# Stability and Wall-Crossing

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## 1 Hearts and tilting

### 1.1 Torsion pair

Let $\mathcal{A}$ be an abelian category.

A "torsion pair" axiomatizes the idea of torsion and torsion-free objects.

*Definition* 1.1. A *torsion pair* $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ is a pair of full categories such that

1. $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}, F \in \mathcal{F}$.
2. For every $E \in \mathcal{A}$, there is a short exact sequence

$$0 \to T \to E \to F \to 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

*Remark* 1.2. The first axiom implies that the sequence in the second axiom is unique. (If you had another sequence with $T'$ and $F'$, then $T'$ couldn't map to $F$, hence its inclusion in $E$ would factor through $T$, etc.)

*Example* 1.3. The standard example is $\mathcal{T}$ being the torsion sheaves on $X$ and $\mathcal{F}$ the torsion-free sheaves.

### 1.2 Hearts of triangulated categories

Let $D$ be a triangulated subcategory.

*Definition* 1.4. A *heart* $\mathcal{A} \subset D$ is a full subcategory such that

1. $\mathrm{Hom}(A[j], B[k]) = 0$ for all $A, B \in \mathcal{A}$ for $j > k$ ("no maps backward").
2. For every object $E \in D$ there is a finite filtration

$$0 = E_m \to E_{m+1} \to \ldots \to E_{n-1} \to E_n = E$$

with factors $F_j = \mathrm{Cone}(E_{j-1} \to E_j) \in \mathcal{A}[-j]$.

The cone is the analog of the "factor" in the usual sense of filtration.

*Example* 1.5. The prototypical example is $\mathcal{A}$ an abelian category insided its bounded derived category, and the filtration being obtained by truncation.

**Properties.**

1. It would be more standard to say that $\mathcal{A} \subset D$ is the "heart of a bounded $t$-structure on $D$." However, the heart is equivalent to the data of a $t$-structure.
2. In analogy, we define $H^j_{\mathcal{A}}(E) = F_j[j] \in \mathcal{A}$. (By a similar argument, this is unique up to isomorphism.)
3. Any such $\mathcal{A}$ is automatically an abelian category.
4. The short exact sequences in $\mathcal{A}$ are precisely the triangles in $D$ all of whose terms lie in $\mathcal{A}$.
5. The inclusion functor gives an identification $K_0(\mathcal{A}) \cong K_0(D)$. (Indeed, any object in $D$ has a filtration by objects in $\mathcal{A}$.)

*Remark* 1.6. It is not necessarily the case that $D^b(\mathcal{A}) \cong D$.

## 1.3 Tilting

Suppose that $\mathcal{A} \subset D$ is a heart, and $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ is a torsion pair. We can define a new, *tilted* heart $A^\# \subset D$ formed out of the torsion pair $(\mathcal{T}, \mathcal{F}[1])$ (thus shifting only the $\mathcal{F}$ part of the previous one). There are enlightening pictures in the slides.

Rigorously, how do we know if $E \in D$ lies in $\mathcal{A}^\#$? We look at its cohomology, so $E \in \mathcal{A}^\#$ if and only if (with respect to the old heart)

$$\begin{cases} H^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \\ H^0_{\mathcal{A}}(E) \in \mathcal{T} \\ H^i_{\mathcal{A}}(E) = 0 \qquad i \neq 0, -1. \end{cases}$$

*Example* 1.7. A threefold flop

$$X_+ \to Y \leftarrow X_-$$

induces a derived equivalence $D^b(X_+) \xrightarrow{\sim} D^b(X_-)$. More precisely, a tilt on each side identifies the *hearts* (which turn out to be the perverse sheaves on $X_+, X_-$).

*Example* 1.8. Consider tilting $\mathcal{A} = \mathbf{Coh}(X) \subset D(X)$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ where $\mathcal{T}$ consists of coherent sheaves with 0-dimensional support, and $\mathcal{F}$ consists of coherent sheaves without any zero-dimensional subsheaves.

Note that $\mathcal{O}_X \in \mathcal{F} \subset A^\#$. Recall that we defined the PT moduli space of stable pairs parametrizing maps

$$\mathcal{O}_X \to E \to \text{coker} \to 0$$

such that $E$ had pure dimension 1 and dim supp coker $f = 0$. This refined the Hilbert scheme parametrizing *surjections*

$$\mathcal{O}_X \to E \to 0.$$

We claim that this moduli space of stable pairs $(\beta, n)$ is the space of surjetions in the *tilt* $\mathcal{A}^\#$

$$\mathcal{O}_X \twoheadrightarrow E.$$

Indeed, suppose that we have a short exact sequence in $\mathcal{A}^\#$

$$0 \to J \to \mathcal{O}_X \xrightarrow{f} E \to 0.$$

Think of this not as a short exact sequence, but as a *triangle* in $D$ all of whose terms are in $\mathcal{A}^\#$. If we take cohomology with respect to $\mathcal{A} \subset D$, then we get a long exact sequence

$$0 \to H^0_{\mathcal{A}}(J) \to \mathcal{O}_X \xrightarrow{f} H^0_{\mathcal{A}}(E) \to H^1_{\mathcal{A}}(J) \to 0 \to H^1_{\mathcal{A}}(E) \to 0.$$

♠♠♠ TONY: [the indices don't seem compatible with what was said before.] It follows that $E \in \mathcal{A} \cap \mathcal{A}^\# = \mathcal{F}$ and $\mathrm{coker}(f) = H^1_{\mathcal{A}}(J) \in \mathcal{T}$.

# 2 Relation to Donaldson-Thomas Theory

## 2.1 Recap

Last time we discussed:

- Hall algebras and correspondences,
- A character map $\mathrm{ch} : K_0(C) \to N \cong \mathbb{Z}^{\oplus n}$.
- A quantum torus, which was a non-commutative deformation of the group algebra (with multiplication twisted by the Euler form).
- An integration map $\mathcal{I}\colon\ \mathrm{Hall}(C) \to \mathbb{C}_q[N]$.

## 2.2 Sketch proof of the DT/PT identity

Use Hall algebras to turn categorical statements (e.g. existence/uniqueness of filtrations) into identities.

1. *(Reineke's identity)* We have

$$\delta^{\mathcal{O}}_{\mathcal{A}} = \mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}} * \delta_{\mathcal{A}}$$

   and

$$\delta^{\mathcal{O}}_{\mathcal{A}} = \mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}} * \delta_{\mathcal{A}}$$

   (basically stated last time, with the meaning that counting maps is the same as counting surjective maps to a given subobject, over all subobjects.)

2. *(Torsion pair identities)* We have $\delta_{\mathcal{A}} = \delta_{\mathcal{T}} * \delta_{\mathcal{F}}$ and
3. Torsion pair identities with sections
4. Reineke's identity with section
5. All maps $\mathcal{O}_X \to \mathcal{T}[-1]$ are zero, so $\delta^{\mathcal{O}}_{\mathcal{T}[-1]} = \delta^{\mathcal{O}}_{\mathcal{T}}$.
6. Some more identities.
7. Restrict to sheaves supported in dimension $\leq 1$. Then the Euler form is trivial so the quantum torus is commutative. Integrate an identity.

### 2.3 Moduli space of framed sheaves

Let $X$ be a Calabi-Yau threefold. So far we have been discussing moduli spaces of objects in the category $D^b\mathbf{Coh}(X)$ equipped with a kind of framing.

1. The framing eliminates stabilizers, making the moduli space a *scheme* (so it has a well-defined Euler characteristic).

What about unframed DT invariants? Fix a polarization of $X$ and a class $\alpha \in N$, and consider the stack

$$\mathcal{M}^{ss}(\alpha) = \{E \in \mathbf{Coh}(X)\colon\ E \text{ semistable}, \operatorname{ch}(E) = \alpha\}.$$

Now we can't define an Euler characteristic of a stack in general, although we can define a Poincaré polynomial.

In a nice situation ($\alpha$ primitive and the polarization in sgeneral), the Euler characteristic makes sense and you can use it to define a naïve DT invariant.

In the general case, Joyce figured out how to define *rational* (naïve) invariants with good properties, and showed that they satisfied wall-crossing formulae as the polarization varies. Kontsevich and Soibelman upgraded this using the Behrend function to genuine DT invariants.

### 2.4 Quantum and classical DT invariants

The generating function for DT invariants is

$$qDT = \mathcal{I}([\mathcal{M}^{ss}(\mu) \subset \mathcal{M}]) \in \mathbb{C}_q[[N_+]].$$

We would like to evaluate this at $q = 1$, but there is a pole. Joyce showed that the right thing to do is to take the logarithm, multiply by $q - 1$ and then take the limit as $q \to 1$:

$$DT_\mu = \lim_{q\to 1}(q-1)\log qDT_\mu \in \mathbb{C}[[N_+]].$$

There is another interpretation. The bottom line is that you should think of DT invariants on a ray as automorphisms of the torus.

*Example* 2.1. This is worked out for the example of a single stable bundle of fixed slope, which corresponds to the Artin stack $\bigsqcup_n B\,\mathrm{GL}_n$.

# 3 Stability conditions

## 3.1 Abelian categories

Let $\mathcal{A}$ be an abelian category.

*Definition* 3.1. A *stability condition* on $\mathcal{A}$ is a map of groups $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ such that $0 \neq E \in \mathcal{E}$ implies that $Z(E) \in \overline{\mathbb{H}}$ where $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}_{<0}$ is the semi-closed upper half plane.

The phase is $\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1]$. We say that $E$ is *semistable* if every subobject of $E$ has a (not necessarily strictly) smaller phase. A stability condition has the *Harder-Narasimhan property* if every $E \in \mathcal{A}$ has a filtration

$$0 \subset E_0 \subset \ldots \subset E_n \subset E$$

such that each factor $F_i = E_i/E_{i-1}$ is $Z$-semistable and $\phi(F_1) > \ldots > \phi(F_n)$.

The argument to show the existence of these generalized Harder-Narasimhan structures is the same as the usual one: given an object, you choose the first step in the filtration by passing to subobjects with larger and larger phase until you stop. This works as long as $\mathcal{A}$ has some property (e.g. Artinian) that prevents this process from going on forever. If the filtration exists, then the standard argument shows that it is unique.

If $C$ has the Harder-Narasimhan property, then we get another Reineke identity, expressing the uniqueness of the Harder-Narasimhan filtration (the flavor is like that of the earlier identity, expressing maps as a sum over surjective maps to subobjects).

Since one side of the identity is *independent* of the stability condition, we get a *wall-crossing formula* for different stability conditions. If appropriate conditions are satisfied (i.e. $C$ has global dimension $\leq 1$), then we can integrate to get an identity in the Hall algebra.

## 3.2 Triangulated categories

*Definition* 3.2. A *stability condition* on $D$ is a pair $(Z, \mathcal{A})$ where

- $\mathcal{A}$ is a heart of $D$,
- $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ is a stability condition on $\mathcal{A}$ with the Harder-Narasimhan property.

$E \in D$ is *semistable* if $E = A[n]$ for some semistable $A \in \mathcal{A}$. We define $\phi(E) = \phi(A)+n$.

We consider only those stability conditions with nice properties:

- The "central charge" $Z\colon K_0(D) \to \mathbb{C}$ factors through $\mathrm{ch} : K_0(D) \to N \cong \mathbb{Z}^{\oplus n}$.
- There is $K > 0$ such that for any semistable $E \in D$,

$$Z(E) \geq K \cdot \|\mathrm{ch}\,(E)\|.$$

**Theorem 3.3.** *Sending a stability condition to Z defines a local homeomorphism*

$$\mathrm{Stab}(D) \to \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}) \cong \mathbb{C}^n.$$

*In particular,* Stab$(D)$ *is a complex manifold.*