BRILL-NOETHER THEORY, II

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This article follows the paper of Griffiths and Harris, "On the variety of special linear systems on a general algebraic curve."

1. Warmup on degenerations

The classic first problem in Schubert calculus is: *how many lines intersect four general lines in $\mathbb{P}^3$?*

First, what does this numerology come from? The space of lines in $\mathbb{P}^3$ is $\mathbb{G}(1,3)$ which has dimension $2 \times 2 = 4$. The locus of lines intersecting a given one in $\mathbb{P}^3$ is a *hypersurface* in $\mathbb{G}(1,3)$(we'll expand more on this later, but there are various easy ways to see it in this case), or in other words the condition that a line intersect a given line is of codimension 1. Therefore, the locus of lines intersecting four general lines in $\mathbb{P}^3$ should be of dimension 0, so one can ask for its degree.

One quick and dirty way to solve such problems is via *degeneration*. Note that by Bertini's theorem, the intersection problem is guaranteed to be transverse for general choices of lines. The idea is that as the fixed line(s) vary in $\mathbb{G}(1,3)$, the answer should vary continuously as long as it is well-defined, i.e. as long as the intersection is transverse. But then we may gain some leverage by picking special configurations of lines.

For instance, suppose that we degenerate two of the lines, say ℓ_1 and ℓ_2, so that they *intersect at a point* p_{12}. Then any line meeting both ℓ_1 and ℓ_2 must either meet them at the same point p_{12}, or at separate points, in which case it lies in the plane Λ_{12} spanned by them.

Similarly, suppose we degenerate the other two lines ℓ_3 and ℓ_4 to intersect at a point p_{34}. Then any line meeting both ℓ_3 and ℓ_4 must pass through p_{34} or lie in the plane Λ_{34} spanned by them (and these conditions are also sufficient).

Now how many lines meet all four of $\ell_1, \ell_2, \ell_3, \ell_4$? There are four cases to check. If the line passes through p_{12} and p_{34}, then it is the unique line spanned by them. If it lies in Λ_{12} and Λ_{34}, then it is the unique line which is their intersection. It *cannot* lie in Λ_{12} and pass through p_{34} as long as p_{34} doesn't lie in Λ_{12}, which can certainly be arranged. Thus we see the answer of $\boxed{2.}$

The tricky thing about degeneration arguments is that when they work, they seem deceptively easy. Often the hard part is to find the *right* degeneration. What would have happened if above we degenerated ℓ_3 to intersect ℓ_2 instead? Then lines meeting ℓ_1, ℓ_2, ℓ_3 would have to meet p_{12} and ℓ_3, or lie in the plane and meet ℓ_3. When we include the condition of meeting ℓ_4, we see that in the first case the lines sweep out a hyperplane, and exactly one line meets ℓ_4 at its point of intersection with this hyperplane. In the second case, there is a unique point of intersection between ℓ_4 and the plane Λ_{12}, so we again see (albeit in a slightly more difficult way) the answer of 2.

What if we had degenerated ℓ_3 to also pass through p_{12}? Well then there would be a *one-dimensional* space of lines through p_{12} and ℓ_4, which is bad news.

By the way, in this case we could have (with some cleverness) just solved the problem directly. I claim that given three skew lines in $\mathbb{P}^3$, there is a *unique* smooth quadric containing them. That's because the space of quadrics in $\mathbb{P}^3$ is 9-dimensional, and vanishing on a line imposes only 3 conditions, so such a quadric exists. On the other hand, the lines (being skew) must be in the *same* ruling. If two such quadrics existed, then the three lines would be contained in their intersection, which should be of type $(2, 2)$.

Now, any line meeting these three lines already has to be contained in the quadric, since it certainly meets the quadric in at least three points. Thus it has to be a line of the other ruling. To meet ℓ_4, it must pass through the intersection points of ℓ_4 with the quadric, and there are exactly two of those.

2. Recap of last time

Last time we were discussing the proof of the Brill-Noether Theorem. Let me recall what was going on. Let $\rho(g, d, r) = g - (r+1)(d - g + r)$. For a smooth projective curve C, we defined $W^r_d(C)$ to be the variety of linear series on C with dimension at least r (a subvariety of $\text{Pic}^d(C)$). We were interested in this space because it describes the existence of maps from C to projective spaces with controlled degree d.

Theorem 2.1. *For a general smooth projective curve C of genus g, we have*

$$\dim W^r_d(C) = \rho(g, d, r).$$

I said that by general non-sense on upper-semicontinuity, and the earlier arguments of Brill-Noether and Kleiman-Laksov, it suffices to exhibit *a single curve* C achieving the equality. Despite the seeming simplicity of this task, it has never been explicitly achieved.

If we take a "meta-mathematical" point of view, then the problem with any potential proof is that it must somehow use all sorts of generality hypotheses in a serious way, since the result simply isn't true without those hypotheses. The "solution" to this sort of problem is to consider a *degeneration*, i.e. a one-parameter family of curves that will "catch" enough curves to include some of the desired sort.

Castelnuovo's suggestion was to degenerate to a g-nodal rational curve. This is singular, so the proof doesn't actually follow immediately from upper-semicontinuity, but that part can be carried out by relatively standard techniques, so we'll ignore it. Last time we saw how to interpret a linear system of degree d and dimension r on such a curve as the linear system of hyperplanes in $\mathbb{P}^d$ containing a fixed Λ^{d-r-1} with the property that it intersects each of g secants lines to the curve.

This is clearly an intersection problem in the Grassmannian $\mathbb{G}(d-r-1, d-1)$. If it were transverse, then by Schubert calculus its dimension would be

$$(d-r)(r+1) - rg = g - (r+1)(g - d + r).$$

This is exactly what we want, so the challenge is to prove that the intersections really are transverse. We would know this to be the case if the lines were general, but they aren't, so some extra argument is needed.

2.1. Examples.

A really bad sign for proving this fact is that it *isn't necessarily true* if the lines are chosen in non-obvious special positions. For example, a twisted cubic in $\mathbb{P}^3$ lies on a quadric surface Q, and is of type $(1, 2)$. That means that *every line* of one of the rulings of Q intersects C in two points, i.e. is a secant. But there is a one-dimensional space of lines intersecting any g such secants, namely the other ruling!

As discussed, this typically means that we have to turn to a degeneration argument. Direct proofs would have to somehow use hypotheses that rule out, for instance, the secants being rulings on the same quadric surface.

2.2. **Schubert varieties.** Let's fix some notation on Schubert calculus, which deals with the geometry of the varieties $\mathrm{Gr}(k, n)$. Fix a flag

$$F_\bullet : 0 = F_0 \subset F_1 \subset \ldots \subset F_n = V.$$

We can define subvarieties of $\mathrm{Gr}(k, n)$ determined by intersection conditions with the members of this flag. These are called *Schubert varieties*. There are many indexing conventions, which I find quite confusing. I'll explain one below, but be warned that it might not be standard.

Definition 2.2. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition with $\lambda_1 \geq \lambda_2 \geq \ldots\ldots \lambda_k \geq 0$ (by padding with zeros, we can always ensure that there are k parts, but we usually drop the ones that are zero). The *Schubert variety* σ_λ is defined as the set of $\Lambda \subset \mathbb{C}^n$ such that

$$\dim \Lambda \cap F_{n-k+i-\lambda_i} \geq i \text{ for each } i.$$

I found this definition impossible to understand for a while. It's helpful to trace out the partition in an $(n-k) \times k$ grid, from bottom left to top right. This takes n steps in total. If the ith step is to the right then it means that there is an additional dimension's worth of intersection with F_i.

Example 2.3. Consider $\mathrm{Gr}(2, 4) = \mathbb{G}(1, 3)$.

- $\sigma_\varnothing$ is the full Grassmanian $\mathrm{Gr}(2, 4)$.
- σ_1 is the Schubert variety of lines meeting the line F_2.
- σ_{11} is the Schubert variety of lines contained in the plane F_3.
- σ_2 is the Schubert variety of lines passing through the point F_1.
- σ_{21} is the Schubert varietyof lines passing through F_1 and contained in F_3.
- σ_{22} is F_2.

Note that $\dim \sigma_\lambda$ is easy to read off as $k(n-k) - \sum \lambda_i$. The reason is that the usual Plücker coordinates, obtained by appending a $k \times k$ identity matrix to the bottom of the $(n-k) \times k$ grid. The columns represent possibilities for entries of the matrix.

$$\sigma_1 = \left(\begin{array}{cc} 0 & * \\ * & * \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right) \quad \sigma_{11} = \left(\begin{array}{cc} 0 & 0 \\ * & * \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right) \quad \sigma_2 = \left(\begin{array}{cc} 0 & * \\ 0 & * \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right) \sigma_{21} = \left(\begin{array}{cc} 0 & * \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right)$$

3. The main degeneration argument

We now arrive at the meat of the argument, which is to degenerate the secant lines and see what happens. Specifically, suppose that we have $p_1, q_1, p_2, q_2, \ldots, p_g, q_g \in C$ and we are considering the secants $\overline{p_1 q_1}, \ldots, \overline{p_g q_g}$. The plan is to degenerate all the points p_1, q_1, p_2, q_2 into *the same point*.

Notation. We will denote by $\overline{p_1 p_2 \ldots}$ the linear subspace spanned by the points $p_1, p_2, \ldots$. If some points are included with multiplicity, e.g. $\overline{2p_1 3p_2 \ldots}$, that means that the points will lie on a distinguished curve C, and we use the tangent line to C at p_1, the osculating plane to C at p_2, etc.

The discussion in §1 essentially establishes the case $d = 3$, as we saw that degenerating until two pairs of lines intersected makes it easy to see that the intersection is transverse. Making a general argument will be more challenging.

3.1. **First example.** To get a feel for how this might work, first consider the case $g = 2$. We are considering the intersection of the two Schubert cycles $\tau(p, q) = \sigma_{d-k-1}(\overline{pq})$ and $\tau(r, s) = \sigma_{d-k-1}(\overline{rs})$. Consider what happens as we degenerate r to p. In the limit, it seems that the intersection becomes the locus of lines passing through pq and ps. This is the union of two Schubert varieties:

(1) the $\{\Lambda\}$ intersecting the plane pqs in a line, which is the Schubert cycle $\sigma_{d-k-1,d-k-1}(\overline{pqs})$ (having the correct codimension $2(d-k-1)$.
(2) The $\{\Lambda\}$ passing through p, which is the Schubert cycle $\sigma_{d-k}(p)$.

The latter space does not have the correct codimension. What went wrong? The situation is analogous to that of asking for the locus of lines passing through p and r, and degenerating r to p. In the limit, one doesn't get all lines through p, but the *tangent line* to p.

The correct way to degenerate is to consider the family $\Sigma \subset \mathbb{G}(k, d)$ defined by

$$\Sigma = \{(\Lambda, r) \colon \Lambda \in \tau(p, q) \cap \tau(r, s), r \neq p\}$$

and then take its closure in $\mathbb{G}(k, d)$. The fiber over p is then the limiting Schubert cycle. That means that we need to look for closed conditions satisfied over $r \neq p$, and extend them to p.

Let's consider the second situation, in which we seemed to obtain a non-transverse intersection. Note that for $r \neq p$, the $d-r-1$-plane Λ meetings the 3-plane $\overline{pqrs}$ in a line. This is a closed condition, and should extend to the fiber over p, where the 3-plane becomes instead $\overline{2pqs}$. So in the second case the degeneration is actually

$$\left\{\Lambda \mid {}_{\dim \Lambda \cap \overline{2pqs} \geq 1}^{\Lambda \ni p}\right\}$$

Henceforth we will use the shorthand $\Lambda \cap \Lambda' \geq \lambda$ to mean $\dim \Lambda \cap \Lambda' \geq \lambda$, including $\Lambda \cap p \geq 0$ to mean $p \in \Lambda$.

Now, this Schubert cycle has codimension $d-k+d-k-2$, which is what we wanted.

What happens if we then degenerate $q \rightsquigarrow p$? Let's examine the first case

$$\{\Lambda \mid {}_{\Lambda \cap \overline{pqs} \geq 1}\}.$$

Now the condition becomes easy to right down: as $q \rightsquigarrow p$, the plane $\overline{pqs}$ degenerates to the plane $\overline{2ps}$, and we are left with

$$\{\Lambda \mid {}_{\Lambda \cap \overline{2ps} \geq 1}\}.$$

In the second case, we similarly obtain

$$\left\{\Lambda \mid {}_{\dim \Lambda \cap \overline{3ps} \geq 1.}^{\Lambda \ni p}\right\}$$

Finally, if we degenerate $s \rightsquigarrow p$ then we obtain the cycles

$$\{\Lambda \mid {}_{\Lambda \cap \overline{3p} \geq 1}\} \quad \text{and} \quad \left\{\Lambda \mid {}_{\dim \Lambda \cap \overline{4p} \geq 1}^{\Lambda \ni p}\right\}$$

3.2. **Second example.** What we've seen from the first example is that degenerating the secants seems to create conditions with respect to the *osculating flag* to C at p.

From this we conjecture that the following line of argument might work. We consider a situation that looks more general: for a partition $a = (a_0 \geq a_1 \geq \ldots \geq a_k \geq 0)$, we consider the Schubert cycle $\sigma_a(p)$ with respect to the osculating flag at p. We then claim that if $p_1, q_1, \ldots, p_g, q_g$ are general points on C, then for all a the intersection $\sigma_a(p) \cap \tau(p_1, q_1) \cap \ldots \cap \tau(p_g, q_g)$ has the right codimension. We prove this by degenerating p_1 and q_1 to p, thus absorbing one "secant condition" into additional "osculating conditions" at p, and then appealing to induction on g. The only thing to check is that the secant condition degenerates to an osculating condition *of the correct codimension*.

Let's do an example to get a better sense of how this will play out. Suppose we have only one secant $\overline{qr}$, and the partition $a = (d-k-1, d-k-2)$ which corresponds to

$$\left\{ \Lambda \mid {}^{\Lambda \cap \overline{2p} \geq 0}_{\Lambda \cap \overline{4p} \geq 1} \right\}.$$

Consider intersecting with $\tau(q, r)$ and then degenerating r to p. The expected codimension is $(d-k-1) + (d-k-2) + (d-k-1) = 3(d-k) - 4$. Again we break up into two immediate cases:

(1) If $\Lambda \not\ni p$, then we have

$$\left\{ \Lambda \mid {}^{\Lambda \cap \overline{2pq} \geq 1}_{\Lambda \cap \overline{4pq} \geq 2} \right\}.$$

(2) If $\Lambda \ni p$, then we have

$$\left\{ \Lambda \mid \begin{matrix} \Lambda \cap p \geq 0 \\ \Lambda \cap \overline{3pq} \geq 1 \\ \Lambda \cap \overline{4p} \geq 1 \\ \Lambda \cap \overline{5pq} \geq 2 \end{matrix} \right\}.$$

Now we have a bit of a problem that this last set doesn't look like a Schubert cell: after all, there are intersection conditions with respect to *two different* 3-planes. What gives?

The crucial observation is that it *is* a union of two Schubert cells. Indeed, let's look at a point where the two flags branch off, which is after the plane $\overline{3p}$. What is $\dim \Lambda \cap \overline{3p}$? Again we have two cases:

(1) $\dim \Lambda \cap \overline{3p} \geq 1$, then we can rewrite the remaining conditions as

$$\left\{ \Lambda \mid \begin{matrix} \Lambda \cap p \geq 0 \\ \Lambda \cap \overline{3p} \geq 1 \\ \Lambda \cap \overline{5pq} \geq 2 \end{matrix} \right\}.$$

(2) If $\dim \Lambda \cap \overline{3p} = 0$, then there is a point in $\Lambda \cap \overline{3pq} \setminus \overline{3p}$, which necessarily lies outside any proper osculating subspace, otherwise q would be in the osculating hyperplane to p. Since $\dim \Lambda \cap \overline{4p} \geq 1$, this means that $\dim \Lambda \cap \overline{4pq} \geq 2$.

$$\left\{ \Lambda \mid \begin{matrix} \Lambda \cap p \geq 0 \\ \Lambda \cap \overline{4p} \geq 1 \\ \Lambda \cap \overline{4pq} \geq 2 \end{matrix} \right\}.$$

As we further degenerate q into p, we find the three Schubert cycles,

$$\left\{ \Lambda \mid {}^{\Lambda \cap \overline{3p} \geq 1}_{\Lambda \cap \overline{5p} \geq 2} \right\} \quad \text{and} \quad \left\{ \Lambda \mid \begin{matrix} \Lambda \cap p \geq 0 \\ \Lambda \cap \overline{3p} \geq 1 \\ \Lambda \cap \overline{6p} \geq 2 \end{matrix} \right\} \quad \text{and} \quad \left\{ \Lambda \mid \begin{matrix} \Lambda \cap p \geq 0 \\ \Lambda \cap \overline{4p} \geq 1 \\ \Lambda \cap \overline{5p} \geq 2 \end{matrix} \right\}.$$

These correspond to the partitions $(d-k-1, d-k-1, d-k-2)$, $(d-k, d-k-1, d-k-3)$, $(d-k, d-k-2, d-k-2)$, all of which have codimension $3(d-k)-4$, which is correct!

3.3. **The general argument.** Now we are ready to tackle the general case. Let $a = (a_0, a_1, \ldots, a_k)$, so that the Schubert cycle $\sigma_a(p)$ has codimension $\sum a_i$.

Lemma 3.1. *For* $\{p, q_i, r_i\}$ *a general collection of* $2g+1$ *points on* C, *and for any* a, *the intersection*

$$\sigma_a(p) \cap \tau(q_1, r_1) \cap \ldots \cap \tau(q_g, r_g)$$

is of codimension $\sum a_i + g(d-k-1)$ *(and has no multiple components).*

Proof. We proceed by induction, and thus focus on degenerating $q := q_1$ and $r := r_1$ into p, starting with r. We know to expect a union of Schubert cycles instead of a single Schubert cycle; the key is to identify the "branching points" in the osculating flag where the Schubert cycles break off. What we see from the examples is that we should consider branching points where there is more intersection with the osculating flag than is dictated by the Schubert index a. For instance, $\Lambda \cap \emptyset = -1$, and the first Schubert cycle breaks off where $\Lambda \cap p \geq 0$. Next one has $\Lambda \cap \overline{2p} \geq 0$, and the second Schubert cycle breaks off where $\Lambda \cap \overline{3p} \geq 1$.

With this in mind, we set i_0 to be the largest index such that

$$\Lambda \cap \overline{(d-k+i+1-a_i+1)p} \geq i+1$$

for all $i < i_0$. Note that the Schubert condition is merely that

$$\Lambda \cap \overline{(d-k+i+1-a_i)p} \geq i.$$

From now on we will denote $b_i = d-k+i+1-a_i$. In particular, this means that

$$\Lambda \cap \overline{b_i p q} \geq i+1 > \Lambda \cap \overline{(b_i+1)p} \geq i+1$$

which means that there is a point t "pointing towards q", not lying in any proper osculating subspace, contained in Λ. Therefore, for $i \geq i_0$ we have

$$\Lambda \cap \overline{b_i p q} \geq i+1.$$

Therefore, we obtain after the degeneration a Schubert cycle

$$\Omega_{i_0}(q) = \left\{ \Lambda \mid \begin{smallmatrix} \Lambda\cap\overline{(b_i+1)p}\geq i+1, & i<i_0 \\ \Lambda\cap\overline{(b_i+1)pq}\geq i+1 & i=i_0 \\ \Lambda\cap\overline{b_i pq}\geq i+1, & i>i_0 \end{smallmatrix} \right\}.$$

As usual the second stage of the degeneration is easy: just turn each q into a p to get

$$\Omega_{i_0}(q) = \left\{ \Lambda \mid \begin{smallmatrix} \Lambda\cap\overline{(b_i+1)p}\geq i+1, & i<i_0 \\ \Lambda\cap\overline{(b_i+2)}\geq i+1 & i=i_0 \\ \Lambda\cap\overline{(b_i+1)p}\geq i+1, & i>i_0 \end{smallmatrix} \right\}.$$

Each such Schubert variety is of the form $\sigma_{a'}$. To get a sense of what a' is, note that if $i_0 \neq 0$, then we are upgrading the dimension of the intersection by 1 for all of the Schubert conditions *except* at i_0, which has the effect of "translating the partition to the right" (and inserting $d-k$ at the front). This means that $a' = (d-k, a_0, \ldots, a_{i_0-1}, \ldots, a_{k-1})$. (This is assuming that $a_k = 0$; if $a_k = 1$, i.e. if Λ is contained in $\overline{dp}$, then the Schubert cycle is necessarily empty.) This has codimension

$$d-k+\sum a_i - 1 = (d-k-1) + \sum a_i$$

which is what we wanted! □