PROBLEM SET 1

Problems about basic affine space:

1. Let G be a semi-simple simply connected algebraic group over an algebraically closed field k of characteristic 0 (the latter assumption is not strictly speaking necessary but things get simpler if we make it). Let B be a Borel subgroup of G and let U be its unipotent radical. The space $X = G/U$ is called the basic affine space of G. X has an action of G on the left and a commuting action of $T = B/U$ on the right.

a) Describe X explicitly for $SL(2)$ and $SL(3)$.

b) Show that the algebra A of global functions on X can be described as follows. Let Λ be the weight lattice of G and let Λ_{dom} denote the dominant weights. For $\lambda \in \Lambda_{dom}$ let $V(\lambda)$ denote the corresponding irreducible representaton of G. Then

$$A = \bigoplus_{\lambda \in \Lambda_{dom}} V(\lambda).$$

The multiplication is described as follows. For $\lambda, \mu \in \Lambda_{dom}$ we the representation $V(\lambda+\mu)$ is naturally a direct summand of $V(\lambda) \otimes V(\mu)$. Then the restriction of the multiplication in A to $V(\lambda) \otimes V(\mu)$ is the corresponding projection $V(\lambda) \otimes V(\mu) \to V(\lambda+\mu)$.

c) Show that A is finitely generated (more precisely, A is generated by $V = \oplus V(\omega_i)$, where ω_i runs through the fundamental weights of G). Set $\overline{X} = \mathrm{Spec}(A)$. Show that X is open in $\overline{X}$. The variety $\overline{X}$ is called the affine closure of X.

d) Describe $\overline{X}$ for $SL(2)$ and $SL(3)$.

e) Show that $\overline{X}$ is singular unless $G = SL(2)$.

f*) (This exercise is more difficult and will not be used in the future). Show that A is a quotient of $\mathrm{Sym}(V)$ by quadratic relations.

2. Let P be a parabolic subgroup of G. Let U_P be its unipotent radical. Show that the spaces G/U_P and $G/[P,P]$ are quasi-affine. Describe all cases when the affine closure of $G/[P,P]$ is smooth.

Problems about formal arcs

3. Let X be an affine scheme over k.

a) Show that for every $n > 0$ there exists a scheme X_n over k such that for any k-algebra R we have $X_n(R) = X(R[t]/t^n)$. Show that if X is of finite type over k then so is X_n.

b) Note that we have a natural map $X_n \to X_m$ for every $n \geq m$. Set X_∞ to be the projective limit of all the X_n. This is typically a scheme of infinite type over k (even if X is of finite type). Show that X_∞ is smooth over k if and only if X is smooth.

c) Let Y be a closed subscheme of X. Show that Y_∞ is a closed subscheme of X_∞.

d) Assume that X is affine. Show that there exists an ind-scheme LX such that $LX(R) = X(R((t)))$. Explain why the assumption that X is affine is important. Show that X_∞ is a closed subscheme of LX.

The following result is due to Drinfeld and Grinberg-Kazhdan.

Theorem: *Let U be a smooth open subset of X and let Y be its complement. Let $\gamma \in X_\infty(k) \backslash Y_\infty(k)$. Then the formal neighbourhood of γ in X_∞ has a decomposition $S \times W$ where S is isomorphic to the formal neighbourhood of 0 in $\mathbb{A}^\infty$ and W is is isomorphic to the formal neighbourhood of a point in a scheme of finite type over k.*

4. Let X be the quadratic cone $x_0^2 = x_1^2 + \cdots x_n^2$. Let $\gamma(t) = (t^i, t^i, 0, \cdots, 0)$ where $i = 1$ or $i = 2$. Describe explicitly how the above theorem works for this γ.

5. Let G be as before and let α be a positive element of the coroot lattice of G. Let $Y^\alpha = \mathrm{QM}^\alpha \times G(\mathcal{O})$ where QM^α denotes the space of quasi-maps $\mathbb{P}^1 \to G/B$ of degree α and $\mathcal{O} = k[[t]]$.

(iii) A trivialization of the T-bundle involved in the definiton of γ to in the formal neighbourhood of 0.

We have an obvious projection map $Y^\alpha \to \mathrm{QM}^\alpha$. This is map is obviously formally smooth since $G(\mathcal{O})$ is.

a) Define a $G(\mathcal{O})$-equivariant map $f_\alpha : Y^\alpha \to \overline{G/U}_\infty/T(\mathcal{O})$ and show that it is formally smooth at any point of $\mathrm{QM}^\alpha \times G(\mathcal{O})$ which has defect only at 0 (hint: restrict your attention to the formal neighbourhood of 0)

b) Show that any $\gamma \in (\overline{G/U}_\infty)/T(\mathcal{O})$ lies in the image of some f_α. How to determine which α to choose?

c) Deduce from a) and b) that for any $\gamma \in (\overline{G/U}_\infty)/T(\mathcal{O})$ you can take W in the Drinfeld-Grinberg-Kazhdan theorem to be the formal neighbourhood of a point in some QM^α. What can you say about α and that point?