Elliptically fibered Calabi-Yau threefolds, Jacobi-Forms, and the topological vertex

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1 Introduction

1.1 Donaldson-Thomas Theory

Let X be a Calabi-Yau threefold, i.e. a smooth threefold over $\mathbb{C}$ with $K_X \cong \mathcal{O}_X$.

Donaldson-Thomas theory is a curve-counting theory, a mathematical analogue of EPS counting in string theory. Although this has its origins in string theory, and has undergone very sophisticated developments, it has now come "full circle" in mathematics, meaning that the mathematical approach is actually quite simple.

The first step is to consider the parameter space for curves:

$$\mathrm{Hilb}^{n,\beta}(X) = \{Z \subset X \mid [Z] = \beta \in H_2(X), n = \chi(\mathcal{O}_Z)\}.$$

So n would be the genus for smooth curves.

Now, the geometric version of counting is taking the Euler characteristic. This naïve thing is almost the right thing to do: we have a Behrend function $\nu\colon \mathrm{Hilb} \to \mathbb{Z}$, and the *Donaldson-Thomas invariant* is a weighted sum of Euler characteristics:

$$DT_{n,\beta}(X) = e(\mathrm{Hilb}^{n,\beta}(X), \nu) = \sum_{k \in \mathbb{Z}} k \cdot e(\nu^{-1}(k)).$$

Today we'll ignore the Behrend function. The amazing thing is that, for somewhat mysterious reasons, there are many contexts in which this doesn't hurt you at all. At then end, if there is time, I'll talk about how to put the Behrend function back in. We define

$$DT_{\beta}(X) = \sum_{n \in \mathbb{Z}} DT_{\beta,n}(X)(-p)^n \in \mathbb{Z}((p)).$$

What makes this so interesting is not so much the individual enumerative geometry answers, but the structure that these partition functions exhibit.

Today we'll assume that there is an *elliptic fibration* $X \to S$.

Slogan. The partition functions $DT_\beta(X)$ are controlled by Jacobi forms.

There is a slew of conjectures and theorems that make this precise in various settings. So instead of trying to formulate a general statement, I want to focus on particular examples and discuss how they generalize.

Example 1.1. The simplest example is that of *trivial fibrations* $X = K3 \times E$ or $X = A \times E$ where A is an abelian surface and E is an elliptic curve.

Non-trivial fibrations: X could be the total space of K_S where $S \to \mathbb{P}^1$ is an elliptic surface.

1.2 Main example

We will focus on the example $X = K3 \times E$. This is nice in many ways (e.g. it is a trivial fibration) but it has some problems. One is that all the invariants $DT_{n,\beta}(X) = 0$. There are two reasons for this:

1. E acts freely on $\mathrm{Hilb}^{n,\beta}(X)$, which implies that $e(\mathrm{Hilb}^{n,\beta}(X))$. Indeed, if an elliptic curve acts and you take the Euler characteristic restricted to the fixed point set, you get 0.

2. (Using that the weighted DT invariant is deformation-invariant) We can deform X sot hat β is not algebraic, so $\mathrm{Hilb}^{n,\beta}(X) = \emptyset$. s

We can solve both of these problems simultaneously by defining a modified DT invariant.

Definition 1.2. Define $\mathbb{D}T_{n,\beta}(X) = e(\mathrm{Hilb}^{n,\beta}(X)/E)$. This is invariant only under deformations leaving β algebraic.

One way to think about this is that there is a deformation in the direction of E, which is dual to the obstruction for deformations in non-algebraic directions, and we are removing both of these at once.

Let's package these into a generating function. We'll need to introduce a little bit of notation. Suppose $\beta_h \in H_2(K3)$ is a primtiive curve class such that $\beta_h^2 = 2h-2$ (so h is the arithmetic genus). Then we get classes $\beta_h + dE \in H_2(X)$. We have two discrete parameters here: d describes the degree of the projection to E, and h is the genus of the projection to K3.

$$\mathbb{D}T_h(X) = \sum_{d,n} \mathbb{D}T_{n,\beta_h+dE}(X)(-p)^n q^{d-1}.$$

We also define

$$\mathbb{D}T(X) = \sum_h \mathbb{D}T_h(X)\widetilde{q}^{h-1}.$$

So p tracks the arithmetic genus, and q tracks the degree of the projection to E, and $\widetilde{q}$ tracks the genus of the projection to K3.

Conjecture 1.3 (Oberdieck-Pandharipande)**.** *$\mathbb{D}T(X) = -1/\chi_{10}$ where χ_{10} is an Igusa cusp form of weight 10, which is a genus 2 Siegel modular form, in terms of the coordinates $p = e^{2\pi i z}, q = e^{2\pi i \tau}, \widetilde{q} = e^{2\pi i \widetilde{z}}$.*

One of the nice things about this Igusa Siegel form is that it admits a Borcherds product

$$\chi_{10} = pq\widetilde{q} \prod_{n,d,h} (1 - p^n q^d \widetilde{q}^h)^{c(4hd-n^2)}$$

Write

$$Z = \sum_{n,d} c(4d - n^2) p^n q^d = -24\wp F^2$$

where

$$\wp = \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_d \sum_{k|d} k(p^k + p^{-k} - 2)q^d$$

and

$$-F^2 = p^{-1}(1-p)^2 \prod_{m=1}^{\infty} (1 - pq^m)^2 (1 - p^{-1}q^m)^2 (1 - q^m)^{-4}.$$

What do these formulas tell us about the geometry?

In this formula, there is a symmetry between q and $\widetilde{q}$, which implies a symmetry between $d \leftrightarrow h$. That's very surprising!

One can ask about $\mathbb{D}T_h(X)$ for a fixed h, which corresponds to expanding $\mathbb{D}T(X)$ in the $\widetilde{q}$ variable. By general theory, these coefficients will be Jacobi forms (of weight -10 and index $h - 1$). One way to think about a Jacobi form is as a meromorphic section of line bundle over the universal elliptic curve (τ is the elliptic curve variable.)

Example 1.4. $\mathbb{D}T_0(X) = \frac{1}{F^2 \Delta}$, $\mathbb{D}T_1(X) = -24\wp/\Delta$, and

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^m)^{24}.$$

Even the first equation (the constant term) encapsulates the KKV equation.

Theorem 1.5 (Bryan)**.** *The conjecture holds for $h = 0$ and $h = 1$.*

The proof is via a new computational technique, develoepd together with M. Kool, which is a mix of motivic and toric methods. In particular, it forms a connection between the topological vertex and these Jacobi forms.

2 Idea of the proof

The Hilbert schemes can be nasty. They can parametrize curves with many irreducible components, non-reduced components (and the "directions" can vary in complicated ways), embedded points, etc. Thus Hilb is very complicated and very singular.

But because we're computing Euler characteristics, we have some tools. For example, they are *motivic*, so we can chop up the space and compute the Euler characteristic on the pieces. Also, if we have a group action then we can use localization to restrict to the fixed point set.

Quick digression on the topological vertex. Suppose that X is a toric CY3 (running example is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$). Then we have a three-dimensional torus action, so

$$\begin{aligned} DT(X) &= \sum e(\mathrm{Hilb}^{n,\beta}(X)^{(\mathbb{C}^*)^3}) q^\beta p^n \\ &= \sum_{\substack{\text{loc. monomial} \\ \text{subschemes Z}}} q^{[Z]} p^{\chi(\mathcal{O}_Z)} \end{aligned}$$

♠♠♠ TONY: [???] There is a picture with two graphs with legs ϕ, ϕ, β, which for some reason implies that this can be written as a sum over partitions of some universal function times a matching monomial

$$= \sum_{\beta \text{ partition}} q^{|\beta|} V_{\beta\phi\phi}(p) V_{\beta\phi\phi}(p) p^{\bullet}.$$

What's going on here? For three partitions α, β, γ we define

$$V_{\alpha\beta\gamma}(p) = \sum p^{|\pi|}$$

where the sum is over 3D partitions with legs α, β, γ. This was computed "explicitly" in terms of Schur functions by Okounkov and collaborators.

Now, in our case $X = K3 \times E$ doesn't have a torus action.

1. First stratify $\mathrm{Hilb}(X)$ into strata that can be written in terms of symmetric products and simpler Hilbert schemes.

2. Find actions of $\mathbb{C}^\times$ on E or individual strata. Then we can restrict to the fixed point set. (This is predicated on a good understanding the *support* of the curves being parametrized.)

3. Iterate the first two steps to reduce to subschemes which are "formally locally monomial" and then use the topological vertex.