Geometry Representation Theory and Quasi-Maps into Flag Varieties

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1 Semi-infinite flag varieties

1.1 The flag variety

Fix a semisimple, simply-connected algebraic group G. The discussion applies to general fields, but for simplicity let's just take everything to be over $\mathbb{C}$. The *flag variety* of G is $\mathrm{Fl} = G/B$ where $B \subset G$ is a Borel subgroup.

The flag variety is a union of *Schubert cells*:

$$\mathrm{Fl} = \bigsqcup_{w \in W} \mathrm{Fl}_w$$

and it is well known that $\dim \mathrm{Fl}_w = \ell(w)$. The closure $\overline{\mathrm{Fl}_w}$ is (usually) singular. If we denote by IC_w the intersection cohomology complex of $\overline{\mathrm{Fl}}_w$, then for $y \in W$ the Poincaré polynomial of IC_w at a point of Fl_y is given by the *Kazhdan-Lusztig polynomials*. For our purposes, you can basically take this to be the definition of the Kazhdan-Lusztig polynomials.

If C is the category of perverse sheaves on Fl which is construtible along the Schubert stratification, then it is known that C is equivalent to category O for $\mathfrak{g} = \mathrm{Lie}\, G$.

There are generalizations of this, e.g. parabolic versions. We're going to be interested in *infinite-dimensional* generalizations. From the finite-dimensional story, we emphasize that the Bruhat cells are singular, so the behavior of their IC sheaves tells us interesting geometric information about their singularities.

1.2 Loop groups

Let $\mathcal{K} = \mathbb{C}((t))$ and $O = \mathbb{C}[[t]] \subset \mathcal{K}$. The loop group of G, at least as an abstract group, is $G(\mathcal{K})$. (We'll turn our attention to describing its algebro-geometric structure later.) We'll call this the *formal loop group*. You can think of this as being like a Kac-Moody group. So what should be the meaning of a flag variety for the loop group?

There are different possible answers, each suitable for its own purposes. As in the finite-dimensional case, the flag is a homogeneous space for the group in question. There are three possible generalizations of B to the infinite-dimensional case:

1. I_+, which gives the *thin flag variety* $\mathrm{Fl}_+ := G(\mathcal{K})/I_+$,
2. I_-, which gives the *thick flag variety* $\mathrm{Fl}_- := G(\mathcal{K})/I_-$,
3. $B_{\infty/2}$, which gives the *semi-infinite flag variety* $G(\mathcal{K})/B_{\infty/2}$.

These are all infinite-dimensional, but the first two are only "mildly infinite-dimensional." We'll see work with them by reducing to finite-dimensional cases. However, the semi-infinite flag variety is "genuinely" infinite-dimensional. So we'll spend a lot of time discussing what it even means to have "singularities" of Schubert varieties for the semi-infinite flag variety.

Remark 1.1. What if we try the naïve generalization $G(\mathcal{K})/G(O)$? In fact this is close to the semi-infinite flag variety. However, the choices I_+ and I_- are natural because they exist for any Kac-Moody group, while $B_{\infty/2}$ is something quite special to the loop group.

1.3 Affine flag varieties

1.3.1 The thin flag variety

Definition 1.2. We define $I_+ \subset G(O)$ to be the pre-image of some Borel subgroup $B \subset G$ under evaluation at $t = 0$:

$$\begin{array}{ccc} I_+ & \hookrightarrow & G(O) \\ \downarrow & & \downarrow \text{evaluation at } t = 0 \\ B & \hookrightarrow & G \end{array}$$

So $G(O)/I_+ \cong G/B$.

We define the *thin flag variety* to be $\mathrm{Fl}_+ = G(\mathcal{K})/I_+$.

Theorem 1.3. *There exists a natural ind-scheme structure on $G(\mathcal{K})/I_+$ such that all I_+-orbits (and their closures) are finite-dimensional.*

This is related to the discussion of the affine Grassmannian from the first week. Recall that $\mathrm{Gr}_G = G(\mathcal{K})/G(O)$. This admits a natural quotient map from Fl_+, whose fiber is G/B.

$$\begin{array}{ccc} G/B & \longrightarrow & \mathrm{Fl}_+ \\ & & \downarrow \\ & & \mathrm{Gr}_G \end{array}$$

Recall that we considered $G(O)$-orbits on $G(\mathcal{K})/G(O)$, and found that they were finite-dimensional with closures being projective varieties. Moreover, they were parametrized by the Cartan decomposition:

$$G(O)\backslash G(\mathcal{K})/G(O) = \Lambda^+$$

where Λ is the coweight lattice of G and Λ^+ is the subset of dominant coweights.

Remark 1.4. $G(O)$ contains I_+, and we are thinking of I_+ as an analogue of the Borel. In the finite-dimensional case, a subgroup containing a Borel subgroup is called parabolic, so we should think of $G(O)$ as a parabolic subgroup of $G(\mathcal{K})$. The $G(O)$-orbits on G_R should be thought of as being like P-orbits on G/P where $P \subset G$ is some parabolic.

We can also say what the I_+ orbits on $G(\mathcal{K})/I_+$ are parametrized by. The answer is the *affine Weyl group*: since W acts on Λ, we can form $W_{\mathrm{aff}} = \Lambda \rtimes W$, and we have

$$\boxed{I_+\backslash G(\mathcal{K})/I_+ \cong W_{\mathrm{aff}}.}$$

An important property of the thin flag variety is that *its Schubert varieties are finite-dimensional.*

1.3.2 The thick flag variety

Now let me comment briefly on the thin flag variety, Fl_-. It is basically like taking an "opposite Borel subgroup" to I_+. In the finite-dimensional world, a Borel is isomorphic to its opposite, but not so in the infinite-dimensional world.

Choose $B_- \subset G$ opposite to B. We will construct $I_- \subset G[t^{-1}]$. Note that $G(\mathcal{O}) \cap G[t^{-1}] = G$ (which you can think of as being the constant loops).

Definition 1.5. We define I_- to be the pre-image in $G[t^{-1}]$ of B_- under evaluation at $t^{-1} = 0$:

$$\begin{array}{ccc} I_- & \hookrightarrow & G[t^{-1}] \\ \downarrow & & \downarrow \text{evaluate at } t^{-1} = 0 \\ B_- & \hookrightarrow & G \end{array}$$

In particular, $I_+ \cap I_- = B \cap B_-$, the maximal torus of G.

We define the *thick flag variety* $\mathrm{Fl}_- = G(\mathcal{K})/I_-$.

Recall that the thin flag variety was an ind-scheme. By contrast, the thick flag variety has an honest scheme structure!

Theorem 1.6. *There exists a natural scheme structure on* Fl_-.

Now we might naïvely again consider I_--orbits on Fl_- in analogy to the thin case, but this turns out to be very bad. For example, $I_- \backslash G(\mathcal{K})/I_-$ is not discrete, and it's certainly not the affine Weyl group, which is what we would like.

Exercise 1.7. Show that $I_- \backslash G(\mathcal{K})/I_-$ is not countable.

The solution turns out to be just to take I_+ orbits again. Then we have an isomorphism of sets

$$\boxed{I_+ \backslash G(\mathcal{K})/I_- \cong W_{\mathrm{aff}}.}$$

This gives a stratification

$$\mathrm{Fl}_- = \bigsqcup_{w \in W_{\mathrm{aff}}} \mathrm{Fl}_{-,w}\,.$$

Theorem 1.8. *The Schubert varieties* $\mathrm{Fl}_{-,w}$ *are infinite-dimensional. However, if* $\mathrm{Fl}_{-,y} \subset \overline{\mathrm{Fl}}_{-,w}$ *then* $\mathrm{Fl}_{-,y}$ *has finite codimension in* $\overline{\mathrm{Fl}}_{-,w}$.

Since the orbit closure of $\mathrm{Fl}_{-,w=1}$ is the whole ind-scheme, this amounts to saying that every Schubert cell has finite codimension.

Remark 1.9. These two constructions can be performed in some generality for Kac-Moody groups.

1.4 The semi-infinite flag variety

We need to define $\mathcal{B}_{\infty/2}$. This will be a subgroup of $\mathcal{B}(\mathcal{K})$, which you can think of as being "almost" equal to it.

Definition 1.10. Let $B = TU$, where U is the unipotent radical of B. Then we define

$$B_{\infty/2} = T(\mathcal{O})U(\mathcal{K})$$

So $B(\mathcal{K})/B_{\infty/2} = T(\mathcal{K})/T(\mathcal{O}) = \Lambda$.

We define $\mathrm{Fl}_{\infty/2} = G(\mathcal{K})/B_{\infty/2}$.

Theorem 1.11. *There is a natural ind-scheme structure on* $\mathrm{Fl}_{\infty/2}$.

Goal: explain what it means to talk about singularities of Schubert varieties in $\mathrm{Fl}_{\infty/2}$.

First, what *are* the Schubert varieties in $\mathrm{Fl}_{\infty/2}$? We've taken I_+ orbits in the two previous cases, and we do the same here.

Lemma 1.12. *We have a natural identifications*

$$I_+ \backslash \mathrm{Fl}_{\infty/2} \cong W_{\mathrm{aff}}$$

In fact, it is easy to see that Λ acts on the right in $\mathrm{Fl}_{\infty/2} = G(\mathcal{K})/B_{\infty/2}$ since it normalizes $\mathcal{B}_{\infty/2}$, and commutes with the left action of I_+.

Lemma 1.13. *We have a natural identification*

$$G(O)\backslash \mathrm{Fl}_{\infty/2} \cong \Lambda$$

in which the Λ*-action is the natural one on* W_{aff}.

In all three cases, we have seen that

- the I_+-orbits are in one-to-one correspondence with W_{aff} and
- the $G(O)$-orbits were in 1-1 correspondence with Λ, the *coweight* lattice of G.

For $T \subset G$ a maximal torus, a coweight $\lambda \in \Lambda$ is a map $\mathbb{G}_m \to T$. Taking $\mathcal{K}$-points, we get $\lambda \colon \mathcal{K}^\times \to T(\mathcal{K}) \subset G(\mathcal{K})$ and we denote $\lambda(t) = t^\lambda$.

Lemma 1.14. *In all cases,* $\lambda \mapsto G(O)t^\lambda$ *is a bijection between* $G(O)\backslash \mathrm{Fl}_?$ *and* Λ.

Exercise 1.15. Formulate the statement for I^+-orbits.

Problem. As before, we have a stratification of $\mathrm{Fl}_{\infty/2}$ into Schubert cells parametrized by the affine Weyl group:

$$\mathrm{Fl} = \bigsqcup_{w \in W_{\mathrm{aff}}} \mathrm{Fl}_{\infty/2,w}\,.$$

In this case, however, the Schubert cells have infinite dimension *and* codimension. This explains why we call $\mathrm{Fl}_{\infty/2}$ the "semi-infinite" flag variety.

You can think of the following toy model. The Laurent power series $\mathbb{C}((t))$ is infinite dimensions in two ways. The space $\mathbb{C}[[t]] \subset \mathbb{C}((t))$ is infinite-dimensional, but it only kills one of the infinite directions, so its codimension is still infinite. That is sort of like what's going on here.

Goal: explain in what sense the singularities of $\overline{\mathrm{Fl}}_{\infty/2,w}$ are finite-dimensional.

Let us end with a remark. Informally, $\mathrm{Fl}_{\infty/2}$ is the universal cover of $G(\mathcal{K})/B(\mathcal{K})$. You can think of $G(\mathcal{K})/B(\mathcal{K}) = G/B(\mathcal{K})$ as the loop space of the flag variety. Then $G/B(\mathcal{K})$ is not simply-connected, as its fundamental group will be isomorphic to the second homology group of the flag variety, which is $H_2(G/B, \mathbb{Z}) \cong \Lambda$ (by Hurewicz's Theorem, since G/B is simply-connected). The semi-infinite flag variety Fl_{-} is a Λ-cover of $G/B(\mathcal{K})$, so in that sense you can think of it as the universal cover.

2 Geometry of the semi-infinite flag variety

The goal of this section is to put the construction we discussed on a set-theoretic level in the first section on a solid algebro-geometric footing. We will build towards formulating a general theorem of Drinfeld and Grinberg-Kazhdan which will imply that at least on the level of formal neighborhoods, the Schuber varieties $\overline{\mathrm{Fl}}_{\infty/2,w}$ have finite-dimensional singularities. This motivates the definition of quasi-maps which will serve as explicit models for these singularities.

Recall that we said that I_+-orbits on Fl_+ are of finite dimension, the I_+-orbits on Fl_- are of finite codimension, and the I_+-orbits on $\mathrm{Fl}_{\infty/2}$ are of both infinite dimension and infinite codimension. The first two situations were at least somewhat reasonable, but concerning the third we ask:

Question. How can we work with Schubert varieties in $\mathrm{Fl}_{\infty/2}$?

Some reductions.

- The closure of the $G(\mathcal{O})$-orbits are "special cases" of the closures of I_+-orbits, in the sense that the closure of every $G(\mathcal{O})$-orbit contains a unique dense I_+-orbit. Since the codimension of I_+ in $G(\mathcal{O})$ is finite, this should tell us that it in order to study I_+-orbits, it "suffices" to study $G(\mathcal{O})$-orbits.
- Recall that we defined $\mathrm{Fl}_{\infty/2} = G(\mathcal{K})/T(\mathcal{O})U(\mathcal{K})$, which admits a right action of $\Lambda = T(\mathcal{K})/T(\mathcal{O})$ since $T(\mathcal{K})$ normalizes $U(\mathcal{K})$. The action of Λ on $G(\mathcal{O})$-orbits turns out to be simply-transitive. Therefore, all of the orbit closures look the same, so we can study them all by studying one of them.
- By the preceding observations, we can basically study the orbit of the image of the identity $e \in \mathrm{Fl}_{\infty/2} = G(\mathcal{K})/T(\mathcal{O})U(\mathcal{K})$. The orbit is obviously a homogeneous space for $G(\mathcal{O})$, and the stabilizer is precisely $G(\mathcal{O}) \cap T(\mathcal{O})U(\mathcal{K}) = B(\mathcal{O})$. So the orbit is the image of the natural injection $G(\mathcal{O})/B(\mathcal{O}) \hookrightarrow \mathcal{F}_{\infty/2}$, and we want to consider its closure.

At this point it's time for us to embark on a more algebro-geometric discussion of everything.

2.1 Arc and loop spaces

Let $D = \operatorname{Spec} \mathbb{C}[[t]] = \operatorname{Spec} \mathcal{O}$ and $D^* = \operatorname{Spec} \mathbb{C}((t)) = \operatorname{Spec} \mathcal{K}$. We think of D as the "formal disc" and D^* as the "formal punctured disc."

Definition 2.1. Let X be any scheme over $\mathbb{C}$. Then we can form an object X_∞ parametrizing maps $D \to X$. Formally, we define the *arc space*

$$X_\infty = \varprojlim_{n>0} X_n,$$

where X_n is the defined by the functor of points

$$X_n(R) = X(R[t]/t^n) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Spec}\ R[t]/t^n, X)$$

for R any $\mathbb{C}$-algebra. (Some refer to X_n as the scheme of *n-jets* over X.) If X is of finite type over $\mathbb{C}$, then so is X_n. The projective limit is over the natural maps $X_n \to X_m$ for $n \geq m$.

In nice situations, one has $X_\infty(R) = X(R[[t]])$. However, we warn that this is not at all obvious - it is not even true for general X! It is true if X is finite type over $\mathbb{C}$, but even then it is fairly difficult to prove. However, if X is affine (or even quasi-affine) then it is true and easy to prove.

Definition 2.2. Assume that X is quasi-affine. Then we define the functor

$$LX(R) = X(R((t))).$$

You should think of LX as being the algebro-geometric analogue of the "loop space on X."

Theorem 2.3. *LX is an ind-scheme, there is a closed embedding $X_\infty \hookrightarrow LX$.*

We're interested in applying this construction to $X = G/U$, which is traditionally called the "basic affine space of G." Note that X admits a T-action on the right, since U is a normal subgroup of B with $T = B/U$.

Example 2.4. When $G = \mathrm{SL}(2)$, we have $X = \mathbb{A}^2 \setminus \{0\}$ (see the problem sheets).

Definition 2.5. We define the *semi-infinite flag variety* to be

$$\mathrm{Fl}_{\infty/2} = LX/T(\mathcal{O}).$$

Exercise 2.6. Define this as an ind-scheme.

2.2 Schubert varieties

Now consider the orbit of $1 \in LX$ under $G(\mathcal{O})$. Since the stabilizer is $G(\mathcal{O}) \cap U(\mathcal{O})$, this can be identified with the natural inclusion $G(\mathcal{O})/U(\mathcal{O}) \hookrightarrow LX$.

Definition 2.7. If $X = G/U$ is the quasi-affine space of G, then define $\overline{X} := \mathrm{Spec}\ H^0(X, \mathcal{O}_X)$ (which is an affine closure of G/U).

Example 2.8. If $G = \mathrm{SL}(2)$, then $X = \mathbb{A}^2 \setminus \{0\}$ (as we saw earlier) and $\overline{X} = \mathbb{A}^2$.

Example 2.9. If $G = \mathrm{SL}(3)$, then $\overline{X}$ is described as follows. Let $V \cong \mathbb{C}^3$ be the standard representation of G. Then $\overline{X} = \{(v, v^*) \mid v \in V, v^* \in V^*, \langle v, v^* \rangle = 0\}$. Here T is two-dimensional, with the two dimensions acting by dilations on v and v^*, respectively.

What is the basic affine space X itself? It's supposed to be an open subset of $\overline{X}$, and it turns out to be the one where neither v nor v^* is equal to 0. Why? We just have to show that G acts transitively on such pairs with stabilizer U.

Definition 2.10. The inclusion $X \hookrightarrow \overline{X}$ induces a map $LX \hookrightarrow L\overline{X}$. We let $L\overline{X}^\circ \hookrightarrow L\overline{X}$ be the subset consisting of loops generically landing in X (i.e. such that the generic point of D lands in X, although the special fiber might not).

The space $L\overline{X}^\circ$ looks very similar to LX (and indeed, is the same on the level of $\mathbb{C}$-points), but the scheme structure is different. It turns out that $L\overline{X}^\circ$ is better, so it should be the basis for the scheme structure on $\mathrm{Fl}_{\infty/2}$. For example, LX is disconnected and $L\overline{X}^\circ$ is connected.

Lemma 2.11. *We have that* $\overline{G(O)/U(O)}$ *(closure in* $L\overline{X}^\circ$*) is* $\overline{X}_\infty \cap L\overline{X}^\circ$.

This is telling us that the orbit of the identity element under $G(O)$ in dense.

Corollary 2.12. *We have* $\overline{G(O)/B(O)}$ *(the closure in* $\mathrm{Fl}_{\infty/2}$*)* $= \overline{X}_\infty \cap L\overline{X}^\circ/T(O)$.

Conclusion: Our Schubert variety is $\overline{X}_\infty \cap L\overline{X}^\circ/T(O)$.

Let's give some examples of the difference between LX and $L\overline{X}^\circ$. Recall that we said that they have the same $\mathbb{C}$-points, which makes a little hard to see any difference, but their scheme structures looks very distinct. Typically, LX is a disjoint union of strata of $L\overline{X}^\circ$.

Example 2.13. If $\overline{X} = \mathbb{A}^1$ and $X = \mathbb{G}_m$, then we claim that LX is disconnected. By definition, $LX(\mathbb{C}) = \mathbb{C}((t))^\times$ and an element of $LX(\mathbb{C})$ is of the form $a(t) = a_n t^n$+(higher order terms) for $n \in \mathbb{Z}$. We can "stratify" by the valuation, obtaining

$$LX_n = \{a(t) \mid a(t) = a_n t^n + a_{n+1} t^{n+1} + \dots, \ a_n \neq 0\}.$$

Lemma 2.14. *The* LX_n *are the connected components of* LX.

If we consider $L\overline{X}(\mathbb{C}) = C((t))$, then it turns out that the components are glued together at 0.To see this, you have to test against arbitrary rings R. Although $L\overline{X}^\circ(\mathbb{C}) = \mathbb{C}((t))^\times$, if you consider general rings then you will find that the components are still glued.

If R is a $\mathbb{C}$-algebra, then

$$\begin{aligned} L\overline{X}(R) &= R((t)) \\ LX(R) &= R((t))^\times \\ L\overline{X}^\circ(R) &= R((t)) - \{0\}. \end{aligned}$$

The point is that if you consider a family parametrized by R, in $LX(R)$ the number n cannot jump (the coefficient of t^n is always forced to be in $R^\times$), but in $L\overline{X}^\circ(R)$ it *can* jump. This reflects the fact that in $L\overline{X} = R((t))$, the "lowest term" n can jump in families, hence $L\overline{X}^\circ$ all the LX_n get glued together.

Example 2.15. Let $Y \subset \mathbb{P}^n$ be a projective variety and $\overline{X}$ the affine cone over Y in $\mathbb{C}^{n+1}$, so $X = \overline{X} \setminus \{0\}$. Then

$$LX = \bigsqcup LX_\alpha, \quad \alpha \in H_2(Y, \mathbb{Z}).$$

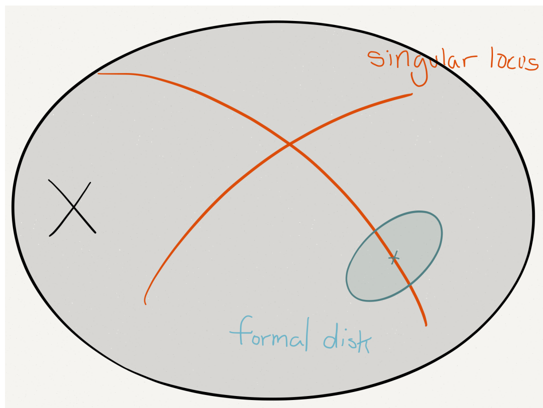
But there is no such decomposition for $L\overline{X}^\circ$.

2.3 Local model of singularities

Definition 2.16. If $\overline{X}$ is any affine variety over $\mathbb{C}$ and X is a non-singular open subscheme, then denote

$$\overline{X}_\infty^\circ = \overline{X}_\infty \cap L\overline{X}^\circ,$$

which is an ind-scheme whose $\mathbb{C}$-points are the maps from D to $\overline{X}$ which formally lie in X.



There is a general theorem about such spaces, which says that in a certain sense the singularities are finite-dimensional. More precisely, any point has a formal neighborhood (a well-defined notion for any point in any scheme) looking like a product $S \times W$, where S is a formal neighborhood in a infinite product of $\mathbb{A}^1$s - in particular, infinite-dimensional but *independent* of γ - and W is a finite-dimensional formal neighborhood.

Theorem 2.17 (Drinfeld, Grinberg-Kazhdan)**.** *Let $\gamma \in \overline{X}_\infty^\circ(\mathbb{C})$. Then the formal neighborhood of γ has a decomposition $S \times W$ where S is the formal neighborhood of 0 in $\mathbb{A}^1_\infty$ and W is a formal neighborhood of a point in a scheme of finite type.*

Remark 2.18. If $\overline{X}$ is non-singular, then the space of formal arcs is also non-singular: we can take S to be the full formal neighborhood. So the singularities of the space of formal arcs come from the singularities of $\overline{X}$. (More precisely, if $\overline{X}$ is formally smooth, then $\overline{X}_\infty$ is formally smooth).

This means that there is a finite-dimensional transversal slice in the space of deformations. That is, the space of deformations has infinitely many free dimensions, and finitely many constrained dimensions. So at least from the perspective of formal neighborhoods, the space of singularities is finite-dimensional.

Now consider $\overline{G/U}(\mathcal{O}) \cap G(\mathcal{K})/U(\mathcal{K})$, i.e. loops which generically go into G/U. This has a stratification by $G(\mathcal{O})$-orbits.

Lemma 2.19. *For a coweight λ, we have $G(\mathcal{O})t^\lambda \subset \overline{G/U}(\mathcal{O})$ if and only if λ is a sum of positive coroots (recall that we assumed G was simply-connected).*

In particular, the closure of the orbit we're interested in contains infinitely many orbits. However, the Theorem tells us that their transversal deformations are finitely constrained.

Question: Is there a nice choice for W?

This is where the space of quasi-maps comes into the picture. Philosophically, we are running into trouble because we are considering infinite loop spaces, which is horribly infinite-dimensional. In these situations, there is a way out of the trouble which always partially works. Namely, the universal way to deal with problems of ∞-dimensionality of $X_\infty, LX, \ldots$ is to replace the formal disk D by a smooth projective curve (e.g. $\mathbb{P}^1$).

This principal works quite generally, but one has to pay a price because it forces us to attack a local problem with a global model.

Definition 2.20. The *space of quasi-maps to the flag variety* $\mathrm{QMaps}(\mathbb{P}^1, G/B)$ is $\mathrm{Maps}(\mathbb{P}^1, \overline{G/U}/T)^\circ$ where the superscript $\circ$ means the generic point of $\mathbb{P}^1$ goes to G/B.

Remark 2.21. Although G/B is projective, we can still compactify it inside a stack $\overline{G/U}/T$. For example $\mathbb{P}^1$ can be compactified within the stack $\mathbb{A}^2/\mathbb{G}_m$.

This provides a finite-dimensional model for W (but we have to consider a compactification). We will undertake a thorough study of quasi-maps in the next section.

3 Quasi-maps

3.1 Initial discussion

Let Y be a projective variety. Then we can stratify the space of maps into Y:

$$\mathrm{Maps}(\mathbb{P}^1, Y) = \bigsqcup_{\alpha \in H_2(Y,\mathbb{Z})} \mathrm{Maps}^\alpha(\mathbb{P}^1, Y).$$

One problem is that the space $\mathrm{Maps}^\alpha(\mathbb{P}^1, Y)$ is usually not compact, so we would like to have a compactification.

3.1.1 Maps to projective space.

Let $X = \mathbb{P}^n$, so $H_2(Y, \mathbb{Z}) = \mathbb{Z}$. For any $\alpha \in \mathbb{Z}_{\geq 0}$, a map $f : \mathbb{P}^1 \to \mathbb{P}^n$ is equivalent to the data of a line sub-bundle $\mathcal{L} \subset \mathcal{O}_{\mathbb{P}^1}^{n+1}$, and under this equivalence we have $\deg f = \alpha \iff \deg \mathcal{L} = -\alpha$.

Now it's easy to see why this space is non-compact. The condition $\deg \mathcal{L} = -\alpha$ forces $\mathcal{L} \cong \mathcal{O}(-\alpha)$, but this line bundle has an automorphism group $\mathbb{C}^\times$. So we get an inclusion

$$\mathrm{Maps}^\alpha(\mathbb{P}^1, \mathbb{P}^n) \hookrightarrow \{\sigma : \mathcal{O}_{\mathbb{P}^1}(-\alpha) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)} \mid \sigma \neq 0\}/\mathbb{C}^\times. \tag{1}$$

Since maps $\mathcal{O}_{\mathbb{P}^1}(-\alpha) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)}$ are equivalent to global sections of $\mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus(n+1)}$, the right hand side is a projective space, of dimension $(n+1)(\alpha+1) - 1$. However, the map is an open embedding but not an isomorphism. The reason is that the right hand side parametrizes subsheaves but *not* subbundles (subbundles are inclusions of subsheaves whose quotients are locally free). The important slogan is that *subsheaf ≠ subbundle.*

Example 3.1. If $n = 1$ and $\alpha = 1$, then the left hand side of (1) is

$$\mathrm{Maps}^\alpha(\mathbb{P}^1, \mathbb{P}^1) = \{\mathbb{P}^1 \to \mathbb{P}^1 \text{ of degree } 1\}.$$

This is precisely the automorphism group PGL(2), which is evidently not compact. On the right hand side of (1), we are considering a non-zero map of the form $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, which of course is equivalent to two maps $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Now, any map $\mathcal{O}(-1) \hookrightarrow \mathcal{O}$ vanishes at one point of $\mathbb{P}^1$. If the two maps vanish at different points, then the map is an injection on fibers, hence the inclusion of a subbundle. However, if the zeros coincide then the corresponding map $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^{\oplus 2}$ is *not* a subbundle.

In other words, (1) describes the inclusion of PGL(2) in $\mathbb{P}^3$ as 2×2 invertible matrices up to scalars. If we took all *non-zero* 2×2 matrices, then we would get all of $\mathbb{P}^3$.

So we see that in this case we can compactify $\mathrm{Maps}^\alpha(\mathbb{P}^1, \mathbb{P}^n)$ by

$$\mathrm{QMaps}^\alpha(\mathbb{P}^1, \mathbb{P}^n) = \left\{ \substack{\text{subsheaves } \mathcal{L} \subset \mathcal{O}_{\mathbb{P}^1}^{n+1} \\ \deg \mathcal{L} = -\alpha} \right\}$$

which will turn out to be some projective space.

Stacky interpretation. There is a fancier definition of this compactification as maps $\mathbb{P}^1 \to [\mathbb{C}^{n+1}/\mathbb{C}^\times]$ (the stacky compactification of projective space) such that the generic point of $\mathbb{P}^1$ lands in $\mathbb{P}^1$.

To digest this, we recall what it means to map a test scheme S to a quotient stack $[W/G]$. By definition, such a map is the data of

- a G-bundle $\mathcal{F}$ on S,
- A G-equivariant map $\mathcal{F} \to W$,

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{G\text{-equiv.}} & W \\ \downarrow & & \downarrow \\ S & \longrightarrow & [W/G] \end{array}$$

Remark 3.2. Such a G-equivariant map is equivalent to the data of a section of $\mathcal{F}_W \to S$, where $\mathcal{F}_W = \mathcal{F} \times^G W = \mathcal{F} \times W/G$ (quotient by the diagonal action) is the associated W-bundle over S.

In our case, $G = \mathbb{G}_m$ and a $\mathbb{G}_m$-bundle is equivalent to a line bundle $\mathcal{L}$. Since $W = \mathbb{C}^{n+1}$, we have $\mathcal{F}_W = \mathcal{O}_{\mathbb{P}^1}^{n+1} \otimes \mathcal{L}^{-1}$. Thus, a section of $\mathcal{F}_W$ is equivalent to a map $\mathcal{L} \to \mathcal{O}_{\mathbb{P}^1}^{n+1}$. We haven't yet imposed the open condition that the generic point of $\mathbb{P}^1$ is sent to $\mathbb{P}^1$. The open condition says that the section is non-zero at the generic point, i.e.
of sheaves.

Generalization to subvarieties. We can generalize to mapping to subvarieties of projective space. Suppose $Y \subset \mathbb{P}^n$ is a closed subvariety. Denote by $\overline{X}$ the afine cone over Y in $\mathbb{C}^{n+1}$ and $X = \overline{X} \setminus \{0\}$. Then

$$\begin{aligned} \mathrm{QMaps}(\mathbb{P}^1, Y) &= \left\{ \begin{matrix} \text{maps } \mathbb{P}^1 \to \mathbb{P}^n \\ \text{generically landing in } Y \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \text{maps } \mathbb{P}^1 \to \overline{X}/\mathbb{C}^\times \\ \text{generically landing in } X/\mathbb{C}^\times = Y \end{matrix} \right\} \end{aligned}$$

Substratification. Given a quasi-map $f\colon \mathbb{P}^1 \to \mathbb{P}^n$, we get an injection of sheaves $\mathcal{L} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{n+1}$ defining an honest map $\mathbb{P}^1 \setminus \{x_1, \ldots, x_n\} \to \mathbb{P}^n$. But since the target is projective, we can complete this to an honest map $\widetilde{f}\colon \mathbb{P}^1 \to \mathbb{P}^n$.

However, this is sort of misleading. It is true set-theoretically, but it doesn't work well in families because $\deg \widetilde{f}$ is smaller than $\deg f$.

Lemma 3.3. *We have an identification of sets*

$$\mathrm{QMaps}^\alpha(\mathbb{P}^1, \mathbb{P}^n) = \bigsqcup_{0 \le \beta \le \alpha} \underbrace{\mathrm{Maps}^\beta(\mathbb{P}^1, \mathbb{P}^n) \times \mathrm{Sym}^{\alpha-\beta}(\mathbb{P}^1)}_{Q^\beta}.$$

Proof. We explained that any quasi-map can be saturated to a map. This lowers the degree of the map, but by how much?

For $x \in \mathbb{P}^1$, and a map $f\colon \mathbb{P}^1 \to \mathbb{P}^n$, corresponding to the line bundle inclusion $\mathcal{L} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{n+1}$, there is a notion of the "defect of f at x" which measures the order of vanishing at x of $\mathcal{L} \hookrightarrow \mathcal{O}^{n+1}$. The defect at x is the maximal integer d such that we get an induced inclusion $\mathcal{L}(dx) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{n+1}$.

Thinking of $\mathrm{Sym}^{\alpha-\beta}(\mathbb{P}^1)$ as effective divisors of degree $\alpha - \beta$ on $\mathbb{P}^1$, the coefficient of $[p]$ in a divisor is precisely the defect at that point. □

This gives a stratification of $\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, \mathbb{P}^n)$. We claim that

$$\overline{Q}^{\beta} = \bigsqcup_{\gamma \le \beta} Q^{\gamma}.$$

In particular, for $\beta = \alpha$ we find that $\mathrm{Maps}^{\beta}(\mathbb{P}^1, \mathbb{P}^n) \times \mathrm{Sym}^{\alpha-\beta}(\mathbb{P}^1)$ is the open stratum.

3.1.2 Products of projective spaces

Now we've defined the notion of quasi-maps into a variety embedded in projective space. Now suppose that we are given an embedding $Y \hookrightarrow \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$. We can define $\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, Y)$ where $\alpha \in \mathbb{Z}^k$.

This is what we'll do for $Y = G/B$ the flag variety, which has a natural closed embedding into a product of projective spaces. Since $Y = G/UT$, we have an open embedding $Y \subset [\overline{G/U}/T]$. Then $\mathrm{QMaps}(\mathbb{P}^1, Y)$ are in bijection with maps from $\mathbb{P}^1$ to the stack $[\overline{G/U}/T]$ which generically land in Y.

We claim that a quasi-map $\mathbb{P}^1 \to Y$ is equivalent to the data:

1. A principal T-bundle $\mathcal{F}_T$ on $\mathbb{P}^1$.
2. For all $\lambda\colon T \to \mathbb{C}^{\times}$ dominant, and $\mathcal{F}_{T,\lambda}$ the induced line bundle on $\mathbb{P}^1$, an injection of sheaves $\kappa_\lambda\colon \mathcal{F}_{T,\lambda} \hookrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes V(\lambda)$. (If this were an injection of *bundles*, we would get a map instead of a quasi-map.) Here $V(\lambda)$ is the irreducible representation of highest weight λ.
3. For all λ, μ the diagram commutes

$$\begin{array}{ccc} \mathcal{F}_{T,\lambda+\mu} & \xrightarrow{\kappa_\lambda \otimes \kappa_\mu} & \mathcal{O} \otimes V(\lambda) \otimes V(\mu) \\ \Big\downarrow{\scriptstyle \kappa_{\lambda+\mu}} & \nearrow & \\ \mathcal{O} \otimes V(\lambda+\mu) & & \end{array}$$

Lemma 3.4. *We have*

$$\mathbb{C}[\overline{G/U}] \cong \bigoplus_{\lambda} V(\lambda)$$

where G acts naturally, T acts by λ on $V(\lambda)$, and

$$V(\lambda)\otimes V(\mu)\to V(\lambda+\mu)$$

is the multiplication.

We claim that if you plug this description into our discussion, then you will see this equivalence.

Exercise 3.5. Check this.

Now we can define a stratum $\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, Y)$ for $\alpha \in \mathrm{Hom}(C^{\times}, T) =: \Lambda$, the coroot lattice of G. There is a natural partial order on Λ, defined by $\alpha \geq \beta$ if and only if $\alpha(\lambda) \geq \beta(\lambda)$ for all dominant weights λ.

Lemma 3.6. *We have*

$$\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, Y) = \bigsqcup_{0\leq\beta\leq\alpha} \mathrm{Maps}^{\beta}(\mathbb{P}^1, Y)\times \mathrm{Sym}^{\alpha-\beta}(\mathbb{P}^1)$$

where $\mathrm{Sym}^{\alpha-\beta}(\mathbb{P}^1) = \left\{\sum \gamma_i x_i \mid {}^{x_i\in\mathbb{P}^1, \gamma_i\in\Lambda}_{\gamma\geq 0, \sum\gamma_i=\alpha-\beta}\right\}$*, now regarding the defect as an element of* Λ.

3.2 Main statement

For describing the singularities of $\mathrm{Fl}_{\infty/2}$ in terms of quasi-maps, the main statement is the following.

Statement. Let $f \in \mathrm{QMaps}^{\alpha}(\mathbb{P}^1, Y)$ be such that $\mathrm{defect}(f) = \alpha\cdot 0$. (Thus f is a constant "map.") We claim that the formal neighborhood of f in $\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, Y)$ serves as a model for the singularities of the closure of a $G(\mathcal{O})$-orbit on $\mathrm{Fl}_{\infty/2}$ at a point of $\mathrm{Fl}_{\infty/2,\alpha}$.

Since $\mathrm{Fl}_{\infty/2} = \bigsqcup \mathrm{Fl}_{\infty/2,\alpha}$ and

$$\overline{\mathrm{Fl}}_{\infty/2,0} = \bigsqcup_{\alpha\geq 0}\mathrm{Fl}_{\infty/2,\alpha}$$

it is tempting to restrict our attention to quasi-maps with defect only at 0. Unfortunately, it turns out that this is not well-defined: it defines a constructible but not locally closed subset. So we can consider a single point where the defect is only at 0, but not the space of such points. If we want to get a reasonable space, then we have to allow the defect be anywhere on our global curve.

Said again differently, the finite-dimensional thing was a transversal slice to the orbit $\mathrm{Fl}_{\infty/2,\alpha}$ but this transversal slice somehow "knows" about all the points on the global curve. This bites us later when trying to define a category of perverse sheaves on $\mathrm{Fl}_{\infty/2}$.

–

Last time we saw that $\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, G/B)$ is a projective variety of dimension $\dim G/B + 2|\alpha|$, where $\alpha = \sum_i\langle\alpha,\omega_i\rangle$, ω_i the fundamental weights, and $\alpha = \sum a_i\alpha_i$ then $|\alpha| = \sum a_i$.

Example 3.7. Let $G = \mathrm{SL}(2)$. Then $\mathrm{QMaps}^\alpha(\mathbb{P}^1, G/B = \mathbb{P}^1) = \mathbb{P}^?$.

Example 3.8. If $G = \mathrm{SL}(3)$, then $\overline{G/U} = \{v \in \mathbb{C}^3, \xi \in (\mathbb{C}^3)* \mid \langle v, \xi \rangle = 0\}$. What is a quasi-map in this case? We have to specify:

- A line bundle $\mathcal{L}_1 \hookrightarrow O_{\mathbb{P}^1} \otimes \mathbb{C}^3$,
- A line bundle $\mathcal{L}_2 \hookrightarrow O_{\mathbb{P}^1} \otimes (\mathbb{C}^3)^*$,
- $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle = 0$.

The locus of maps are where these sheaf injections are actually bundle injections. The quasimaps condition relaxes them to be merely injections at the generic point.

What is the degree α? It is specified by $\deg \mathcal{L}_1$ and $\deg \mathcal{L}_2$. That means that if we fix the degreese, then both maps are projective spaces, hence QMaps^α is given by quadratic equations in $\mathbb{P}^? \times \mathbb{P}^{??}$. In fact, there will be many equations, if you think about what is involved in $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle = 0$. Indeed, you can think of the pairing as giving a map $\mathcal{L}_1 \otimes \mathcal{L}_2 \to O_{\mathbb{P}^1}$. Think of $\mathcal{L}_1, \mathcal{L}_2$ as both being pretty negative, so their tensor product is even more negative, and we demand that this is the zero map.

4 Other applications of quasi-maps

We motivated the spaces QMaps through their application to the singularities of $\mathrm{Fl}_{\infty/2}$, but in this section we touch on their relation to other interesting objects.

4.1 IC sheaves

In the previous section we stated that

$$\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, G/B) = \bigsqcup \mathrm{Maps}^{\beta}(\mathbb{P}^1, G/B) \times \mathrm{Sym}^{\alpha-\beta}(\mathbb{P}^1).$$

The deepest stratum is when $\beta = 0$, so the map is constant. Then there is a substratification refining this one, based on the number of distinct points appearing in $\mathrm{Sym}^{\alpha}(\mathbb{P}^1)$. The deepest stratum of this substratification is where all points coincide, and we see only $\mathbb{P}^1$. So the deepest stratum of $\mathrm{QMaps}^{\alpha}(\mathbb{P}^1, G/B)$ is $G/B \times \mathbb{P}^1$.

Question. How can we describe the stalks of the IC sheaf of QMaps^{α}?

It turns out that you can compare them to some "periodic Kazhdan-Lustzig polynomials" which were defined by Lustzig.

Next we turn our attention to the IC-sheaves of closures of $G(\mathcal{O})$-orbits. How can we model singularities of *all* semi-infinite Schubert varieties (the closures of I^+-orbits)? It turns out that we can do this just by considering products of the form $\bigsqcup_{\alpha} \mathrm{QMaps}^{\alpha}(\mathbb{P}^1, G/B) \times G/B$.

Exercise 4.1. Define a stratification such that the strata are in bijection with elements $w \in W_{\mathrm{aff}}$. You can use IC sheaves of these strata to get all periodic Lusztig polynomials.

4.2 Variant of QMaps

Now let C be *any* smooth projective curve. Consider the moduli stack $\mathrm{Bun}_G(C)$. Then we have a map

$$\mathrm{Bun}_B(C) \xrightarrow{p} \mathrm{Bun}_G(C)$$

(more generally, any homomorphism $G \to H$ induces a map $\mathrm{Bun}_G \to \mathrm{Bun}_H$ by change of structure group). This morphism has the property that $p^{-1}(\mathcal{F}_{\mathrm{triv}}) = \mathrm{Maps}(C, G/B)$.

We also have a map

$$\mathrm{Bun}_B(C) \xrightarrow{q} \mathrm{Bun}_T(C) \cong \mathrm{Pic}(C)^{\mathrm{rank}\ G}.$$

The connected components of $\mathrm{Bun}_T(C)$ are in bijection with $\alpha \in \Lambda$. Moreover, if $q_\alpha \colon \mathrm{Bun}_B^{\alpha}(C) \to \mathrm{Bun}_T^{\alpha}(C)$ is the restriction of the map over the connected component corresponding to α, then q_α is representable.

Thus we have a diagram

$$\begin{array}{ccc} \mathrm{Bun}_B^\alpha(C) & \xrightarrow{q_\alpha} & \mathrm{Bun}_T^\alpha(C) \\ \big\downarrow{\scriptstyle p_\alpha} & & \\ \mathrm{Bun}_G(C) & & \end{array}$$

If $G = \mathrm{GL}(n)$, then $\mathcal{F} \in \mathrm{Bun}_G(C)$ can be viewed equivalently as a vector bundle of rank n, and the possible reductions to B are obtained by choosing a flag $0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}$ where rank $\mathcal{F}_i = i$. From this it is easy to see that the fibers of p_α are quasi-projective schemes. So a natural question is if there is a relative compactification of this map.

We've already seen that the fiber over the trivial bundle is $\mathrm{Maps}(C, G/B)$, and this has a compactification by the space of quasi-maps $\mathrm{QMaps}(C, G/B)$.

Theorem 4.2. *There exists a relative compactification of p_α,*

$$\overline{p}_\alpha \colon \overline{\mathrm{Bun}_B} \to \mathrm{Bun}_G$$

such that

1. $(\overline{p}_\alpha)^{-1}(\mathcal{F}_{triv}) = \mathrm{QMaps}^\alpha(C, G/B)$.
2. $\overline{\mathrm{Bun}}_B^\alpha(C) = \bigsqcup_\beta \mathrm{Bun}_B^\beta(C) \times \mathrm{Sym}^{\alpha-\beta}(C)$.

The singularities depend *only* on the defect (not even on the curve!). So a description of the IC sheaves for $\mathrm{QMaps}^\alpha(\mathbb{P}^1, G/B)$ would yield a description of the IC sheaves for $\overline{\mathrm{Bun}}_B$.

Remark 4.3. The space $\overline{\mathrm{Bun}}_B$ first appeared in the context of the geometric Langlands correspondence. The geometric Langlands correspondence is about perverse nice perverse sheaves on Bun_G, which are analogues of classical automorphic forms, and the "geometric Eisenstein series" are geometric analogues of classical Eisenstein series. The classical Eisenstein series can be thought of as obtained by $p_!$ for $p\colon \mathrm{Bun}_G \to \mathrm{Bun}_G$.

In geometric situations, it was known that it is bad to take the direct image under a non-proper map, which is how the question for compactifications was inspired.

In this problem, the description of the IC sheaves plays an important role, for example in the appearance of certain L-functions in the functional equation.

Remark 4.4. Consider the map $\overline{\mathrm{Bun}}_B(C) \to \mathrm{Bun}_G(C)$. These will have the same singularities on an open set. If $C = \mathbb{P}^1$, then $\mathcal{F}_{\mathrm{triv}}$ is open in $\mathrm{Bun}_G(C)$, because it contains the open substack $[\mathrm{pt}/G]$, so in that case is suffices to study the single *fiber* over the trivial bundle. However, on a general curve this is not true.

We discussed quasi-maps into a target of the form $Y \subset \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$. The flag variety has a canonical embedding of this form. Namely, if $\omega_1, \ldots, \omega_\ell$ are the fundamental weights of G, then we have a canonical embedding

$$G/B \hookrightarrow \prod_i \mathbb{P}(V(\omega_i)).$$

So we can define $\mathrm{QMaps}^\alpha(C, G/B)$ coming from this embedding.

4.3 Kontsevich moduli spaces

We want to relate quasi-maps to Kontsevich's moduli spaces, so fix $C = \mathbb{P}^1$. How can we relate QMaps to Kontsevich's spaces? The thing is, Kontsevich's spaces parametrize not maps from a fixed $\mathbb{P}^1$, but from varying domain curves. This seems orthogonal to our situation.

The remedy is through the "graph space." For $\alpha \in H_2(Y, \mathbb{Z})$ we define

$$\text{Graph}^\alpha(Y) = \overline{\mathcal{M}_{0,0}}^{(\alpha,1)}(Y \times \mathbb{P}^1).$$

Here the first 0 in $\mathcal{M}_{0,0}$ means genus 0, the second 0 means no marked points, and $(\alpha, 1)$ means that the image of the fundament class is of type $(\alpha, 1) \in H_*(Y \times \mathbb{P}^1)$.

We claim that $\text{Graph}^\alpha(Y)$ contains the space $\text{Maps}^\alpha(\mathbb{P}^1, Y)$ as an open subset. Why? The point is that we can take the space where the curve has no singularities, hence is abstractly isomorphic to $\mathbb{P}^1$. If we then make it have degree 1 over $\mathbb{P}^1$, then projecting to the $\mathbb{P}^1$ component gives a *canonical* isomorphism with $\mathbb{P}^1$. Then this will be the graph of a morphism $\mathbb{P}^1 \to Y$.

Lemma 4.5. *There exists a proper map* $\text{Graphs}^\alpha(Y) \to \text{QMaps}^\alpha(Y)$ *which is the identity on* $\text{Maps}^\alpha(\mathbb{P}^1, Y)$.

This is not trivial. The proof is by reduction to the case of $Y = \mathbb{P}^n$.

This has some nice applications. Sometimes, if you want to compute something in Gromov-Witten theory then it is easier to work on QMaps.

Exercise 4.6. Describe this map set-theoretically.

Theorem 4.7. *If $Y = G/B$, then* $\text{QMaps}^\alpha(\mathbb{P}^1, G/B)$ *is normal and Cohen-Macaulay. If G is simply-laced, then it is also Gorenstein, and has rational singularities.*

Remark 4.8. For the flag variety, the Graph space is even smooth (as a Deligne-Mumford stack), so it is in some sense a resolution of singularities.

4.4 Zastava space

We consider $\text{QMaps}^\alpha(\mathbb{P}^1, G/B)^\infty$, meaning the space of quasi-maps with no defect at ∞. Since we have no defect at infinity, any such quasi-map defines an honest map at ∞, so we can "evaluate it" to obtain a map to G/B:

$$\begin{array}{c} \text{QMaps}^\alpha(\mathbb{P}^1, G/B)^\infty \\ \Big\downarrow {\scriptstyle \eta^\alpha} \\ G/B \end{array}$$

This is a locally trivial fibration. Pick a point on the base, say $1 \in G/B$ (but it doesn't really matter) and set $Z^\alpha = (\eta^\alpha)^{-1}(1)$. This is called the space of *based* quasi-maps $\mathbb{P}^1 \to G/B$. Inside it we have Z_0^α, the space of based maps. Then you can easily convince yourself that $\dim Z^\alpha = 2|\alpha|$. This is even, so it has a chance to be symplectic. In fact, we have:

Proposition 4.9. *There is a canonical Poisson structure on Z^α, which is generically symplectic*

However, the symplectic locus is of codimension 1, so the singularities are not symplectic (which would imply that the symplectic locus had codimension 2).

Now, since based quasi-maps have no defect at ∞, they have a "defect map" to $\mathbb{A}^1$.

Proposition 4.10. *There exists a map $\pi^\alpha : Z^\alpha \to \mathrm{Sym}^\alpha(\mathbb{A}^1)$, which is an "integrable system."*

Finally, the map π^α has a "factorization property" in the following sense. Suppose you have a divisor $D = \sum \beta_i x_i + \sum \gamma_j y_j$ where β_i and γ_j are in Λ, and $x_i, y_j \in \mathbb{A}^1$ are such that $x_i \neq y_j$ for all i, j. Denote $\beta = \sum \beta_i$ and $\gamma = \sum \gamma_j$. Then:

Proposition 4.11. *We have*

$$(\pi^\alpha)^{-1}(D) \cong (\pi^\beta)^{-1}(D_\beta) \times (\pi^\gamma)^{-1}(D_\gamma).$$

Remark 4.12. Z^α and Z_0^α are familiar spaces in Gauge theory. Z_0^α is the space of "framed magnetic monopoles" on S^3, and Z^α is a natural partial compactification. (This is due to Donaldson for $G = \mathrm{SL}(2)$.)

5 Parabolic and affine generalizations and IC sheaves

5.1 Parabolic generalization

Let G be a simply-connected, simple reductive group and $P \subset G$ a parabolic subgroup. We can consider the partial flag variety G/P, and then the space $\mathrm{Maps}^\theta(\mathbb{P}^1, G/P)$ where θ is in some lattice Λ_P depending on P (the bigger P is the smaller this lattice; for maximal parabolics it has rank 1).

One way to think about this canonically is that if you write $P = MU_P$, then Λ_P is the cocharacter lattice of $M/[M, M]$:

$$\Lambda_P \cong \mathrm{Hom}(\mathbb{G}_m, M/[M, M]) = \mathrm{Hom}(Z(M^\vee), \mathbb{G}_m)$$

where $M^\vee$ is the Langlands dual to M.

What would be a notion of *quasi-map into G/P*?

Here is one possibility, which is the one that we'll adopt. The quotient $G/[P, P]$ is quasi-affine. In analogy to the case of Borels, we could define

$$\mathrm{QMaps}^\theta(\mathbb{P}^1, G/P) = \mathrm{Maps}^\theta(\mathbb{P}^1, \overline{G/[P, P]}/?)_0$$

where the subscript 0 means that the generic point goes to G/P. For choosoing ?, note that G/P is a quotient of $G/[P, P]$ by some torus, namely $T_M := M/[M, M]$.

Definition 5.1. We define $\mathrm{QMaps}^\theta(\mathbb{P}^1, G/P) = \mathrm{Maps}^\theta(\mathbb{P}^1, \overline{G/[P, P]}/T_M)_0$.

Then $\mathrm{QMaps}^\theta(\mathbb{P}^1, G/P)$ is a projective variety, and $\mathrm{Maps}^\theta(G/P) \subset \mathrm{QMaps}^\theta(\mathbb{P}^1, G/P)$ is open and dense.

Remark 5.2. There is another choice, which is also important. Recall from the problem sheet that the quotient G/U_P is also quasi-affine. We can embed $G/P \hookrightarrow (\overline{G/U_P})/M$. This can also be used to define quasi-maps, and you end up with a different space, $\widetilde{\mathrm{QMaps}}^\theta(\mathbb{P}^1, G/P)$.

Exercise 5.3. Show that there exists a projective birtaional morphism $\widetilde{\mathrm{QMaps}}^\theta(\mathbb{P}^1, G/P) \to \mathrm{QMaps}^\theta(\mathbb{P}^1, G/P)$. The fibers are certain closed subsets inside Gr.

It's especially instructive to look at $G = \mathrm{SL}(n)$ and P a maximal parabolic. Then G/P is a Grassmannian.

Stratification. We have a stratification

$$\mathrm{QMaps}^\theta(\mathbb{P}^1, G/P) = \bigsqcup_{0 \le \theta' \le \theta} \mathrm{Maps}^{\theta'}(\mathbb{P}^1, G/P) \times \mathrm{Sym}^{\theta-\theta'}(\mathbb{P}^1).$$

The spaces $\mathrm{QMaps}^\theta(\mathbb{P}^1, G/P)$ serve as finite-dimensional models for the singularities of closures of $G(\mathcal{O})$-orbits in $G(\mathcal{K})/[P, P](\mathcal{K})M(\mathcal{O})$, which is a parabolic version of the semi-infinite *partial* flag variety $\mathrm{Fl}_{P,\infty/2}$. This admits an action of $\Lambda_P = M(\mathcal{K})/[M, M](\mathcal{K})M(\mathcal{O})$.

Problem. Describe stalks of IC sheaves of $\mathrm{QMaps}^\theta(\mathbb{P}^1, G/P)$.

In the previous section we discussed quasi-maps and based quasi-maps, and we saw that they had some nice properties. We can consider analogous parabolic versions: let $Z^{\theta}_{P,0}$ be the space of based maps $\mathbb{P}^1 \to G/P$ sending $\infty \mapsto 1$. This lies inside the space Z^{θ}_P of based *quasi*-maps which are not allowed to have a defect at ∞. Thus, this projects to $\operatorname{Sym}^\theta \mathbb{A}^1$ because nothing bad is allowed to happen at ∞, and has a factorization property similar to the one we discussed last time.

5.2 Affine generalizations

We want to generalize this even further, to relate it to other objects which have been discussed here.

Let G be simple and simply-connected. Define

$$\operatorname{Bun}^d_G(\mathbb{A}^2) = \left\{ (\mathcal{F}, \phi) \mid \substack{G\text{-bundles } \mathcal{F} \text{ on } \mathbb{P}^2 \\ c_2(\mathcal{F})=d \\ \phi=\text{trivialization of } \mathcal{F}|_{PS_\infty}} \right\}.$$

Why do we pass to the compactification $\mathbb{P}^2$ when defining a moduli space of bundles on $\mathbb{A}^2$? We can view a bundle on $\mathbb{A}^2$ as a bundle on the compactification $\mathbb{P}^2$, together with a trivialization at ∞ What would happen if we used a different compactificaton? It's easy to see that this actually doesn't depend on the compactification. If we replace $\mathbb{P}^2$ by another smooth projetive surface S and a divisor D_∞ on S such that $S \setminus D_\infty = \mathbb{A}^2$, then we get the same space.

Example 5.4. One could take $S = \mathbb{P}^1 \times \mathbb{P}^1$. This is the only other choice that we use in practice.

Theorem 5.5. *The space* $\operatorname{Bun}^d_G(\mathbb{A}^2)$ *is a smooth variety of dimension* $2dh^\vee$ *where* $h^\vee$ *is the dual Coxeter number, and has a symplectic structure.*

Example 5.6. If $G = SL(n)$, then $h^\vee = n$ so $\dim \operatorname{Bun}^d_G(\mathbb{A}^2) = 2dn$.

Lemma 5.7. $\operatorname{Bun}^d_G(\mathbb{A}^2)$ *may be identified with the space of based maps of degree d from* $\mathbb{P}^1$ *to* $G(\mathcal{K})/G(k[t^{-1}])$ *(the thick affine Grassmannian as a scheme).*

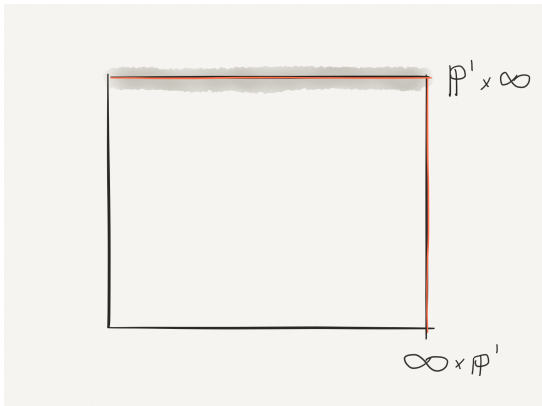
Here $\mathcal{K} = \mathbb{C}((t))$ as always. The notion of degree is that the thick affine Grassmannian has a canonical line bundle ♠♠♠ TONY: [what is it?], and the degree is the degree of its pullback. Alternatively, one can show that H_2 of the affine Grassmannian is $\mathbb{Z}$, and then demand that the image of the fundamental class of $\mathbb{P}^1$ maps to d.

Remark 5.8. The thick affine Grassmannian should be thought of as a partial flag variety for the loop group. Although it is wildly infinite-dimensional, the space of *based maps* into is *finite*-dimensional. (Of course, without the basing it is infinite-dimensional, as even the space of constant maps is infinite-dimensional.) In fact, the finite-dimensionality is true for any partial flag variety of any symmetrizable Kac-moody group.

Proof. We use the fact that $\mathrm{Gr}_G := G(\mathcal{K})/G[t^{-1}]$ is the moduli space of G-bundles on $\mathbb{P}^1$ trivialized at the formal neighborhood of $\infty \in \mathbb{P}^1$.

Exercise 5.9. Prove this. (Beware that this is the thick affine Grassmannian, not the thin one that Xinwen talked about.)

Then a map $\mathbb{P}^1 \xrightarrow{f} \mathrm{Gr}_G$ is equivalent to the data of a G-bundle on $\mathbb{P}^1$ trivialized at the formal neighborhood of $\mathbb{P}^1 \times \infty$.



Now what about based maps? The based condition says that $\mathcal{F}|_{\infty \times \mathbb{P}^1}$ is trivial, and the trivialization is tautological. So the result boils down to the following sublemma.

□

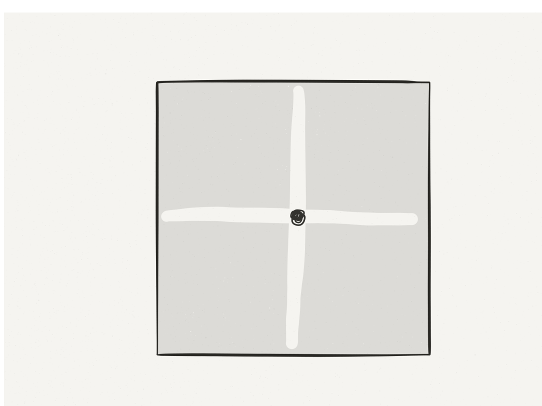
Lemma 5.10. *If $\mathcal{F} \in \mathrm{Bun}_G(\mathbb{P}^1 \times \mathbb{P}^1)$ is trivialized on $\mathbb{P}^1_{ver} \cup \mathbb{P}^1_{hor}$, then the trivialization extends uniquely to $\widehat{\mathbb{P}^1_{hor}}$ in a compatible way.*

These spaces have significance in gauge theory. There is an "Uhlenbeck partial compactification" $\mathcal{U}^d_G(\mathbb{A}^2)$, with a stratification

$$\mathcal{U}^d_G(\mathbb{A}^2) = \bigsqcup_{0 \le d' \le d} \mathrm{Bun}^{d'}_G(\mathbb{A}^2) \times \mathrm{Sym}^{d-d'}(\mathbb{A}^2).$$

In this case quasi-maps are well-defined, but the result is not a scheme of finite type.

Example 5.11. Conside the affine plane with the two axes removed, and then the origin put back in.



This is a constructible subset of a scheme, which is not itself a scheme.

Exercise 5.12. Define an affine scheme of infinite type over $\mathbb{C}$ whose set of $\mathbb{C}$-points is this.

Compactifying $\mathbb{A}^2$ by $\mathbb{P}^1 \times \mathbb{P}^1$ involves choosing vertical and horizontal directions. The "right" thing to do is to *work with all possible choices of coordinates at the same time*! This rectifies the problem that appears in the exercise. This is how one construct $\mathcal{U}_G^d(\mathbb{A}^2)$, which is an affine scheme of finite type containing $\mathrm{Bun}_G^d(\mathbb{A}^2)$ as a dense open subset.

Example 5.13. If $G = \mathrm{SL}(n)$, then $\mathcal{U}_G^d(\mathbb{A}^2)$ is a (singular) Nakajima quiver variety.

Question. Describe the IC sheaves on Z^α, Z_P^θ, and $\mathcal{U}_G^d(\mathbb{A}^2)$.

Remark 5.14. If you can describe the IC sheaf then you can describe the intersection cohomology, because all of these spaces have a $\mathbb{C}^\times$-action which contracts the whole space to a unique fixed point.

Example 5.15. In the Uhlenbeck space of $\mathbb{A}^2$, the fixed point is in the deepest stratum $d' = 0$, with defect $d \cdot 0$, and bundle being the trivial bundle.

It is a general phenomenon that if you have an equivariant sheaf on a space ? with an action contracting everything to a point, then the stalk of the IC-sheaf at that point is $H^\bullet(IC_?)$. The fixed point is the "most singular" point, and he answer for the IC stalk at the most singular point is:

- In the Borel case $P = B$ and Z^α the space of based maps to G/B, we have the inclusion of dual Lie algebras $\mathfrak{g}^\vee \supset \mathfrak{n}^\vee$. The answer is $\mathrm{Sym}(\mathfrak{n}^\vee[2])_\alpha$ up to some universal shift.

  So the total dimension, ignoring the grading, is equal to the number of "Kostant partitions" of α, i.e. the number of partitions of α into sums of positive roots. This is the value of the Poincaré polynomial at 1. If you want the polynomial itself, then you count partitions with a certain weight:

$$\sum_{P \in \mathrm{Kostant}(\alpha)} q^{-|P|}.$$

- In the parabolic case, you have $P \subset G$ hence $\mathfrak{g}^\vee \supset \mathfrak{n}_P^\vee$, which has an action of $M^\vee$. Then it turns out that the answer is that the stalk of $IC(Z_P^\theta)$ at the most singular point is $\mathrm{Sym}(\mathfrak{n}_{P^\vee}^{f_{M^\vee}}[z])_\theta$ where $(e_{M^\vee}, h_{M^\vee}, f_{M^\vee})$ is the principal $\mathfrak{sl}_2$ triple in $\mathfrak{m}^\vee = \mathrm{Lie}(M^\vee)$. The grading is by the eigenvalues of $h_{M^\vee}$.

- If G is simply laced then $(\mathfrak{g}_{\mathrm{aff}})^\vee = (\mathfrak{g}^\vee)_{\mathrm{aff}}$ (but not in general). If G is simply-laced, then the stalk of the IC sheaf of $\mathcal{U}_G^d(\mathbb{A}^2)$ at the most singular point is

$$\mathrm{Sym}(t\mathfrak{g}^f[t][2])_d.$$

  Here the d is the degree in t, and (e, n, f) is the principal triple.