Problems for "Geometric Representation Theory and Quasi-Maps into Flag Varieties"

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1 Problem Sheet 1

1.1 Problem 1

(a) Let $G = \mathrm{SL}_2$. We first proceed naïvely by computing with explicit matrix entries. Choose the upper-triangular Borel B. To understand the coset space G/U we observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} a & ax+b \\ c & cx+d \end{pmatrix}.$$

Thus we see that we can cover G/U by two affine charts:

- If $a \neq 0$, then by column operations we have a unique representative of the form $\begin{pmatrix} a & \\ c & a^{-1} \end{pmatrix}$, and the set of such matrices has coordinate ring is $k[a^{\pm 1}, c]$.
- If $c \neq 0$, then we have a unique representative of the form $\begin{pmatrix} a & -c^{-1} \\ c & 0 \end{pmatrix}$, whose coordinate ring is $k[a, c^{\pm 1}]$.

These are glued along an open set of the form $k[a^{\pm 1}, c^{\pm 1}]$ via $a \leftrightarrow a$ and $c \leftrightarrow c$, which we recognize as the description of $\mathbb{A}^2 - \{(0,0)\}$.

The analogous computation for SL_3 is daunting, so can we see in retrospect another way to obtain this result? Aha! Note that SL_2 acts transitively on $\mathbb{A}^2 - 0$, and the stabilizer of the point $(1,0)$ is precisely U, which immediately describes SL_2/U as $\mathbb{A}^2 - 0$.

Inspired by this, let's try think of what SL_3 might act on with U as a stabilizer. Given the similarity between U and B, we might think to find an action on something like a flag. However, since there is no longer a $k^\times$-ambiguity on the diagonal, we are choosing a *point* instead of a line. Explicitly, SL_3/U can be modeled by (v_1, v_2) where

1. A non-zero vector $v_1 \in V$,
2. A non-zero vector $v_2 \in V/\langle v_1\rangle$.

Note that SL_3 acts transitively on such pairs, since if we take v_1 and v_2 to be the first two columns of the matrix then there is always a way to fill in the third column to make the determinant equal to 1. Also, it is easy to see that the stabilizer is precisely U, since the third vector is determined uniquely mod $\langle v_1, v_2\rangle$ by the requiring of the matrix having determinant 1.

Since the second choice depends on the first one, it is not so obvious how to write down algebraically coordinates for this set cleanly. In this case, there is a convenient trick: since $V/\langle v_1\rangle$ is dual to $\langle v_1\rangle^\perp \subset V^*$ under the duality pairing, a the choice of non-zero v_2 is the same as a choice of a non-zero vector in $\langle v_1\rangle^\perp \subset V^*$. Therefore,

$$\mathrm{SL}_3 /U = \{(v, \phi) \in V \times V^* \mid \langle v, \phi\rangle = 0, v \neq 0, \phi \neq 0\}.$$

(b) We use an algebraic analogue of the Peter-Weyl Theorem:

$$\mathbb{C}[G] \cong \bigoplus_{(V,\rho)\in \mathrm{Irr}(G)} \mathrm{End}(V) \cong \bigoplus_{(V,\rho)\in \mathrm{Irr}(G)} V^* \otimes V. \tag{1}$$

Why is this true? Since G is reductive and we are in characteristic 0, $\mathbb{C}[G]$ will be a direct sum of finite-dimensional irreducible representations. (Any algebraic representation must be locally finite, i.e. any vector must be contained in a finite-dimensional subrepresentation, by the usual argument using the Hopf algebra structure.) So it suffices to compute the V-isotypic component, which can be determined by Frobenius reciprocity:

$$\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G]) = \mathrm{Hom}_{\mathbb{C}[G]}(V, \mathrm{Ind}_e^G \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \cong V^*.$$

This tells us that the V-isotypic component of $\mathbb{C}[G]$ as a left G-module is precisely V^*. Now, we are interested in the group algebra $\mathbb{C}[G/U] = \mathbb{C}[G]^U$. Using (1), we see that

$$\begin{aligned} \mathbb{C}[G]^U &\cong \bigoplus_{(V,\rho)\in \mathrm{Irr}(G)} V^* \otimes V^U \\ &\cong \bigoplus_{(V,\rho)\in \mathrm{Irr}(G)} V^* \end{aligned}$$

because V^U is the space of highest weight vectors of V, which is one-dimensional. Again by highest weight theory,

$$\bigoplus_{(V,\rho)\in \mathrm{Irr}(G)} V^* \cong \bigoplus_{(V,\rho)\in \mathrm{Irr}(G)} V \cong \bigoplus_{\lambda\in\Lambda_+} V(\lambda).$$

In order to understand the multiplication, we must understand the multiplicative structure of (1). This is via *matrix coefficients*: for $v \otimes v^* \in V \otimes V^*$, the corresponding function on $\mathbb{C}[G]$ is

$$g \mapsto \langle v^*, g \cdot v\rangle.$$

Therefore, if $u \otimes u^* \in U \otimes U^*$, the product of the functions corresponding to $v \otimes v^*$ and $u \otimes u^*$

$$g \mapsto \langle u^*, g \cdot u\rangle\langle v^*, g \cdot v\rangle = \langle u^* \otimes v^*, g \cdot u \otimes v\rangle.$$

Now we consider what happens when we take G-invariants. Let v_λ denote the highest weight vector of $V(\lambda)$. Then under the identification

$$A = \bigoplus_{\lambda \in \Lambda_+} V(\lambda) \tag{2}$$

we have that $v \in V(\lambda)$ corresponds to $v \otimes v_{\lambda^*}$, so the product $v \cdot u$ is $(v \otimes u) \otimes (v_{\lambda^*} \otimes u_{\mu^*})$. Since $v_{\lambda^*} \otimes u_{\mu^*}$ is the highest weight vector of $V(\lambda + \mu) \subset V(\lambda) \otimes V(\mu)$, the multiplication restricted to $V(\lambda) \times V(\mu)$ is

$$V(\lambda) \times V(\mu) \to V(\lambda) \otimes V(\mu) \xrightarrow{\text{project}} V(\lambda + \mu).$$

(c) We show that A is generated by $V := \bigoplus V(\omega_i)$ where the ω_i are the fundamental weights. Since every dominant weight is a positive integral linear combination of fundamental weights, the tensor products of $V(\omega_i)$ contain every $V(\lambda)$ for λ dominant.

Since the morphism $A \to \operatorname{Sym}^\bullet V$ is injective, the map $X \to \overline{X} = \operatorname{Spec} A$ is dominant. Furthermore, the action of G on A obviously extends to a compatible action on $\operatorname{Sym}^\bullet V$, so that the image of X in $\overline{X}$ is a G-orbit.

It is a fact that any G-orbit is necessarily be locally closed, hence open in its closure (which in this case is the whole space $\overline{X}$, as we just said that the morphism is dominant). Indeed, it is constructible by general considerations (Chevalley's Theorem). Therefore, it contains an open dense orbit, so which is then open as it is acted on transitively by G.

It only remains to show that $X \hookrightarrow \overline{X}$ is injective. It suffices to check this after extending scalars to $\overline{k}$, so that we may check it on k-points. Then we want to show that G/U injects into $\bigoplus V(\omega_i)$. The collection of highest weight vectors $(v_{\omega_1}, \ldots, v_{\omega_n})$ is acted on by G, fixed by U. They cannot be fixed by anything larger subgroup, since such a subgroup would fix *all* highest weight vectors because all such are tensor products of these ω_i.

(d) First using our description in (a), we find that

- for $G = \mathrm{SL}_2$, $\overline{X} = \mathbb{A}^2$. In this case $A \cong k[x, y]$.
- for $G = \mathrm{SL}_3$, $\overline{X} = \{(v, \phi) \in V \times V^* \mid \langle v, \phi \rangle = 0\}$. In this case

$$A \cong k[x_1, y_1, z_1, x_2, y_2, z_2]/(x_1x_1 + y_1y_2 + z_1z_2).$$

Let's see if we can see this using our description in (b).

- For $G = \mathrm{SL}_2$, we have $\Lambda = \mathbb{Z}$. By (b), we know that A is generated by $V(1) = k\langle x\rangle \oplus k\langle y\rangle$ where x has weight 1 and y has weight -1. Therefore, we can realize A as a quotient of $\operatorname{Sym}^\bullet V(1) = k[x, y]$.

  The relations come from decomposing the tensor powers of $V(1)$, as described in (b). Since
  $$V(1) \otimes V(1) \cong \operatorname{Sym}^2 V(1) \oplus \wedge^2 V(1) \cong V(2) \oplus \wedge^2 V(1),$$

the relation comes from killing $\bigwedge^2 V(1)$, which is generated by $x \otimes y - y \otimes x$. Therefore, the only relation is that $xy = yx$, which we already knew. This confirms that $A \cong k[x, y]$.

- For $G = \mathrm{SL}_3$, we have fundamental weights ω_1, ω_2 such that $V(\omega_1) \cong V(\omega_2)^*$. Thus, we expect to find A as a quotient of $k[x_1, y_1, z_1, x_2, y_2, z_2]$. What relations are there? We have three types to consider: the relations for multiplication within $V(\omega_1)$, which come from decomposing its tensor square, the relations for multiplication within $V(\omega_2)$, and the relations for multiplication between $V(\omega_1)$ and $V(\omega_2)$, which come from decomposing $V(\omega_1) \otimes V(\omega_2)$.

  The tensor product $V(\omega_1) \otimes V(\omega_1)$ decomposes as $\mathrm{Sym}^2 V(\omega_1) \oplus \bigwedge^2 V(\omega_1)$, and $\mathrm{Sym}^2 V(\omega_1) = V(2\omega_1)$. The relations obtained by killing $\bigwedge^2 V(\omega)$ simply encode that the algebra generated by $V(\omega_1)$ should be commutative, which we already knew. Thus, there are no new relations for multiplication within x_1, y_1, z_1. Dually, there are no new relations for multiplication with x_2, y_2, z_2.

  Finally, $V(\omega_1) \otimes V(\omega_2)$ decomposes, since there is obviously a one-dimensional copy of the trivial representation. The irreducible representation $V(\omega_1 + \omega_2)$ may be identified with the kernel of the map $V(\omega_1) \otimes V(\omega_2) \to \mathbb{C}$ coming from the trace pairing, which is under an appropriate choice of coordinates is

  $$(x_1, y_1, z_1) \otimes (x_2, y_2, z_2) \mapsto x_1 x_2 + y_1 y_2 + z_1 z_2.$$

  A priori we should also consider the relations from decomposing $V(\omega_2) \otimes V(\omega_1)$, but since these are also obtained by projecting to $V(\omega_1 + \omega_2)$ it is clear that this well tell us nothing new except that the "variables from $V(\omega_1)$" and the "variables from $V(\omega_2)$" commute. Thus, we have confirmed the calculation

  $$A \cong k[x_1, y_1, z_1, x_2, y_2, z_2]/(x_1 x_1 + y_1 y_2 + z_1 z_2).$$

(e) We recognize from the examples that the equations cutting out $\overline{X}$ in $\bigoplus V(\omega_i)$ seem to always be homogeneous. If this is the case, and there are non-trivial higher degree homogeneous equations, then of course $\overline{X}$ will be a cone with cone point at the origin. So first we confirm the homogeneity of the defining ideal, and then we investigate when there will be more relations.

Since the relations come from spaces of the form $V(\omega_1)^{i_1} \otimes \ldots \otimes V(\omega_n)^{i_n}$, which have pure degree inside $T^\bullet(V(\omega_1) \oplus \ldots \oplus V(\omega_n))$, the statement that any subspaces killed must be homogeneous in the coordinates of the $V(\omega_i)$ turns out to be tautological, after some thought.

Furthermore, thinking about this will make it clear that there are no linear relations (this is somehow part of the statement that the $V(\omega_i)$ are already irreducible). If $V(\omega_i) \otimes V(\omega_j)$ is reducible (i.e. not already equal to $V(\omega_i + \omega_j)$, then we will already obtain non-trivial relations. However, killing the subspace $\bigwedge^2 V(\omega_i) \subset V(\omega_i)^{\otimes 2}$ only tells us that the algebra is commutative, which of course introduces no equations in the homogeneous ideal.

The upshot is that the homogeneous ideal of $\overline{X}$ in $\bigoplus V(\omega_i)$ is zero, hence $\overline{X}$ is not singular (and in fact isomorphic to affine space), if and only if:

- $\mathrm{Sym}^2 V(\omega_i)$ is irreducible for each i, and
- $V(\omega_i) \otimes V(\omega_j)$ is irreducible for each $i \neq j$.

It obviously suffices to restrict our attention to simple G, since a product of varieties is singular if and only if at least one of the factors is singular. In that case, we can attempt to rule out most G using the second condition. For instance, if any of the $V(\omega_i)$ is not self-dual then we already lose. Anyway, one way to rule it out is to compare the dimension of the two sides. One way to do so is to use the Weyl dimension formula.

Theorem 1.1 (Weyl dimension formula)**.** *If $\lambda \in \Lambda_+$, then*

$$\dim L(\lambda) = \frac{\prod_{\alpha>0}\langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha>0}\langle \rho, \alpha^\vee \rangle}.$$

Therefore,

$$\dim V(\omega_i) \otimes V(\omega_j) = \prod_{\alpha>0} \frac{\langle \omega_i + \rho, \alpha^\vee \rangle \langle \omega_j + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle \langle \rho, \alpha^\vee \rangle}$$

and

$$\dim V(\omega_i + \omega_j) = \prod_{\alpha>0} \frac{\langle \omega_i + \omega_j + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

Let's compare the products factor-by-factor. Let $a_i = \langle \omega_i, \alpha^\vee \rangle$ and $c = \langle \rho, \alpha^\vee \rangle$. Since the ω_i are dominant, we have $a_i, a_j > 0$. Then we are comparing

$$(a_i + c)(a_j + c) \stackrel{?}{>} (a_i + a_j + c)c.$$

Multiplying out, we see that the inequality is strict as long as $a_i a_j > 0$, which is always the case. Therefore, $\overline{X}$ is singular unless X has only one fundamental weight. In that case, we know that G is a form of SL_2, and $\overline{X}$ is actually non-singular.

1.2 Problem 2

We adapt the usual argument for constructing G/U_P as an algebraic variety. We can find some finite-dimensional faithful representation V of G, such that U_P is the stabilizer of a subspace $W \subset V$. By replacing V with $\bigwedge^{\dim W} V$, we have that G acts faithfully on V with U_P stabilizing a one-dimensional subspace.

Ordinarily, this argument would be used to embed G/U_P as a quasiprojective variety. However, in this case U_P acts on the line by a character, which *must be the trivial character* because U_P is unipotent. Thus we see that G/U_P actually embeds in the affine space V/W. Moreover, this affine space has an action of G, in which G/U_P is a G-orbit, thus locally closed. Thus G/U_P is open in its affine closure, i.e. quasi-affine.

The argument for $G/[P,P]$ is similar. We have a factorization $P = M_P U_P$ of P into a Levi subgroup and a parabolic subgroup, such that M_P normalizes U_P. It is easy to check that $[P,P] = [M_P, M_P]U_P$, a product of a semisimple group and a unipotent group, so it again has no non-trivial characters. Then we may apply the same argument as above.

Ring of functions. Now, to investigate the singularity of the parabolic analogues we want to generalize the analysis from Problem 1. What happens when we take U_P or $[P,P]$ invariants on

$$\mathbb{C}[G] = \bigoplus_{\lambda\in\Lambda_+} V(\lambda)\otimes V(\lambda)^*?$$

As a sanity check, note that $U_P \subset U \subset [P,P]$, so we should find that

$$\mathbb{C}[G]^{[P,P]} \subset \mathbb{C}[G]^U \subset \mathbb{C}[G]^{U_P}.$$

Noting that $[P,P] = [M_P, M_P]U_P$, we will find that

$$\mathbb{C}[G] = \bigoplus_{\lambda\in\Lambda_+} V(\lambda)^{[M_P,M_P]U_P}\otimes V(\lambda)^*.$$

Thus, $V(\lambda)^*$ appears (necessarily with multiplicity 1) if and only if the highest weight vector v_λ is fixed by $[M_P, M_P]$ (since it is automatically fixed by $U_P \subset U$; on the other hand the only vector fixed by the subgroup $U \subset [P,P]$ is the highest weight vector). If $\{\omega_j\}$ are fundamental weights such that $\langle \alpha_i^\vee, \omega_j\rangle = 0$ for all i, then the conclusion is that

$$A = \bigoplus_{\lambda in \mathbb{Z}^+\langle\{\omega_j\}\rangle} V(\lambda) \tag{3}$$

with multiplication being the restriction of that described in §1.1.

Now suppose that P corresponds to the simple roots $\{\alpha_i : i \in I\}$, i.e.

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_i \mathfrak{g}_{-\alpha_i}.$$

Then we have that $V(\lambda)^{[P,P]\neq 0}$ if and only if each $\lambda - \alpha_i$ is not a weight of $V(\lambda)$, i.e. $\operatorname{ad} F_{\alpha_i}(v_\lambda) = 0$. Since also $\operatorname{ad} E_{\alpha_i}(v_\lambda) = 0$, we see that the highest space is trivial as a representation of $\mathfrak{sl}_2$-triple $(E_{\alpha_i}, F_{\alpha_i}, H_{\alpha_i})$ corresponding to α_i, or equivalently $\operatorname{ad} H_{\alpha_i}(v_\lambda) = [\alpha_i, \lambda]v_\lambda = 0$. In other words, the λ for which $V(\lambda)^{[P,P]\neq 0}$ are those orthogonal to all of the simple coroots corresponding to P.

On the other end, the space $V(\lambda)^{U_P}$ will always contain the highest weight vector, but it might contain more. It contains a vector v if and only if the raising operators corresponding to the complement of the $\{\alpha\} \leftrightarrow P$ in the roots all kill v. In this case, I do not see such a clean description of the fixed subspace.

Singularities. By an analogous discussion to that in §1.1, $G/[P,P]$ will be singular if there is any non-trivial relation in (3). By the same analysis as in §1.1, such a non-trivial relation will occur as soon as there are at least two distinct fundamental weights. Therefore, it must be the case that P is a maximal parabolic, hence there is only a single ω_j. However, this is not sufficient. We would also need that $\mathrm{Sym}^n V(\omega_j)$ is irreducible for each n. It probably suffices for $\mathrm{Sym}^2 V(\omega_j)$ by the analogue of part (e) of Problem 1, which however I didn't solve.

1.3 Problem 3

(a) We break the argument up into several steps.

The case $\mathbb{A}^1$. We first show the result for $X = \mathbb{A}^1$. We are seeking a scheme X_n such that

$$\mathrm{Hom}(\mathrm{Spec}\ R, X_n) = \mathrm{Hom}(\mathrm{Spec}\ R[t]/t^n, Y) = \mathrm{Hom}(k[x], R[t]/t^n) \cong R[t]/t^n.$$

Since the identification of sets $R[t]/t^n \cong R^n$ is reasonably natural, we guess that $X_n = \mathbb{A}^n$ works. It is easy to check that the identification $\mathrm{Hom}(\mathrm{Spec}\ R, X_n) = \mathrm{Hom}(\mathrm{Spec}\ R[t]/t^n, X)$ with this choice is functorial.

The construction commutes with limits. Now we observe that the construction $X \mapsto X_n$ is compatible with limits. Indeed, suppose that we have $X = \varprojlim X_\alpha$. Then

$$\begin{aligned}
\mathrm{Hom}(\mathrm{Spec}\ R, \varprojlim (X_\alpha)_n) &\cong \varprojlim \mathrm{Hom}(\mathrm{Spec}\ R, (X_\alpha)_n) \\
&\cong \varprojlim \mathrm{Hom}(\mathrm{Spec}\ R[t]/t^n, X_\alpha) \\
&\cong \mathrm{Hom}(\mathrm{Spec}\ R[t]/t^n, \varprojlim X_\alpha)
\end{aligned}$$

This means that if we can construct $(X_\alpha)_n$ then we will have constructed $\left(\varprojlim X_\alpha\right)_n$.

Limits of affine are affine. Moreover, we recall that limits of affine schemes are affine. Indeed, recall that if X is an *affine* scheme and T is any scheme, then there is a natural equivalence

$$\mathrm{Hom}(T, X) \cong \mathrm{Hom}(\Gamma(X, \mathcal{O}_X), \Gamma(T, \mathcal{O}_T))$$

Then if $X_\alpha = \mathrm{Spec}\ A_\alpha$, we claim that $\varprojlim X_\alpha = \mathrm{Spec}\ \varinjlim A_\alpha$. Indeed, if T is a test scheme

$$\begin{aligned}
\mathrm{Hom}(T, \mathrm{Spec}\ \varinjlim A_\alpha) &\cong \mathrm{Hom}(\varinjlim A_\alpha, \Gamma(T, \mathcal{O}_T)) \\
&\cong \varprojlim \mathrm{Hom}(A_\alpha, \Gamma(T, \mathcal{O}_T)) \\
&\cong \varprojlim \mathrm{Hom}(T, \mathrm{Spec}\ A_\alpha) \\
&\cong \mathrm{Hom}(T, \varprojlim \mathrm{Spec}\ A_\alpha).
\end{aligned}$$

In particular, if $(X_\alpha)_n$ is affine for every α then so is $\left(\varprojlim X_\alpha\right)_n$.

All affines are limits of $\mathbb{A}^1$. The existence of X_n for all affine schemes X will be established once we know that all affine schemes over k are limits of $\mathbb{A}^1_k$. But indeed, any such scheme can be presented as $\operatorname{Spec} k[\{x_i : i \in I\}]/(\{f_j : j \in J\})$ where I and J are arbitrary index sets, and this is the limit of the diagram

$$(\mathbb{A}^1)^J \xrightarrow{t_j \mapsto f_j} (\mathbb{A}^1)^I \to 0.$$

Inspecting the argument easily reveals that if X was of finite type, then so is X_n.

(b) We first show that X_∞ has the property that

$$\operatorname{Hom}(\operatorname{Spec} R, X_\infty) \cong \operatorname{Hom}(\operatorname{Spec} R[[t]], X_\infty).$$

(This was the point of the construction!) Indeed, we may assume $X = \operatorname{Spec} A$, and then

$$\begin{aligned} \operatorname{Hom}(\operatorname{Spec} R, X_\infty) &= \operatorname{Hom}(\operatorname{Spec} R, \varprojlim X_n) \\ &= \varprojlim \operatorname{Hom}(\operatorname{Spec} R, \operatorname{Spec} A_n) \\ &= \varprojlim \operatorname{Hom}(A, R[t]/t^n) \\ &= \operatorname{Hom}(A, R[[t]]) \end{aligned}$$

Now we study the smoothness. Since my definition of smooth includes "locally of finite type," it seems that what's meant is formal smoothness.

Definition 1.2. We say that $X \to Y$ is *formally smooth* if for every k-algebra A and square-zero ideal $I \subset A$, and every map $\operatorname{Spec}(A/I) \to X$ there exists a lift $\operatorname{Spec} A \to X$. (Think of this as meaning that we can extend a map into a formally smooth scheme to a formal neighborhood of the domain.)

$$\begin{array}{ccc} \operatorname{Spec} A/I & \longrightarrow & X \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} k \end{array}$$

Now this definition is well adapted to checking formal smoothness.

In one direction, suppose that X is formally smooth. Then we want to show that for every diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I & \longrightarrow & X_\infty \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} k \end{array}$$

there is a lift $\operatorname{Spec} A \to X_\infty$. By what we just proved, a lift $\operatorname{Spec} A \to X_\infty$ for this diagram is equivalent to a lift of the diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I[[t]] & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ \operatorname{Spec} A[[t]] & \longrightarrow & \operatorname{Spec} k \end{array}$$

But since $A/I[[t]] \cong A[[t]]/I[[t]]$, and $I[[t]]$ has square zero in $A[[t]]$ since I does in A, such a lift does exist by the formal smoothness of X.

Now for the other direction, suppose that X_∞ is formally smooth. We want to find a lift for a diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} k \end{array}$$

Of course, such a diagram induces, via the natural maps $\operatorname{Spec} A[[t]] \to \operatorname{Spec} A$ and $\operatorname{Spec} A[[t]]/I[[t]] \to \operatorname{Spec} A/I$, a diagram

$$\begin{array}{ccccc} \operatorname{Spec} A[[t]]/I[[t]] & \longrightarrow & \operatorname{Spec} A/I & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} A[[t]] & \longrightarrow & \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} k \end{array}$$

This gives a lift $l\colon \operatorname{Spec} A[[t]] \to X$, but it need not factor through $\operatorname{Spec} A$. However, note that $\operatorname{Spec} A[[t]] \to \operatorname{Spec} A$ has a section s, induced by the quotient map $A[[t]] \to A$, so we *do* get a map $\operatorname{Spec} A \to X$ by pre-composing with s. This still lifts $\operatorname{Spec} A/I \to X$, since there is similarly a section $\operatorname{Spec} A/I \to \operatorname{Spec} A/I[[t]]$.

(c) We note that our general recipe produces the following description of X_∞ if $X = \operatorname{Spec} k[\{x_i\}]/(\{f_j\})$. For each x_i, we introduce new variables $x_{i(0)}, x_{i(1)}, \ldots$ which we think of as describing the coefficients of a power series

$$x_{i[[t]]} := x_{i(0)} + x_{i(1)}t + x_{i(2)}t^2 + \ldots$$

For each f_j, we quotient by the relations corresponding to $f_j(x_{i[[t]]}) = 0$.

This description makes it clear that if we have a closed embedding $Y \hookrightarrow X$, which at the level of rings means that $X = \operatorname{Spec} k[\{x_i\}]/(\{f_j\})$ and Y is the spectrum of a ring with more relations, then $Y_\infty \to X_\infty$ is a closed embedding.

(d) Since the construction $X \rightsquigarrow LX$ is qualitatively identical to the construction $X \rightsquigarrow X_\infty$, we see that LX is compatible with limits, and hence it suffices to show that LX exists if

$X = \mathbb{A}^1$. In that case, we have

$$LX(R) = \mathrm{Hom}(k[t], R((t))) \cong R((t)) \cong \varprojlim_n t^{-n} R[[t]].$$

This suggests that we can build LX as an inductive limit over X_∞. Specifically, we set

$$LX = \varinjlim(\ldots \to X_\infty \xrightarrow{t} X_\infty \xrightarrow{t} X_\infty \to \ldots)$$

where by t we mean the map corresponding to "multiplication by t." More explicitly, we have $X_\infty = \mathrm{Spec}\; k[x_0, x_1, x_2, \ldots]$ which we think of as modeling a power series $x_0 + x_1 t + x_2 t^2 + \ldots$. The "multiplication by t map" is then obviously $x_i \mapsto x_{i+1}$.

1.4 Problem 4.

As practice, let's attempt to compute X_∞ explicitly. First off,

$$\mathbb{A}^\infty = k[\{x_{0(i)}\}_{i=0}^\infty, \ldots \{x_{n(i)}\}_{i=0}^\infty].$$

To find X_∞, we introduce relations modelling the equation

$$\left(\sum_{i=0}^\infty x_{0(i)} t^i\right)^2 = \left(\sum_{i=0}^\infty x_{1(i)} t^i\right)^2 + \ldots + \left(\sum_{i=0}^\infty x_{n(i)} t^i\right)^2.$$

The relations are

$$\begin{aligned} x_{0(0)}^2 &= x_{1(0)}^2 + \ldots + x_{n(0)}^2 \\ 2x_{0(0)}x_{0(1)} &= \sum_{j=1}^n 2x_{j(0)}x_{j(1)} \\ &\vdots \qquad \vdots \end{aligned}$$

Then the formal neighborhood of γ is the spectrum of the completion of X_∞ at the ideal generated by $x_{0(1)} - 1, x_{1(1)} - 1$, and all other variables. Unfortunately, it doesn't seem like we are achieving the desired factorization by pursuing this presentation.

♠♠♠ TONY: [to be continued...]

1.5 Problem 5.

Meow...

2 Problem Set 2

2.1 Problem 1

(a) If $G = \mathrm{SL}(2)$, then $G/B \cong \mathbb{P}^1$ so Z^α is just the space of based quasi-maps $\mathbb{P}^1 \to \mathbb{P}^1$. Now, by a definition a quasi-map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree α is a subsheaf

$$O_{\mathbb{P}^1} \subset O_{\mathbb{P}^1}^{\oplus 2}(\alpha).$$

This amounts to an injection $O_{\mathbb{P}^1} \subset O_{\mathbb{P}^1}^{\oplus 2}(\alpha)$ up to scalar, which consists of two maps $O_{\mathbb{P}^1} \to O_{\mathbb{P}^1}(\alpha)$, not both zero, up to scalar. Now, a map $O_{\mathbb{P}^1} \hookrightarrow O_{\mathbb{P}^1}(\alpha)$ is a non-zero polynomial of degree α.

Now, we can choose a basing by sending $\infty \mapsto \infty$. This means that if the two polynomials restricted to $\mathbb{A}^1$ are $f(t)$ and $g(t)$, corresponding to $\frac{f(t)}{g(t)}$, then $\deg f < \deg g$. We can use the freedom of scaling to make $\deg g$ monic, so that a based quasi-map will be represented uniquely by a pair (f, g) where $f = a_{n-1}t^{n-1} + \dots$ and $g = t^n + b_{n-1}t^{n-1} + \dots$. The coefficients $(a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1})$ describe an isomorphism between this space and $\mathbb{A}^{2n}$.

The subset Z_0^α describing actual maps corresponds to the subset of pairs (f, g) such that f and g have no common factors. This can be described as the open subset where the resultant and f and g is non-zero.

(b) We first choose an embedding of G/B. Since G/B parametrizes flags $0 \subset V_1 \subset V_2 \subset V$, we have an embedding

$$G/B \hookrightarrow \{V_1\} \times \{V_2\} = G(1,3) \times G(2,3) = \mathbb{P}^2 \times (\mathbb{P}^2)^\vee.$$

The condition cutting out G/B that $V_2 \supset V_1$ translates into the point in $(\mathbb{P}^2)^\vee$ being orthogonal to the point in $\mathbb{P}^2$ (see §1.1). In terms of coordinates $[x_0, x_1, x_2]$ and $[y_0, y_1, y_2]$ on the two copies of $\mathbb{P}^2$, this says that G/B is the subset cut out by

$$x_0y_0 + x_1y_1 + x_2y_2 = 0.$$

We can describe a quasi-map to G/B as a quasi-map to $\mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ landing in G/B.

So we have to describe two maps $\mathbb{P}^1 \to \mathbb{P}^2$, or two injections of sheaves up to scalars:

$$O_{\mathbb{P}^1} \hookrightarrow O_{\mathbb{P}^1}(1)^{\oplus 3}$$

(The reason that we are considering $O(1)$ is because $\alpha = 1 \cdot \alpha_1 + 1 \cdot \alpha_2$, so 1 is the coefficient of both the fundamental weights.) Each such injection is given by three choices of global sections of $O_{\mathbb{P}^1}(1)$, not all zero, up to simultaneous scaling.

If we consider *based* maps where the map $\mathbb{P}^1 \to \mathbb{P}^2$ must send ∞ to the point $[0 : 0 : 1]$, then the third linear polynomial must have degree 1 and the others must have degree 0. By renormalizing, any such choice can be uniquely represented as $[a : b : c + t]$.

Similarly, a choice quasi-map $\mathbb{P}^1 \to (\mathbb{P}^2)^\vee$ that sends ∞ to, say, $[1 : 0 : 0]$ must be of the form $[d + t, e, f]$. Therefore, we find that Z^α has generators a, b, c, d, e, f such to the relations coming from

$$a \cdot (d + t) + b \cdot e + (c + t) \cdot f = 0$$

which are

$$\begin{aligned} ad + be + cf &= 0 \\ a + f &= 0. \end{aligned}$$

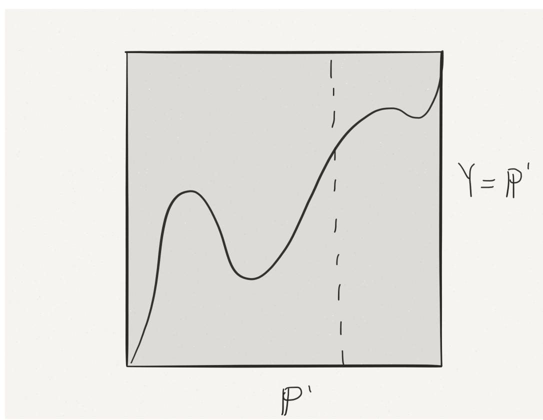
♠♠♠ TONY: [compare this with our actual definition of quasimap...]

2.2 Problem 2

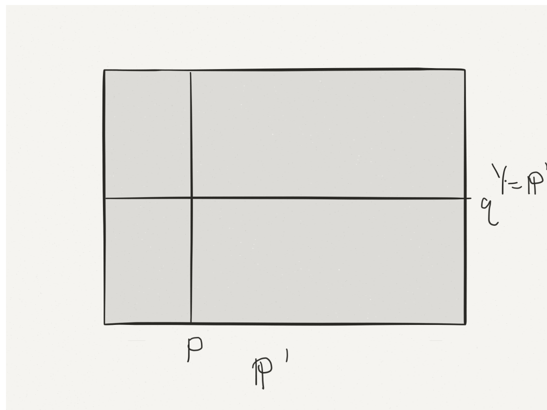
We study the example $Y = \mathbb{P}^1$ first.

$\alpha = 1$. The k-points of $\mathrm{Graph}^\alpha(Y)$ consists of maps from a rational curve to $\mathbb{P}^1 \times \mathbb{P}^1$ with the homology fundamental class pushing forward to a class of type $(1,1) \in H_2(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$. In fact, we know the Chow ring of $\mathbb{P}^1 \times \mathbb{P}^1$: the Picard group is generated by the two rulings, represented by lines e and f satisfying $e \cdot e = f \cdot f = 0$ and $e \cdot f = 1$.

Generically, the image of such a curve is a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the graph of a morphism $\mathbb{P}^1 \to \mathbb{P}^1$, since its intersection with any "vertical" fiber is 1 (by our description of the Chow ring).



Such a graph should obviously be sent to its corresponding map in $\mathrm{QMaps}(\mathbb{P}^1, \mathbb{P}^1)$. There is another type of map in $\mathrm{Graph}(\mathbb{P}^1)$, namely a map from the union of two $\mathbb{P}^1$ to the union of two rulings in $\mathbb{P}^1 \times \mathbb{P}^1$.



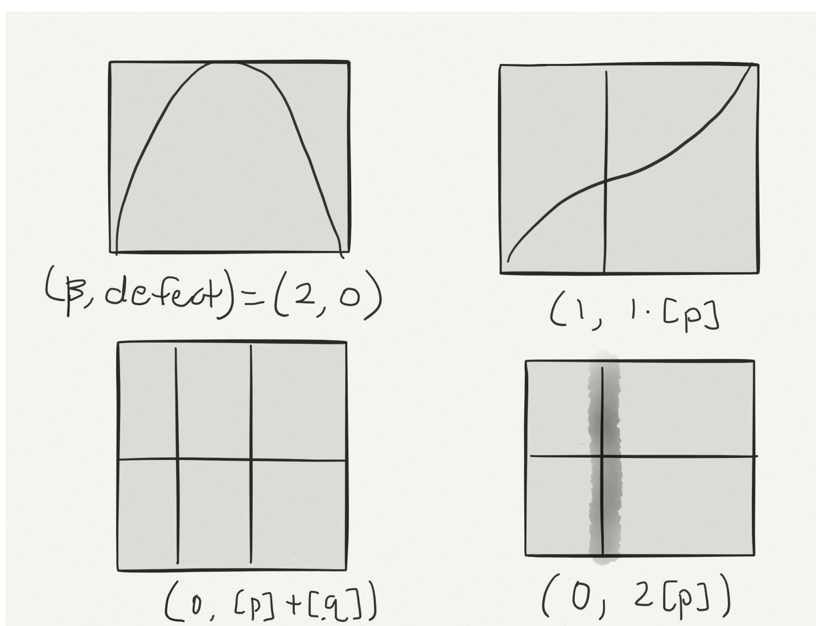
What quasi-maps do these go to? Recall that we had a stratification

$$\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, Y) = \bigsqcup_{\beta \le \alpha} \mathrm{Maps}^{\beta}(\mathbb{P}^1, Y) \times \mathrm{Sym}^{\alpha-\beta} Y.$$

In the case $\alpha = 1$, the only possibility is $\beta = 0$, i.e. constant maps into $\mathbb{P}^1$. The location of the vertical fiber describes the defect, while the location of the horizontal map describes the constant map.

You can easily check that this indeed describes everything in $\mathrm{Graph}^{\alpha}(Y)$.

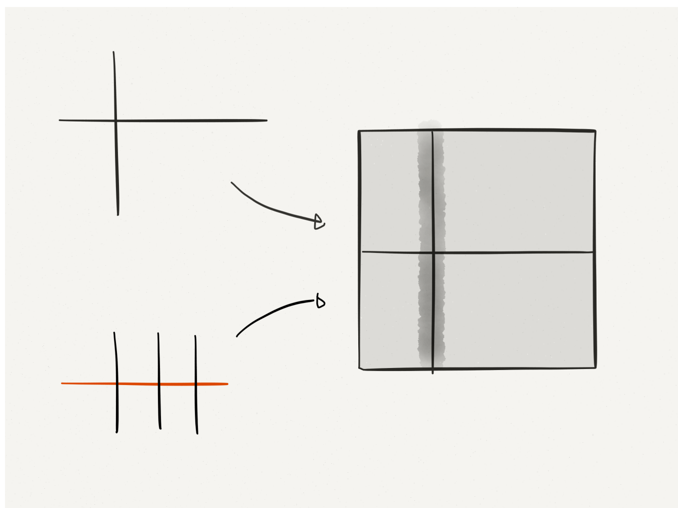
$\alpha = 2$. In this case, we have four strata.



The open stratum $\beta = \alpha$ corresponds to honest maps. The stratum $\beta = 0$ describes maps from a rational curve with two components meeting at a node into $\mathbb{P}^1 \times \mathbb{P}^1$, with image of type $(0, 1)$ and $(1, 1)$. The stratum $\beta = 0$ can be divided into two substrata. The first consists of constant maps with defect $[p] + [q]$ where $p \neq q$, and the second consists of constant maps with defect $2 \cdot [p]$.

So why does the map fail to be an automorphism? For each image curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is described, we have described a unique quasi-map, and it is evident that every quasi-map is hit by this correspondence.

The punchline is that in this case a stable map $\mathrm{Graph}^\alpha(Y)$ is *not* determined by the image. A curve in the deepest stratum corresponds to two possible stable maps. It could be the image of a union of two copies of $\mathbb{P}^1$ glued together along a node, with one mapping 2:1 onto its image. Alternatively, it could be the image of a union of *four* copies of $\mathbb{P}^1$, with three components glued to a central one via nodes, which is then contracted.



Now returning to the general case, it is fairly clear how to define the map $\mathrm{Graph}^\alpha(Y) \to \mathrm{QMaps}^\alpha(Y)$. To each stable map in $\mathrm{Graph}^\alpha(Y)$ we associate the image curve C, which must be a union of a graph of an honest morphism (the component with this property will be the unique one whose fundamental class pushes forward to $[\mathbb{P}^1]$ under the first projection map) and vertical components (which you can see by the excision exact sequence plus homotopy invariance, for example). In terms of the stratification on $\mathrm{QMaps}^\alpha(Y)$, the graph describest the map and the vertical components describe the defect.