroup with maximal torus T and root system $\Phi_{r_{i-1}+}$. We define $G^{-1} \coloneqq T$, so we have

$$T = G^{-1} \subset G^0 \subset \cdots \subset G^d = G.$$

Note that if p is good for G and does not divide the order of $\pi_1(G_{\mathrm{der}})$, then G^i is a twisted Levi subgroup of G by [Kal19, Lemma 3.6.1]. In fact, Lemma 7.2.9 will show that this is true for arbitrary p when $(T,\theta) \in \mathcal{T}(\Psi)$ for a normalized Yu datum Ψ.

Definition 7.2.6. We say that $\theta\colon T^\dagger \to k^\times$ is **F**-*non-singular* (resp. *F-non-singular*) if T is maximally unramified in G^0 and, when T is considered inside G^0, the character θ (which satisfies the hypotheses of Definition 7.2.5) is **F**-non-singular (resp. F-non-singular) with respect to G^0 in the sense of Definition 7.2.5. These notions and the groups $G^i(T,\theta)$ depend only on $\theta|_{T(F)_{\mathrm{b}}}$.

As in [Kal21b, Fact 3.1.4(1)], an F-non-singular character is automatically **F**-non-singular.

Remark 7.2.7. Definitions 7.2.5 and 7.2.6 differ slightly from [Kal21b, Definition 3.1.1, Definition 3.4.1]: indeed, [Kal21b, Definition 3.4.1] requires that θ is non-singular with respect to G^0 in the sense of [Kal21b, Definition 3.1.1], which strictly speaking implies that θ is of depth 0; however, this is clearly an unintended restriction.

Definition 7.2.8. We will say that an irreducible Yu datum $\Psi = (\vec{G}, x, \vec{r}, \tau, \vec{\phi})$ over $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Q}}_\ell\}$ is **F**-*non-singular* if τ is non-singular in the sense of [Cot26, Definition 2.9.3].

Lemma 7.2.9. *Assume $p \neq 2$, let $\Psi = (\vec{G}, x, \vec{r}, \tau, \vec{\phi})$ be an irredundant normalized irreducible Yu datum for G over a field k among $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$, and let $[(T,\theta)] \in \mathcal{T}(\Psi)$.*

(1) $G^i = G^i(T,\theta)$ for all $0 \leq i \leq d$, where $G^i(T,\theta)$ is defined from the filtration $\Phi_\bullet(T,\theta)$ preceding Definition 7.2.6.

(2) The canonical ($\widehat{G}(k)$-conjugacy class of) embedding(s) $\widehat{G^0} \to \widehat{G}$ realizes $\widehat{G^0}$ as the ($\widehat{G}(k)$-conjugacy class of) connected centralizer(s) $Z_{\widehat{G}}(({}^L j_{T,G} \circ {}^L\theta)(P_F))^\circ$.

*(3) Ψ is **F**-non-singular if and only if θ is **F**-non-singular. In this case, T is elliptic and $\mathcal{T}(\Psi)$ consists of the single conjugacy class $[(T,\theta)]$.*

Proof. We begin with (1). Let E/F be a finite tamely ramified Galois extension splitting T. It is enough to prove

$$\Phi_{r_{i-1}+}(T,\theta) = \Phi(G_E^i, T_E) \qquad (0 \leq i \leq d).$$

If $j < i$, then ϕ_j has depth at most r_{i-1}, so we have

$$\theta(\mathrm{Nm}_{E/F}(\alpha^\vee(E_{r_{i-1}+}^\times))) = \prod_{j=i}^{d} \phi_j(\mathrm{Nm}_{E/F}(\alpha^\vee(E_{r_{i-1}+}^\times))). \tag{7.2.3}$$

If $\alpha \in \Phi(G_E^i, T_E)$, then the coroot $\alpha^\vee\colon \mathbf{G}_m \to G_E^i$ factors through a map $\alpha_{\mathrm{sc}}^\vee\colon \mathbf{G}_m \to (G_{\mathrm{sc}}^i)_E$, where $\pi\colon G_{\mathrm{sc}}^i \to G_{\mathrm{der}}^i$ is the simply connected cover. Since Ψ is normalized, functoriality of the norm on the inverse-image torus in G_{sc}^j gives

$$\phi_j \circ \mathrm{Nm}_{E/F} \circ \alpha^\vee = 1 \qquad (i \leq j \leq d). \tag{7.2.4}$$

By (7.2.3), this means that $\alpha \in \Phi_{r_{i-1}+}(T,\theta)$.

Conversely, suppose that $\alpha \in \Phi_{r_{i-1}+}(T,\theta)$. If $\alpha \in \Phi(G_E^j, T_E)$ for $j \le i$, then $\alpha \in \Phi(G_E^i, T_E)$ immediately. Otherwise there is a unique $i \le j < d$ such that

$$\alpha \in \Phi(G_E^{j+1}, T_E) - \Phi(G_E^j, T_E).$$

Equation (7.2.4) shows that $\phi_m \circ \mathrm{Nm}_{E/F} \circ \alpha^\vee$ is trivial for $m \ge j+1$, while $\phi_m \circ \mathrm{Nm}_{E/F} \circ \alpha^\vee$ has depth smaller than r_j for $m < j$. The assumption $\alpha \in \Phi_{r_{i-1}+}(T,\theta)$ and the inequality $r_j > r_{i-1}$ therefore give

$$\phi_j(\mathrm{Nm}_{E/F}(\alpha^\vee(E^\times_{r_j}))) = 1. \tag{7.2.5}$$

Let v_E denote the extension to E of the normalized valuation of F. Let $X_j^* \in \mathrm{Lie}^*(G^j)^{G^j}(F)_{x,-r_j}$ represent $\phi_j|_{G^j(F)_{x,r_j}}$ through the Moy–Prasad isomorphism and ψ. Genericity means that $v_E(\langle X_j^*, H_\alpha\rangle) = -r_j$. Let $z \in E$ be such that $v_E(z) = r_j$ and

$$\psi\big(\mathrm{Tr}_{E/F}(z\langle X_j^*, H_\alpha\rangle)\big) \ne 1;$$

such a z exists because E/F is tame. The Moy–Prasad isomorphism $T(F)_{r_j}/T(F)_{r_j+} \xrightarrow{\sim} \mathfrak{t}(F)_{r_j}/\mathfrak{t}(F)_{r_j+}$ and its analogue for E sends $\mathrm{Nm}_{E/F}\big(\alpha^\vee(1+z)\big)$ to $\mathrm{Tr}_{E/F}(zH_\alpha)$. Since X_j^* is F-rational and represents ϕ_j on $\mathfrak{g}^j(F)_{x,r_j}/\mathfrak{g}^j(F)_{x,r_j+}$, it follows that

$$\begin{aligned}\phi_j\big(\mathrm{Nm}_{E/F}(\alpha^\vee(1+z))\big) &= \psi\left(\langle X_j^*, \mathrm{Tr}_{E/F}(zH_\alpha)\rangle\right) \\ &= \psi\big(\mathrm{Tr}_{E/F}(z\langle X_j^*, H_\alpha\rangle)\big) \ne 1,\end{aligned}$$

contradicting Equation (7.2.5). This proves $\Phi_{r_{i-1}+}(T,\theta) = \Phi(G_E^i, T_E)$.

For (2), note that since T is tamely ramified, the restriction ${}^L j_{T,G}|_{\widehat{T}\rtimes P_F}$ is simply the canonical embedding $\widehat{T}\times P_F \to \widehat{G}\times P_F$. Thus $Z_{\widehat{G}}(({}^L j_{T,G}\circ {}^L\theta)(P_F))^\circ$ is the reductive k-subgroup of $\widehat{G}$ with maximal k-torus $\widehat{T}$ and root system consisting of those $\widehat{\alpha}$ (for $\alpha \in \Phi(G_{\overline{F}}, T_{\overline{F}})$) such that $\widehat{\alpha}\circ\widehat{\theta}|_{P_F} = 1$. By duality, this means $\theta(\mathrm{Nm}_{E/F}(\alpha^\vee(E^\times_{0+}))) = 1$, i.e., $\alpha \in \Phi(G^0_{\overline{F}}, T_{\overline{F}})$. So the claim follows from (1).

We now prove (3). Let $S \subset T$ be the maximal unramified subtorus, and choose a finite unramified extension F'/F splitting S. Part (1) shows that the restriction of θ to every coroot occurring in G^0 has depth 0. More precisely, for each restricted coroot $\alpha_0^\vee$ of $S_{F'}$ in $G^0_{F'}$, the same argument which led to Equation (7.2.4) gives

$$\phi_i \circ \mathrm{Nm}_{F'/F}\circ\alpha_0^\vee = 1 \qquad (0 \le i \le d).$$

Thus θ is $\mathbf{F}$-non-singular if and only if θ_0 is non-singular. Since τ lies in $\mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}, \theta_0])$, [Cot26, Definition 2.9.3 and Lemma 2.9.2] show that τ is non-singular if and only if θ_0 is non-singular. Under these equivalent conditions, [Cot26, Lemma 2.9.4] shows that $\overline{T}^\circ$ is elliptic. Hence T is elliptic in G^0; the anisotropy of $Z(G^0)/Z(G)$ then makes T elliptic in G. Moreover, [Cot26, Lemma 2.9.2] shows that $(\overline{T}, \theta_0)$ is unique up to $\overline{G}^0_{[x]}(\mathbf{F}_q)$-conjugacy. Lemma 7.1.3 therefore shows that $\mathcal{T}(\Psi)$ consists of the single orbit class $[(T,\theta)]$. $\square$

Definition 7.2.10. Let Ψ be an irreducible Yu datum over $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Q}}_\ell\}$. We say that Ψ is *F-non-singular* (or just *non-singular*) if some class $[(T,\theta)] \in \mathcal{T}(\Psi)$ has the property that θ is F-non-singular in the sense of Definition 7.2.6. We will then say that $\pi = \pi(\Psi)$ is non-singular.

Remark 7.2.11. If Ψ is a normalized irreducible Yu datum with $G^0 = T$, then Ψ is F-non-singular.

Note that Lemma 7.2.9(3) and [Kal21b, Fact 3.1.4(1)] show that any F-non-singular Yu datum is $\mathbf{F}$-non-singular.

Definition 7.2.12. Let $\Psi = (\vec{G}, x, \vec{r}, \tau, \vec{\phi})$ be an irreducible Yu datum for G over $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Q}}_\ell\}$, let Ψ^{irr} be the irredundant datum obtained by iterating Lemma 6.6.1, and assume that Ψ^{irr} is G-equivalent to a normalized Yu datum. Choose a class $[(T,\theta)] \in \mathcal{T}(\Psi)$. When Ψ is F-non-singular, Lemma 7.2.9(3) shows that $\mathcal{T}(\Psi)$ consists of this single $G(F)$-conjugacy class, with T elliptic and $\theta\colon T(F) \to k^\times$ non-singular. Define $\rho^{\mathrm{Kal}}(\Psi)$ to be the $\widehat{G}(k)$-conjugacy class represented by

$$W_F \xrightarrow{{}^L\theta} {}^LT(k) \xrightarrow{{}^L j_{T,G}} {}^LG(k) \tag{7.2.6}$$

where ${}^L\theta\colon W_F \to {}^LT(k)$ is dual to θ under the Local Langlands Correspondence for tori, and ${}^L j_{T,G}$ is the embedding from Definition 4.5.1.

Under the same hypothesis on Ψ^{irr}, without assuming that Ψ is F-non-singular, define $\rho_I^{\mathrm{Kal}}(\Psi)$ to be the $\widehat{G}(k)$-conjugacy class represented by

$$I_F \xrightarrow{{}^L(\theta|_{T(F)_{\mathrm{b}}})} {}^LT(k) \xrightarrow{{}^Lj_{T,G}} {}^LG(k) \tag{7.2.7}$$

where ${}^L(\theta|_{T(F)_{\mathrm{b}}})\colon I_F \to {}^LT(k)$ is the inertial L-parameter corresponding to $\theta|_{T(F)_{\mathrm{b}}}$ under the inertial Local Langlands Correspondence for tori, as in the discussion preceding Lemma 5.1.2. The fact that this is well-defined is proven in Lemma 7.2.13.

The superscript "Kal" is used here in recognition of Kaletha's work, although strictly speaking Kaletha only defines $\rho^{\mathrm{Kal}}(\Psi)$ in [Kal21b] in the case that k is of characteristic 0 and Ψ is non-singular. Note that if Ψ is $\mathbf{F}$-non-singular but not F-non-singular, then the recipe defining $\rho^{\mathrm{Kal}}(\Psi)$ still makes sense, but it does not yield the "correct" semisimple L-parameter.

Lemma 7.2.13. *The conjugacy classes $\rho_I^{\mathrm{Kal}}(\Psi)$ and, when Ψ is F-non-singular, $\rho^{\mathrm{Kal}}(\Psi)$, only depend on the G-equivalence class of Ψ.*

Proof. Independence of the chosen pair $[(T,\theta)] \in \mathcal{T}(\Psi)$ for the inertial parameter is Lemma 7.2.3; when Ψ is F-non-singular, there is a unique such pair. The fact that G-equivalent Yu data yield the same (inertial) L-parameter is simply a matter of unraveling the definitions recalled in §6.6.2, which we leave to the reader. □

Remark 7.2.14. If $k = \overline{\mathbf{Q}}_\ell$, then it follows from Lemma 7.2.13 and [Kal19, Corollary 3.5.5] (which builds on [HM08, Theorem 6.6]) that $\rho^{\mathrm{Kal}}(\Psi)$ and $\rho_I^{\mathrm{Kal}}(\Psi)$ only depend on $\pi(\Psi)$. We expect that the analogous assertion over $\overline{\mathbf{F}}_\ell$ is true, and that it can be established by similar methods, but it will not be used or proven here. Our eventual results compare $\rho^{\mathrm{Kal}}(\Psi)$ and $\rho_I^{\mathrm{Kal}}(\Psi)$ to the parameters arising from the Fargues–Scholze correspondence, establishing a certain amount of independence a posteriori.

Remark 7.2.15. We regard $\rho_I^{\mathrm{Kal}}(\Psi)$ as our guess for the restriction to I_F of the L-parameter associated to $\pi(\Psi)$ under the "true" semisimple Local Langlands Correspondence. As a sanity check, we will show in Lemma 10.1.11(1) that $\rho_I^{\mathrm{Kal}}(\Psi)$ is semisimple; this is nontrivial if k is of positive characteristic.

It seems to be a difficult problem to give a reasonably concrete definition of a full (semisimple) L-parameter $\rho^{\mathrm{Kal}}(\Psi)$ when Ψ is a Yu datum over $\overline{\mathbf{Q}}_\ell$ which is not F-non-singular, even when $\pi(\Psi)$ is of depth 0.

The following lemma is a straightforward generalization of [Kal21b, Lemma 4.1.10] beyond the case of characteristic 0 coefficients.

Lemma 7.2.16. *Let $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$, let Ψ be an irredundant normalized irreducible Yu datum for G with coefficients in k, and choose $[(T,\theta)] \in \mathcal{T}(\Psi)$. Let $\rho\colon I_F \to {}^LG(k)$ be the representative of $\rho_I^{\mathrm{Kal}}(\Psi)$ given by (7.2.7) using (T,θ), and identify $\widehat{T}$ with its image under ${}^Lj_{T,G}$. The following are equivalent:*

(1) $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}}$ is a torus,

(2) θ is non-singular.

Under these hypotheses, $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}} = (\widehat{T}^{I_F})^\circ_{\mathrm{red}}$.

Proof. The proof of [Kal21b, Lemma 4.1.10] can be read nearly verbatim in our setting, so we omit it. □

7.3. **Modular reduction.** Our ultimate aim is to prove results about the L-parameters of supercuspidal $\overline{\mathbf{Q}}_\ell$-representations of $G(F)$ obtained from Yu's construction. However, in order to analyze these we need to relate such supercuspidal representations to those constructed over $\overline{\mathbf{F}}_\ell$.

Lemma 7.3.1. *Let $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ be a Yu datum over $\overline{\mathbf{Z}}_\ell$. Put*

$$\overline{\tau} = \tau \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell, \qquad \overline{\Psi} = ((G^i), x, (r_i), \overline{\tau}, (\overline{\phi}_i)),$$

where $\overline{\phi}_i$ is the reduction of ϕ_i. The tuple $\overline{\Psi}$ is a (possibly reducible) Yu datum, and there is a natural isomorphism

$$\pi(\overline{\Psi}) \cong \pi(\Psi) \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell. \tag{7.3.1}$$

Moreover:

(1) Every irreducible subquotient $\overline{\tau}_0$ of $\overline{\tau}$ is cuspidal, and the tuple

$$\overline{\Psi}_0 = ((G^i), x, (r_i), \overline{\tau}_0, (\overline{\phi}_i))$$

is an irreducible Yu datum. The representation $\pi(\overline{\Psi}_0)$ is an irreducible subquotient of $\pi(\Psi) \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell$.

(2) *Every irreducible subquotient of* $\pi(\Psi) \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell$ *is isomorphic to* $\pi(\overline{\Psi}_0)$ *for some irreducible subquotient* $\overline{\tau}_0$ *of* $\overline{\tau}$*.*

If Ψ *is irredundant (resp. normalized), then every* $\overline{\Psi}_0$ *is irredundant (resp. normalized).*

Proof. First observe that [Cot26, Lemma 2.10.1] and exactness of the formation of Jacquet modules (since $\ell \neq p$) imply that $\overline{\tau}$ and each of its irreducible subquotients are cuspidal. Next, note that modular reduction preserves the depth and genericity of each ϕ_i, since $\ell \neq p$. Thus $\overline{\Psi}$ is a Yu datum and every $\overline{\Psi}_0$ is an irreducible Yu datum. Now (7.3.1) and (1) follow from unraveling the definitions.

Since Yu's construction is exact as a functor of the depth 0 representation, applying it to a composition series of $\overline{\tau}$ proves part (2); the resulting representations are irreducible by [Fin22, Theorem 3.1]. If Ψ is normalized, its reductions are normalized because each $\overline{\phi}_i$ remains normalized. Preservation of irredundancy is also clear. □

Lemma 7.3.2. *Retain the notation of Lemma 7.3.1, and assume that* $\Psi \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{Q}}_\ell$ *is irreducible. For an irreducible subquotient* $\overline{\tau}_0$ *of* $\overline{\tau}$ *let* $\overline{\Psi}_0$ *be the associated normalized irreducible Yu datum. For every class* $[(T,\theta)] \in \mathcal{T}(\Psi_{\overline{\mathbf{Q}}_\ell})$*, the character* θ *takes values in* $\overline{\mathbf{Z}}_\ell^\times$*, and its reduction* $\overline{\theta}$ *satisfies* $[(T,\overline{\theta})] \in \mathcal{T}(\overline{\Psi}_0)$*. In particular,*

(1) $\rho_I^{\mathrm{Kal}}(\overline{\Psi}_0)$ *is the (semisimplified) modular reduction of* $\rho_I^{\mathrm{Kal}}(\Psi_{\overline{\mathbf{Q}}_\ell})$*.*
(2) *If* $\overline{\Psi}_0$ *is* F*-non-singular, then* $\Psi_{\overline{\mathbf{Q}}_\ell}$ *is* F*-non-singular and* $\rho^{\mathrm{Kal}}(\overline{\Psi}_0)$ *is the (semisimplified) modular reduction of* $\rho^{\mathrm{Kal}}(\Psi_{\overline{\mathbf{Q}}_\ell})$*.*

Proof. Fix $[(T,\theta)] \in \mathcal{T}(\Psi_{\overline{\mathbf{Q}}_\ell})$, arising from a depth 0 character θ_0. The group $\overline{T}(\mathbf{F}_q)$ is finite modulo $Z(\overline{G}^0_{[x]})(\mathbf{F}_q)$. On this central subgroup, [Cot26, Definition 2.8.2] identifies θ_0 with the central character of $\tau \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{Q}}_\ell$, which is $\overline{\mathbf{Z}}_\ell^\times$-valued because τ is a finite free $\overline{\mathbf{Z}}_\ell$-lattice. Hence θ_0 is $\overline{\mathbf{Z}}_\ell^\times$-valued on $\overline{T}(\mathbf{F}_q)$, while each ϕ_i is $\overline{\mathbf{Z}}_\ell^\times$-valued by hypothesis. Thus

$$\theta = \theta_0 \prod_{i=0}^{d} \phi_i|_{T(F)_{[x]}}$$

is $\overline{\mathbf{Z}}_\ell^\times$-valued. Since $\overline{\tau}_0$ is an irreducible subquotient of the reduction of τ, [Cot26, Definition 2.8.2] shows $\overline{\tau}_0 \in \mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}, \overline{\theta}_0])$. Hence $[(T,\overline{\theta})] \in \mathcal{T}(\overline{\Psi}_0)$. From this, (1) and (2) are both clear from the definitions. □

7.4. **Compatibility with central isogenies.** Finally, we record a basic compatibility, which is improved considerably for non-singular supercuspidal representations with characteristic 0 coefficients in [BM25] under some technical conditions. Fix an F-homomorphism $f\colon G \to G'$ inducing an isomorphism $G_{\mathrm{ad}} \cong G'_{\mathrm{ad}}$, let $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$, and let $\Psi' = ((G'^i), x', (r'_i), \tau', (\phi'_i))$ be a normalized irreducible irredundant Yu datum for G' with coefficients in k. Choose a Yu datum $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ as follows:

(1) $G^i = f^{-1}(G'^i)$ for all i,
(2) $x \in \mathcal{B}(G)$ is any point with the same image as x' in $\mathcal{B}(G_{\mathrm{ad}}) = \mathcal{B}(G'_{\mathrm{ad}})$,
(3) $r_i = r'_i$ for all i,
(4) τ is an irreducible constituent of τ' as a $\overline{G}^0_{[x]}(\mathbf{F}_q)$-representation,
(5) $\phi_i = \phi'_i \circ f$ for all i.

Suppose that each ϕ_i has the same depth as ϕ'_i. In this case, the fact that Ψ is a normalized irredundant irreducible Yu datum is clear.

Lemma 7.4.1. *If* ${}^L f\colon {}^L G' \to {}^L G$ *is the dual L-homomorphism, then* $\rho_I^{\mathrm{Kal}}(\Psi) \sim {}^L f \circ \rho_I^{\mathrm{Kal}}(\Psi')$*. If* Ψ *is* F*-non-singular, then* Ψ' *is also* F*-non-singular and* $\rho^{\mathrm{Kal}}(\Psi) \sim {}^L f \circ \rho^{\mathrm{Kal}}(\Psi')$*.*

Proof. If $[(T',\theta')] \in \mathcal{T}(\Psi')$ and we define $T = f^{-1}(T')$, $\theta = \theta' \circ f|_{T(F)_{[x]}}$, then $[(T,\theta)] \in \mathcal{T}(\Psi)$: indeed, the definition of $\epsilon_{G'}$ involves passing to G'_{ad}, so it is clear that $\epsilon_G = \epsilon_{G'} \circ f$, and it is enough to invoke [Cot26, Lemma 2.10.3]. Moreover, we have ${}^L j_{T,G} \circ {}^L(f|_T) \sim {}^L f \circ {}^L j_{T',G'}$ by Lemma 4.5.2(2). This implies the claim. □

8. Base change functoriality for Yu data

Throughout this section, assume that $p \neq 2$. We will study the behavior of ρ^{Kal} and ρ_I^{Kal} under base change in two cases: base change of "large" prime degree for arbitrary cuspidal representations, base change of "small" prime degree for "toral" representations (i.e., those arising from Yu data such that G^0 is a torus). Both cases have important simplifying features.

- In the first case, individual apartments in the Bruhat–Tits building are unchanged after base change of large prime degree (see [Cot26, Proposition 4.1.2]).
- In the second case, the reduced building of a torus is a point, so no issues from Bruhat–Tits theory arise. Furthermore, the irreducible representations of an abelian group are 1-dimensional, so no complications from Deligne–Lusztig theory arise.

8.1. **Preliminaries on characters.** In this section, we collect a number of technical results which will allow us to extend characters of unramified twisted Levi subgroups after base change. Throughout this subsection, let $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Z}}_\ell\}$, and let $G^i \subset G^{i+1} \subset G$ be tamely ramified twisted Levi subgroups.

Let E/F be a tamely ramified cyclic extension of prime degree $\ell \neq p$, let $\phi\colon G^i(F) \to k^\times$ be a normalized character (in the sense of Definition 6.6.4), and let $\phi_E\colon G^i(E) \to k^\times$ be the character defined in Proposition 7.1.2.

Lemma 8.1.1. *The characters ϕ and ϕ_E are of the same depth.*

Proof. Suppose that ϕ is of depth r. By definition, this means that

(1) there exists a point $x \in \mathcal{B}(G^i)$ such that $\phi|_{G^i(F)_{x,r+}}$ is trivial,
(2) if $r > 0$, then for any point $y \in \mathcal{B}(G^i)$, the restriction $\phi|_{G^i(F)_{y,r}}$ is nontrivial.

First, we show that $\phi_E|_{G^i(E)_{x,r+}}$ is trivial. Let $x \in \mathcal{B}(G^i)$ be a point such that condition (1) holds for ϕ, and let $T \subset G^i$ be a tamely ramified maximal F-torus with $x \in \mathcal{B}(T)$. By Lemma 7.2.4, the restriction $\phi|_{T(F)}$ is of depth r. The norm map $\mathrm{Nm}_{E/F}\colon T(E)_{x,r+} \to T(F)_{x,r+}$ is surjective by [Kal19, Lemma 3.1.3], so indeed (1) holds.

Now suppose $r > 0$, let $y \in \mathcal{B}(G^i_E)$ be any point, and suppose for the sake of contradiction that $\phi_E|_{G^i(E)_{y,r}}$ is trivial. Let c be the center of mass of the $\mathrm{Gal}(E/F)$-orbit of y, so $c \in \mathcal{B}(G^i)$ by [KP23, Theorem 12.9.2]. Since ϕ_E is a character, it is trivial on $G^i(E)_{g\cdot y,r}$ for every $g \in G^i(E)$. Choose an apartment A containing c and a point $y_0 \in A \cap G^i(E) \cdot y$, let S be the maximal E-split torus defining A, and put $Z = Z_{G^i_E}(S)$. For each relative root α, choose a vector v_α in the image of $S(E)$ in $\mathrm{Aut}(A)$ with $\langle \alpha, v_\alpha\rangle > 0$. Then $y_0 + nv_\alpha$ lies in $A \cap G^i(E) \cdot y$, and [KP23, Lemma 6.1.6] gives

$$U_\alpha(E)_{c,r} \subset U_\alpha(E)_{y_0+nv_\alpha,r}$$

for all sufficiently large n: indeed, once $\alpha(y_0) + n\alpha(v_\alpha) \geq \alpha(c)$, we have

$$r - \alpha(c) \geq r - \alpha(y_0) - n\alpha(v_\alpha),$$

so the condition defining $U_\alpha(E)_{c,r}$ is stronger than the one defining $U_\alpha(E)_{y_0+nv_\alpha,r}$.

By [KP23, Definition 13.2.1], the group $G^i(E)_{c,r}$ is generated by $Z(E)_r$ and these $U_\alpha(E)_{c,r}$. Hence ϕ_E is trivial on $G^i(E)_{c,r}$. By [KP23, Proposition 12.9.4], this implies that ϕ^ℓ is trivial on $G^i(F)_{c,r}$, contradicting the depth of ϕ. Thus $\phi_E|_{G^i(E)_{y,r}}$ is nontrivial, as desired. $\square$

Lemma 8.1.2. *If ϕ is of depth $r > 0$, then ϕ is G^{i+1}-generic of depth r if and only if ϕ_E is G^{i+1}_E-generic of depth r.*

Proof. Choose a z-extension $\widetilde{G} \to G$. By [Kal19, Lemma 3.5.3], the map $\widetilde{G}(F)_{x,r} \to G(F)_{x,r}$ is surjective, so by construction of ϕ_E we may pass from G to $\widetilde{G}$ to assume that G_{der} is simply connected. Therefore hypothesis $\mathrm{C}(\vec{G})$ of [HM08, §2.6] holds by [Kal19, Lemma 3.5.2]. Recall that we have fixed a choice of additive character $\psi\colon F \to k^\times$. Let $x \in \mathcal{B}(G^i)$ be a point such that $\phi|_{G^i(F)_{x,r}}$ factors through a character of $G^i(F)_{x,r}/G^i(F)_{x,r+}$. Under the Moy–Prasad isomorphism

$$G^i(F)_{x,r}/G^i(F)_{x,r+} \cong \mathfrak{g}^i(F)_{x,r}/\mathfrak{g}^i(F)_{x,r+},$$

Hypothesis $\mathrm{C}(\vec{G})$ shows that there is an element $X^* \in \mathrm{Lie}^*(G^i)^{G^i}(F)_{-r}$ such that $\phi|_{G^i(F)_{x,r}}$ is induced by X^* and ψ. We may consider X^* as an element of $\mathrm{Lie}^*(G^i)^{G^i}(E)_{-r}$, in which case we will denote it by X^*_E

for clarity. Then the restriction of ϕ_E to $G^i(E)_{x,r}$ is induced by X_E^* and the additive character $\psi \circ \mathrm{Tr}_{E/F}$, since both of these characters are the ℓth powers of the unique $\mathrm{Gal}(E/F)$-stable extension of $\phi|_{G^i(F)_{x,r}}$ to $G^i(E)_{x,r}$.

For a twisted Levi $M \subset G$, let M_{sc} denote the preimage of M in the universal cover G_{sc} of G_{der}, and fix a maximal F-torus T of G^i. For a root $\alpha \in \Phi(G^{i+1}_{\overline{F}}, T_{\overline{F}})$, let $\alpha^\vee_{\mathrm{sc}}$ denote the $\overline{F}$-coroot $\mathbf{G}_m \to (G^{i+1}_{\mathrm{sc}})_{\overline{F}}$, and let $H_\alpha = \mathrm{d}\alpha^\vee_{\mathrm{sc}}(1) \in \mathrm{Lie}(G^{i+1})(\overline{F})$. Recall from [Yu01, §8] (noting the mild correction of [FKS23, Remark 4.1.3]) that genericity of ϕ is equivalent to the conjunction of the following two conditions:

(1) We have $v(\langle X^*, H_\alpha\rangle) = -r$ for all $\alpha \in \Phi((G^{i+1}/G^i)_{\overline{F}}, T_{\overline{F}})$.
(2) Recall that there is a natural isomorphism

$$\mathrm{Lie}^*(G^i_{\mathrm{sc,ab}}) \otimes_F \overline{F} \cong X^*((G^i_{\mathrm{sc,ab}})_{\overline{F}}) \otimes_{\mathbf{Z}} \overline{F} \subset X^*((T_{\mathrm{sc}})_{\overline{F}}) \otimes_{\mathbf{Z}} \overline{F}.$$

Let $\varpi_r \in \overline{F}$ denote an element of valuation r, and let $\overline{X}^*$ denote the residue class of $\varpi_r X^*$ in $X^*((G^i_{\mathrm{sc,ab}})_{\overline{F}}) \otimes_{\mathbf{Z}} \overline{\mathbf{F}}_q$. Then the stabilizer of $\overline{X}^*$ in the absolute Weyl group of G^{i+1} is equal to the absolute Weyl group of G^i.

The conditions for genericity of ϕ_E are completely similar. It is clear that these two conditions hold for ϕ if and only if they hold for ϕ_E. □

8.2. **Large prime degree base change.** In this section, let $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Z}}_\ell, \overline{\mathbf{Q}}_\ell\}$ be the coefficient ring. Suppose that ℓ is a banal prime for G such that $\ell > \mathrm{rk}\, G + 1$, and note that ℓ is also banal for G_{F_ℓ} by [Cot26, Proposition 4.1.2(2)]. Throughout the subsection, let

$$\Psi = ((G^i)_{0\le i\le d}, x, (r_i), \tau, (\phi_i))$$

be a normalized irreducible Yu datum for G over k.

Lemma 8.2.1. *There is a character $\chi\colon G^0(F) \to k^\times$, trivial on $G^0(F)^1$, such that $\tau \otimes \chi^{-1}$ factors through a finite quotient of $G^0(F)_{[x]}$ of order prime to ℓ.*

Proof. Choose a splitting $Z(G)(F) \cong Z(G)(F)_{\mathrm{b}} \times M$, where M is a finitely generated free abelian group. The induced map $M \to G^0(F)/G^0(F)^1$ is injective, and since $k^\times$ is divisible there exists an extension of the central character of $\tau|_M$ to $G^0(F)/G^0(F)^1$. Since ℓ is banal for $G(F)$, it follows that the central character of $\tau \otimes \chi^{-1}$ is finite of order prime to ℓ, and in particular $\tau \otimes \chi^{-1}$ has finite image. □

Definition 8.2.2. For $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Z}}_\ell, \overline{\mathbf{Q}}_\ell\}$, the *base change of* τ is the k-representation τ_ℓ of $G^0(F_\ell)_{[x]}$ defined as follows: first, choose a character χ as in Lemma 8.2.1. Let $\overline{\widetilde{\tau}}_\chi$ denote the Frobenius twist of the ℓ-modular reduction of the Glauberman correspondent (in the sense of [Cot26, §3.5.3]) of $\tau \otimes \chi^{-1}$, and let $\widetilde{\tau}_\chi$ denote the unique irreducible lift of $\overline{\widetilde{\tau}}_\chi$ which factors through a finite quotient of $G^0(F_\ell)_{[x]}$ of order prime to ℓ. Note that $\widetilde{\tau}_\chi$ is irreducible by [Ser77, Part III, no. 15.5, Proposition 43]. Then set

$$\tau_\ell := \widetilde{\tau}_\chi \otimes \chi_{F_\ell}\big|_{G^0(F_\ell)_{[x]}}. \tag{8.2.1}$$

[Cot26, Remark 3.5.10] shows that τ_ℓ does not depend on the choice of χ; in particular, if τ itself factors through a finite quotient of order prime to ℓ, one may take $\chi = 1$. If $G^0 = T$ is a torus, then (8.2.1) shows that τ_ℓ is the character τ_{F_ℓ} of Proposition 7.1.2.

We define a Yu datum $\Psi_\ell = ((G^i_\ell), x_\ell, (r_{\ell,i}), \tau_\ell, (\phi_{\ell,i}))$ as follows.

(1) $G^i_\ell = (G^i)_{F_\ell}$ for $0 \le i \le d$.
(2) $x_\ell = x$, considered as a point of $\mathcal{B}(G^0_{F_\ell})$; its image in $\mathcal{B}((G^0_{F_\ell})_{\mathrm{der}})$ is a vertex by [Cot26, Proposition 4.1.2(3)].
(3) $r_{\ell,i} = r_i$ for all i.
(4) Let τ_ℓ be the representation of Definition 8.2.2; this is irreducible and cuspidal by [Cot26, Lemma 3.5.9 and Proposition 4.2.2].
(5) Let $\phi_{\ell,i}$ be the character of $G^i(F_\ell)$ defined as $(\phi_i)_{F_\ell}$ in Proposition 7.1.2. Note that $\phi_{\ell,i}$ is $(G^{i+1})_{F_\ell}$-generic of depth r_i for $0 \le i < d$ by Lemma 8.1.1 and Lemma 8.1.2.

Observe that the isomorphism class of $\pi(\Psi_\ell)$ is $\mathrm{Gal}(F_\ell/F)$-stable. If $k = \overline{\mathbf{Z}}_\ell$ and $\overline{\Psi}$ denotes the modular reduction of Ψ, then the modular reduction of Ψ_ℓ is equal to $(\overline{\Psi})_\ell$.

Proposition 8.2.3. *Let ℓ be a banal prime for G such that $\ell > \mathrm{rk}\, G + 1$. Assume that $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Q}}_\ell\}$, and let Ψ_ℓ be as above. Then*

$$\rho_I^{\mathrm{Kal}}(\Psi_\ell) \sim \rho_I^{\mathrm{Kal}}(\Psi)|_{I_{F_\ell}}.$$

If Ψ is moreover F-non-singular, then so is Ψ_ℓ, and

$$\rho^{\mathrm{Kal}}(\Psi_\ell) \sim \rho^{\mathrm{Kal}}(\Psi)|_{W_{F_\ell}}.$$

Proof. If $k = \overline{\mathbf{F}}_\ell$, then Lemma 8.2.1, [Cot26, Lemma 3.5.16], and Teichmüller lifting give a normalized integral lift of Ψ with cuspidal generic fiber. After applying Lemma 6.6.1 to assume Ψ is irredundant, Lemmas 7.3.1 and 7.3.2 reduce both assertions to the case $k = \overline{\mathbf{Q}}_\ell$. Let $[(T, \theta)] \in \mathcal{T}(\Psi)$ as in §7.2. For the first statement, write $\theta := \theta_0 \prod_{i=0}^d \phi_i|_{T(F)_{[x]}}$ as in the construction of $\mathcal{T}(\Psi)$, so $\tau \in \mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}, \theta_0])$. By [Cot26, Proposition 4.3.4(2)] and the definitions, the representation τ_ℓ lies in the semi-rational Lusztig series $\mathcal{E}((\overline{G}^0_{[x]})_{\mathbf{F}_{q^\ell}}, [\overline{T}_{\mathbf{F}_{q^\ell}}, \theta_{\ell,0}])$, where $\theta_{\ell,0} = \theta_0 \circ \mathrm{Nm}_{\mathbf{F}_{q^\ell}/\mathbf{F}_q}$.

Let

$$\theta_\ell := \theta_{\ell,0} \prod_{i=0}^d \phi_{\ell,i}|_{T(F_\ell)_{[x]}}.$$

The definition of Ψ_ℓ gives

$$[(T_{F_\ell}, \theta_\ell)] = [(T_{F_\ell}, \theta \circ \mathrm{Nm}_{F_\ell/F})] \in \mathcal{T}(\Psi_\ell).$$

Norm functoriality for the inertial Local Langlands Correspondence for tori identifies the parameter of $\theta \circ \mathrm{Nm}_{F_\ell/F}$ with ${}^L\theta|_{I_{F_\ell}}$. Thus

$$\begin{aligned} \rho_I^{\mathrm{Kal}}(\Psi)|_{I_{F_\ell}} &\sim {}^L j_{T,G} \circ {}^L\theta|_{I_{F_\ell}} \\ &\sim {}^L j_{T_{F_\ell}, G_{F_\ell}} \circ {}^L(\theta \circ \mathrm{Nm}_{F_\ell/F})|_{I_{F_\ell}} \\ &\sim \rho_I^{\mathrm{Kal}}(\Psi_\ell). \end{aligned}$$

The second relation follows from Lemma 7.2.3, and the third follows from Lemma 5.2.4(2) and norm functoriality. If Ψ is F-non-singular, then so is Ψ_ℓ, and the same calculation holds on all of W_{F_ℓ}. □

8.3. **Small prime degree base change.** We do not currently have a robust general theory of low degree base change for Yu data, but for our main results we need only the following special case. Let $k \in \{\overline{\mathbf{F}}_\ell, \overline{\mathbf{Q}}_\ell\}$, and let

$$\Psi = ((G^i)_{0 \le i \le d}, x, (r_i), \tau, (\phi_i))$$

be a normalized irreducible Yu datum for G over k such that $G^0 = T$ is a torus. In particular, τ is a character. Let E/F be a Galois extension of prime degree $\ell \neq p$, and assume that T_E is an elliptic maximal E-torus of G_E. This implies that $T(E) = T(E)_{[x]}$ and $T(F) = T(F)_{[x]}$.

Define a Yu datum

$$\Psi_E = ((G_E^i), x_E, (r_{E,i}), \tau_E, (\phi_{E,i}))$$

as follows.

(1) $G_E^i = (G^i)_E$ for $0 \le i \le d$.
(2) $x_E = x$, considered as a point of $\mathcal{B}(G_E^0)$; its image in $\mathcal{B}((G_E^0)_{\mathrm{der}})$ is the unique point, hence a vertex.
(3) $r_{E,i} = r_i$ for $0 \le i \le d$.
(4) $\tau_E = ((\tau \epsilon_F \epsilon_{F,\sharp,x}) \circ \mathrm{Nm}_{E/F}) \epsilon_E \epsilon_{E,\sharp,x}$, where ϵ_F, $\epsilon_{F,\sharp,x}$ and ϵ_E, $\epsilon_{E,\sharp,x}$ are the sign characters associated to $(\vec{G}, x, \vec{r}, \vec{\phi})$ and $(\vec{G}_E, x, \vec{r}, \vec{\phi}_E)$, respectively, as in §3.3; this is trivially irreducible and cuspidal.
(5) $\phi_{E,i} = (\phi_i)_E$ as in Proposition 7.1.2; this is G_E^{i+1}-generic of depth r_i for each $0 \le i < d$ by Lemmas 8.1.1 and 8.1.2.

Observe that Ψ_E is normalized.

Proposition 8.3.1. *If ℓ is odd, then*

$$\rho^{\mathrm{Kal}}(\Psi)|_{W_E} \sim \rho^{\mathrm{Kal}}(\Psi_E).$$

The same holds if ℓ is even and $k = \overline{\mathbf{F}}_2$.

Proof. Ellipticity of T_E implies that $T(E) = T(E)_{[x]}$ and $T(F) = T(F)_{[x]}$. Since $G^0 = T$, the torus–character pairs associated to Ψ and Ψ_E are (T, θ) and (T_E, θ_E), where

$$\theta = \tau \prod_{i=0}^{d} \phi_i|_{T(F)}, \qquad \theta_E = \tau_E \prod_{i=0}^{d} \phi_{E,i}|_{T(E)}.$$

By construction, we have $\theta_E = ((\theta\epsilon_F\epsilon_{F,\sharp,x}) \circ \mathrm{Nm}_{E/F}) \cdot \epsilon_E\epsilon_{E,\sharp,x}$. By Proposition 5.2.5, we therefore have

$$\rho^{\mathrm{Kal}}(\Psi)|_{W_E} \sim {}^L j_{T,G} \circ {}^L\theta|_{W_E} \sim {}^L j_{T_E,G_E} \circ {}^L\theta_E \sim \rho^{\mathrm{Kal}}(\Psi_E),$$

as desired. □

9. Unramified twisted Levi functoriality for Yu data

Throughout this section, continue to assume $p \neq 2$ and let $\Psi = ((G^i)_{0\le i\le d}, x, (r_i)_{0\le i\le d}, \tau, (\phi_i)_{0\le i\le d})$ be a normalized irreducible Yu datum for G with coefficients in a field $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$. Fix an unramified twisted Levi subgroup $H \subset G$ for which there exists $[(T, \theta)] \in \mathcal{T}(\Psi)$ with $T \subset H$, let $H^i = H \cap G^i$, and fix an embedding $\mathcal{B}(H^0) \subset \mathcal{B}(G^0)$.

This section constructs a Yu datum Ψ_L on a Levi subgroup L of an unramified twisted Levi subgroup $H \subset G$ for which

$$\rho_I^{\mathrm{Kal}}(\Psi) \sim {}^L j_{L,G} \circ \rho_I^{\mathrm{Kal}}(\Psi_L).$$

When Ψ is F-non-singular, the construction gives $L = H$ and proves the full comparison

$$\rho^{\mathrm{Kal}}(\Psi) \sim {}^L j_{L,G} \circ \rho^{\mathrm{Kal}}(\Psi_L).$$

In general, one cannot take $L = H$; this reflects the fact that functoriality for twisted Levis does not always preserve cuspidality. For instance, if θ_{10} is the unipotent supercuspidal $\overline{\mathbf{Q}}_\ell$-representation of $\mathrm{Sp}_4(F)$ and $H \subset G$ is a twisted Levi such that $H \cong \mathrm{SL}_2 \times S$ for an anisotropic F-torus S, then each smooth representation of $H(F)$ functorially associated to θ_{10} is unipotent, and $H(F)$ has no supercuspidal unipotent representations.

9.1. **Constructing the Levi.** Let $(\overline{T}', \theta_0')$ be a pair consisting of a generalized maximal torus $\overline{T}'$ of $\overline{H}^0_{[x]_G}$ and a character $\theta_0' \colon \overline{T}'(\mathbf{F}_q) \to k^\times$ such that τ lies in the semi-rational Lusztig series $\mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}', \theta_0'])$, with notation as in [Cot26, Definition 2.8.2]. By [Cot26, Lemma 2.8.4], this set is uniquely determined by τ.

We first extract an irreducible k-representation of $\overline{H}^0_{[x]_G}(\mathbf{F}_q)$.[24] Let $\tau_{H,0}$ be an irreducible k-representation of $\overline{H}^0_{[x]_G}(\mathbf{F}_q)$ occurring in $\mathcal{E}(\overline{H}^0_{[x]_G}, [\overline{T}', \theta_0'])$. Note that $\tau_{H,0}$ might not be cuspidal (see [Cot26, Remark 3.4.4]), so it cannot be the depth 0 representation in a Yu datum for H. Instead we will use parabolic induction to descend further to a Levi subgroup of H.

By [Cot26, Lemma 2.10.2], there is a split $\mathbf{F}_q$-subtorus $\overline{S}^\circ \subset \overline{H}^0_{[x]_G}$ which is the maximal central split $\mathbf{F}_q$-torus of $\overline{L} := Z_{\overline{H}^0_{[x]_G}}(\overline{S}^\circ)$, a parabolic $\mathbf{F}_q$-subgroup $\overline{P}^\circ \subset (\overline{H}^0_{[x]_G})^\circ$ with Levi factor $\overline{L}^\circ$, and an irreducible cuspidal k-representation $\tau_{L,0}$ of $\overline{L}(\mathbf{F}_q)$ such that $\tau_{H,0}$ is an irreducible subrepresentation of $\mathrm{ind}_{\overline{P}(\mathbf{F}_q)}^{\overline{H}^0_{[x]_G}(\mathbf{F}_q)}(\tau_{L,0})$, where $\overline{P} = \overline{L} \cdot \overline{P}^\circ$.

Let $\mathcal{H}^0_{[x]_G}$ be the smooth separated $\mathcal{O}_F$-model of H^0 with $\mathcal{H}^0_{[x]_G}(\mathcal{O}_F) = H^0(F)_{[x]_G}$, so $\mathcal{H}^0_{[x]_G}$ is an open $\mathcal{O}_F$-subgroup scheme of $\mathcal{H}^0_{[x]_H}$. Let $\mathcal{S} \subset \mathcal{H}^0_{[x]_G}$ be an $\mathcal{O}_F$-subtorus lifting $\overline{S}^\circ$, as in the construction §7.2.2. Let S be the generic fiber of $\mathcal{S}$ and set $L = Z_H(S)$. For all $0 \le i \le d$, put

$$L^i := L \cap G^i = L \cap H^i.$$

Since $\mathcal{S} \subset \mathcal{H}^0_{[x]_G}$, [KP23, §9.7.6(2)] shows that there is a maximally unramified maximally split maximal F-torus $T_x \subset H^0$ containing S whose apartment in $\mathcal{B}(H^0)$ contains x; we will also regard x as a point of $\mathcal{B}(T_x)$ via a choice of embedding $\mathcal{B}(T_x) \subset \mathcal{B}(L^0)$. By definition of L, we must then have $T_x \subset L^0$. Fixing an embedding of $\mathcal{B}(L^0)$ in $\mathcal{B}(H^0)$, we may regard x as a point of $\mathcal{A}(T_x) \subset \mathcal{B}(L^0)$. Note that the Moy–Prasad filtration depends only on the image $[x]_L$ of x in $\mathcal{B}(L^0_{\mathrm{der}})$, hence is independent of this choice of lift. Recall the notation $\mathcal{L}^0_{[x]_L}$ and $\overline{L}^0_{[x]_L}$ from §2.

[24] We are careful to write $\overline{H}^0_{[x]_G}$ instead of $\overline{H}^0_{[x]}$ for clarity, because in this generality it may happen that $[x]_H \neq [x]_G$.

Lemma 9.1.1. *The torus S is the maximal F-split central subtorus of L^0, and thus Z_{L^0}/Z_L is F-anisotropic.*

Proof. Let A be the maximal F-split central subtorus of L^0, so $S \subset A$. By [KP23, Axiom 4.1.20], if $\mathcal{A}$ is the split $\mathcal{O}_F$-torus with generic fiber A, then the inclusion $A \subset L^0$ extends to an $\mathcal{O}_F$-homomorphism $\mathcal{A} \to \mathcal{L}^0_{[x]_L}$ which is a closed embedding. But then the special fiber $\overline{A}^\circ$ is an $\mathbf{F}_q$-split central subtorus of $(\overline{L}^0_{[x]_L})^\circ$, so $\overline{A}^\circ = \overline{S}^\circ$ by choice of $\overline{S}^\circ$. For dimension reasons, it follows that $S = A$. □

Note that Lemma 9.1.1 shows that $\mathcal{B}(L^0_{\mathrm{der}}) \subset \mathcal{B}(L_{\mathrm{der}})$, so the notation $[x]_L$ above is consistent with §2.

Lemma 9.1.2. *The point $[x]_L$ is a vertex in $\mathcal{B}(L^0_{\mathrm{der}})$.*

Proof. Let $\mathcal{F}$ be the facet of $\mathcal{B}(L^0_{\mathrm{der}})$ containing $[x]_L$, and let y be a vertex in the closure of $\mathcal{F}$; we aim to show $y = [x]_L$. By [KP23, Axiom 4.1.22], there is an associated $\mathcal{O}_F$-homomorphism $(\mathcal{L}^0_{[x]_L})^\circ \to (\mathcal{L}^0_y)^\circ$ between the associated parahoric models, and the associated map on special fibers gives rise to an isomorphism of $(\overline{L}^0_{[x]_L})^\circ$ with a Levi factor of a proper parabolic $\mathbf{F}_q$-subgroup of $(\overline{L}^0_y)^\circ$. In particular, if $[x]_L \neq y$ then the maximal $\mathbf{F}_q$-split central torus of $(\overline{L}^0_{[x]_L})^\circ$ is of strictly larger dimension than that of $(\overline{L}^0_y)^\circ$.

By construction, the $\mathbf{F}_q$-split central torus $\overline{S}^\circ$ of $(\overline{L}^0_{[x]_L})^\circ$ is maximal, and $\overline{S}^\circ$ is the special fiber of an $\mathcal{O}_F$-torus $\mathcal{S} \subset \mathcal{L}^0_{[x]_L}$ with generic fiber S. Since S is an F-split central torus in L^0, it follows that the maximal $\mathbf{F}_q$-split central torus of $(\overline{L}^0_y)^\circ$ is of dimension at least $\dim S$, concluding the proof. □

9.2. **Constructing the depth 0 representation.** We are almost ready to define a Yu datum Ψ_L, but there are two remaining technical subtleties:

- $\tau_{L,0}$ is a representation of $\overline{L}^0_{[x]_G}(\mathbf{F}_q)$, which is generally smaller than $\overline{L}^0_{[x]_L}(\mathbf{F}_q)$.
- Even when $\overline{L}^0_{[x]_G}(\mathbf{F}_q) = \overline{L}^0_{[x]_L}(\mathbf{F}_q)$, the representation $\tau_{L,0}$ must be twisted by a sign character to give the "correct" Ψ_L.

Let $\mathcal{L}^0_{[x]_G} := Z_{\mathcal{H}^0_{[x]_G}}(\mathcal{S})$, a smooth separated $\mathcal{O}_F$-model of L^0 with

$$\mathcal{L}^0_{[x]_G}(\mathcal{O}_F) = L^0(F)_{[x]_G} := L^0(F) \cap G^0(F)_{[x]_{G^0}}.$$

Let $\overline{L}^0_{[x]_G}$ denote the quotient of $(\mathcal{L}^0_{[x]_G})_{\mathbf{F}_q}$ by the unipotent radical of $(\mathcal{L}^0_{[x]_G})^\circ_{\mathbf{F}_q}$, so $\overline{L}^0_{[x]_G}$ is a paraductive $\mathbf{F}_q$-group scheme.

By [Cot26, Proposition 2.6.2], there is a generalized maximal torus-character pair $(\overline{T}_L, \eta_0)$ in $\overline{L}^0_{[x]_G}$ such that

$$\tau_{L,0} \in \mathcal{E}(\overline{L}^0_{[x]_G}, [\overline{T}_L, \eta_0]).$$

By [Cot26, Proposition 2.8.5], for either $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$ we have

$$\tau_{H,0} \in \mathcal{E}(\overline{H}^0_{[x]_G}, [\overline{T}_L, \eta_0]).$$

9.2.1. *The sign twisting.* Fix an F-parabolic subgroup $P \subset H$ with Levi factor L. Since $Z(G^0)/Z(G)$ is F-anisotropic, the groups G^0 and G share a maximal F-split central subtorus. By compatibility of the Moy–Prasad filtration with respect to Levi F-subgroups, we have

$$L^0(F)_{x,0+} = L^0(F) \cap H^0(F)_{x,0+}.$$

Let $\epsilon^{\vec{G},L}_{\sharp,x}$ be the (depth 0) sign character of $L^0(F)_{[x]_G}$ defined in (3.3.4). Define a sign character $\kappa_{G,L,x}$ on $L^0(F)_{[x]_G}$ by

$$\kappa_{G,L,x} = \epsilon_G \epsilon_L \epsilon^{\vec{G},L}_{\sharp,x}. \tag{9.2.1}$$

9.2.2. *The extension.* By [Cot26, Lemma 2.1.3], we have

$$\overline{L}^0_{[x]_G} = Z_{\overline{H}^0_{[x]_G}}(\overline{S}^\circ). \tag{9.2.2}$$

By construction, the character $\kappa_{G,L,x}$ of (9.2.1) factors through $\overline{L}^0_{[x]_G}(\mathbf{F}_q)$; we use the same notation for the resulting quotient character. Let $\overline{T}_{L,1} = Z_{\overline{L}^0_{[x]_L}}(\overline{T}^\circ_L)$. By [Cot26, Lemma 2.10.3], if τ_L is a representation of

$\overline{L}^0_{[x]_L}(\mathbf{F}_q)$ whose restriction to $\overline{L}^0_{[x]_G}(\mathbf{F}_q)$ admits $\tau_{L,0} \otimes \kappa_{G,L,x}$ as an irreducible subquotient, then there exists a character $\eta_1 \colon \overline{T}_{L,1}(\mathbf{F}_q) \to k^\times$ and an irreducible cuspidal k-representation τ_L of $\overline{L}^0_{[x]_L}(\mathbf{F}_q)$ such that

$$\eta_1|_{\overline{T}_L(\mathbf{F}_q)} = \eta_0\, \kappa_{G,L,x}|_{\overline{T}_L(\mathbf{F}_q)}, \tag{9.2.3}$$

$$\tau_{L,0} \otimes \kappa_{G,L,x} \subset \tau_L|_{\overline{L}^0_{[x]_G}(\mathbf{F}_q)}, \qquad \tau_L \in \mathcal{E}(\overline{L}^0_{[x]_L}, [\overline{T}_{L,1}, \eta_1]). \tag{9.2.4}$$

9.3. **The descended Yu datum and functoriality.** With notation as above, we also use τ_L to denote the inflation of τ_L to $L^0(F)_{[x]_L}$. For every $0 \le i \le d$, define $\phi_{L,i} \coloneqq \phi_i|_{L^i(F)}$. We may now define the 5-tuple

$$\Psi_{L,\tau_L} = ((L^i)_{0\le i\le d}, x, (r_i)_{0\le i\le d}, \tau_L, (\phi_{L,i})_{0\le i\le d}). \tag{9.3.1}$$

We proceed to verify that it is a Yu datum.

Lemma 9.3.1. *For every $0 \le i < d$, the character $\phi_{L,i}$ is L^{i+1}-generic of depth r_i. Moreover, if $r_{d-1} < r_d$, then $\phi_{L,d}$ has depth r_d.*

Proof. Since each L^i contains a tamely ramified maximal F-torus of G^0, the depth claims follow from Lemma 7.2.4 and the fact that the Moy–Prasad filtration restricts in the obvious way from G^i to L^i. For the genericity assertion, fix $0 \le i < d$. If $L^i = L^{i+1}$, then genericity is vacuous, so we may and do assume that $L^i \ne L^{i+1}$.

Observe that $X_i^*|_{\mathrm{Lie}(L^i)}$ is L^{i+1}-generic of depth $-r_i$ by definition of genericity. Functoriality of the Moy–Prasad isomorphisms shows that $X_i^*|_{\mathrm{Lie}(L^i)}$ induces $\phi_{L,i}$ on the depth r_i quotient at x, so $\phi_{L,i}$ is L^{i+1}-generic. □

Proposition 9.3.2. *The 5-tuple Ψ_{L,τ_L} from* (9.3.1) *is a normalized irreducible Yu datum for L over k.*

Proof. It is clear that the constituents of Ψ_L are of the kind described in (a)-(e) of §6.1, the depth requirement in (e) for $\phi_d|_{L^d(F)}$ holding by Lemma 9.3.1. Condition (1) follows from Lemma 9.1.1. Condition (2) is the statement of Lemma 9.1.2. For (3), note that τ_L is an irreducible cuspidal k-representation of $\overline{L}^0_{[x]_L}(\mathbf{F}_q)$ by definition. Finally, condition (4) follows from Lemma 9.3.1. It is obvious that Ψ_{L,τ_L} is normalized and irreducible. □

Since L is an unramified twisted Levi F-subgroup of G, Definition 4.5.1 shows that there is a canonical L-embedding ${}^L j_{L,G} \colon {}^L L \to {}^L G$.

Theorem 9.3.3. *Suppose that $p \ne 2$. We have*

$$\rho_I^{\mathrm{Kal}}(\Psi) \sim {}^L j_{L,G} \circ \rho_I^{\mathrm{Kal}}(\Psi_{L,\tau_L}). \tag{9.3.2}$$

If Ψ is F-non-singular, then $L = H$, the datum Ψ_{L,τ_L} is F-non-singular, and

$$\rho^{\mathrm{Kal}}(\Psi) \sim {}^L j_{L,G} \circ \rho^{\mathrm{Kal}}(\Psi_{L,\tau_L}). \tag{9.3.3}$$

Proof. We consider the general case first. We will use the notation of the preceding sections freely. Choose, by the construction of §7.2, an $\mathcal{O}_F$-torus $\mathcal{S}' \subset \mathcal{H}^0_{[x]_G}$ whose special fiber maps onto $\overline{T}'^{\circ}$ (with notation as in §9.1), and let S' be its generic fiber. Let

$$T' \coloneqq Z_{G^0}(S') = Z_{H^0}(S') \subset H^0,$$

so T' is a maximally unramified maximal F-torus of both G^0 and H^0. Let θ_0' also denote its inflation along the reduction map from $T'(F)_{[x]_G}$ to $\overline{T}'(\mathbf{F}_q)$. The pair $[(T', \theta')] \in \mathcal{T}(\Psi)$ associated to $(\overline{T}', \theta_0')$ satisfies

$$\theta' = \theta_0' \prod_{i=0}^{d} \phi_i|_{T'(F)_{[x]}}.$$

Note that $(\mathcal{L}^0_{[x]_G})^\circ = (\mathcal{L}^0_{[x]_L})^\circ$. Let $T_L \subset L^0$ be the torus associated to $\overline{T}_L$ in the same way that T' is associated to $\overline{T}'$. Choose compatible embeddings $\mathcal{B}(T_L) \subset \mathcal{B}(L^0) \subset \mathcal{B}(H^0)$, so we may regard x as a point of $\mathcal{B}(T_L)$ and $\mathcal{B}(L^0)$.

Define

$$\widetilde{\eta}_0 = \eta_0 \prod_{i=0}^{d} \phi_i|_{T_L(F)_{[x]_G}}, \quad \widetilde{\eta}_1 = \eta_1 \prod_{i=0}^{d} \phi_{L,i}|_{T_L(F)_{[x]_L}}$$

on $T_L(F)_{[x]_G}$ and $T_L(F)_{[x]_L}$, respectively. By (9.2.3), we have

$$\widetilde{\eta}_1|_{T_L(F)_{[x]_G}} = (\widetilde{\eta}_0\, \kappa_{G,L,x})|_{T_L(F)_{[x]_G}} = (\widetilde{\eta}_0\, \epsilon_G \epsilon_L \epsilon_{G,\sharp,x}\epsilon_{L,\sharp,x})|_{T_L(F)_{[x]_G}}.$$

By construction, T, T', and T_L are maximally unramified in G^0. By [Cot26, Lemma 2.8.4], we have $\mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}, \theta_0]) = \mathcal{E}(\overline{G}^0_{[x]}, [\overline{T}', \theta_0'])$ since both sets contain τ. Meanwhile, the definition of $\tau_{H,0}$ shows that $\mathcal{E}(\overline{H}^0_{[x]_G}, [\overline{T}', \theta_0']) = \mathcal{E}(\overline{H}^0_{[x]_G}, [\overline{T}_L, \eta_0])$. Thus Lemma 7.1.3 shows that (T, θ), (T', θ'), and $(T_L, \widetilde{\eta}_0)$ are almost equivalent in G^0, and hence they are almost equivalent in G.

By Lemma 7.2.3, it follows that

$$^L j_{T,G} \circ {}^L(\theta|_{T(F)_{\mathrm{b}}})|_{I_F} \sim {}^L j_{T_L,G} \circ {}^L(\widetilde{\eta}_0|_{T_L(F)_{\mathrm{b}}})|_{I_F}.$$

Hence (9.3.2) reduces to the claim

$$^L j_{L,G} \circ {}^L j_{T_L,L} \circ {}^L(\widetilde{\eta}_1|_{T_L(F)_{\mathrm{b}}})|_{I_F} \sim {}^L j_{T_L,G} \circ {}^L(\widetilde{\eta}_0|_{T_L(F)_{\mathrm{b}}})|_{I_F}, \tag{9.3.4}$$

where $^L j_{T_L,L}$ and $^L j_{T_L,G}$ are the L-embeddings from Definition 4.5.1. Since $k^\times$ is divisible and $T_L(F)_{\mathrm{b}}$ is open, $\widetilde{\eta}_0|_{T_L(F)_{\mathrm{b}}}$ extends to a smooth character of $T_L(F)$. Then (9.3.4) follows from Proposition 5.1.5 applied to any such extension.

Now suppose that Ψ is F-non-singular. The proof of (9.3.3) is similar to the above except easier, since no parabolic induction is needed, so we will be concise.

Lemma 7.2.13, Lemma 7.2.9(3), and [Kal21b, Fact 3.1.4(1)] show that the chosen θ is F-non-singular and that τ is non-singular. [Cot26, Lemma 2.9.4], applied to $\mathcal{E}(\overline{H}^0_{[x]_G}, [\overline{T}', \theta_0'])$, implies that $\tau_{H,0}$ is non-singular and cuspidal; thus $L = H$ and $\tau_H = \tau_L$ is non-singular and cuspidal by definition. We now follow the same argument as above for the general case, except that we use [Cot26, Lemma 2.9.2] in place of [Cot26, Lemma 2.8.4]. Since Ψ and Ψ_{H,τ_H} are $\mathbf{F}$-non-singular as above, Lemma 7.2.9 and Lemma 7.1.3 imply that (T', θ') is $G^0(F)$-conjugate to (T, θ). From the definitions, it follows that Ψ_{H,τ_H} is F-non-singular. Note that Lemma 7.2.9(3) implies T is elliptic in G, so T' is also elliptic in G. Then Proposition 5.1.5 and (9.2.3) yield

$$\rho^{\mathrm{Kal}}(\Psi) \sim {}^L j_{T_L,G} \circ {}^L(\widetilde{\eta}_0) \sim {}^L j_{L,G} \circ {}^L j_{T_L,L} \circ {}^L(\widetilde{\eta}_1) = {}^L j_{L,G} \circ \rho^{\mathrm{Kal}}(\Psi_L).$$

This proves (9.3.3). □

9.4. **An example.** The following example justifies the pains to which we have gone in defining sign twistings.

Example 9.4.1. It can happen that $\kappa_{G,L,x} \neq 1$. For example, let $S_1, S_2 \subset \mathrm{SL}_2$ be elliptic F-tori such that S_1 is unramified and S_2 is ramified. Let $G = \mathrm{Sp}_4$, and let $T = S_1 \times S_2 \subset \mathrm{SL}_2 \times \mathrm{SL}_2 \subset G$. Let $H = S_1 \times \mathrm{SL}_2$, so H is an unramified twisted Levi F-subgroup of G. Let

$$\Psi = ((G^0 \subsetneq G^1), x, (r_0 = r_1 = r > 0), \tau, (\phi_0, 1))$$

be a normalized Yu datum for G over $\overline{\mathbf{Q}}_\ell$, where $G^0 = \mathrm{SL}_2 \times S_2$. Choose compatible embeddings $\mathcal{B}(T) \subset \mathcal{B}(G^0) \subset \mathcal{B}(G)$, denote the images of the unique point of $\mathcal{B}(T)$ by x, and suppose that τ is non-singular. Every irreducible constituent of $^*R^{\overline{G}^0_{[x]}}_{\overline{H}^0_{[x]_G}}(\tau)$ is non-singular and cuspidal by [Cot26, Lemma 2.9.4], so the construction of this section gives $L = H$ and a Yu datum Ψ_H with $H^0 = T$ and $H^1 = H$. If ϵ_G and ϵ_H are constructed from Ψ and Ψ_H as in §6.5, then we claim $\epsilon_G \epsilon_H \epsilon_{G,\sharp,x} \epsilon_{H,\sharp,x}|_{T(F)_x} \neq 1$. By Lemma 5.1.1, we have

$$\epsilon_G \epsilon_H \epsilon_{G,\sharp,x} \epsilon_{H,\sharp,x} = \epsilon_{G,\flat} \epsilon_{H,\flat},$$

where $\epsilon_{G,\flat} = \epsilon^{G^1/G^0}_{\flat,0}\, \epsilon^{G^1/G^0}_{\flat,1}\, \epsilon^{G^1/G^0}_{\flat,2}$, and similarly for $\epsilon_{H,\flat}$.

Let $e_i \in X^*((S_i)_{\overline{F}})$ be a $\mathbf{Z}$-basis element for $i = 1, 2$ such that the absolute roots of $T_{\overline{F}}$ are (up to sign) given by

$$\alpha = e_1 - e_2, \qquad \beta = 2e_2, \qquad \alpha + \beta = e_1 + e_2, \qquad 2\alpha + \beta = 2e_1.$$

Thus $\Phi(H_{\overline{F}}, T_{\overline{F}}) = \{\pm\beta\}$ and $\Phi(G^0_{\overline{F}}, T_{\overline{F}}) = \{\pm(2\alpha+\beta)\}$. Let E/F be the totally ramified quadratic splitting field of S_2, and let F_2/F be the unramified quadratic splitting field of S_1, so E_2 is the splitting field of T. Let E'/F be the other ramified quadratic subfield of E_2/F. Thus both E_2/E and E_2/E' are unramified quadratic extensions. Let σ_1 be the nontrivial element of $\mathrm{Gal}(E_2/E)$ and σ_2 the nontrivial element of $\mathrm{Gal}(E_2/F_2)$.

The elements σ_1 and σ_2 alter the signs of e_1 and e_2, respectively, and the $\mathrm{Gal}(\overline{F}/F)$-orbits on $\Phi(G_{\overline{F}}, T_{\overline{F}})$ are as follows:

$\mathrm{Gal}(\overline{F}/F)$-orbit	$F_\gamma/F_{\pm\gamma}$	type	location
$\{\pm\beta\}$	E/F	symmetric ramified	G^1/G^0 and H^1/H^0
$\{\pm(2\alpha+\beta)\}$	F_2/F	symmetric unramified	G^0
$\{\pm\alpha, \pm(\alpha+\beta)\}$	E_2/E'	symmetric unramified	G^1/G^0

(9.4.1)

We now compute the three sign factors. Fix $t \in \overline{T}^\circ(\mathbf{F}_q)$. For $\epsilon_{\flat,0}^{G^1/G^0}$, the restriction α_0 of α to $1 \times S_2$ is symmetric ramified, and $F_{\alpha_0} = E$, so $e(\alpha/\alpha_0) = 1$. Thus (3.2.3) gives

$$\epsilon_{\flat,0}^{G^1/G^0}(t) = \mathrm{sgn}_{\mathbf{F}_\alpha^1}(\alpha(t)), \qquad \epsilon_{\flat,0}^{H^1/H^0}(t) = 1.$$

The two $\epsilon_{\flat,1}$ factors come from the orbit of β and, by (3.2.6), are equal to

$$\epsilon_{\flat,1}^{G^1/G^0}(t) = \epsilon_{\flat,1}^{H^1/H^0}(t) = \begin{cases} \mathrm{sgn}_{\mathbf{F}_q^\times}(2) & \text{if } \beta(t) = -1, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, (3.2.7) gives

$$\epsilon_{\flat,2}^{G^1/G^0}(t) = \epsilon_{\flat,2}^{H^1/H^0}(t).$$

Thus we conclude

$$(\epsilon_{G,\flat}\epsilon_{H,\flat})(t) = \mathrm{sgn}_{\mathbf{F}_\alpha^1}(\alpha(t)).$$

Note that this character is nontrivial: the restriction $\alpha|_{S_1 \times \{1\}}$ identifies S_1 with $\mathrm{Res}^1_{F_2/F}\, \mathbf{G}_m$, and hence identifies $S_1(F)$ with $\ker\bigl(\mathrm{Nm}_{F_2/F} : F_2^\times \to F^\times\bigr)$. Reduction identifies $S_1(F)/S_1(F)_{0+}$ with $\mathbf{F}_\alpha^1$, on which $\mathrm{sgn}_{\mathbf{F}_\alpha^1}$ is nontrivial.

Remark 9.4.2. One consequence of Example 9.4.1 is that, if $\theta\colon T(F) \to \overline{\mathbf{Q}}_\ell^\times$ is a regular character (in the sense of [Kal19, Definition 3.7.5]) and π is the $\overline{\mathbf{Q}}_\ell$-representation constructed from (T, θ) as in [Kal19, Corollary 3.7.10], then the representation $\pi_{(T,\theta)}^H$ does not always satisfy the obvious guess $\rho^{\mathrm{Kal}}\left(\pi_{(T,\theta)}^G\right) \sim {}^L j_{H,G} \circ \rho^{\mathrm{Kal}}\left(\pi_{(T,\theta)}^H\right)$. By Theorem 9.3.3, we have

$$\rho^{\mathrm{Kal}}\left(\pi_{(T,\theta)}^G\right) \sim {}^L j_{H,G} \circ \rho^{\mathrm{Kal}}\left(\pi_{(T,\theta\epsilon)}^H\right),$$

where $\epsilon := \epsilon_{G,\flat}\epsilon_{H,\flat}\colon T(F) \to \{\pm 1\}$. This character can be nontrivial, as Example 9.4.1 shows. If θ is of odd order, then we claim

$${}^L j_{H,G} \circ {}^L j_{T,H} \circ {}^L\theta \not\sim {}^L j_{H,G} \circ {}^L j_{T,H} \circ {}^L(\theta\epsilon)$$

To see this, note that if these two L-parameters were $\widehat{G}(\overline{\mathbf{Q}}_\ell)$-conjugate, then they must be $\widehat{T}(\overline{\mathbf{Q}}_\ell)$-conjugate: indeed, any element of $\widehat{G}(\overline{\mathbf{Q}}_\ell)$ not lying in $\widehat{T}(\overline{\mathbf{Q}}_\ell)$ changes the restrictions of these L-parameters to W_E for some quadratic extension E/F since $\theta \circ \mathrm{Nm}_{E/F}$ is regular.[25] Since θ and $\theta\epsilon$ are different characters, the Local Langlands Correspondence for tori implies that these two L-parameters cannot be $\widehat{T}(\overline{\mathbf{Q}}_\ell)$-conjugate.

10. A partial characterization of the semisimple Local Langlands Correspondence

This section contains the main theorem of this paper (Theorem 10.2.1), which gives a partial characterization of ρ^{Kal} in terms of the various functoriality results that we have established in the previous sections.

[25] Note that x was chosen to have hyperspecial image in $\mathcal{B}(G^0_{\mathrm{der}})$, so the rational Weyl group appearing in [Kal19, Definition 3.7.5] does not change after base extension.

10.1. **Generalities on L-parameters.** We begin with a number of technical results on reductive groups. The following lemma generalizes [DM94, Lemme 1.10].

Lemma 10.1.1. *If α is an automorphism of a connected reductive group H over a field k such that $H^\alpha/Z(H)^\alpha$ is finite, then H is a torus.*

Proof. By passing from H to H_{der}, we may assume that H is semisimple. Every automorphism of H lifts to an automorphism of the universal cover of H, and using this fact one reduces easily to the case that H is simply connected. Let α_0 denote a pinning-preserving automorphism of H lying in the same component of the automorphism scheme $\mathrm{Aut}_{H/k}$ as α. By [Spr06, Proposition 1] (which applies since H is simply connected), the group H^{α_0} is semisimple, say of rank r. In fact, $r > 0$: for instance, if α_0 preserves the pinning $(B, T, \{X_\alpha\})$ and $\beta^\vee$ is a simple coroot with respect to this pinning, then $\sum_{\gamma^\vee \in \alpha_0^{\mathbf{Z}}(\beta^\vee)} \gamma^\vee \colon \mathbf{G}_{\mathrm{m}} \to H$ is a nontrivial cocharacter factoring through H^{α_0}.

Since the identity component of $\mathrm{Aut}_{H/k}$ is $H/Z(H)$, acting via inner automorphisms, it follows that $\alpha = c_h \circ \alpha_0$ for some $h \in H(k)$, where c_h denotes the inner automorphism of H induced by h. In the notation of [XZ19, §5.2], we have $H^\alpha = I_h$, and it follows from [XZ19, Remark 5.2.2(1)] that $\dim H^\alpha \geq r$. This proves the lemma. $\square$

Lemma 10.1.2. *Let k be a field of characteristic $\neq p$, let H be a reductive k-group such that $\pi_0(H)$ is a finite p-group, and let A be a finite p-group acting on H. Then $\pi_0(H^A)$ is a finite p-group.*

Proof. By induction, we may assume that A is cyclic, and we may further assume that H is connected. In this case, the result is [Ste75, Corollary 2.16(b)]. $\square$

Lemma 10.1.3. *Let $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$ with $\ell \neq p$, and let H be a (possibly disconnected) reductive k-group. Let I be a finite group with a normal p-subgroup P such that I/P is cyclic of order prime to p. Let $\rho_1, \rho_2 \colon I \to H(k)$ be homomorphisms such that*

(1) $\rho_1|_P = \rho_2|_P$,
(2) the induced maps $I \to \pi_0(H)(k)$ are equal,
(3) $Z_H(\rho_1(P))^\circ$ is a torus.

Then there is some $c \in Z_{H^\circ}(\rho_1(P))(k)$ such that $c\rho_1 c^{-1}$ and ρ_2 induce the same action of I on $Z_H(\rho_1(P))^\circ$.

Proof. By (1), the map
$$x \mapsto \rho_1(x)\rho_2(x)^{-1}$$
is a 1-cocycle $\varphi \colon I/P \to Z_H(\rho_1(P))(k)$; by (2), the map φ is valued in $Z_{H^\circ}(\rho_1(P))(k)$. The map φ induces a 1-cocycle $\overline{\varphi} \colon I/P \to \pi_0(Z_{H^\circ}(\rho_1(P)))(k)$, and the lemma is equivalent to the statement that $\overline{\varphi}$ is a coboundary (since $Z_H(\rho_1(P))^\circ$ is commutative by (3)). But I/P is of order prime to p and $\pi_0(Z_{H^\circ}(\rho_1(P)))(k)$ is a p-group by Lemma 10.1.2, so this is automatic. $\square$

10.1.1. *Semisimple and irreducible L-parameters.* Recall from [CGP15, Lemma 2.1.4 and the following paragraph] that if k is a field, H is a reductive k-group, and $\lambda \colon \mathbf{G}_{\mathrm{m}} \to H$ is a cocharacter, then there are associated subgroups $Z_H(\lambda)$ and $P_H(\lambda)$ of H: by definition, $Z_H(\lambda)$ is the centralizer of the cocharacter λ, while $P_H(\lambda)$ represents the functor parameterizing local sections h of H such that the limit $\lim_{t\to 0} \lambda(t)h\lambda(t)^{-1}$ exists. Following [BMR05, §6], we call $P_H(\lambda)$ an *R-parabolic subgroup* of H, and we call $Z_H(\lambda)$ an *R-Levi subgroup* of $P_H(\lambda)$. By [CGP15, Proposition 2.2.9], the identity component $P_H(\lambda)^\circ$ is a parabolic subgroup of H°. Moreover, by [CGP15, Proposition 2.1.8] we have $P_H(\lambda) = Z_H(\lambda) \ltimes U_H(\lambda)$, where $U_H(\lambda)$ is the unipotent radical of $P_H(\lambda)^\circ$; in particular, there is a natural projection map $P_H(\lambda) \to Z_H(\lambda)$, given informally by "sending h to $\lim_{t\to 0} \lambda(t)h\lambda(t)^{-1}$".

Definition 10.1.4. Let Γ be a(n "abstract") group, and let H be a reductive group over a field k. If $\rho \colon \Gamma \to H(k)$ is a homomorphism, then ρ is *semisimple* if for every R-parabolic $\overline{k}$-subgroup $P \subset H_{\overline{k}}$ such that $\rho(\Gamma) \subset P(\overline{k})$, there is an R-Levi $\overline{k}$-subgroup $L \subset P$ such that $\rho(\Gamma) \subset L(\overline{k})$.

If $\rho \colon \Gamma \to H(k)$ is any homomorphism, then a *semisimplification* of ρ is a semisimple homomorphism $\rho^{\mathrm{ss}} \colon \Gamma \to H(k)$ such that there exists an R-parabolic $P \subset H$ with R-Levi $L \subset P$ such that ρ factors through $P(k)$ and ρ^{ss} factors as $\Gamma \xrightarrow{\rho} P(k) \to L(k) \subset H(k)$. We will use ρ^{ss} to denote a choice of semisimplification of ρ.

The proof of the following lemma is surprisingly complicated, and we do not know whether one should expect a simpler proof.

Lemma 10.1.5. *Let k be a field, and let H be a reductive k-group. Let Γ be a group which admits a normal subgroup Γ_0 such that Γ/Γ_0 is cyclic. If $\rho\colon \Gamma \to H(k)$ is a homomorphism, then the following are equivalent:*

(1) $\rho|_{\Gamma_0}$ is semisimple and there is a $\rho(\Gamma)$-stable Borel-torus pair (B, T) in $Z_H(\rho(\Gamma_0))^\circ_{\mathrm{red}}$,
(2) ρ is semisimple.

Proof. We may and do assume $k = \overline{k}$. Passing from Γ to the Zariski closure of $\rho(\Gamma)$, we may assume that ρ is the inclusion of a smooth closed subgroup Γ such that Γ/Γ_0 is topologically generated by an element $\gamma \in \Gamma(k)$. If ρ is semisimple, then $\rho|_{\Gamma_0}$ is semisimple by (the disconnected version of) [BMR05, Theorem 3.10]. Thus by [BMR08, Corollary 3.7], the homomorphism ρ is semisimple if and only if $\rho|_{\Gamma_0}$ is semisimple and the induced homomorphism $\alpha\colon \Gamma/\Gamma_0 \to N_H(\Gamma_0)_{\mathrm{red}}/\Gamma_0$ is semisimple. The identity component of $N_H(\Gamma_0)_{\mathrm{red}}/\Gamma_0$ admits a central isogeny from $Z_H(\Gamma_0)^\circ_{\mathrm{red}}$, so α is semisimple if and only if Γ preserves a Borel-torus pair of $Z_H(\Gamma_0)^\circ_{\mathrm{red}}$ (if and only if conjugation by γ induces a quasi-semisimple automorphism of $Z_H(\Gamma_0)^\circ_{\mathrm{red}}$) by [Spr06, Proposition 3]. □

Lemma 10.1.6. *Let $k \in \{\overline{\mathbf{Q}}_\ell, \overline{\mathbf{F}}_\ell\}$ be a field.*

(1) If E/F is finite Galois, $G = \mathrm{Res}_{E/F}(H_E)$, and $\rho\colon W_F \to {}^L H(k)$ is semisimple, then the composition $\Delta \circ \rho$ is semisimple, where $\Delta\colon {}^L H \to {}^L G$ is the canonical homomorphism arising from base change.
(2) If $H \subset G$ is an unramified twisted Levi and $\rho\colon W_F \to {}^L H(k)$ is semisimple, then $({}^L j_{H,G} \circ \rho)|_{I_F}$ is semisimple. Consequently,

$$\left(z \cdot {}^L j_{H,G} \circ \rho\right)^{\mathrm{ss}}|_{I_F} \sim ({}^L j_{H,G} \circ \rho)|_{I_F}$$

for every unramified cocycle z valued in $Z(\widehat{H})(k)^{I_F}$.
(3) In (2), if $\ell > \mathrm{rk}\, G + 1$, then $z \cdot {}^L j_{H,G} \circ \rho$ itself is semisimple.

Proof. In (1), the Shapiro isomorphism identifies $\Delta \circ \rho$ with $\rho|_{W_E}$. This isomorphism is easily checked to preserve semisimplicity, so (1) follows from the fact that $\rho|_{W_E}$ is semisimple by [BMR05, Theorem 3.10, §6.3].

For (2) and (3), write $H = Z_G(S)$ for an unramified F-torus S, and choose an unramified extension E/F splitting S. Lemmas 5.2.2 and 4.5.2(1) identify ${}^L j_{H,G}|_{\widehat{H} \rtimes W_{E_2}}$ with the standard embedding, which preserves semisimplicity by [BMR08, Corollary 2.10]. Since $I_F \subset W_{E_2}$, this proves the first assertion of (2); the second follows from [BMR05, Theorem 3.10, §6.3].

For (3), we may assume G is not a torus, so $\ell > 2$. Let E/F be splitting field of S, so every prime r dividing $[E : F]$ satisfies $r - 1 \le \dim S \le \mathrm{rk}\, G$. Thus $[E_2 : F]$ is prime to ℓ. Twisting ρ by the central cocycle z preserves semisimplicity, so $z \cdot {}^L j_{H,G} \circ \rho|_{W_{E_2}}$ is semisimple. The image of this restriction is normal of index prime to ℓ in the image of $z \cdot {}^L j_{H,G} \circ \rho$, so (3) follows from [BMR08, Corollary 3.7(ii)]. □

Lemma 10.1.7. *Let k be a field, and let H be a reductive k-group. Let Γ be a group which admits a normal subgroup Γ_0 such that Γ/Γ_0 is cyclic. If $\rho\colon \Gamma \to H(k)$ is a homomorphism, then the following are equivalent:*

(1) $\rho|_{\Gamma_0}$ is semisimple and $Z_{H^\circ}(\rho(\Gamma))/Z(H^\circ)^{\rho(\Gamma)}$ is finite,
(2) ρ is irreducible.

Proof. For (1) $\Rightarrow$ (2), note that $Z_H(\rho(\Gamma_0))^\circ_{\mathrm{red}}$ is connected reductive by [BMR05, Proposition 3.12], so applying Lemma 10.1.1 to $\alpha = \rho(\gamma)$, we find that $S = Z_H(\rho(\Gamma_0))^\circ_{\mathrm{red}}$ is a torus. Thus ρ is semisimple by Lemma 10.1.5.

For (2) $\Rightarrow$ (1), note that $Z_{H^\circ}(\rho(\Gamma))^\circ_{\mathrm{red}}$ is connected reductive by [BMR05, Proposition 3.12]. Thus if $Z_{H^\circ}(\rho(\Gamma))/Z(H^\circ)^\Gamma$ is not finite then there exists a non-central torus $S_0 \subset H^\circ$ centralized by Γ. This implies that $\rho(\Gamma) \subset Z_H(S_0)(k)$ and $Z_H(S_0) \subsetneq H$, contradicting irreducibility of ρ. □

10.1.2. *Non-singular characters.* We now use Lemma 10.1.7 to study non-singular characters of tori. Recall that an F-torus T is *totally ramified* if T does not admit a nontrivial unramified F-subtorus.

Lemma 10.1.8. *If $T \subset G$ is a maximal F-torus, then T is elliptic if and only if $\widehat{T}^{W_F}/Z(\widehat{G})^{W_F}$ is finite. Moreover, $T \cap G_{\mathrm{der}}$ is totally ramified if and only if $\widehat{T}^{I_F}/Z(\widehat{G})^{I_F}$ is finite.*

Proof. We may and do assume that G is semisimple, so $\widehat{T}^{W_F}/Z(\widehat{G})^{W_F}$ is finite if and only if $\widehat{T}^{W_F}$ is finite. The cocharacter group $X_*(\widehat{T}^{W_F})$ is equal to $X_*(\widehat{T})^{W_F} = X^*(T)^{W_F}$ and this is clearly trivial if and only if T is elliptic. The same argument works for the second claim. $\square$

Lemma 10.1.9. *If k is either $\overline{\mathbf{F}}_\ell$ or $\overline{\mathbf{Q}}_\ell$ and $\rho\colon W_F \to {}^LG(k)$ is an L-parameter, then ρ is irreducible if and only if $\rho|_{I_F}$ is semisimple and $Z_{\widehat{G}}(\rho(W_F))/Z(\widehat{G})^{W_F}$ is finite, and in this case $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}}$ is a torus. Similarly, $\rho|_{I_F}$ is irreducible if and only if $Z_{\widehat{G}}(\rho(I_F))/Z(\widehat{G})^{I_F}$ is finite, and in this case $Z_{\widehat{G}}(\rho(P_F))^\circ$ is a torus.*

Proof. The first claim is immediate from Lemma 10.1.7 applied to $\Gamma = \rho(W_F)$ and $\Gamma_0 = \rho(I_F)$. For the second claim, note that $\rho|_{P_F}$ is automatically semisimple since $\ell \neq p$ [BMR05, Lemma 2.6], and $Z_{\widehat{G}}(\rho(P_F))$ is automatically reductive (and in particular smooth) by [PY02, Theorem 2.1]. Thus the result again follows immediately from Lemma 10.1.7. $\square$

The following lemma is similar to [Kal21b, Lemma 4.1.3]. The main difference is that our lemma handles L-parameters ρ for which $\rho|_{P_F}$ may not factor through a maximal torus of $\widehat{G}$; this generality will show up when studying the Fargues–Scholze correspondence, for which this property is not known a priori when p is small.

Lemma 10.1.10. *Let k be a field among $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$, and let $\rho\colon W_F \to {}^LG(k)$ be a semisimple L-parameter. There exists a $\rho(I_F)$-stable Borel-torus pair $(\widehat{B}, \widehat{T})$ in $Z_{\widehat{G}}(\rho(P_F))$, and for any such pair the torus $(\widehat{T}^{I_F})^\circ_{\mathrm{red}}$ is a maximal k-torus of $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}}$. Moreover, $\widehat{T}$ can be chosen to be $\rho(W_F)$-stable.*

Proof. The first claim follows from Lemma 10.1.5 applied to suitable finite quotients Γ and Γ_0 of I_F and P_F, respectively. The second claim follows from [DM94, Théorème 1.8(iii)]. For the final claim, note that another application of Lemma 10.1.5 shows that there is a $\rho(W_F)$-stable Borel-torus pair $(\widehat{B}_0, \widehat{T}_0)$ in $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}}$. By [DM94, Théorème 1.8(iv)], the pair $(\widehat{B}, \widehat{T})$ can be chosen such that $\widehat{T} = Z_{Z_{\widehat{G}}(\rho(P_F))^\circ}(\widehat{T}_0)$ and $\widehat{B}_0 \subset \widehat{B}$, proving the claim. $\square$

Lemma 10.1.11. *Let k be a field among $\overline{\mathbf{F}}_\ell$ and $\overline{\mathbf{Q}}_\ell$, let Ψ be a normalized irreducible Yu datum for G with coefficients in k, and let $[(T,\theta)] \in \mathcal{T}(\Psi)$.*

(1) $\rho_I^{\mathrm{Kal}}(\Psi)$ is semisimple.

(2) The k-group scheme $Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(I_F))^\circ_{\mathrm{red}}$ is a torus if and only if Ψ is non-singular, and in this case

$$Z_{\widehat{G}}(\rho^{\mathrm{Kal}}(\Psi)(I_F))^\circ_{\mathrm{red}} = (\widehat{T}^{I_F})^\circ_{\mathrm{red}}. \tag{10.1.1}$$

In particular, in this case $\rho^{\mathrm{Kal}}(\Psi)$ is irreducible.

(3) $\rho_I^{\mathrm{Kal}}(\Psi)$ is irreducible if and only if Ψ is non-singular and $T \cap G_{\mathrm{der}}$ is totally ramified.

Proof. By Lemma 6.6.1, we may assume that Ψ is irredundant. For (1), note that if (G^i) is the twisted Levi sequence of Ψ, then by construction T is a maximally unramified maximal F-torus of G^0. By Lemma 7.2.9(2), the canonical embedding $\widehat{G^0} \to \widehat{G}$ identifies $\widehat{G^0}$ with $Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(P_F))^\circ$. Since T is maximally unramified in G^0, there is a $\rho_I^{\mathrm{Kal}}(\Psi)(I_F)$-stable Borel subgroup $\widehat{B} \subset Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(P_F))^\circ$ containing $\widehat{T}$. Since $\rho_I^{\mathrm{Kal}}(\Psi)|_{P_F}$ is automatically semisimple by [BMR05, Lemma 2.6], the claim follows from Lemma 10.1.5.

The first claim of (2) follows from Lemma 7.2.16 and Definition 7.2.10. To prove (10.1.1), it is enough to show that $(\widehat{T}^{\rho_I^{\mathrm{Kal}}(\Psi)(I_F)})^\circ_{\mathrm{red}}$ is a maximal torus of $Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(I_F))^\circ_{\mathrm{red}}$, and this follows from the previous paragraph and Lemma 10.1.10.

Now assume that Ψ is non-singular, so θ is non-singular and thus T is elliptic by Lemma 7.2.9. Let $\rho = \rho^{\mathrm{Kal}}(\Psi)$. First, (1) shows that $\rho|_{I_F}$ is semisimple. Moreover, $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}}$ is a torus by Lemma 7.2.16, so $Z_{\widehat{G}}(\rho(I_F))^\circ_{\mathrm{red}} \subset \widehat{T}^{I_F}$ by Lemma 10.1.10 and thus $Z_{\widehat{G}}(\rho(W_F))^\circ_{\mathrm{red}} \subset \widehat{T}^{W_F}$. Since T is elliptic, it follows that $Z_{\widehat{G}}(\rho(W_F))/Z(\widehat{G})^{W_F}$ is finite. Thus Lemma 10.1.9 shows that ρ is irreducible, proving the final claim of (2).

For (3), first suppose that $T \cap G_{\mathrm{der}}$ is totally ramified. Then $G^0 = T$ since the elliptic torus $T_{F^{\mathrm{unr}}}$ is a maximally split torus of $G^0_{F^{\mathrm{unr}}}$, which is quasi-split by Steinberg's theorem. Thus Ψ is non-singular by Remark 7.2.11. By (2), we have $Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(I_F))^\circ_{\mathrm{red}} = (\widehat{T}^{I_F})^\circ_{\mathrm{red}}$. By Lemma 10.1.8, the statement that $T \cap G_{\mathrm{der}}$ is totally ramified is equivalent to finiteness of $\widehat{T}^{I_F}/Z(\widehat{G})^{I_F}$. It follows that $Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(I_F))/Z(\widehat{G})^{I_F}$ is finite, and Lemma 10.1.9 implies that $\rho_I^{\mathrm{Kal}}(\Psi)$ is irreducible. Conversely, if $\rho_I^{\mathrm{Kal}}(\Psi)$ is irreducible, then

Lemma 10.1.9 shows that $Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(I_F))/Z(\widehat{G})^{I_F}$ is finite; by (2), this implies that Ψ is non-singular. Since $\widehat{T}^{I_F} \subset Z_{\widehat{G}}(\rho_I^{\mathrm{Kal}}(\Psi)(I_F))$, the quotient $\widehat{T}^{I_F}/Z(\widehat{G})^{I_F}$ is finite, and thus $T \cap G_{\mathrm{der}}$ is totally ramified by Lemma 10.1.8, as desired. □

10.2. **The main theorem.** For a field k among $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$, let $\Pi_k(G)_{\mathrm{tc}}$ denote the set of isomorphism classes of k-representations of $G(F)$ which arise from Yu's construction applied to a normalized irreducible Yu datum for G over k, and let $\Phi_k^{\mathrm{ss}}(G)$ denote the set of semisimple L-parameters $W_F \to {}^LG(k)$ up to $\widehat{G}(k)$-conjugacy.

Theorem 10.2.1. *Let $\mathcal{S}$ be a set of prime numbers containing p, let F be a non-archimedean local field of residue characteristic $p \neq 2$, let G be a connected reductive F-group, and let k be a field among $\overline{\mathbf{Q}}_\ell$ and $\overline{\mathbf{F}}_\ell$. Suppose that we are given, for each finite tamely ramified extension E/F and each connected reductive E-group H of semisimple rank at most that of G, a map of sets*

$$\rho\colon \Pi_k(H)_{\mathrm{tc}} \to \Phi_k^{\mathrm{ss}}(H).$$

Abbreviate $\rho_I := \rho|_{I_F}$. Suppose that ρ enjoys the following properties.

(1) When G is a torus, ρ agrees with the Local Langlands Correspondence for tori.

(2) ρ is compatible with central characters.

(3) ρ is compatible with homomorphisms which are isomorphisms up to center, i.e., Lemma 7.4.1 holds with ρ_I (resp. ρ) in place of ρ_I^{Kal} (resp. ρ^{Kal})

(4) for every $\ell \notin \mathcal{S}$, if $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ is a normalized irreducible Yu datum for G over k such that G^0 is a torus, and every tamely ramified extension E/F of degree ℓ such that G^0_E is elliptic, we have

$$\rho(\pi(\Psi))|_{P_F} \sim \rho(\pi(\Psi_E))|_{P_F},$$

with notation as in Proposition 8.3.1.

(5) If $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ is a normalized irreducible Yu datum for G over k such that $G^0 \cap G_{\mathrm{der}}$ is not a totally ramified torus, then there exists a proper unramified twisted Levi $L \subsetneq G$ and a Yu datum Ψ_L for L over k such that the conclusion of Theorem 9.3.3 holds for both ρ_I and ρ_I^{Kal}. Moreover,

(a) if $S_\Psi = Z_{\widehat{G}}(\rho_I(\pi(\Psi))(I_F))^\circ_{\mathrm{red}}$ is a torus, then one can choose representatives such that $\rho_I(\pi(\Psi)) = {}^Lj_{L,G} \circ \rho_I(\pi(\Psi_L))$ and the actions of W_F on S_Ψ through $\rho(\pi(\Psi))$ and ${}^Lj_{L,G} \circ \rho(\pi(\Psi_L))$ are the same.

(b) if $\rho(\pi(\Psi))$ is irreducible, so $L = H$, then there exists a cocycle $z\colon W_F/I_F \to Z(\widehat{H} \cap \widehat{G}_{\mathrm{der}})(k)^{I_F}$ such that

$$\rho(\pi(\Psi)) \sim z \cdot {}^Lj_{H,G} \circ \rho(\pi(\Psi_H)).$$

Let Ψ be a normalized irreducible Yu datum for G with k-coefficients, and let $[(T,\theta)] \in \mathcal{T}(\Psi)$ (with notation as in §7.2). Let T_0 be the maximal unramified subtorus of T, and let n be the ramification degree of the splitting field of the torus $T \cap Z_G(T_0)_{\mathrm{der}}$.

(A) If no prime factor of $n \cdot [F(\mu_n) : F]$ lies in $\mathcal{S}$, then

$$\rho_I(\pi(\Psi)) \sim \rho_I^{\mathrm{Kal}}(\Psi). \tag{10.2.1}$$

Therefore Ψ is non-singular if and only if $S_\Psi = Z_{\widehat{G}}(\rho_I(\pi(\Psi))(I_F))^\circ_{\mathrm{red}}$ is a torus.

(B) Suppose moreover that p is good for $Z_G(T_0)$ and Ψ is non-singular, and let

$$X = X_*((T/Z(Z_G(T_0))^\circ_{\mathrm{red}})_{\overline{F}})_{I_F} \quad \text{and} \quad X_{\mathrm{ad}} = X_*((T/Z(Z_G(T_0)))_{\overline{F}})_{I_F}.$$

If N is the least integer such that the map

$$\mathrm{H}^1(W_F/I_F, \mathrm{Hom}(X_{\mathrm{ad}}, k^\times)) \to \mathrm{H}^1(W_{F_N}/I_F, \mathrm{Hom}(X, k^\times))$$

is trivial, then

$$\rho(\pi(\Psi))|_{W_{F_N}} \sim \rho^{\mathrm{Kal}}(\Psi)|_{W_{F_N}}. \tag{10.2.2}$$

Moreover, there exist $\widehat{G}(k)$-conjugates such that $\rho(\pi(\Psi))|_{I_F} = \rho^{\mathrm{Kal}}(\Psi)|_{I_F}$ and the actions of $\rho(\pi(\Psi))$ and $\rho^{\mathrm{Kal}}(\Psi)$ on S_Ψ are the same. In particular, a normalized irreducible Yu datum Ψ_0 is non-singular if and only if $\rho(\pi(\Psi_0))$ is irreducible.

Remark 10.2.2. The main case of interest for Theorem 10.2.1 is the Fargues–Scholze correspondence $\rho = \rho^{\mathrm{FS}}$. This is known to satisfy hypotheses (1), (2), and (3) of Theorem 10.2.1 by [FS21, Theorem I.9.6]. In [CF26b], we will verify (5) using the existing [Fen24, Theorem 1.3.1], and (4) with $\mathcal{S} = \{p\}$ assuming [CF26b, Hypothesis 4.1.4] for cyclic base change at all primes $\ell \neq p$, as discussed in Remark 1.3.3.

10.2.1. *Some bounds on the index.* Before proving Theorem 10.2.1, we explain how one can get some handle on the complicated-looking integer N appearing in (B). First, if T is maximally unramified, then $X = 0$ in Theorem 10.2.1(B), and thus $N = 1$.

Example 10.2.3. Suppose $G_{F^{\mathrm{unr}}} \cong \mathrm{SL}_r$ and $T \subset \mathrm{SL}_r$ is a totally ramified maximal F-torus, so I_F/P_F acts on $X_*(T_{\overline{F}})$ through an elliptic Weyl element. The only elliptic Weyl element of the absolute Weyl group of G is the Coxeter element, so $X \cong \mathbf{Z}/r\mathbf{Z} \cong X_{\mathrm{ad}}$, and the map $X \to X_{\mathrm{ad}}$ is trivial. Thus $N = 1$ in Theorem 10.2.1(B). In fact, we will see in [CF26b, Theorem 10.4.3] (by a longer argument) that one can take $N = 1$ for arbitrary Ψ in this case.

Now we establish some elementary methods for computing N.

Lemma 10.2.4. *Let Y be a finite abelian group equipped with an automorphism α. Let m be an integer such that for each prime ℓ_0 dividing $|Y|$, the action of α^m on $Y/\ell_0 Y$ is of order prime to ℓ_0, and let n_0 be the exponent of the image Z of $Y \to Y$, $y \mapsto \sum_{i=0}^{m-1} \alpha^i(y)$. Then the restriction map*

$$\mathrm{H}^1(\alpha, Y) \to \mathrm{H}^1(\alpha^{mn_0}, Y)$$

is trivial.

Proof. Note that

$$\mathrm{H}^1(\alpha, Y) = Y_\alpha = \bigoplus_{\ell_0 \mid |Y|} Y[\ell_0^\infty]_\alpha,$$

and similarly for $\mathrm{H}^1(\alpha^{mn_0}, Y)$, so we may pass to $Y[\ell_0^\infty]$ to assume that $|Y| = \ell_0^e$ for some e. The image of the map $\mathrm{H}^1(\alpha, Y) \to \mathrm{H}^1(\alpha^m, Y)$ lies in the image of $\mathrm{H}^1(\alpha^m, Z)$, so it is enough to show that the map $\mathrm{H}^1(\alpha^m, Z) \to \mathrm{H}^1(\alpha^{mn_0})$ is trivial. Note that the elements α^m and $\alpha^{\ell_0 m}$ generate the same cyclic subgroup of $\mathrm{Aut}(Y/\ell_0 Y)$, so the restriction map $\mathrm{H}^1(\alpha^m, Y/\ell_0 Y) \to \mathrm{H}^1(\alpha^{\ell_0 m}, Y/\ell_0 Y)$, which is induced by the map $Y \to Y$, $y \mapsto \sum_{i=0}^{\ell_0 - 1} \alpha^{mi}(y)$, is identified with multiplication by ℓ_0 on $(Y/\ell_0 Y)_{\alpha^m}$ and hence vanishes. Thus the restriction map $\mathrm{H}^1(\alpha^m, Z) \to \mathrm{H}^1(\alpha^{\ell_0 m}, Y)$ factors through $\ell_0 \mathrm{H}^1(\alpha^{\ell_0 m}, Y)$, and the result follows by induction. $\square$

The following special case is useful in examples.

Lemma 10.2.5. *Let $Y = (\mathbf{Z}/\ell)^a \oplus (\mathbf{Z}/\ell^2)^b$ for some $a, b \in \mathbf{Z}_{\geq 0}$, and let α be an automorphism of Y. If $r = \ell^{\lceil \log_\ell(\max(1, a+b, \ell b)) \rceil + 1}$, then the restriction map*

$$\mathrm{H}^1(\alpha, Y) \to \mathrm{H}^1(\alpha^r, Y)$$

is trivial.

Proof. By passing from Y to Y_{α^r}, we may and do assume that $\alpha^r = 1$. Let Y_0 be the image of $Y[\ell]$ in $Y/\ell Y$, and let Y_1 be the image of $\{1\} \oplus (\mathbf{Z}/\ell^2)^b$ in $Y/\ell Y$, so $Y/\ell Y = Y_0 \oplus Y_1$ and α stabilizes Y_0. If $r_1 = \ell^{\lceil \log_\ell(\max(1, b)) \rceil}$, then since $\dim_{\mathbf{F}_\ell} Y_1 = b$, the automorphism α_1 of $Y/(\ell Y + Y[\ell])$ induced by α satisfies $\alpha_1^{r_1} = 1$. On the other hand, if $r_2 = \ell^{\lceil \log_\ell(\max(1, a+b)) \rceil}$, then the automorphism $\overline{\alpha}$ of $Y/\ell Y$ induced by α satisfies $\overline{\alpha}^{r_2} = 1$. It follows that $\alpha^{\max(\ell r_1, r_2)} = 1$. By choice of r_1, the map $Y \to Y$, $y \mapsto \sum_{i=0}^{\ell r_1 - 1} \alpha^i(y)$, factors through $Y[\ell]$, and the claim therefore follows from Lemma 10.2.4. $\square$

Example 10.2.6. Let G be a connected reductive F-group such that $G_{F^{\mathrm{unr}}}$ is isomorphic to one of GL_r, SO_{2r+1}, Sp_{2r}, and SO_{2r}. Suppose $p \neq 2$ and ρ is a map as in Theorem 10.2.1.

Assume that $T \cap G_{\mathrm{der}}$ is totally ramified. In this case, T corresponds to an elliptic Weyl element of the absolute Weyl group of G (namely, the element through which I_F/P_F acts on $X_*(T_{\overline{F}})$; here we use that G splits after an unramified extension). We split into cases:

(1) If $G_{F^{\mathrm{unr}}}$ is isomorphic to GL_r, then the only elliptic Weyl element is the Coxeter element, and in the notation of Theorem 10.2.1(B) we have $X \cong X_{\mathrm{ad}} \cong \mathbf{Z}/r$. By Lemma 10.2.4 applied to $Y = \mathrm{Hom}(X, k^\times)$, we may take $N = r$: indeed, $Y/\ell_0 Y$ is either trivial or isomorphic to $\mathbf{Z}/\ell_0$ for all primes ℓ_0, and $\mathbf{Z}/\ell_0$ has no automorphisms of order ℓ_0.

(2) If $G_{F^{\mathrm{unr}}}$ is either SO_{2r+1} or Sp_{2r}, then [GP00, §3.4] shows that every elliptic Weyl element is the Coxeter element of a Weyl group of type $\mathrm{B}_{a_1} \times \cdots \times \mathrm{B}_{a_s}$, where $(a_1, \ldots, a_s)$ is a partition of r. For such an element, the group X of Theorem 10.2.1(B) is isomorphic to $(\mathbf{Z}/2)^s$, so m divides $2^{\lceil \log_2(s) \rceil}$ in the notation of Lemma 10.2.4 and $n_0|2$. In particular, since $s \leq r$ we may take $N \leq 2^{\lceil \log_2(2r) \rceil}$ to be a power of 2.
(3) If $G_{F^{\mathrm{unr}}}$ is isomorphic to SO_{2r}, then by [GP00, §3.4] every elliptic Weyl element is a twisted Coxeter element of a Weyl group of type $\mathrm{D}_{a_1} \times \cdots \times \mathrm{D}_{a_s}$, where $(a_1, \ldots, a_s)$ is a partition of r with s even. For such an element, the group X of Theorem 10.2.1(B) is $(\mathbf{Z}/2)^a \oplus (\mathbf{Z}/4)^b$, where $a + 2b \leq s$, and Lemma 10.2.5 shows $N \leq 2^{\lceil \log_2(2r) \rceil}$ is a power of 2.

Note that in all cases above, every unramified twisted Levi F-subgroup of G becomes isomorphic to a product of groups, each of which is either isomorphic to GL_s $(s \leq r)$ or of the same type as G (of smaller rank). From this, if one assumes that ρ respects products and Weil restriction then one can obtain similar upper bounds to the above for general T. However, the bounds are of a more combinatorial nature and appear tricky to state cleanly.

Using Lemma 10.2.4 and the classification of elliptic (twisted) Weyl elements as in [GP00, §3.4], [GM97, §3], [GKP00, §§3 and 4], and [He07, §7], it should be possible to give reasonably sharp explicit bounds on the integer N in Theorem 10.2.1(B) in all cases. Indeed, it is possible to bound both integers m and n_0 in Lemma 10.2.4 using the characteristic polynomial of α, as the following well-known elementary lemma (inspired by [Ree11, Lemma 4.1]) shows.

Lemma 10.2.7. *Let Y_0 be a finite free $\mathbf{Z}$-module, and let α_0 be an automorphism of Y_0 such that $(Y_0)_{\alpha_0}$ is finite. Let μ (resp. χ) be the minimal (resp. characteristic) polynomial of α_0. Then*

(1) $(Y_0)_{\alpha_0}$ *is killed by* $\mu(1)$*,*
(2) $(Y_0)_{\alpha_0}$ *is of order* $|\chi(1)|$.

Proof. For (1), observe that $\mu(1)$ acts by $\mu(\alpha_0) = 0$ on $(Y_0)_{\alpha_0}$. For (2), note that Y_0 admits a finite index α_0-stable submodule Y_1 which is a direct sum of $\mathbf{Z}$-submodules on which α_0 acts by a matrix in rational canonical form. Applying Tate cohomology for α_0 to the exact sequence $0 \to Y_1 \to Y_0 \to Y_0/Y_1 \to 0$ and noting that $(Y_0)^{\alpha_0} = (Y_1)^{\alpha_0} = 0$ and $|\mathrm{T}^i(\alpha_0, Y_0/Y_1)|$ is independent of i (by standard results on Herbrand quotients), we thereby reduce from Y_0 to Y_1. But the claim is easily checked by direct calculation for Y_1. □

Example 10.2.8. Let G be an unramified absolutely simple F-group of exceptional type, suppose that p is good for G, and suppose that ρ is a map as in Theorem 10.2.1 with $\mathcal{S} = \{p\}$ which is compatible with Weil restrictions and products. Let π be a non-singular supercuspidal $\overline{\mathbf{Q}}_\ell$-representation of $G(F)$ with associated torus-character pair (T, θ). The torus $T/Z(Z_G(T_0))$ is a totally ramified torus of (a semisimple quotient of) a Levi subgroup of G, so to bound the integer N in Theorem 10.2.1(B) it suffices to bound N when T is totally ramified and G has the same type as a Levi subgroup of an exceptional group.

For each element of the Weyl group Ω, there is a Weyl subgroup Ω_0 of Ω (i.e., the Weyl group of a root subsystem of $\Phi(G_{\overline{F}}, T_{\overline{F}})$, as in [Car72, §3, Example (iii)]) such that $w \in \Omega_0$ and w is not contained in a proper Weyl subgroup of Ω_0. If w is elliptic, then this root subsystem must span $X^*(T_{\overline{F}}) \otimes \mathbf{Q}$. The set of possible Ω_0 which occur can be described using Borel–de Siebenthal theory, as explained in [Car72, §4]. Moreover, given Ω_0, the set of w satisfying this property is classified in [Car72, §5, Theorem A], and the possible characteristic polynomials of such w are described in [Car72, §6, Proposition 21].

Using these considerations and Lemmas 10.2.4 and 10.2.7, we can bound the integer N in Theorem 10.2.1(B) effectively. The calculations are straightforward but tedious, and we will not use them, so we only summarize here:

(1) If G is of type G_2, then $N \leq 4$.
(2) If G is of type F_4, then $N \leq 9$.
(3) If G is of type E_6, then $N \leq 18$.
(4) If G is of type E_7, then $N \leq 24$.
(5) If G is of type E_8, then $N \leq 36$.

By inspecting [Ree11, Table 1], one sees that if G is of type E_8 and T is totally ramified, then there are 30 possible elliptic Weyl elements w corresponding to T, and for most such elements N is considerably smaller. For instance, $N = 1$ for nine possible such w (those with $X_w = 0$, with notation as in [Ree11, Table 1]), and N divides 4 for seven more (those with $X_w = 2^2$). There are only five such w for which we cannot ensure $N \leq 9$ (those labeled A_1^8, $A_3D_5(a_1)$, A_4^2, A_2^4, and $A_5A_2A_1$, for which N divides 16, 16, 25, 27, and 36, respectively).

10.3. **Proof of Theorem 10.2.1.** Fix a connected reductive F-group G and a normalized irreducible Yu datum $\Psi = ((G^i), x, (r_i), \tau, (\phi_i))$ for G with k-coefficients. We start by proving (A).

10.3.1. *Reductions.* Let $T_{\mathrm{der}} \coloneqq (T \cap G_{\mathrm{der}})^\circ$, and let $T_{\mathrm{der},0}$ be its maximal unramified subtorus. Let $\pi = \pi(\Psi)$. We will prove (10.2.1) and the first statement of (B) by induction on the dimension of G_{der} and on the size of the image of I_F in $\mathrm{Aut}(X^*((T_{\mathrm{der}})_{\overline{F}}))$. The base case in which G is a torus is clear from (1). If $T_{\mathrm{der},0} \neq 1$, then $H \coloneqq Z_G(T_{\mathrm{der},0})$ is a proper unramified twisted Levi subgroup of G, so the induction hypothesis applies to H, and it proves (A) for G by (5) and Theorem 9.3.3. We may therefore assume $T_{\mathrm{der},0} = 1$, so T_0 is central in G and T_{der} is totally ramified. Since T is maximally unramified in G^0, it follows that $Z_{G^0}(T_0) = T$ and thus $G^0 = T$ since T_0 is central in G^0. In this case, observe that Ψ is automatically F-non-singular by Remark 7.2.11.

Since T_{der} is totally ramified, it follows from Lemma 10.1.11(3) that $\rho^{\mathrm{Kal}}(\Psi)|_{I_F}$ is irreducible. Lemma 10.1.9 then shows that $Z_{\widehat{G}}(\rho^{\mathrm{Kal}}(\Psi)(P_F))^\circ$ is a torus, and since it contains the maximal torus $\widehat{T}$ through which $\rho^{\mathrm{Kal}}(\Psi)|_{P_F}$ factors, it equals $\widehat{T}$. Irreducibility of $\rho^{\mathrm{Kal}}(\Psi)|_{I_F}$ implies that $\widehat{T}^{I_F/P_F}/Z(\widehat{G})^{I_F/P_F}$ is finite.

We first prove

$$\rho(\pi)|_{P_F} \sim \rho^{\mathrm{Kal}}(\Psi)|_{P_F}. \tag{10.3.1}$$

10.3.2. *Killing Frobenius.* Note that Lemma 10.1.11(3) implies that $\rho^{\mathrm{Kal}}(\Psi)|_{I_F}$ is irreducible. Let M be a positive integer such that the image of W_{F_M} in $\mathrm{Aut}(X^*((T_{\mathrm{der}})_{\overline{F}}))$ is equal to the image of I_F. Write $M = M_0M_1$, where M_0 divides a power of n and M_1 is prime to n. Write $M_0 = \ell_1 \cdots \ell_b$ for primes ℓ_i, with $b = 0$ if $M_0 = 1$. If $b > 0$, then since T_{der} is totally ramified, $T_{F_{\ell_1}}$ is still elliptic. If $\Psi_{F_{\ell_1}}$ is the Yu datum from (4), then

$$\rho(\pi(\Psi_{F_{\ell_1}}))|_{P_F} \sim \rho(\pi(\Psi))|_{P_F}, \qquad \rho^{\mathrm{Kal}}(\Psi_{F_{\ell_1}})|_{P_F} \sim \rho^{\mathrm{Kal}}(\Psi)|_{P_F}. \tag{10.3.2}$$

and thus we may pass from F to F_{ℓ_1} in order to prove (10.3.1). Repeating this process for the b prime factors of M_0, we may pass from F to F_{M_0} and thus reduce to the case that M is prime to n. Repeating this procedure for the prime factors of $[F(\mu_n) : F]$, all of which lie outside $\mathcal{S}$ by assumption, we may further assume that $\mu_n \subset F$.

10.3.3. *Killing ramification.* The torus T_{der} is now totally ramified, and n is the ramification degree of its splitting field. We will now make further base changes to kill the ramification.

By the choice of M, the image of W_{F_M} in $\mathrm{Aut}(X^*((T_{\mathrm{der}})_{\overline{F}}))$ is the same as the (cyclic) image of I_F, and it is of order n. Hence the splitting field E/F_M of $(T_{\mathrm{der}})_{F_M}$ is a totally ramified cyclic extension of degree n. We may therefore write $E = F_M(\sqrt[n]{\varpi})$, where $\varpi \in F_M^\times$ is a uniformizer. Let $\varpi_0 \in F^\times$ be a uniformizer. We claim that there is an element $c \in F^\times$ such that the extension $E' = F(\sqrt[n]{c\varpi_0})$ (which is Galois since F contains all nth roots of unity in $\overline{F}$) is a tamely ramified degree n extension of F, is linearly disjoint from E, and has the property that EE'/E' is unramified.

$$\begin{tikzcd}
& EE' \arrow[dl, dash] \arrow[drr, dash, "\text{unramified}"] & & \\
E = F_M(\sqrt[n]{\varpi}) \arrow[ddr, dash] & & & E' = F(\sqrt[n]{c\varpi_0}) \arrow[dl, dash, "\deg n/\ell_0"] \\
& & E'_0 \arrow[dl, dash, "\deg \ell_0"] & \\
& F & &
\end{tikzcd}$$

To see this, let $n = \ell_1'^{e_1} \cdots \ell_t'^{e_t}$ be the prime factorization of n, and for each i choose a unit $c_i' \in \mathcal{O}_F^\times$ which is an $n/\ell_i'^{e_i}$th power but not an ℓ_i'th power; such a unit exists because $\mu_n \subset F$. Since $\ell_i' \nmid M$, the extensions $F_M(\sqrt[\ell_i']{\varpi_0})/F_M$ and $F_M(\sqrt[\ell_i']{c_i'\varpi_0})/F_M$ are linearly disjoint, so at least one is linearly disjoint from E/F_M.

Choose $c_i \in \{1, c'_i\}$ giving that extension and set $c = \prod_{i=1}^{t} c_i$. One checks that E and E' are linearly disjoint and EE' is unramified by Abhyankar's lemma, proving the claim.

10.3.4. *Congruence on wild inertia.* Let E'_0/F be a cyclic subextension of E'/F of prime degree ℓ_0. Since E'_0 is linearly disjoint from E, base change to E'_0 preserves the image of W_F in $\mathrm{Aut}(X^*((T_{\mathrm{der}})_{\overline{F}}))$ and thus T is still elliptic. Thus (10.3.2) holds with E'_0 in place of F_{ℓ_1}. By construction, the image of $I_{E'_0}$ in $\mathrm{Aut}(X^*((T_{\mathrm{der}})_{\overline{F}}))$ is of order n/ℓ_0, so the induction hypothesis applied to $\Psi_{E'_0}$ therefore gives

$$\rho(\pi(\Psi_{E'_0}))|_{P_F} \sim \rho^{\mathrm{Kal}}(\Psi_{E'_0})|_{P_F}. \tag{10.3.3}$$

Combining the two comparisons in (10.3.2) (with E'_0 in place of F_{ℓ_1}) with the induction hypothesis gives

$$\rho(\pi(\Psi))|_{P_F} \overset{(10.3.2)}{\sim} \rho(\pi(\Psi_{E'_0}))|_{P_F} \overset{(10.3.3)}{\sim} \rho^{\mathrm{Kal}}(\Psi_{E'_0})|_{P_F} \overset{(10.3.2)}{\sim} \rho^{\mathrm{Kal}}(\Psi)|_{P_F}.$$

Thus (10.3.1) follows by induction on the size of the image of I_F in $\mathrm{Aut}(X^*((T_{\mathrm{der}})_{\overline{F}}))$.

10.3.5. *Promoting from wild inertia to all of inertia.* With (10.3.1) established, we may pass to $\widehat{G}(k)$-conjugates to assume that

$$\rho(\pi)|_{P_F} = \rho^{\mathrm{Kal}}(\Psi)|_{P_F}.$$

Identify the dual torus $\widehat{T}$ with its image in $\widehat{G}$ via ${}^L j_{T,G}|_{\widehat{T}\times\{1\}}$. We have already seen $Z_{\widehat{G}}(\rho(\pi)(P_F))^\circ = Z_{\widehat{G}}(\rho^{\mathrm{Kal}}(\Psi)(P_F))^\circ = \widehat{T}$, so Lemma 10.1.3 implies that the two actions on $\widehat{T}$ agree after conjugating $\rho(\pi)$ by some element of $Z_{\widehat{G}}(\rho(\pi)(P_F))$. After making this conjugation and renaming $\rho(\pi)$, let $s \in I_F$ lift a topological generator of I_F/P_F, define $h_1 = \rho(\pi)(s)$ and $h_2 = \rho^{\mathrm{Kal}}(\Psi)(s)$.

Set $\widehat{T}_{\mathrm{der}} \coloneqq \widehat{T} \cap \widehat{G}_{\mathrm{der}}$, so $\widehat{T}_{\mathrm{der}}^{h_1}$ is finite. Let $f\colon {}^L G \to {}^L G/\widehat{G}_{\mathrm{der}}$ be the quotient map. Compatibility with central characters in hypothesis (2) shows that there is an element $\bar{t} \in (\widehat{G}/\widehat{G}_{\mathrm{der}})(k)$ such that $f(h_2) = \bar{t} f(h_1)\bar{t}^{-1}$; let $t \in \widehat{T}(k)$ lift $\bar{t}$, so $d \coloneqq h_2^{-1} t h_1 t^{-1} \in \widehat{G}_{\mathrm{der}}(k)$. Since h_1 and h_2 induce the same action on $\widehat{T}$ and have the same image in W_F, we have $d \in Z_{\widehat{G}_{\mathrm{der}}}(\widehat{T})(k) = \widehat{T}_{\mathrm{der}}(k)$. This means that the images of h_1 and h_2 in $N_{\widehat{G}\rtimes W_F}(\widehat{T})/\widehat{T}_{\mathrm{der}}$ are conjugate under $\widehat{T}/\widehat{T}_{\mathrm{der}}$, so [Cot26, Lemma 5.3.4] implies that h_1 and h_2 are $\widehat{T}(k)$-conjugate. This means precisely that $\rho_I(\pi) \sim \rho_I^{\mathrm{Kal}}(\Psi)$, proving (10.2.1). The final claim of (A) follows from Lemma 10.1.11(2).

10.3.6. *Proof of (B).* Next, we prove (B), so assume p is good for $Z_G(T_0)$ and Ψ is non-singular. In particular, S_Ψ is a torus in this case.

We begin with some reductions. Let $H \subset \widetilde{H}$ be an embedding with $H_{\mathrm{der}} = \widetilde{H}_{\mathrm{der}}$ such that $Z(\widetilde{H})$ is an induced torus, and let $\widetilde{T} = T \cdot Z(\widetilde{H})$. If $\widetilde{X}$ and $\widetilde{X}_{\mathrm{ad}}$ are the $\mathbf{Z}[W_F/I_F]$-modules attached to $(\widetilde{H}, \widetilde{T})$ as in (B), then $\widetilde{X} = \widetilde{X}_{\mathrm{ad}} = X_{\mathrm{ad}}$. By (3) and Lemma 7.4.1, we may and do therefore assume that $Z(G)$ is an induced torus, which implies that $\widehat{G}$ has simply connected derived group.

Note that the torus S_Ψ lies in the image of ${}^L j_{T,G}|_{\widehat{T}\rtimes 1}$ by Lemma 10.1.11(2), and thus it does not change when applying Theorem 9.3.3 in the above reduction to the case that $T \cap G_{\mathrm{der}}$ is totally ramified. By condition (5)(a), the action of $\rho(\pi(\Psi))$ on S_Ψ is also unchanged by this reduction step. Thus to check that there are $\widehat{G}(k)$-conjugates such that $\rho(\pi(\Psi))|_{I_F} = \rho^{\mathrm{Kal}}(\Psi)|_{I_F}$ and the two actions of W_F on S_Ψ are the same, we may argue as above to reduce to the case that $G^0 = T$ and $T_{\mathrm{der},0} = 1$.

Once we have established the action claim of (B), we can also reduce (10.2.2) to the case that $T \cap G_{\mathrm{der}}$ is totally ramified: indeed, if the result is known in this case and we let $H = Z_G(T_0)$, then condition (5)(b) shows that we may choose representatives for $\rho(\pi(\Psi))$ and $\rho^{\mathrm{Kal}}(\Psi_H)$ such that there is a cocycle $z\colon W_F/I_F \to Z(\widehat{H} \cap \widehat{G}_{\mathrm{der}})(k)^{I_F}$ such that

$$\rho(\pi(\Psi))|_{W_{F_N}} = \left(z \cdot {}^L j_{H,G} \circ \rho^{\mathrm{Kal}}(\Psi_H)\right)|_{W_{F_N}}.$$

Thus for the reduction step it suffices to show that $z|_{W_{F_N}/I_F}$ is a coboundary.

For this, note the short exact sequence

$$1 \to Z(\widehat{H}_{\mathrm{der}}) \to Z(\widehat{H} \cap \widehat{G}_{\mathrm{der}}) \to (\widehat{H} \cap \widehat{G}_{\mathrm{der}})/\widehat{H}_{\mathrm{der}} \to 1,$$

where $(\widehat{H} \cap \widehat{G}_{\mathrm{der}})/\widehat{H}_{\mathrm{der}}$ is dual to $Z(H)/Z(G)$. Since $Z(H)/Z(G)$ is the center of an unramified twisted Levi subgroup of G_{ad}, it follows that $(Z(H)/Z(G))_{F^{\mathrm{unr}}}$ is an induced torus and thus $((\widehat{H} \cap \widehat{G}_{\mathrm{der}})/\widehat{H}_{\mathrm{der}})^{I_F}$ is a torus. Since $((\widehat{H} \cap \widehat{G}_{\mathrm{der}})/\widehat{H}_{\mathrm{der}})^{W_F}$ is finite, [Cot26, Lemma 5.3.4] implies that z is cohomologous to a

$Z(\widehat{H}_{\mathrm{der}})(k)^{I_F}$-valued cocycle. If $\widehat{T}_{\mathrm{der}} = \widehat{T} \cap \widehat{H}_{\mathrm{der}}$, then $\widehat{T}_{\mathrm{der}}(k)^{I_F} \cong \mathrm{Hom}(X, k^\times)$, and thus indeed $z|_{W_{F_N}}$ is a coboundary by definition. We have thus reduced (B) to the case that $T \cap G_{\mathrm{der}}$ is totally ramified. Since T_0 is now central in G, our assumption implies that p is good for G.

By (10.2.1), we may pass to $\widehat{G}(k)$-conjugates to assume $\rho(\pi)|_{I_F} = \rho^{\mathrm{Kal}}(\Psi)|_{I_F}$. Let $C_P \coloneqq Z_{\widehat{G}}(\rho(\pi)(P_F))$, so $C_P^\circ = \widehat{T}$ as above. By [GH91, Proposition 2.1] and [Ste75, Theorem 2.15] (using that $\widehat{G}$ has simply connected derived group, G is tamely ramified[26], and p is good for G), if $\gamma \in P_F$ is any element then $Z_{\widehat{G}}(\rho(\pi)(\gamma))$ is a Levi k-subgroup of $\widehat{G}$. Applying this inductively to a composition series of $\rho(\pi)(P_F)$ with cyclic subquotients, we learn that C_P is connected, and thus $C_P = \widehat{T}$.

Let $\widehat{S} = Z_{\widehat{G}}(\rho(\pi)(I_F))^\circ_{\mathrm{red}}$, so $\widehat{S}$ is a (central) torus. Let $\rho_1 = \rho(\pi)$, and let $\rho_2 = \rho^{\mathrm{Kal}}(\Psi) = {}^L j_{T,G} \circ {}^L\theta$. Let $w \in W_F$ be any element. For $\gamma \in I_F$, we have

$$ {}^{\rho_2(w)}\rho_2(\gamma) = \rho_2({}^w\gamma) = \rho_1({}^w\gamma) = {}^{\rho_1(w)}\rho_1(\gamma) = {}^{\rho_1(w)}\rho_2(\gamma), \tag{10.3.4}$$

so $\rho_1(w)$ and $\rho_2(w)$ have the same effect on $\rho_1(I_F) = \rho_2(I_F)$. Since $Z_{\widehat{G}}(\rho_1(P_F)) = \widehat{T}$, the parameters ρ_1 and ρ_2 differ by a cocycle valued in $\widehat{T}(k)$ and induce the same action on $\widehat{T}$, hence also on $\widehat{S}$, as desired. The final two claims of (B) now follow, the latter from Lemma 10.1.11(2).

Finally, we turn to the proof of (10.2.2). For this, we first pin down the images of ρ_1 and ρ_2 in ${}^LG/\widehat{G}_{\mathrm{der}}$. As above, let $s \in I_F$ be a lift of a pro-generator of tame inertia. Let $f\colon {}^LG \to {}^LG/\widehat{G}_{\mathrm{der}}$ be the natural quotient map, and note that $f \circ \rho_1$ and $f \circ \rho_2$ are $(\widehat{G}/\widehat{G}_{\mathrm{der}})(k)$-conjugate by (1) and (2). Since $Z(G)$ is an induced torus, the coinvariant group $X_*(Z(G_{\overline{F}}))_{I_F}$ is torsion-free and thus the k-group scheme $(\widehat{G}/\widehat{G}_{\mathrm{der}})^{I_F}$ is connected. For dimension reasons, it follows that the map $Z(\widehat{G})^{I_F} \to (\widehat{G}/\widehat{G}_{\mathrm{der}})^{I_F}$ is surjective. Thus, after conjugating by an element of $Z(\widehat{G})^{I_F}(k)$, we may assume that the central characters of ρ_1 and ρ_2 agree.

For $w \in W_F$, we define $\varphi(w) \coloneqq \rho_1(w)\rho_2(w)^{-1} \in \widehat{G}_{\mathrm{der}}(k)$. Now, (10.3.4) implies that $\rho_1(w)$ and $\rho_2(w)$ induce the same automorphism of the common subgroup $\rho_1(I_F) = \rho_2(I_F)$, so in fact we have $\varphi(w) \in Z_{\widehat{G}_{\mathrm{der}}}(\rho_1(I_F))(k)$. The map φ is constant on the right cosets of I_F and hence factors through a map, again denoted by φ,

$$\varphi\colon W_F/I_F \longrightarrow Z_{\widehat{G}_{\mathrm{der}}}(\rho_1(I_F))(k) = \widehat{T}_{\mathrm{der}}^{\rho_1(I_F)}(k), \qquad \varphi(\overline{w}) = \rho_1(w)\rho_2(w)^{-1},$$

and it is straightforward to check that this defines a cocycle. Note that the character group of $\widehat{T}_{\mathrm{der}}^{\rho_1(I_F)}$ is isomorphic to $X = X_*((T/Z(Z_G(T_0))^\circ_{\mathrm{red}})_{\overline{F}})_{I_F}$, so $\widehat{T}_{\mathrm{der}}^{\rho_1(I_F)}(k) \cong \mathrm{Hom}(X, k^\times)$. By hypothesis on N, the restriction $\varphi|_{W_{F_N}/I_F}$ is a 1-coboundary, proving (10.2.2). □

Appendix A. Some calculations with sign characters

In this appendix, we establish certain technical lemmas related to the FKS sign character and χ-data.

A.1. **Proof of Lemma 5.1.2.**

Proof. By passing to an inner twist as in the construction of ${}^Lj_\chi$, we may assume that G is quasi-split. We only prove the assertion for G when T is elliptic; if T is not elliptic, the same calculation works after restricting the characters to $T(F)_{\mathrm{b}}$ and the corresponding cocycles to I_F. Write ${}^L\theta(w) = (\widehat{\theta}(w), w)$ for a 1-cocycle $\widehat{\theta}\colon W_F \to \widehat{T}(k)$. By definition (4.3.2), our claim reduces to the following equality of cohomology classes:

$$[Q_{\chi''_G} Q^{-1}_{\min \chi''_G}] = [\widehat{\epsilon}_{G,\flat,0}] \quad \text{in } \mathrm{H}^1(W_F, \widehat{T}(k)). \tag{A.1.1}$$

Observe that in (4.3.1), the elements $v_0(u_i(w))$ and $w_i^{-1}\widehat{\alpha}^\vee$ do not depend on the choice of set of χ-data. Moreover, the terms $\chi''_{G,\alpha}$ and $\min \chi''_{G,\alpha}$ only differ if α is either asymmetric or symmetric unramified. Thus we have

$$Q_{\chi''_G} Q^{-1}_{\min \chi''_G} = \prod_{\overline{\alpha} \in (\Phi(G_{\overline{F}}, T_{\overline{F}})_{\mathrm{asym}} \cup \Phi(G_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,unram}})/\Sigma} Q_{\chi''_G, \overline{\alpha}} Q^{-1}_{\min \chi''_G, \overline{\alpha}}.$$

Now set $G^{-1} \coloneqq T$ and fix a root $\alpha \in \Phi((G^{j+1}/G^j)_{\overline{F}}, T_{\overline{F}})$, where $0 \le j \le d-1$, which is either asymmetric or symmetric unramified, and let $\alpha_j = \alpha|_{(Z_{G^j})_{\overline{F}}} \in \Phi((G^{j+1}/G^j)_{\overline{F}}, (Z_{G^j})_{\overline{F}})$ denote the corresponding restricted character. There are five cases, as in the proof of [Kal21a, Proposition 5.27]:

[26] This follows from the existence of a Yu datum for G.

(1) α and α_j are both asymmetric,
(2) α is asymmetric and α_j is symmetric unramified,
(3) α is asymmetric and α_j is symmetric ramified,
(4) α and α_j are both symmetric unramified,
(5) α is symmetric unramified and α_j is symmetric ramified.

Define a character $\epsilon_{\flat,0,\alpha}\colon T(F) \to k^\times$ as follows: in cases (1), (2), and (4), let $\epsilon_{\flat,0,\alpha}$ be the trivial character. In case (3), define

$$\epsilon_{\flat,0,\alpha}(t) = \operatorname{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(t))^{e(\alpha/\alpha_j)}.$$

In case (5), define

$$\epsilon_{\flat,0,\alpha}(t) = \operatorname{sgn}_{\mathbf{F}_\alpha^1}(\alpha(t))^{e(\alpha/\alpha_j)}.$$

By definition, we have

$$\epsilon_{\flat,0} = \prod_{\alpha\in(\Phi(G_{\overline{F}},T_{\overline{F}})_{\mathrm{asym}}\cup\Phi(G_{\overline{F}},T_{\overline{F}})_{\mathrm{sym,unram}})/\Sigma} \epsilon_{\flat,0,\alpha},$$

so it suffices to show the following equality of cohomology classes for each α:

$$[Q_{\chi''_G,\overline{\alpha}} Q^{-1}_{\min\chi''_G,\overline{\alpha}}] = [\widehat{\epsilon}_{\flat,0,\alpha}] \quad \text{in } \mathrm{H}^1(W_F, \widehat{T}(k)). \tag{A.1.2}$$

In cases (1) and (4), observe that $\chi''_{G,\alpha} = \min\chi''_{G,\alpha}$, so (A.1.2) is immediate from the definitions. Next, suppose that α is asymmetric; in this case, we have $\min\chi''_{G,\overline{\alpha}} = 1$, and by definition of Q, for $w \in W_F$ we have

$$Q_{\chi''_{G,\overline{\alpha}}}(w) Q_{\min\chi''_{G,\overline{\alpha}}}(w)^{-1} = \prod_{i=1}^m (w_i^{-1}\widehat{\alpha}^\vee)(\chi''_{G,\alpha}(\operatorname{rec}_{F_\alpha}(v_0(u_i(w))))). \tag{A.1.3}$$

Suppose first that we are in case (2). Then χ''_{G,α_j} is the unique unramified quadratic character of $F_{\alpha_j}^\times$, so $\chi''_{G,\alpha}$ is also unramified of order ≤ 2. Lemma 4.4.2 identifies the cohomology class represented by the left side of (A.1.3) with the cocycle associated to $\chi''_{G,\alpha}\circ\alpha$, where $\alpha\colon T \to J_\alpha = \operatorname{Res}_{F_\alpha/F}\mathbf{G}_m$ is the F-homomorphism arising from the root α. Since T is elliptic, $\alpha(T(F)) \subset \mathcal{O}_{F_\alpha}^\times$, on which $\chi''_{G,\alpha}$ is trivial. The resulting cohomology class is therefore trivial in case (2).

Next suppose that we are in case (3). Once again, $\alpha(T(F)) \subset \mathcal{O}_{F_\alpha}^\times$. On $\mathcal{O}_{F_\alpha}^\times$, the character $\chi''_{G,\alpha}$ is equal to

$$\operatorname{sgn}_{\mathbf{F}_{\alpha_j}^\times} \circ \operatorname{Nm}^{e(\alpha/\alpha_j)}_{\mathbf{F}_\alpha/\mathbf{F}_{\alpha_j}} = \operatorname{sgn}^{e(\alpha/\alpha_j)}_{\mathbf{F}_\alpha^\times}.$$

Lemma 4.4.2 therefore identifies the class represented by (A.1.3) with the cohomology class dual to $\epsilon_{\flat,0,\alpha}$, which proves (A.1.2) in this case.

Finally, suppose we are in case (5). Since both $\chi''_{G,\alpha}$ and $\min\chi''_{G,\alpha}$ restrict to κ_α on $F_{\pm\alpha}^\times$, the quotient $\chi''_{G,\alpha}\cdot(\min\chi''_{G,\alpha})^{-1}$ factors through the map $F_\alpha^\times \to F_\alpha^1$, $t \mapsto t\cdot v_1(t)^{-1}$. By the same argument as in the asymmetric case, the factored map is $\operatorname{sgn}^{e(\alpha/\alpha_j)}_{\mathbf{F}_\alpha^1}\colon F_\alpha^1 \to k^\times$. Therefore Lemma 4.4.2, applied to this quotient character, gives the following identity of classes:

$$[\widehat{\epsilon}_{\flat,0,\alpha}] = \left[w \longmapsto \prod_{i=1}^m (w_i^{-1}\widehat{\alpha}^\vee)((\chi''_{G,\alpha}\cdot(\min\chi''_{G,\alpha})^{-1})(\operatorname{rec}_{F_\alpha}(v_0(u_i(w))))) \right]$$

This is exactly (A.1.2) in case (5). □

A.2. **Proof of Lemma 5.1.3.**

Proof. By twisting, we may and do assume that G is quasi-split. We will assume that T is elliptic; the general case is the same after restricting characters to $T(F)_{\mathrm{b}}$ and cocycles to I_F. As in the proof of Lemma 5.1.2, the proof reduces to the following equality of cohomology classes:

$$[Q_{\chi''_{H,G}} Q^{-1}_{\min\chi''_G}] = [\widehat{\epsilon}_{G,\flat,1}\widehat{\epsilon}_{H,\flat,1}\widehat{\epsilon}_{G,\flat,2}\widehat{\epsilon}_{H,\flat,2}] \quad \text{in } \mathrm{H}^1(W_F, \widehat{T}(k)). \tag{A.2.1}$$

Since $\chi''_{H,G}$ and $\min\chi''_G$ are both minimally ramified, these sets of χ-data only differ at symmetric ramified roots, at which they agree with χ''_H and χ''_G, respectively. Thus we have

$$Q_{\chi''_{H,G}} Q^{-1}_{\min\chi''_G} = \prod_{\overline{\alpha}\in\Phi(G_{\overline{F}},T_{\overline{F}})_{\mathrm{sym,ram}}/\Gamma} Q_{\chi''_H,\overline{\alpha}} Q^{-1}_{\chi''_G,\overline{\alpha}}.$$

If we further decompose $\epsilon_{G,\flat,i} = \prod_{\alpha\in\Phi(G_{\overline{F}},T_{\overline{F}})_{\mathrm{sym,ram}}/\Gamma} \epsilon_{G,\flat,i,\alpha}$ for $i = 1, 2$ as in the definitions (3.2.6), (3.2.7), then it suffices to prove the following equality for all $\alpha \in \Phi(G_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,ram}}$:

$$[Q_{\chi''_H,\overline{\alpha}} Q^{-1}_{\chi''_G,\overline{\alpha}}] = [\widehat{\epsilon}_{G,\flat,1,\alpha}\widehat{\epsilon}_{H,\flat,1,\alpha}\widehat{\epsilon}_{G,\flat,2,\alpha}\widehat{\epsilon}_{H,\flat,2,\alpha}] \quad \text{in } \mathrm{H}^1(W_F, \widehat{T}(k)). \tag{A.2.2}$$

Set $G^{-1} := T$ and fix a root $\alpha \in \Phi((G^{j+1}/G^j)_{\overline{F}}, T_{\overline{F}})_{\mathrm{sym,ram}}$. By Lemma 3.2.1, we have $\alpha \in \Phi(H_{\overline{F}}, T_{\overline{F}})$. For $w \in W_F$, we have

$$Q_{\chi''_H,\overline{\alpha}}(w) Q_{\chi''_G,\overline{\alpha}}(w)^{-1} = \prod_{i=1}^{m} (w_i^{-1}\widehat{\alpha}^\vee)((\chi''_{H,\alpha}\cdot(\chi''_{G,\alpha})^{-1})(\mathrm{rec}_{F_\alpha}(v_0(u_i(w))))).$$

Let $\alpha_{G,j}$ (resp. $\alpha_{H,j}$) be the restriction of α to Z_{G^j} (resp. Z_{H^j}). Note that $\chi''_{G,\alpha}$ is a tamely ramified character of $F_\alpha^\times$ whose restriction to $\mathcal{O}_{F_\alpha}^\times$ is $\mathrm{sgn}_{\mathbf{F}^\times_{\alpha_{G,j}}} \circ \mathrm{Nm}_{F_\alpha/F_{\alpha_{G,j}}}$, and whose restriction to $F^\times_{\pm\alpha}$ is $\kappa_{\alpha_{G,j}} \circ \mathrm{Nm}_{F_{\pm\alpha}/F_{\pm\alpha_{G,j}}}$. By the final paragraph of the proof of [FKS23, Lemma 4.2.7] (the statement of which assumes, but the proof of which does not really use, that p does not divide the order of any bond in the Dynkin diagram), replacing $\ell_G(\alpha^\vee)$ from *loc. cit.* by $\ell_{G,p'}(\alpha^\vee)$, we have

$$\chi''_{G,\alpha}(\ell_{G,p'}(\alpha^\vee)a_\alpha) = (-1)^{f_\alpha+1}\cdot \mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0)\cdot \kappa_\alpha(-1)^{(e(\alpha/\alpha_{G,j})-1)/2}. \tag{A.2.3}$$

These properties uniquely characterize $\chi''_{G,\alpha}$; there is a similar characterization of $\chi''_{H,\alpha}$. Note that the definition of a_α does not change when one passes from G to H, justifying the notation.

We have

$$\mathrm{sgn}_{\mathbf{F}^\times_{\alpha_{G,j}}} \circ \mathrm{Nm}_{F_\alpha/F_{\alpha_{G,j}}} = \mathrm{sgn}^{e(\alpha/\alpha_{G,j})}_{\mathbf{F}^\times_\alpha} = \mathrm{sgn}_{\mathbf{F}^\times_\alpha} \quad \text{on } \mathcal{O}^\times_{F_\alpha},$$

because $e(\alpha/\alpha_{G,j})$ is odd by [FKS23, Lemma 5.6.5]. The same calculation applies to H, so the quotient

$$\zeta_\alpha := \chi''_{H,\alpha}(\chi''_{G,\alpha})^{-1}$$

is unramified. Moreover, both $\chi''_{H,\alpha}$ and $\chi''_{G,\alpha}$ restrict to κ_α on $F^\times_{\pm\alpha}$, so $\zeta_\alpha|_{F^\times_{\pm\alpha}} = 1$. Applying (A.2.3) to G and H, we see that ζ_α is the unique unramified character with these properties and satisfying

$$\zeta_\alpha(a_\alpha) = \mathrm{sgn}_{\mathbf{F}^\times_\alpha}(\ell_{G,p'}(\alpha^\vee)\ell_{H,p'}(\alpha^\vee))\cdot \kappa_\alpha(-1)^{(e(\alpha/\alpha_{G,j})-1)/2+(e(\alpha/\alpha_{H,j})-1)/2}. \tag{A.2.4}$$

Because ζ_α is trivial on $F^\times_{\pm\alpha}$, it factors uniquely through $t \mapsto tv_1(t)^{-1}$, giving a character $\zeta^1_\alpha\colon F^1_\alpha \to \{\pm 1\}$. Since a_α has odd valuation (with respect to the normalized valuation on F_α), we have $a_\alpha v_1(a_\alpha)^{-1} \in -1+\mathfrak{p}_\alpha$, and (A.2.4) therefore yields, directly from the definitions,

$$\zeta^1_\alpha \circ \alpha = \epsilon_{G,\flat,1,\alpha}\epsilon_{H,\flat,1,\alpha}\epsilon_{G,\flat,2,\alpha}\epsilon_{H,\flat,2,\alpha}$$

on $T(F)$, or on $T(F)_{\mathrm{b}}$ in the inertial case. Lemma 4.4.2 identifies the dual cohomology class of the left side with $[Q_{\chi''_H,\overline{\alpha}} Q^{-1}_{\chi''_G,\overline{\alpha}}]$, proving (A.2.2) and hence the lemma. ☐

A.3. **Rootwise formulation of the FKS sign character.** Recall the notation $\overline{T}$ from §3. There is another useful way of organizing $\epsilon^{G/M}_x|_{\overline{T}(\mathbf{F})}$ as a product of characters: namely, one can write

$$\epsilon^{G/M}_x|_{\overline{T}(\mathbf{F})} = \prod_{\alpha\in\Phi((G/M)_{\overline{F}},T_{\overline{F}})/\Sigma} \epsilon^{G/M}_{x,\alpha},$$

where $\epsilon^{G/M}_{x,\alpha}$ is a $\{\pm1\}$-valued character of $\overline{T}(\mathbf{F})$, defined as follows. If α is asymmetric, then we define

$$\epsilon^{G/M}_{x,\alpha}(\gamma) = \begin{cases} \mathrm{sgn}_{\mathbf{F}^\times_\alpha}(\alpha(\gamma)) & \begin{array}{l}\text{if } \frac{r}{2} \in \mathrm{ord}_x(\alpha) \text{ xor} \\ \quad \big(\alpha_M \in \Phi((G/M)_{\overline{F}}, (Z_M)_{\overline{F}})_{\mathrm{sym,ram}} \\ \qquad \text{and } 2 \nmid e(\alpha/\alpha_M)\big),\end{array} \\ 1 & \text{otherwise.}\end{cases}$$

If α is symmetric unramified, then we define

$$\epsilon^{G/M}_{x,\alpha}(\gamma) = \begin{cases} \mathrm{sgn}_{\mathbf{F}^1_\alpha}(\alpha(\gamma)) & \begin{array}{l}\text{if } \frac{r}{2} \in \mathrm{ord}_x(\alpha) \text{ xor} \\ \quad \big(\alpha_M \in \Phi((G/M)_{\overline{F}}, (Z_M)_{\overline{F}})_{\mathrm{sym,ram}} \\ \qquad \text{and } 2 \nmid e(\alpha/\alpha_M)\big),\end{array} \\ 1 & \text{otherwise.}\end{cases}$$

Finally, if α is symmetric ramified, then we define

$$\epsilon^{G/M}_{x,\alpha}(\gamma) = \begin{cases} f_{(G,T)}(\alpha)(-1)^{[\mathbf{F}_\alpha\colon \mathbf{F}]+1} & \\ \quad \cdot \operatorname{sgn}_{\mathbf{F}_\alpha^\times}\left(e_\alpha \ell_{p'}(\alpha^\vee)(-1)^{\frac{e(\alpha/\alpha_M)-1}{2}}\right) & \text{if } \alpha(\gamma) \in -1+\mathfrak{p}_\alpha, \\ 1 & \text{otherwise.} \end{cases}$$

We will use this notation freely in the next two subsections.

A.4. **Proof of Lemma 5.2.1.**

Proof. By definition of the sign characters, we may assume that G is of adjoint type and thus $T(E)_{[x]} = T(E)_{\mathrm{b}}$. Filtering the extension E/F by finite extensions of prime degree, we may assume that E/F is cyclic of odd prime degree ℓ (possibly equal to p). For each Σ-orbit in $\Phi((G/M)_{\overline{F}}, T_{\overline{F}})$, choose a Γ-orbit ω contained in it. Then either

(1) ω is a single $\mathrm{Gal}(\overline{F}/E)$-orbit, or
(2) $\omega = \bigcup_{i=0}^{\ell-1} \omega_i$ for pairwise distinct $\mathrm{Gal}(\overline{F}/E)$-orbits $\omega_0, \dots, \omega_{\ell-1} \subset \omega$.

In case (1), $E \cap F_\alpha = F$ and $[E_\alpha : F_\alpha] = \ell$, while in case (2), $E \subset F_\alpha$ and $E_\alpha = F_\alpha$. Fix some such ω and fix $\alpha \in \omega$. In case (2), fix $\beta_i \in \omega_i$ for $0 \le i < \ell$. We will show

$$\epsilon^{G/M}_{x,\alpha} \circ \mathrm{Nm}_{E/F}|_{T(E)_{\mathrm{b}}} = \begin{cases} \epsilon^{G_E/M_E}_{x,\alpha} & \text{in case (1),} \\ \prod_{i=0}^{\ell-1} \epsilon^{G_E/M_E}_{x,\beta_i} & \text{in case (2).} \end{cases} \tag{A.4.1}$$

We separate further into cases according to whether α is asymmetric, symmetric unramified, or symmetric ramified. Since E/F is unramified of odd degree, it follows that α is asymmetric (resp. symmetric unramified, resp. symmetric ramified) for the action of $\mathrm{Gal}(\overline{F}/F)$ if and only if it is the same for the action of $\mathrm{Gal}(\overline{F}/E)$. In case (2), each such condition for α is equivalent to the corresponding condition for each β_i. Completely similar remarks apply to α_M in place of α.

Suppose first that α is asymmetric. Note that the good element X remains good over E, so the real number r remains the same. Since E/F is unramified, we have $\mathrm{ord}_x(\alpha)_F = \mathrm{ord}_x(\alpha)_E$. Moreover,

$$e(\alpha/\alpha_M)_F \equiv e(\alpha/\alpha_M)_E \pmod 2.$$

Thus by definition, in case (1) above it suffices to show that for $t \in T(E)_{\mathrm{b}}$, we have

$$\operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}(\overline{\alpha(t)}) = \operatorname{sgn}_{\mathbf{F}_\alpha^\times}(\overline{\alpha(\mathrm{Nm}_{E/F}(t))}),$$

where the overline denotes reduction modulo the maximal ideal. Note that $\overline{\alpha(\mathrm{Nm}_{E/F}(t))} = \alpha(\mathrm{Nm}_{\mathbf{F}_E/\mathbf{F}}(\bar{t}))$ since E/F is unramified. Since $[\mathbf{F}_{E,\alpha} \colon \mathbf{F}_\alpha]$ is odd, we have $\operatorname{sgn}_{\mathbf{F}_\alpha^\times} = \operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}|_{\mathbf{F}_\alpha^\times}$. If γ is a generator for $\mathrm{Gal}(\mathbf{F}_{E,\alpha}/\mathbf{F}_\alpha)$, then for all $\bar{t} \in \overline{T}(\mathbf{F}_E)$ we have

$$\begin{aligned} \operatorname{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(\mathrm{Nm}_{\mathbf{F}_E/\mathbf{F}}(\bar{t}))) = \operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}(\alpha(\mathrm{Nm}_{\mathbf{F}_E/\mathbf{F}}(\bar{t}))) &= \prod_{i=0}^{\ell-1} \operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}(\gamma^i(\alpha(\bar{t}))) \\ &= \operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}(\alpha(\bar{t}))^\ell \\ &= \operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}(\alpha(\bar{t})), \end{aligned}$$

where the second equality follows from the fact that $\mathrm{Gal}(\mathbf{F}_{E,\alpha}/\mathbf{F}_\alpha)$ stabilizes α and the third follows from the fact that $\operatorname{sgn}_{\mathbf{F}_{E,\alpha}^\times}$ is $\mathrm{Gal}(\mathbf{F}_{E,\alpha}/\mathbf{F}_\alpha)$-invariant. This proves (A.4.1) in case (1) when α is asymmetric.

Using the same reasoning as above, in case (2) (still assuming that α is asymmetric) we reduce to showing that for $\bar{t} \in \overline{T}(\mathbf{F}_E)$ we have

$$\prod_{i=0}^{\ell-1} \operatorname{sgn}_{\mathbf{F}_{E,\beta_i}^\times}(\beta_i(\bar{t})) = \operatorname{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(\mathrm{Nm}_{\mathbf{F}_E/\mathbf{F}}(\bar{t}))).$$

In this case, we have $E \subset F_\alpha$, so $\mathbf{F}_{E,\beta_i} = \mathbf{F}_\alpha$ for $0 \leq i < \ell$. Fix $\bar{t} \in \overline{T}(\mathbf{F}_E)$. Let $\sigma_i \in \mathrm{Gal}(\overline{F}/F)$ be such that $\sigma_i^{-1}\alpha = \beta_i$ for all i, and note

$$\prod_{i=0}^{\ell-1} \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\beta_i(\bar{t})) = \prod_{i=0}^{\ell-1} \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\sigma_i^{-1}(\alpha(\sigma_i \bar{t}))) = \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\alpha(\mathrm{Nm}_{\mathbf{F}_E/\mathbf{F}}(\bar{t}))),$$

and (A.4.1) holds in general when α is asymmetric.

The case that α is symmetric unramified is entirely similar to the previous case; in fact, the previous two paragraphs may be read verbatim in this case if one replaces each instance of $\mathbf{F}_\alpha^\times$ (resp. $\mathbf{F}_{E,\alpha}^\times$, resp. $\mathbf{F}_{E,\beta_i}^\times$) by $\mathbf{F}_\alpha^1$ (resp. $\mathbf{F}_{E,\alpha}^1$, resp. $\mathbf{F}_{E,\beta_i}^1$).

Finally, suppose that α is symmetric ramified. Unramified base change preserves e_α, $e(\alpha/\alpha_M)$, and the element c_α from (3.2.4) defining $f_{(G,T)}(\alpha)$. Moreover,

$$\kappa_{E,\alpha} = \kappa_{F,\alpha} \circ \mathrm{Nm}_{E_{\pm\alpha}/F_{\pm\alpha}},$$

and $[E_{\pm\alpha} : F_{\pm\alpha}]$ is odd, so the $f_{(G,T)}$-factors in (3.2.5) agree. The quadratic characters in (3.2.6) agree for the same reason, while the factors in (3.2.7) agree because $e(\alpha/\alpha_M)$ is unchanged upon passage from F to E. Since ℓ is odd, the exponents occurring in (3.2.6) have the same parity in both cases. Since the elements α and β_i take values in $\{\pm 1\}$ on $\overline{T}(\mathbf{F}_E)$, by definition of $\epsilon_{x,\alpha}^{G/M}$ it suffices to show that for each $\bar{t} \in \overline{T}(\mathbf{F}_E)$ we have

$$\alpha(\mathrm{Nm}_{\mathbf{F}_E/\mathbf{F}}(\bar{t})) = \begin{cases} \alpha(\bar{t}) & \text{in case (1),} \\ \prod_{i=0}^{\ell-1} \beta_i(\bar{t}) & \text{in case (2).} \end{cases}$$

In both cases, one shows this via exactly the same calculations as before, using that E/F is of odd degree. □

A.5. **Proof of Proposition 5.2.5.**

Proof. We will assume that E/F is totally ramified and ℓ is odd. We may and do assume that G is of adjoint type. By [Kal21a, Lemma 6.4], we may pass to an inner twist of G to assume that G and each of the G^i are quasi-split: indeed, $G^0(F)_{[x]}/G^0(F)_{x,0+}$ is unaffected by such twisting since $G^0 = T$ is a torus. We will only prove the claim when T is elliptic, in which case $T(F) = T(F)_{[x]}$; the general case is obtained by replacing all cocycles in the argument below by their restrictions to I_F. Write ${}^L\theta(w) = (\widehat{\theta}(w), w)$ for a 1-cocycle $\widehat{\theta}\colon W_F \to \widehat{T}(k)$. By definition (4.3.2), our claim reduces to the following equality of cohomology classes:

$$[Q_{\chi_F''} Q_{\chi_E''}^{-1}] = [\widehat{\epsilon_F \epsilon_{F,\sharp,x}}][\widehat{\epsilon_E \epsilon_{E,\sharp,x}}] \quad \text{in } \mathrm{H}^1(W_E, \widehat{T}(k)). \tag{A.5.1}$$

Indeed, since $[E : F]$ is odd, the character s of §4.3.1 does not change upon passage from F to E. Let $\Sigma_F = \mathrm{Gal}(\overline{F}/F) \times \{\pm 1\}$ and similarly for Σ_E. By definition of $Q_{\chi_F''}$ and $Q_{\chi_E''}$, it suffices to show that for each $\alpha \in \Phi((G^{j+1}/G^j)_{\overline{F}}, T_{\overline{F}})$ one has the following equality of cohomology classes:

$$\left[Q_{\chi_{F,\alpha}''} \cdot \prod_{\beta \in \Sigma_F \cdot \alpha/\Sigma_E} Q_{\chi_{E,\beta}''}^{-1} \right] = \left[\widehat{\epsilon_{x,\alpha}^{G^{j+1}/G^j} \epsilon_{\sharp,x,\alpha}^{G^{j+1}/G^j}} \right] \cdot \prod_{\beta \in \Sigma_F \cdot \alpha/\Sigma_E} \left[\widehat{\epsilon_{x,\beta}^{G_E^{j+1}/G_E^j} \epsilon_{\sharp,x,\beta}^{G_E^{j+1}/G_E^j}} \right] \tag{A.5.2}$$

in $\mathrm{H}^1(W_E, \widehat{T}(k))$.

Fix a root $\alpha \in \Phi((G^{j+1}/G^j)_{\overline{F}}, T_{\overline{F}})$, and let $\alpha_j = \alpha|_{Z_{\overline{F}}^j}$. Since ℓ is odd, the root α is asymmetric (resp. symmetric unramified, resp. symmetric ramified) for the action of $\mathrm{Gal}(\overline{F}/F)$ if and only if it is the same for the action of $\mathrm{Gal}(\overline{F}/E)$. The same equivalence holds for α_j. By precisely the same argument as in the proof of Lemma 5.2.1, we have

$$(\epsilon_{x,\alpha}^{G^{j+1}/G^j} \epsilon_{\sharp,x,\alpha}^{G^{j+1}/G^j}) \circ \mathrm{Nm}_{E/F} = \prod_{\beta \in \Sigma_F \cdot \alpha/\Sigma_E} \epsilon_{x,\beta}^{G_E^{j+1}/G_E^j} \epsilon_{\sharp,x,\beta}^{G_E^{j+1}/G_E^j} \tag{A.5.3}$$

whenever α is asymmetric or symmetric unramified. Moreover, if α is symmetric ramified, $\bar{t} \in \overline{T}(\mathbf{F}_E)$, and m is the order of $\Sigma_F \cdot \alpha/\Sigma_E$, then the formulas (3.2.5)–(3.2.7) give

$$\begin{aligned} &((\epsilon_{x,\alpha}^{G^{j+1}/G^j} \circ \mathrm{Nm}_{E/F}) \cdot \prod_{\beta \in \Sigma_F \cdot \alpha/\Sigma_E} \epsilon_{x,\beta}^{G_E^{j+1}/G_E^j})(\bar{t}) \\ &= \prod_{\substack{\beta \in \Sigma_F \cdot \alpha/\Sigma_E \\ \beta(\bar{t}) = -1}} \mathrm{sgn}_{\mathbf{F}_\beta^\times} \left(e_{\beta,F} e_{\beta,E} (-1)^{\frac{e(\beta/\beta_j)_E - e(\beta/\beta_j)_F}{2}} \right). \end{aligned} \tag{A.5.4}$$

Here $e_{\alpha,F}$ (resp. $e_{\alpha,E}$) is the ramification degree of F_α/F (resp. E_α/E), and similarly for $e(\alpha/\alpha_j)_F$ and $e(\alpha/\alpha_j)_E$. If α_j is asymmetric or symmetric unramified, then $\chi''_{F,\alpha_j} \circ \mathrm{Nm}_{E_{\alpha_j}/F_{\alpha_j}} = \chi''_{E,\alpha_j}$ by definition since $[E_{\alpha_j} : F_{\alpha_j}]$ is odd, and thus

$$\chi''_{F,\alpha} \circ \mathrm{Nm}_{E_\alpha/F_\alpha} = \chi''_{E,\alpha}. \tag{A.5.5}$$

Note that the element a_α does not change upon passage from F to E, so if α_j is symmetric ramified then by (A.2.3) we have

$$\begin{aligned} \chi''_{F,\alpha}(\ell_{p'}(\alpha^\vee)a_\alpha) &= (-1)^{f_{F,\alpha}+1} \mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0)\, \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(-1)^{\frac{e(\alpha/\alpha_j)_F - 1}{2}}, \\ \chi''_{E,\alpha}(\ell_{p'}(\alpha^\vee)a_\alpha) &= (-1)^{f_{E,\alpha}+1} \mathfrak{G}_{\mathbf{F}_{E,\alpha}}(\psi_E^0)\, \mathrm{sgn}_{\mathbf{F}_{E,\alpha}^\times}(-1)^{\frac{e(\alpha/\alpha_j)_E - 1}{2}}. \end{aligned} \tag{A.5.6}$$

There are now six cases, namely cases (1)–(5) in the proof of Lemma 5.1.2, and an additional case that α (and hence also α_j) is symmetric ramified. In cases (1), (2), and (4), the displayed equations (A.5.3) and (A.5.5) immediately imply (A.5.2) by definition (4.3.1). Cases (3) and (5) proceed in almost precisely the same way as in the proof of Lemma 5.1.2, so we omit them.

It remains to understand the case that α (and hence also α_j) is symmetric ramified. Choosing the representatives in (4.3.1) compatibly with the Σ_E-orbits, Lemma 4.4.2 and norm functoriality identify the left side of (A.5.2) with the class dual to $\prod_{\beta \in \Sigma_F \cdot \alpha/\Sigma_E} (\xi_\beta^1 \circ \beta)$, where $\xi_\beta^1 \colon E_\beta^1 \to k^\times$ is induced by the unramified character

$$(\chi''_{F,\beta} \circ \mathrm{Nm}_{E_\beta/F_\beta}) \cdot (\chi''_{E,\beta})^{-1} \colon E_\beta^\times \to k^\times \tag{A.5.7}$$

through $c \mapsto c v_1(c)^{-1}$. The character ξ_β^1 is trivial on $E_\beta^1 \cap (1 + \mathfrak{p}_{E,\beta})$. Since $\ell_{p'}(\beta^\vee)a_\beta$ maps to -1, equation (A.5.4) reduces the claim to the following identity for each orbit, written for $\beta = \alpha$:

$$\mathrm{sgn}_{\mathbf{F}_\alpha^\times}\big(e_{\alpha,F} e_{\alpha,E} (-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}}\big) = \big((\chi''_{F,\alpha} \circ \mathrm{Nm}_{E_\alpha/F_\alpha}) \cdot (\chi''_{E,\alpha})^{-1}\big)(\ell_{p'}(\alpha^\vee)a_\alpha). \tag{A.5.8}$$

First, we compute the right side of (A.5.8). Since ℓ is odd we have $(-1)^{f_{F,\alpha}+1} = (-1)^{f_{E,\alpha}+1}$. We compute

$$\mathfrak{G}_{\mathbf{F}_{E,\alpha}}(\psi_E^0) = \begin{cases} \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell) \mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0) & \text{if } \mathbf{F}_{E,\alpha} = \mathbf{F}_\alpha, \\ \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell) \mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0)^\ell & \text{if } \mathbf{F}_{E,\alpha} \neq \mathbf{F}_\alpha, \end{cases}$$

where the first case comes from the fact that $\psi_E^0(x) = \psi^0(x)^\ell$ by definition, and the second case comes from the Hasse–Davenport relation. Since $[E_\alpha : F_\alpha] = \ell/m$, in the case $\mathbf{F}_{E,\alpha} = \mathbf{F}_\alpha$ we have

$$\begin{aligned} &\big((\chi''_{F,\alpha} \circ \mathrm{Nm}_{E_\alpha/F_\alpha}) \cdot (\chi''_{E,\alpha})^{-1}\big)(\ell_{p'}(\alpha^\vee)a_\alpha) = \\ &= \mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0)^{\ell/m}\, \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(-1)^{\frac{e(\alpha/\alpha_j)_F - 1}{2}} \mathfrak{G}_{\mathbf{F}_{E,\alpha}}(\psi_E^0)^{-1}\, \mathrm{sgn}_{\mathbf{F}_{E,\alpha}^\times}(-1)^{\frac{e(\alpha/\alpha_j)_E - 1}{2}} \\ &= \mathrm{sgn}_{\mathbf{F}_\alpha^\times}\big(\ell \cdot (-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F + \ell/m - 1}{2}}\big) \end{aligned} \tag{A.5.9}$$

since $\mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0)^2 = \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(-1)$. If instead $\mathbf{F}_{E,\alpha} \neq \mathbf{F}_\alpha$, then $m = 1$ and thus similarly

$$\begin{aligned} &\big((\chi''_{F,\alpha} \circ \mathrm{Nm}_{E_\alpha/F_\alpha}) \cdot (\chi''_{E,\alpha})^{-1}\big)(\ell_{p'}(\alpha^\vee)a_\alpha) = \\ &= \mathfrak{G}_{\mathbf{F}_\alpha}(\psi^0)^{\ell}\, \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(-1)^{\frac{e(\alpha/\alpha_j)_F - 1}{2}} \mathfrak{G}_{\mathbf{F}_{E,\alpha}}(\psi_E^0)^{-1}\, \mathrm{sgn}_{\mathbf{F}_{E,\alpha}^\times}(-1)^{\frac{e(\alpha/\alpha_j)_E - 1}{2}} \\ &= \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell \cdot (-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}}) \end{aligned} \tag{A.5.10}$$

Since E/F is a *Galois* extension of degree ℓ, it follows that $|\mathbf{F}_\alpha| \equiv 1 \pmod{\ell}$ and thus

$$\mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell \cdot (-1)^{\frac{\ell-1}{2}}) = 1 \tag{A.5.11}$$

by quadratic reciprocity.[27] Thus by (A.5.9) and (A.5.10), we have

$$\begin{aligned}\big((\chi''_{F,\alpha} \circ \mathrm{Nm}_{E_\alpha/F_\alpha}) \cdot (\chi''_{E,\alpha})^{-1}\big)(\ell_{p'}(\alpha^\vee)a_\alpha) = \\ = \begin{cases} \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}} & \text{if } E_\alpha/F_\alpha \text{ is ramified,} \\ \mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell \cdot (-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}}) & \text{otherwise.}\end{cases}\end{aligned} \tag{A.5.12}$$

Next, we compute the left side of (A.5.8). There are three cases:

(A) $E \subset F_\alpha$,
(B) E_α/F_α is unramified of degree ℓ,
(C) E_α/F_α is ramified of degree ℓ.

In case (A), we have $e_{\alpha,F} = \ell e_{\alpha,E}$ and $m = \ell$. Thus the left side of (A.5.8) is

$$\mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell \cdot (-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}}),$$

which agrees with (A.5.12) since $E_\alpha = F_\alpha$.

In case (B), we have $e_{\alpha,F} = \ell e_{\alpha,E}$ and $m = 1$. Thus the left side of (A.5.8) is

$$\mathrm{sgn}_{\mathbf{F}_\alpha^\times}(\ell \cdot (-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}}),$$

which again agrees with (A.5.12) since $\mathbf{F}_{E,\alpha} \neq \mathbf{F}_\alpha$.

Finally, in case (C) we have $e_{\alpha,F} = e_{\alpha,E}$ and $m = 1$. Thus the left side of (A.5.8) is

$$\mathrm{sgn}_{\mathbf{F}_\alpha^\times}(-1)^{\frac{e(\alpha/\alpha_j)_E - e(\alpha/\alpha_j)_F}{2}},$$

which again agrees with (A.5.12) since E_α/F_α is ramified of degree ℓ. □

References

[AL10] Jeffrey D. Adler and Joshua M. Lansky, *Depth-zero base change for ramified* $\mathrm{U}(2,1)$, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5569–5599. MR 2657691

[AS09] Jeffrey D. Adler and Loren Spice, *Supercuspidal characters of reductive p-adic groups*, Amer. J. Math. **131** (2009), no. 4, 1137–1210. MR 2543925

[Art13] James Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[ABPS16] Anne-Marie Aubert, Paul Baum, Roger Plymen, and Maarten Solleveld, *The local Langlands correspondence for inner forms of* SL_n, Res. Math. Sci. **3** (2016), Paper No. 32.

[AX22] Anne-Marie Aubert and Yujie Xu, *The explicit local Langlands correspondence for* G_2, 2022, Revised April 2023.

[BMR05] Michael Bate, Benjamin Martin, and Gerhard Röhrle, *A geometric approach to complete reducibility*, Invent. Math. **161** (2005), no. 1, 177–218. MR 2178661

[BMR08] ———, *Complete reducibility and commuting subgroups*, J. Reine Angew. Math. **621** (2008), 213–235. MR 2431255

[BZCHN24] David Ben-Zvi, Harrison Chen, David Helm, and David Nadler, *Coherent Springer theory and the categorical Deligne–Langlands correspondence*, Invent. Math. **235** (2024), 255–344.

[BMHN24] Alexander Bertoloni Meli, Linus Hamann, and Kieu Hieu Nguyen, *Compatibility of the Fargues-Scholze correspondence for unitary groups*, Math. Ann. **390** (2024), no. 3, 4729–4787. MR 4803488

[BPHT25] Raphaël Beuzart-Plessis, Michael Harris, and Jack Thorne, *Inductive construction of supercuspidal l-packets*, 2025.

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012

[BM25] Adèle Bourgeois and Paul Mezo, *A functoriality property for supercusipidal L-packets*, Pacific J. Math. **339** (2025), no. 1, 73–131. MR 4954965

[Car72] R. W. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. **25** (1972), 1–59. MR 318337

[CO25] Charlotte Chan and Masao Oi, *Geometric L-packets of Howe-unramified toral supercuspidal representations*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 4, 1465–1526. MR 4875610

[CZ21] Rui Chen and Jialiang Zou, *Local Langlands correspondence for even orthogonal groups via theta lifts*, Selecta Math. (N.S.) **27** (2021), no. 5, Paper No. 88, 71.

[Cho17] Kwangho Choiy, *The local Langlands conjecture for the p-adic inner form of* Sp_4, Int. Math. Res. Not. IMRN (2017), no. 6, 1830–1889.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad, *Pseudo-reductive groups*, second ed., New Mathematical Monographs, vol. 26, Cambridge University Press, Cambridge, 2015. MR 3362817

[Cot26] Sean Cotner, *Modular functoriality for finite groups*, 2026.

[CF26b] Sean Cotner and Tony Feng, *Local Langlands functoriality for Yu's supercuspidals II: Fargues–Scholze's parametrization*, 2026.

[27] This can be proven by splitting into cases according to the classes of $|\mathbf{F}_\alpha|$ and ℓ modulo 4.

[DvHKZ26] Patrick Daniels, Pol van Hoften, Dongryul Kim, and Mingjia Zhang, *Igusa Stacks and the Cohomology of Shimura Varieties II*, 2026.

[DL26] Jean-François Dat and Thomas Lanard, *Depth zero representations over* $\mathbb{Z}[1/p]$, Algebra Number Theory **20** (2026), no. 7, 1451–1478. MR 5118155

[DR09] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901. MR 2480618

[DS18] Stephen DeBacker and Loren Spice, *Stability of character sums for positive-depth, supercuspidal representations*, J. Reine Angew. Math. **742** (2018), 47–78. MR 3849622

[SGA3$_{\mathrm{II}}$] Michel Demazure and Alexandre Grothendieck (eds.), *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Lecture Notes in Mathematics, vol. Vol. 152, Springer-Verlag, Berlin-New York, 1970, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Dirigé par M. Demazure et A. Grothendieck. MR 274459

[DM94] François Digne and Jean Michel, *Groupes réductifs non connexes*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 3, 345–406. MR 1272294

[Ete23] Arnaud Eteve, *Faisceaux monodromiques, théorie de Deligne–Lusztig, et cohomologie des champs de chtoucas en profondeur 0*, 2023.

[FS21] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, 2021, To appear in Asterisque.

[Fen24] Tony Feng, *Modular functoriality in the Local Langlands Correspondence*, 2024.

[Fin21a] Jessica Fintzen, *On the construction of tame supercuspidal representations*, Compos. Math. **157** (2021), no. 12, 2733–2746. MR 4357723

[Fin21b] ———, *Types for tame p-adic groups*, Ann. of Math. (2) **193** (2021), no. 1, 303–346. MR 4199732

[Fin22] ———, *Tame cuspidal representations in non-defining characteristics*, Michigan Math. J. **72** (2022), 331–342. MR 4460255

[FKS23] Jessica Fintzen, Tasho Kaletha, and Loren Spice, *A twisted Yu construction, Harish-Chandra characters, and endoscopy*, Duke Math. J. **172** (2023), no. 12, 2241–2301. MR 4654051

[Fu26] Chenji Fu, *Depth-zero local Langlands via shtukas in mixed characteristic*, 2026.

[GHSBP24] Wee Teck Gan, Michael Harris, Will Sawin, and Raphaël Beuzart-Plessis, *Local parameters of supercuspidal representations*, Forum Math. Pi **12** (2024), Paper No. e13, 41. MR 4794807

[GL18] Wee Teck Gan and Luis Lomelí, *Globalization of supercuspidal representations over function fields and applications*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 11, 2813–2858.

[GS23] Wee Teck Gan and Gordan Savin, *The local Langlands conjecture for* G_2, Forum Math. Pi **11** (2023), Paper No. e28, 42.

[GT10] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for* Sp_4, Int. Math. Res. Not. IMRN (2010), no. 15, 2987–3038.

[GT11] ———, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2) **173** (2011), no. 3, 1841–1882.

[GT14] Wee Teck Gan and Welly Tantono, *The local Langlands conjecture for* $\mathrm{GSp}(4)$*, II: the case of inner forms*, Amer. J. Math. **136** (2014), no. 3, 761–805.

[Gan15] Radhika Ganapathy, *The local Langlands correspondence for* GSp_4 *over local function fields*, Amer. J. Math. **137** (2015), no. 6, 1441–1534.

[GV17] Radhika Ganapathy and Sandeep Varma, *On the local Langlands correspondence for split classical groups over local function fields*, J. Inst. Math. Jussieu **16** (2017), no. 5, 987–1074.

[GH91] Meinolf Geck and Gerhard Hiss, *Basic sets of Brauer characters of finite groups of Lie type*, J. Reine Angew. Math. **418** (1991), 173–188. MR 1111205

[GKP00] Meinolf Geck, Sungsoon Kim, and Götz Pfeiffer, *Minimal length elements in twisted conjugacy classes of finite Coxeter groups*, J. Algebra **229** (2000), no. 2, 570–600. MR 1769289

[GM97] Meinolf Geck and Jean Michel, *"Good" elements of finite Coxeter groups and representations of Iwahori-Hecke algebras*, Proc. London Math. Soc. (3) **74** (1997), no. 2, 275–305. MR 1425324

[GP00] Meinolf Geck and Götz Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press, Oxford University Press, New York, 2000. MR 1778802

[GL18] Alain Genestier and Vincent Lafforgue, *Chtoucas restreints pour les groupes réductifs et paramétrisation de langlands locale*, 2018.

[Gér77] Paul Gérardin, *Weil representations associated to finite fields*, J. Algebra **46** (1977), no. 1, 54–101. MR 460477

[Gla68] George Glauberman, *Correspondences of characters for relatively prime operator groups*, Canadian J. Math. **20** (1968), 1465–1488. MR 232866

[GHI+26] Ian Gleason, Linus Hamann, Alexander B. Ivanov, João Lourenço, and Konrad Zou, *On the Schematic and Analytic Constructions of the Local Langlands Category*, 2026.

[HM08] Jeffrey Hakim and Fiona Murnaghan, *Distinguished tame supercuspidal representations*, Int. Math. Res. Pap. IMRP (2008), no. 2, Art. ID rpn005, 166. MR 2431732

[Ham25] Linus Hamann, *Compatibility of the Fargues-Scholze and Gan-Takeda local Langlands*, Compos. Math. **161** (2025), no. 9, 2202–2271. MR 4978375

[Han26] David Hansen, *Excursion operators and the stable Bernstein center*, Forum Math. Pi **14** (2026), Paper No. e10, 14. MR 5042225

[HKW22] David Hansen, Tasho Kaletha, and Jared Weinstein, *On the Kottwitz conjecture for local shtuka spaces*, Forum Math. Pi **10** (2022), Paper No. e13, 79. MR 4430954

[HM26] David Hansen and Lucas Mann, *The categorical local langlands conjecture*, 2026.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

[He07] Xuhua He, *Minimal length elements in some double cosets of Coxeter groups*, Adv. Math. **215** (2007), no. 2, 469–503. MR 2355597

[HSX26] David Helm, Maarten Solleveld, and Yujie Xu, *A generic categorical local Langlands correspondence for quasi-split reductive groups*, 2026.

[Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps* p*-adique*, Invent. Math. **139** (2000), no. 2, 439–455.

[HS12] Kaoru Hiraga and Hiroshi Saito, *On* L*-packets for inner forms of* SL_n, Mem. Amer. Math. Soc. **215** (2012), no. 1013, vi+97.

[JS03] Dihua Jiang and David Soudry, *The local converse theorem for* $\mathrm{SO}(2n+1)$ *and applications*, Ann. of Math. (2) **157** (2003), no. 3, 743–806.

[Kal15] Tasho Kaletha, *Epipelagic* L*-packets and rectifying characters*, Invent. Math. **202** (2015), no. 1, 1–89. MR 3402796

[Kal19] ———, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32** (2019), no. 4, 1071–1170. MR 4013740

[Kal21a] Tasho Kaletha, *On* L*-embeddings and double covers of tori over local fields*, 2021, Revised author version.

[Kal21b] ———, *Supercuspidal* L*-packets*, 2021.

[KMSW14] Tasho Kaletha, Alberto Mínguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: inner forms of unitary groups*, 2014.

[KP23] Tasho Kaletha and Gopal Prasad, *Bruhat-Tits theory—a new approach*, New Mathematical Monographs, vol. 44, Cambridge University Press, Cambridge, 2023. MR 4520154

[Kha01] Chandrashekhar Khare, *On the local Langlands correspondence mod* ℓ, J. Number Theory **88** (2001), no. 2, 357–365.

[Kim07] Ju-Lee Kim, *Supercuspidal representations: an exhaustion theorem*, J. Amer. Math. Soc. **20** (2007), no. 2, 273–320. MR 2276772

[Laf18] Vincent Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, J. Amer. Math. Soc. **31** (2018), no. 3, 719–891. MR 3787407

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271. MR 909227

[LRS93] G. Laumon, M. Rapoport, and U. Stuhler, $\mathscr{D}$*-elliptic sheaves and the Langlands correspondence*, Invent. Math. **113** (1993), no. 2, 217–338. MR 1228127

[LH23] Siyan Daniel Li-Huerta, *Local-global compatibility over function fields*, 2023.

[MR18] Colette Mœglin and David Renard, *Sur les paquets d'Arthur des groupes classiques et unitaires non quasi-déployés*, Relative aspects in representation theory, Langlands functoriality and automorphic forms, Lecture Notes in Mathematics, vol. 2221, Springer, Cham, 2018, pp. 341–361.

[Mok15] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248. MR 3338302

[Oi26] Masao Oi, *Twisted endoscopic character relation for toral supercuspidal l-packets of classical groups*, 2026.

[OT21] Masao Oi and Kazuki Tokimoto, *Local Langlands correspondence for regular supercuspidal representations of* $GL(n)$, Int. Math. Res. Not. IMRN (2021), no. 3, 2007–2073. MR 4206603

[Pen26] Hao Peng, *Fargues-Scholze correspondence and endoscopic classification for special orthogonal and unitary groups*, 2026.

[PY02] Gopal Prasad and Jiu-Kang Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. **147** (2002), no. 3, 545–560. MR 1893005

[Ree11] Mark Reeder, *Elliptic centralizers in Weyl groups and their coinvariant representations*, Represent. Theory **15** (2011), 63–111. MR 2765477

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, vol. Vol. 42, Springer-Verlag, New York-Heidelberg, 1977, Translated from the French by Leonard L. Scott. MR 450380

[Spr06] T. A. Springer, *Twisted conjugacy in simply connected groups*, Transform. Groups **11** (2006), no. 3, 539–545. MR 2264465

[Ste75] Robert Steinberg, *Torsion in reductive groups*, Advances in Math. **15** (1975), 63–92. MR 354892

[Tok23] Kazuki Tokimoto, *Local Jacquet-Langlands correspondence for regular supercuspidal representations*, 2023.

[TV16] David Treumann and Akshay Venkatesh, *Functoriality, Smith theory, and the Brauer homomorphism*, Ann. of Math. (2) **183** (2016), no. 1, 177–228. MR 3432583

[Vig01] Marie-France Vignéras, *Correspondance de Langlands semi-simple pour* $\mathrm{GL}(n,F)$ *modulo* $\ell \neq p$, Invent. Math. **144** (2001), no. 1, 177–223. MR 1821157

[XZ19] Liang Xiao and Xinwen Zhu, *On vector-valued twisted conjugation invariant functions on a group*, Representations of reductive groups, Proc. Sympos. Pure Math., vol. 101, Amer. Math. Soc., Providence, RI, 2019, With an appendix by Stephen Donkin, pp. 361–425. MR 3930024

[Yu01] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622. MR 1824988

[Zhu25] Xinwen Zhu, *Tame categorical local Langlands correspondence*, 2025.