Stability and Wall-Crossing

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The basic problem is to calculate "motivic" invariants of moduli spaces of coherent sheaves on Calabi-Yau threefolds. This might sound like a rather specialized topic, but I hope to convince you that it's not. Actually, I'm more interested in the *dependence* of the invariants on the stability parameters. The parameter space is divided up by "walls," and interesting things happen when you jump across a wall.

These computations are interesting to physicists, and Calabi-Yau threefolds are an interesting mathematical frontier as well.

1 Invariants on Moduli spaces

1.1 Motivic invariants

By "motivic invariants" we mean invariants of varieties which satisfiy the "scissor relation"

$$\chi(X) = \chi(Y) + \chi(U)$$

whenever Y is a closed subset of X and $U = X \setminus Y$.

Example 1.1. The Euler characteristic is a motivic invariant.

There is a *universal* motivic invariant, provided by the Grothendieck ring.

Definition 1.2. The *Grothendieck group* $K(\mathbf{Var}/\mathbb{C})$ is the free abelian group on the set of isomorphism classes of varieties, modluo the "scissor" relations.

This is a ring with product coming from the fibered product, but we won't use the ring structure.

1.2 Curve counting invariants

Let X be a Calabi-Yau threefold. Fix $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. We can look at the part of the Hilbert scheme of curves in X parametrizing closed subschemes of dimension ≤ 1 satisfying $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$. We define the "naive DT invariant" to be the Euler characteristic of $\mathrm{Hilb}(\beta, n)$.

There are also "genuine DT invariants" which are a *weighted* Euler characteristic. This is a construction of Behrend. He defined a constructible function, depending only on the singularities of the variety, and the weighted Euler characteristic is the weighted sum of the Euler characteristic of the strata with weights coming from this function.

We're going to talk about the naïve DT invariants for simplicity. The genuine ones are better, though, mainly because they are *deformation invariant.* The basic example if you are considering the moduli space of two points, which has Euler characteristic 2, but if the points "collide" then you get Euler characteristic 1 using the naïve counting, while the genuine count weights it with 2.

1.3 Effect of a flop

Suppose that you have two Calabi-Yau threefolds related by a flop.

$$\begin{array}{ccc} X & & X' \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

Toda proved a theorem, which basically says that if you take the generating function of DT invariants, and you dvide by the generating function for curves *contra ted to a point*, then this is invariant on the two sides. (A priori, the things on the two sides live in different rings, so you have to identify the homology groups in order to make this comparison.)

This is an example of "wall-crossing." It was extended to genuine DT invariants by Calabrese.

This is a very *clean* theorem, with no clause about -1 curves or anything. That comes from the fact that we work in the *derived category*, where things are true in great generality.

It's not quite clear why a flop fits into the framework of "variation of stability." I want to elaborate on this later.

1.4 Stable pair invariants

With the same data β, n as above, consider the moduli space of coherent sheaves E on X equipped with a section

$$f: \mathcal{O}_X \to E.$$

If we asked for f to be surjective, then we would get the Hilbert scheme. We relax that condition in return for being able to require E to have pure dimension 1. So we are parametrizing coherent sheaves E such that

1. E is pure dimension 1 with $\operatorname{ch}(E) = (0, 0, \beta, n)$ and

2. $\dim_{\mathbb{C}} \operatorname{supp} \operatorname{coker}(f) = 0.$

This removes "wandering points" from consideration in the Hilbert scheme.

This definition was considered by Pandariphande-Thomas, who constructed compactifications.

Again, Toda proved a theorem relating the generating function for stable pair invariants and the generating function for ordinary DT variants. On the DT side you divide by the generating function for "wandering points."

2 Strategy of Proofs

2.1 Outline

1. Describe the relevant phenomenon as a kind of change of stability condition.
2. The main ingredient is the "Hall algebra" which turns categorical statements into algebra. The Hall algebra is a crazy ring which you can't write down.
3. There is a ring homomorphism $\mathcal{I}$ from the Hall algebra to a relatively concrete ring, which if you apply it to the identity gives an identity of generating functions. This is the deep part.

The first two steps are completely general, but the existence of the "integration map" $\mathcal{I}$ requires assumpstions, either:

1. $\mathcal{C}$ has global dimension ≤ 1 (sheaves on a curve).
2. $\mathcal{C}$ satisfies the "Calabi-Yau condition."

2.2 Hall algebras

Let $\mathcal{C}$ be an abelian category (coherent sheaves on a variety). We need to introduce

1. The stack $\mathcal{M}$ of objects of $\mathcal{C}$
2. The stack $\mathcal{M}^{(2)}$ of short exact sequences in $\mathcal{C}$.

We get a map

$$\begin{array}{ccc} & \mathcal{M}^{(2)} & \\ \swarrow & & \searrow \\ \mathcal{M} & & \mathcal{M} \end{array}$$

sending $0 \to A \to B \to C \to 0$ to (A, C) and B. We think of this as a correspondence. Then for any suitable "cohomology theory," i.e. a functor satisfying certain functoriality properties (we aren't think of anything strict like the Steenrod axioms), we get a multiplication

$$m\colon H^*(\mathcal{M}) \otimes H^*(\mathcal{M}) \to H^*(\mathcal{M}).$$

The important thing is that this is an associative multiplication. It is a good exercise to check the associativity, which comes down to the axioms of an abelian category (specifically the second and third isomorphism theorems). So this is like encoding the axioms of an abelian category in an associative algebra.

2.3 Grothendieck groups of stacks

For the cohomology theory, you take a "relative Grothendieck group of stacks." This means that if $\mathcal{M}$ is the moduli stack of objects, then you take $H^*(\mathcal{M})$ to be the free vector space of stacks mapping to $\mathcal{M}$, modulo some scissor relations.

This is functorial in an obvious way. Pushforward is composition, and pullback is the fibered product.

Remark 2.1. The small print is that all stacks are Artin stacks, locally of finite type over $\mathbb{C}$ with affine stabilizers.

Fibers of the corrspondence. Consider

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow{\scriptstyle (a,c)} & & \\ \mathcal{M} \times \mathcal{M} & & \end{array}$$

The fiber of b over $B \in \mathcal{M}$ is the choice of a subobject or quotient object, hence is precisely the Quot scheme of B.

The fiber of (a, c) over $(A, C) \in \mathcal{M} \times \mathcal{M}$ is the quotient stack $[\mathrm{Ext}^1(C, A)/\mathrm{Hom}(C, A)]$.

Remark 2.2. This might be interesting if you're a derived algebraic geometer. I mentioned that this definition works best in the global dimension 1 case, because the higher Exts vanish, so this thing varies "smoothly." (Both parts can jump, but they must jump equally, i.e. the quotient doesn't jump, because the Euler characteristic is constant.) For higher dimensional things this jumps like crazy. I suspect that in that case it is appropriate to use derived algebraic geometry.

2.4 A Toy Model

Suppose we're working in an abelian category C such that

1. every object has only finitely many subobjects,
2. all groups $\mathrm{Ext}^j(E, F)$ are finite.

The categories we're really interested in (sheaves on varieties over $\mathbb{C}$) don't satisfy this, but there are some interesting categories that do. For example, if A is a finite-dimensional algebra over $k = \mathbb{F}_q$ then $C = \mathbf{Mod}(A)$ fits the bill (everything is question is finite even as a set).

In this toy model you can really do explicit computation with Hall algebras. In this case the Hall algebra is the algebra of functions from isomorphism classes of objects of C with a convolution product:

$$(f_1 * f_2)(B) = \sum_{A \subset B} f_1(A) \cdot f_2(B/A).$$

You can define a "finitary Hall algebra", consisting of "compactly (i.e. finitely) supported" functions.

Exercise 2.3. Prove that this is associative. You'll see that you need the second and third isomorphism theorems.

Example 2.4. Let C be the category of finite-dimensional vector spaces over $\mathbb{F}_q$. Let δ_n be the characteristic function of vector spaces of dimension n. Then

$$\delta_n * \delta_m = |\operatorname{Gr}(n, n+m)(\mathbb{F}_q)| \cdot \delta_{n+m}.$$

You can compute

$$|\operatorname{Gr}_{n,n+m}(\mathbb{F}_q)| = \frac{(q^{n+m)-1)\dots(q^{m+1}-1)}}{(q^n-1)\dots(q-1)}$$

In fact, there is an isomorphism of algebras $\mathcal{I} \colon \operatorname{Hall}_{\text{finite}}(C) \to \mathbb{C}[x]$ sending $\delta_n \mapsto \frac{x^n}{(q^n-1)\dots(q-1)}$.

There is a distinguished element δ_C which is 1 on all objects (it is not finitely supported). This maps to

$$\sum_{n\geq 0} \frac{x^n}{(q^n-1)\dots(q-1)}$$

which is a well-known object called the "quantum dilogarithm."

Let me give an example of how categorical identities are turned into algebraic identities in the Hall algebra. Define $\delta_C^P(E) = |\operatorname{Hom}_C(P, E)|$ and $\operatorname{Quot}_C^P(E) = |\operatorname{Hom}_C^{\twoheadrightarrow}(P, E)|$ (the number of surjective morphisms.

We claim that

$$\boxed{\delta_C^P = \operatorname{Quot}_C^P * \delta_C.}$$

Proof. For an object E, the left hand side is the number of maps from P to E, and the right hand side is a sum over subobjects of E of the number of surjective maps onto that subobject. □

2.5 The Geometric Case

Let us consider the case $C = \mathbf{Coh}(X)$ and $P = \mathcal{O}_X$. Define

1. The stack $\mathcal{M}^{\mathcal{O}}$ paramerizing sheaves E equipped with a section $s \colon \mathcal{O}_X \to E$,
2. The scheme Hilb parametrizing sheaves E equiped with a *surjective* section $s \colon \mathcal{O}+_X \twoheadrightarrow E$.

In analogy to the result from the toy case, we have:

Theorem 2.5. *There is an identity*

$$[\mathcal{M}^{\mathcal{O}} \to \mathcal{M}] = [\operatorname{Hilb} \to \mathcal{M}] * [\mathcal{M} \xrightarrow{\operatorname{Id}} \mathcal{M}].$$

2.6 Integration map

From now on, we work in an abelian category linear over a field k with finite Ext-groups.

Example 2.6. We can take $\mathcal{C} = \mathbf{Coh}(X)$ with X smooth and projective.

There is an Euler form on $K_0(\mathcal{C})$ (the Grothendieck group) by the alternating sum of Ext groups. It is often convenient to fix a group homomorphism $\mathrm{ch} : K_0(\mathcal{C}) \to N$ where N is a free abelian group of finite rank. This is a bit of a hack, but it works.

Example 2.7. The Chern character is such a choice of homomorphims.

We always make some assumptions:

1. The Euler form descends to a bilinear form $N \times N \to \mathbb{Z}$.
2. The character $\mathrm{ch}(E)$ is locally constant in families. This gives a decomopsition of $\mathcal{M}$ into clopen substacks, parametrizing objects with a given Chern character, which induces a grading on the Hall algebra.

Quantum Torus. Define a non-commutative algebra over $\mathbb{C}(t)$ by

$$\mathbb{C}_t[N] = \bigoplus_{\alpha \in N} \mathbb{C}(t) \cdot x^\alpha$$

with the twisted multipolication

$$x^\alpha * x^\gamma = t^{-(\gamma,\alpha)} \cdot x^{\alpha+\gamma}.$$

You should think of this as a non-commutative deformation of the group ring $\mathbb{C}[N] \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, which is the coordinate ring of the algebraic torus $T = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$. That is why this is called a "quantum torus." We often set $q = t^2$.

Virtual Poincaré invariant. There is an algebra homomrphism $\chi_t \colon K(St/\mathbb{C}) \to \mathbb{Q}(t)$ (domain is the Grothendieck group of stacks over $\mathbb{C}$) uniquely defined by the properties:

1. If V is smooth projective, then
$$\chi_t(V) = \sum \dim_{\mathbb{C}} H^i(V^{an}, \mathbb{C})(-t)^i \in \mathbb{Z}[t].$$
2. If V has an action of $\mathrm{GL}(n)$, then
$$\chi_t([V/\,\mathrm{GL}(n)]) = \chi_t(V)/\chi_T(\mathrm{GL}(n)).$$

This is the called the *(virtual) Poincaré polynomial* of the stack.

If we had put varieties instead of stacks, then this would already be a deep theorem of Deligne. However, the step up from schemes to stacks is not hard.

Now, stacks have Poincaré polynomials but not Euler characteristics. Stacks don't have Euler characteristics, because $\chi_t(\mathrm{GL}(n))$ has χ_t with a pole at $t = 1$.

Theorem 2.8 (Joyce)**.** *When $C = \mathbf{Coh}(X)$ with X a curve, there is an algebra map*

$$\mathcal{I}\colon Hall_{mot}(C) \to \mathbb{C}_{t^2}[N]$$

such that $\mathcal{I}(S \to M_\alpha) = \chi_t(S)x^\alpha$.

This comes from working out the Poincaré invariants of the fibers of $(a, c)\colon \mathcal{M}^{(2)} \to \mathcal{M} \times \mathcal{M}$. The curve case is not that hard.

What is harder is that Kontsevich and Soibelman also constructed an algebra map $\mathcal{I}\colon Hall_{mot}(C) \to \mathbb{C}_t[N]$ in the case that X is a Calabi-Yau threefold.

It is hard to say what this map is. However, if S is a scheme then the answer as $t \to \pm 1$ is basically an Euler characteristic ($t = 1$) or a weighted Euler characteristic ($t = -1$).

There are still technical problems with applying this. For our applications, we can specialize to a "classical limit" (commutative) and then consider a first-order "Poisson deformation." We call this a "semiclassical limit." You can understand moraly why this works. By the Calabi-Yau condition

$$\mathrm{Ext}^1(C, A) - \mathrm{Hom}(C, A) - (\mathrm{Ext}^1(A, C) - \mathrm{Hom}(A, C)) = \chi(A, C).$$

This suffices for applications to classical invariants.