THE BRAVERMAN-KAZHDAN-NGO APPROACH TO L-FUNCTIONS

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Contents

1. Local L-factors and γ-factors

Let F be a local field of characteristic 0. The Local Langlands Correspondence gives a finite-to-one map from irreducible admissible representation of $G(F)$ to Langlands parameters:

$$\mathrm{Irr}(G) \twoheadrightarrow \Phi(G).$$

We want to recall the definition of L-factors and ϵ-factors.

1.1. Representations of the Weil group.

Let W_F be the Weil group of F.

Definition 1.1. A representation $\sigma \colon W_F \to \mathrm{GL}(V)$ is *admissible* if it is smooth and $\sigma(W_F)$ consists of semi-simple elements.

1.1.1. $F = \mathbf{C}$. The Weil group is $W_{\mathbf{C}} = \mathbf{C}^\times$, so irreducible admissible representations are characters of $\mathbf{C}^\times$, which are parametrized by pairs $(\ell \in \mathbf{Z}, t \in \mathbf{C})$, with

$$(\ell, t) \leftrightarrow \left(\sigma_{\ell,t} \colon z \mapsto |z|^t \left(\frac{z}{|z|} \right)^\ell \right).$$

1.1.2. $F = \mathbf{R}$. The Weil group is $W_{\mathbf{R}} = \mathbf{C}^\times \rtimes \mathrm{Gal}(\mathbf{C}/\mathbf{R})$. We let j be the image of the non-trivial element of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ under the splitting, so we have a presentation

$$W_{\mathbf{R}} = \mathbf{C}^\times \oplus \mathbf{C}^\times j, \quad j^2 = -1,\ jzj^{-1} = \overline{z}.$$

All irreducible representations of $W_{\mathbf{R}}$ have dimension 1 or 2, since $W_{\mathbf{R}}$ has an abelian index-2 subgroup.

- Irreducible characters of $W_{\mathbf{R}}$ are parametrized by $\{\pm\} \times \{t \in \mathbf{C}\}$, with

$$t \leftrightarrow \left(\sigma_{\pm,t} \colon z \mapsto |z|^t, \quad j \mapsto \pm 1\right).$$

The Local Langlands correspondence with representations of $\mathrm{GL}_1(\mathbf{R})$ matches

$$\sigma_{+,t} \leftrightarrow 1 \otimes |\cdot|^t$$
$$\sigma_{-,t} \leftrightarrow \mathrm{sgn} \otimes |\cdot|^t$$

- Next we discuss the irreducible 2-dimensional representations of $W_{\mathbf{R}}$. These are parametrized by $\{\ell \in \mathbf{Z}, t \in \mathbf{C}\}$. A model for $\sigma_{\ell,t}$ can be presented with basis e_1, e_2 such that

$$\sigma_{\ell,t}(z)e_1 = \left(\frac{z}{|z|}\right)^{\ell} |z|^{2t} e_1$$
$$\sigma_{\ell,t}(z)e_2 = \left(\frac{z}{|z|}\right)^{-\ell} |z|^{2t} e_2$$
$$\sigma_{\ell,t}(j)e_1 = e_2$$
$$\sigma_{\ell,t}(j)e_2 = (-1)^{\ell} e_1.$$

Fact 1.1. Admissible representations of W_F are automatically semi-simple.

Under the local Langlands correspondence, the $\sigma_{\ell,t}$ correspond to discrete series representations of $\mathrm{GL}_2(\mathbf{R})$, namely $\sigma_{\ell,t} \leftrightarrow D_\ell \otimes |\cdot|^t$ where D_ℓ is the discrete series of $\mathrm{SL}_2(\mathbf{R})$.

1.1.3. *The p-adic case.* The p-adic theory is much more complicated. We'll just describe some low rank examples. In addition, we'll assume that the residue characteristic is odd.

Definition 1.1. An *admissible pair* (E, χ): consists of:

- a quadratic extension E/F, and
- a character χ of $E^\times$, satisfying
  (1) χ doesn't factor through $\mathrm{Nm} \colon E^\times \to F^\times$.
  (2) If $\chi|_{1+\varpi_E \mathcal{O}_E}$ factors through $N_{E/F}$, then E/F is unramified.

From such an admissible pair, we can construct an irreducible 2-dimensional representation of W_E. Let $a_E \colon W_E \to E^\times$ be the Artin reciprocity map. Then $\chi \circ a_E^{-1}$ is the character associated to W_E by local class field theory, and we can form $\mathrm{Ind}_{W_E}^{W_F}(\chi \circ a_E^{-1})$, a 2-dimensional representation of W_F. The conditions in Definition 1.1 imply that this is irreducible. The assumption that the residue characteristic is ≥ 3 implies that all irreducible admissible representations of W_F come from this construction, so we get a bijection.

Under the local Langlands correspondence, admissible pairs are bijection with the supercuspidal representations of $\mathrm{GL}_2(F)$.

1.2. **Local L-factors.** Now we're going to define the L-factor associated to Weil-Deligne representations.

Definition 1.1. Let $\sigma\colon W_F \to \mathrm{GL}(V)$. The *$L$-factor* associated to σ is

$$L(s,\sigma) := \det(1 - q^{-s}\sigma(\mathrm{Frob}_v) \mid V^I)^{-1}.$$

We have multiplicativity of both L-factors and ϵ-factors in direct sums:

$$L(s, \sigma_1 \oplus \sigma_2) = L(s,\sigma_1)L(s,\sigma_2)$$

and

$$\epsilon(s, \sigma_1 \oplus \sigma_2) = \epsilon(s,\sigma_1)\epsilon(s,\sigma_2).$$

1.2.1. $F = \mathbf{R}$. Refer to the parametrization of L-parameters in §1.1.2. The L-factor is

$$L(s,\sigma_{\ell,t}) = \begin{cases} \pi^{-(s+t)/2}\Gamma((s+t)/2) & \sigma = \sigma_{(+,t)} \\ \pi^{-(s+t+1)/2}\Gamma((s+t+1)/2) & \sigma = \sigma_{(-,t)} \\ 2(2\pi)^{-(s+t+\frac{\ell}{2})}\Gamma(s+t+\frac{\ell}{2}) & \sigma = \sigma_{(\ell,t)} \end{cases}$$

1.2.2. $F = \mathbf{C}$. Refer to the parametrization of L-parameters in §1.1.1. The L-factor is

$$L(s,\sigma_{\ell,t}) = 2(2\pi)^{-(s+t+\frac{|\ell|}{2})}\Gamma\left(s+t+\frac{|\ell|}{2}\right).$$

1.3. **ϵ-factors.** We now discuss the local ϵ-factors.

1.3.1. $F = \mathbf{R}$. We define

$$\epsilon(s,\sigma,\psi) = \begin{cases} 1 & \sigma = \sigma_{(+,t)} \\ i & \sigma = \sigma_{(-,t)} \\ i^{\ell+1} & \sigma = \sigma_{(\ell,t)} \end{cases}$$

where we choose $\psi = e^{2\pi i x}$.

1.3.2. $F = \mathbf{C}$. We define

$$\epsilon(s, \sigma_{(\ell,t)}, \psi) = i^{|\ell|}$$

where we choose $\psi = e^{2\pi i(z+\overline{z})}$.

1.3.3. *p-adic.* We now consider the case where F is a p-adic field.

Theorem 1.1. *Let ψ be a non-trivial additive character of F. As E ranges over all finite extensions of F, there exists a unique family of functions*

$$\{\textit{admissible rep'ns of } W_E\} \to \mathbf{C}[q^{-s}, q^s]^\times$$

denoted

$$\sigma \mapsto \epsilon(s,\sigma,\psi_E := \psi \circ \mathrm{Tr}_{E/F})$$

satisfying the following properties:

(1) *(GL_1-normalization) If χ is a character of $E^\times$, then $\epsilon(s, \chi \circ a_E, \psi_E) = \epsilon(s,\chi,\psi_E)$ from Tate's thesis.*
(2) *(Additivity) We have $\epsilon(s, \sigma_1 \oplus \sigma_2, \psi_E) = \epsilon(s,\sigma_1,\psi_E)\epsilon(s,\sigma_2,\psi_E)$.*

(3) (Inductive in degree 0) If $E \supset K \supset F$, then

$$\frac{\epsilon(s, \mathrm{Ind}_{W_E}^{W_K} \sigma, \psi_K)}{\epsilon(s, \sigma, \psi_E)} = \frac{\epsilon(s, \mathrm{Ind}_{W_E}^{W_K} 1_{W_E}, \psi_K)^n}{\epsilon(s, 1_{W_E}, \psi_E)^n}$$

where $n = \dim \sigma$.

Proposition 1.2. *This ϵ-factor enjoys the following properties:*

(1) $\epsilon(s, \sigma, \psi) = q^{n(\sigma)(1/2-s)}\epsilon(1/2, \sigma, \psi)$ where $n(\sigma) \in \mathbf{Z}$ is the Artin conductor.
(2) Functional equation

$$\epsilon(s, \sigma, \psi)\epsilon(1 - s, \sigma^\vee, \psi^{-1}) = 1.$$

1.4. **Local factors for L-parameters.** Since G is split, ${}^LG = \widehat{G} \times \Gamma_F$. Suppose we have an L-parameter

$$\phi \colon \mathrm{WD}_F \to {}^LG.$$

We denote by ϕ_N the Weil representation $\phi_N \colon W \to {}^LG$ which is the summand of ϕ corresponding to $\ker N$.

Let

$$\rho \colon \widehat{G} \to \mathrm{GL}(V)$$

be any (algebraic) representation.

Definition 1.1. Given an L-parameter ϕ, we define

$$L(s, \phi, \rho) := L(s, \rho \circ \phi)$$

and

$$\epsilon(s, \phi, \rho) := \epsilon(s, \rho \circ \phi).$$

Suppose

$$\rho \circ \phi = \bigoplus_{n \geq 0} \sigma_n \otimes \mathrm{Sym}^n$$

as representations of $\mathrm{WD}_F \times \mathrm{SL}_2$. Denote the space of σ_n by V_n. Then

$$L(s, \varphi, \rho) = \prod_{n \geq 0} \det(1 - q^{-\frac{n}{2}-s}\sigma_n(\mathrm{Frob}) \mid (V_n)^I)^{-1}$$

and

$$\epsilon(s, \phi, \rho) = \epsilon(1/2, \phi, \rho)q^{a(\rho\circ\phi)(\frac{1}{2}-s)}$$

where

$$a(\rho \circ \varphi) := \sum_{n \geq 0}(n + 1)a(\sigma_n) + \sum_{n \geq 0} n \dim((V_n)^I_N)$$

with $a(\rho \circ \varphi)$ the *Artin conductor* of σ_n, and

$$\epsilon(1/2, \phi, \rho) = \prod_{n \geq 0} \epsilon(1/2, \sigma_n)^{n+1} \prod_{n \geq 0} \det(-\phi(\mathrm{Frob}) \mid (V_n)^I_N)^n.$$

1.5. **γ-factors.** Let ϕ, ρ be as before. We define the associated *γ-factor* to be

$$\gamma(s,\phi,\rho,\psi) = \frac{L(1-s,\phi,\rho^\vee)\epsilon(s,\rho\circ\phi,\psi)}{L(s,\varphi,\rho)}.$$

From the functional equation for ϵ, we get

$$\gamma(s,\varphi,\rho,\psi)\gamma(1-s,\varphi,\rho^\vee,\psi^{-1}) = 1.$$

We can apply this to our previous examples.

Example 1.1. Suppose F is p-adic. If σ is irreducible admissible of W_F with $\dim\sigma > 1$, we have $L(s,\sigma) = 1$ but the ϵ factor is complicated.

Unramified representations of GL_n correspond to parameters φ which are trivial on SL_2 and inertia, so they are completely specified by $\varphi(\mathrm{Frob})$, a semi-simple conjugacy class in $\mathrm{GL}_n(\mathbf{C})$. Such representations are $\pi < \mathrm{Ind}_B^{\mathrm{GL}_n}(\chi_1\otimes\ldots\otimes\chi_n)$ with all χ_i unramified, and the corresponding $\varphi(\mathrm{Frob})$ is $\mathrm{diag}(\chi_1(\varpi),\ldots,\chi_n(\varpi))$. Then

$$L(s,\sigma) = \prod_{i=1}^n (1-\chi_i(\varpi)q^{-s})^{-1}$$

and $\epsilon(s,\sigma,\psi) = 1$.

Exercise 1.2. For $F = \mathbf{R}$ or $\mathbf{C}$, pick your favorite G and $\widehat{G}$ and compute the poles of $L(s,\varphi,\rho)$ in terms of the parametrization.

Exercise 1.3. Suppose F is p-adic. Compute $L(s,\mathrm{Ind}_{W_E}^{W_F}\chi,\mathrm{Sym}^n)$ and $\epsilon(s,\mathrm{Ind}_{W_E}^{W_F}\chi,\mathrm{Sym}^n)$. [The ϵ-factor is complicated for $n=1$, but less bad for $n\geq 2$.]

2. The local Langlands correspondence

Let F be a non-archimedean field of char 0. Let $\mathcal{O}$ be the ring of integers of F and $\varpi\in\mathcal{O}$ a uniformizer. Let $q = |\mathcal{O}/\varpi\mathcal{O}|$.

2.1. **Characterization of LLC.** Let $\mathscr{L}\colon \mathrm{Irr}(\mathrm{GL}_n)\to\Phi(\mathrm{GL}_n)$, sending $\pi\mapsto\varphi = \mathscr{L}(\pi)$. We defined L-factors and ϵ-factors on the Galois side. How do we define them in terms of representation theory?

Godement-Jacquet explained how to go construct them directly from π, in a compatible way.

$$L(s,\pi) = L(s,\varphi)$$
$$\epsilon(s,\pi,\psi) = \epsilon(s,\pi,\psi)$$

Furthermore, the conductor of π should agree with the Artin conductor of φ. Recall that $c(\pi) = 0$ iff π is unramified. In general,

$$c(\pi) = \min_{t\geq 0}\{t\colon V^{k_t}\neq 0\}$$

where

$$K_t = \left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\mathrm{GL}_n(\mathcal{O})\colon {\scriptstyle C\in M_{1\times(n-1)}(\varpi^t\mathcal{O}) \atop \scriptstyle D\equiv 1 \mod \varpi^t\mathcal{O}}\right\}$$

This normalization assumes $\psi|_{\mathcal{O}} = \mathrm{Id}$ and $\mathrm{vol}(\mathcal{O}) = 1$.

The L-factors and ϵ-factors do not determine the representations. (For supercuspidals the L-factor is always 1, while the ϵ-factor is more complicated, but still not rich enough to determine π.)

Theorem 2.1. *There is a unique map*

$$\mathscr{L}\colon \mathrm{Irr}(\mathrm{GL}_2) \to \Phi(\mathrm{GL}_2)$$

satisfying:

(1) $L(s, \pi \otimes \chi) = L(s, \mathscr{L}(\pi) \otimes \chi)$ *for all* χ *of* $F^\times$.
(2) $\epsilon(s, \pi \otimes \chi, \psi) = \epsilon(s, \mathscr{L}(\pi) \otimes \chi, \psi)$ *for all* χ *of* $F^\times$.

Why is this enough? One line of reasoning comes from the converse theorem. Another comes from functoriality. For G/F connected, split, reductive, there should be a map from

$$\{\pi \in \mathrm{Irr}(G)\} \mapsto \{\varphi\colon WD_F \to \widehat{G}\}.$$

Given a representation $\rho\colon \widehat{G} \xrightarrow{\rho} \mathrm{GL}_N(\mathbf{C})$, we get a Langlands parameter for GL_N, hence $\ell_\rho(\pi) := \Pi \in \mathrm{Irr}(\mathrm{GL}_N)$. Furthermore, it is easy to show that the γ-factors enjoy

$$\gamma(s, \varphi, \rho, \psi) = \gamma(s, \rho \circ \varphi, \psi) = \gamma(s, \ell_\rho(\pi), \psi).$$

(The same holds for the ϵ and L-factors.)

In the setting of the theorem, think $G = \mathrm{GL}_2 \times \mathrm{GL}_1$ and ρ is the tensor product of the standard representations.

2.2. **The Hecke algebra.** Let B be a Borel subgroup over F. Suppose a Levi decomposition $B = TU$. Let $X^*(T), X_*(T)$ be the character/cocharacter groups. Let $K = G(\mathcal{O})$ be a hyperspecial maximal compact subgroup of $G(F)$.

Definition 2.1. The *Hecke algebra* is $\mathcal{H}(G, K) = (C_c^\infty(K\backslash G/K), \star)$ where the multiplication $\star$ is given by

$$f_1 \star f_2(g) = \int_G f_1(x) f_2(x^{-1}g)\, dx.$$

We have a decomposition

$$G(F) = \bigcup_{\mu \in X_*^+(T)} K\mu(\varpi)K$$

where $X_\mu^+(T) = \{\lambda \in X_*(T)\colon \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$. Let $\mathbf{I}_{K\mu(\varpi)K}$ be the characteristic function of $K\mu(\varpi)K$. Then $\mathcal{H}(G, K)$ is commutative.

2.3. **Satake transform.** Define the *Satake transform*

$$\mathrm{Sat}\colon \mathcal{H}(G, K) \to \mathcal{H}(T, T(\mathcal{O}))$$

by

$$\mathrm{Sat}(f)(t) \mapsto \delta_B^{1/2}(t) \int_{U(F)} f(tu)\, du$$

where δ_B is the modular character of B.

We have a map $\gamma\colon T(F)/T(\mathcal{O}) \xrightarrow{\sim} X_*(T) \cong X^*(\widehat{T})$ sending

$$t \mapsto \langle \gamma(t), \chi \rangle := \operatorname{ord} \chi(t) \text{ for all } \chi \in X^*(T).$$

Theorem 2.1 (Harish-Chandra)**.** *For regular $t \in T(F)$, we have*

$$\operatorname{Sat}(f)(t) = D(t) \int_{G(F)/T(F)} f(gtg^{-1}) dg$$

where $D(t) = \delta_B^{1/2}(t) |\det \operatorname{Ad}_U(t) - \operatorname{Id}_U|$.

Theorem 2.2 (Satake)**.** *The Satake transform induces*

$$\operatorname{Sat}\colon \mathcal{H}(G,K) \xrightarrow{\sim} \mathbf{C}[X_*(T)]^W \cong \mathbf{C}[X^*(\widehat{T})]^W$$

where W is the Weyl group of G.

2.4. **Change of basis.** For $\lambda \in X_*^+(T)$, we have

$$\operatorname{Sat}(\mathbf{I}_{K\lambda(\varpi)K}) = q^{\langle \lambda, \rho_B \rangle} \chi_\lambda + \sum_{\mu < \lambda} a_\lambda(\mu) \chi_\mu$$

where ρ_B is the half sum of positive roots, $a_\lambda(\mu) \in \mathbf{Z}$, and $\chi_\lambda = \operatorname{Tr} V_\lambda$.

We can invert this to express χ_λ in terms of the Satake basis.

Theorem 2.1 (Lusztig-Kato)**.** *We have*

$$\chi_\lambda = q^{-\langle \lambda, \rho_B \rangle} \sum_{\mu \leq \lambda, \mu \in X_*^+(T)} P_{\mu,\lambda}(q) \operatorname{Sat}(\mathbf{I}_{K\mu(\varpi)K})$$

where $P_{\mu,\lambda}$ is a Kazhdan-Lusztig polynomial.

2.5. **Satake parameter.** Let (π, V_π) be an irreducible admissible representation of $G(F)$. For all $f \in C_c^\infty(G)$, we define an operator $\pi(f)$ by

$$\pi(f)v := \int_{G(F)} f(g)\pi(g)v \, dg, \quad v \in V.$$

We define the trace of π as a distribution,

$$\operatorname{Tr}_\pi \colon C_c^\infty(G) \to \mathbf{C}$$

sending $f \mapsto \operatorname{Tr} \pi(f)$.

For $v \in V_\pi$, assume π is unramified, and $f \in \mathcal{H}(G,K)$. If $v \in V^K$, which is 1-dimensional, then it must be an eigenvector for the $\mathcal{H}(G,K)$-action, i.e. $\pi(f)v = \omega(f)v$. This ω defines a character of $\mathcal{H}(G,K)$, which is of the form

$$\omega(f) = \int_{T(F)} \operatorname{Sat}(f)(t)\theta(t) \, dt$$

where $\pi < \operatorname{Ind}_{B(F)}^{G(F)} \theta$.

Definition 2.1. We define the *Satake parameter of* π, denoted c_π, as the semi-simple conjugacy class of $\widehat{G}$ determined by the property that

$$\operatorname{Tr} \pi(f) = \operatorname{Sat}(f)(c_\pi).$$

3. The basic function

3.1. **Goal.** Assume that we have a group G satisfying the following properties.

(1) There exists a short exact sequence,

$$1 \to G_0 \to G \xrightarrow{\det} \mathbf{G}_m \to 1 \tag{3.1}$$

hence also a short exact sequence

$$0 \to X_*(T_0) \to X_*(T) \xrightarrow{\det} X_*(\mathbf{G}_m) \to 0.$$

(2) We have a representation $\rho\colon \widehat{G} \to \mathrm{GL}(V)$ such that $\rho(z) = z \cdot \mathrm{Id}$ for $z \in \mathbf{C}^\times$ for $\mathbf{G}_m \hookrightarrow \widehat{G}$ induced by (3.1).

Our goal is to define the *basic function* $\mathbf{L}_{\rho,s}$ such that

$$\operatorname{Tr} \pi(\mathbf{L}_{\rho,s}) = L(s, \pi, \rho).$$

3.2. **Example.** Let $G = \mathrm{GL}_2$, ρ the standard representation of $\widehat{G} = \mathrm{GL}_2(\mathbf{C})$.

Consider a representation of the form $\pi < \operatorname{Ind}_B^{\mathrm{GL}_2}(\chi_1 \otimes \chi_2)$, where χ_1, χ_2 are unramified characters. The Satake parameter is

$$c_\pi = \begin{pmatrix} \chi_1(\varpi) & \\ & \chi_2(\varpi) \end{pmatrix} =: \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

The associated L-factor is then

$$L(s, \pi, \rho) := \frac{1}{(1 - \alpha q^{-s})(1 - \beta q^{-s})}. \tag{3.2}$$

We Taylor-expand (3.2)

$$L(s, \pi, \rho) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \alpha^i \beta^{k-i} \right) q^{-ks},$$

which we can rewrite as

$$L(s, \pi, \rho) = \sum_{k=0}^{\infty} \operatorname{Tr} \left(\operatorname{Sym}^k \rho \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \right) \cdot q^{-ks}. \tag{3.3}$$

Now, we have

$$\operatorname{Tr}(\operatorname{Sym}^k \rho) = q^{-k/2} \sum_{\substack{a+b=k \\ a \geq b \geq 0}} \operatorname{Sat}(\mathbf{I}_{K \operatorname{diag}(\varpi^a, \varpi^b) K}) \tag{3.4}$$

Inserting (3.4) into (3.3) above, we find that

$$\begin{aligned} L(s, \pi, \rho) &= \sum_{k \geq 0} \sum_{\substack{a+b=k \\ a \geq b \geq 0}} \operatorname{Sat}(\mathbf{I}_{K \operatorname{diag}(\varpi^a, \varpi^b) K}) q^{-k(s+1/2)} \\ &= \sum_{a \geq b \geq 0} \operatorname{Sat}(\mathbf{I}_{K \operatorname{diag}(\varpi^a, \varpi^b) K}) q^{-(s+1/2)(a+b)}. \end{aligned}$$

We define a new function

$$\mathbf{L}_{\rho,s} := \sum_{a \geq b \geq 0} \mathbf{I}_{K\operatorname{diag}(\varpi^a,\varpi^b)K} q^{-(s+1/2)(a+b)}.$$

Recalling that $\operatorname{Tr}\pi(f) = \operatorname{Sat}(f)(c_\pi)$ for all $f \in \mathcal{H}(G,K)$, the upshot is that

$$\operatorname{Tr}\pi(\mathbf{L}_{\rho,s}) = L(s,\pi,\rho).$$

Notice that the support of $\mathbf{L}_{\rho,s}$ is $M_{2\times 2}(\mathcal{O}) \cap \mathrm{GL}_2(F)$.

Suppose we take $s = -1/2$. Then we get

$$\mathbf{L}_{\rho,-1/2} = \sum_{a \geq b \geq 0} \mathbf{I}_{K\operatorname{diag}(\varpi^a,\varpi^b)K}.$$

This is just the characteristic function of $M_{2\times 2}(\mathcal{O}) \cap \mathrm{GL}_2(F)$.

3.3. **Definition of the basic function.** Let π be an unramified representation of G, with Satake parameter c_π. Suppose we're given a representation

$$\rho \colon \widehat{G} \to \mathrm{GL}_N(\mathbf{C}).$$

By definition,

$$L(s,\pi,\rho) = \det(I_N - \rho(c_\pi)q^{-s})^{-1}.$$

We again want to rewrite this in terms of Satake transforms. First, we have

$$\det(I_N - \rho(c_\pi)q^{-s})^{-1} = \sum \operatorname{Tr}\left(\operatorname{Sym}^k \rho(c_\pi)\right) q^{-ks}.$$

We then decompose $\operatorname{Sym}^k \rho$ in terms of irreducibles.

$$\operatorname{Sym}^k \rho = \sum_{k \geq 0} \sum_{\lambda \in X_+^*(\widehat{T})} m(\operatorname{Sym}^k \rho : V_\lambda) V_\lambda$$

hence

$$\operatorname{Tr}\left(\operatorname{Sym}^k \rho(c_\pi)\right) q^{-ks} = \sum_{k \geq 0} \sum_{\lambda \in X_+^*(\widehat{T})} m(\operatorname{Sym}^k \rho : V_\lambda) \chi_\lambda(c_\pi) q^{-ks}.$$

To express χ_π in terms of Satake transforms, we use the Lusztig-Kato formula from Theorem 2.1:

$$\chi_\lambda = q^{-\langle \lambda, \rho_B \rangle} \sum_{\mu \leq \lambda} P_{\mu,\lambda}(q) \operatorname{Sat}(\mathbf{I}_{K\mu(\varpi)K}).$$

We then plug this in to get a complicated formula for the L-function in terms of the Satake transform of some function.

Definition 3.1. Define

$$\mathbf{L}_k^\rho := \operatorname{Sat}^{-1}(\operatorname{Tr}\operatorname{Sym}^k \rho) =: \sum_{\mu \in X_*^+(T)} a_\mu(q,s,k) \mathbf{I}_{K\mu(\varpi)K}.$$

Define the *basic function* to be

$$\mathbf{L}_{\rho,s} := \sum_{k \geq 0} \mathbf{L}_\rho^k q^{-ks} \in C^\infty(K \backslash G / K).$$

By construction, we have

$$\operatorname{Tr}\pi(\mathbf{L}_{\rho,s}) = L(s,\pi,\rho).$$

Several natural questions present themselves:

(1) What is the support of $\mathbf{L}_{\rho,s}$? (It will be a union of double cosets $K\mu(\varpi)K$, but which ones?)
(2) Find M_ρ to replace $M_{2\times 2}$ in the example of §3.2 such that the support of $\mathbf{L}_{\rho,s}$ is contained in $G(F)\cap M_f(\mathcal{O})$ and $\overline{\operatorname{supp}(\mathbf{L}_{\rho,s})}$ is compact in $M_\rho(F)$.

4. Reductive monoids

4.1. Linear algebraic monoids.

Definition 4.1. A *linear algebraic monoid* M is an affine algebraic variety together with an associative morphism

$$\mu\colon M\times M\to M$$

and an identity element $1\in M$ for μ.

We define the *unit group* of a linear algebraic monoid M to be

$$G(M) := \{g\in M\colon g^{-1}\in M\}.$$

Theorem 4.2. *Let G be an irreducible algebraic group. There exists an irreducible algebraic monoid M with $G(M)=G$ such that $G\neq M$ if and only if $X^*(G)\neq\{1\}$.*

Let M be reductive (i.e. $G(M)$ is reductive), normal, with $0\in M$ and one-dimensional center. We will generally restrict our attention to such monoids, possibly relaxing the last condition. Under these conditions, have the following facts:

(1) If M is smooth, then $M\cong M_{n\times n}(\overline{F})$ as algebraic monoids.
(2) $(M-0)/\mathbf{G}_m$ is projective.

4.2. Example: Vinberg's universal monoid of SL_3.

Let $G'=\mathrm{SL}_3$,

$$T' := \left\{ t := \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & (t_1t_2)^{-1}\end{pmatrix}\right\}\subset G'.$$

Let Z' be the center of G'.

Define $G^+=(G'\times T')/Z'$. We have a short exact sequence

$$1\to\mathrm{SL}_3\to G^+\to T'/Z'\to 1.$$

We will construct a monoid M for G^+. The idea is to take a large faithful representation, and take the affine closure in the space of endomorphisms.

Define two representations of $G'=\mathrm{SL}_2$: $\rho_1=\mathrm{Id}$ and $\rho_2=g\mapsto(g^{-1})^T$, corresponding to the fundamental weights $\omega_1(t)=t_1$ and $\omega_2(t)=t_1t_2$, respectively.

We will define new representations $\rho_i^+\colon G^+\to\mathrm{GL}_3$, by

$$(t,g)\mapsto\omega_i(w_0(t^{-1}))\rho_i(g).$$

where w_0 be the longest Weyl element, which specializes in this case to

$$\rho_1^+\colon(t,g)\mapsto(t_1t_2g)$$

and

$$\rho_2^+ \colon (t, g) \to (t_1(g^{-1})^T).$$

Let $\alpha_1(t) = t_1 t_2^{-1}$ and $\alpha_2(t) = t_1 t_2^2$. Hence we have $(\alpha^+, \rho^+) \colon G^+ \to \mathbf{G}_m^2 \times \mathrm{GL}_3 \times \mathrm{GL}_3$ by

$$(t, g) \mapsto (t_1 t_2^{-1}, t_1 t_2^2, t_1 t_2 g, t_1(g^{-1})^T).$$

Let M^+ be the affine closure of the image of (α^+, ρ^+) in $\mathbf{G}_a^2 \times M_{3\times 3}^2$. We can describe the image as

$$\left\{ (x, y, A_1, A_2) \in \mathbf{G}_m^2 \times \mathrm{GL}_3^2 \colon \substack{A_1(A_2^T) = (A_1^T)A_2 = xy\,\mathrm{Id}_3 \\ \wedge^2 A_1 = xA_2 \\ \wedge^2 A_2 = yA_1} \right\}.$$

We have $G^+ = G(M^+)$. Writing $M^+ = \bigcup_{e \in \Lambda} G^+ e G^+$, we have $\#\Lambda = 11$ in this case. There is an action of $G' \times G'$ on M^+. Let

$$\pi \colon M^+ \to M^+ // G' \times G'$$

be the GIT quotient.

Example 4.1. For $G' = \mathrm{SL}_2$, $M^+ = M_{2\times 2}$ and $\pi \colon M^+ \to M^+ // \mathrm{SL}_2 \times \mathrm{SL}_2$ and $\pi(g) = \det g$.

In the previous example, $M^+ / G' \times G' \cong \mathbf{G}_a^2$. In general, the quotient is an affine space of dimension rank equal to $\operatorname{rank} G'$.

4.3. **Example: $G' = \mathrm{SL}_2$.** Let's go back to the $G' = \mathrm{SL}_2$ case. Let $\lambda_n \colon \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t^n$. This is the highest weight of the representation $\rho_n = \mathrm{Sym}^n$ of SL_2.

Consider $Z' \subset T' \subset G' = \mathrm{SL}_2$. We define $\lambda_n \in X^*(\widehat{T}^{\mathrm{sc}}) = X_*(T^{\mathrm{ad}})$ by $\lambda_n(a) = a^n$. This extends to a morphism $\mathbf{G}_a \to \mathbf{G}_a$.

We then define M^{λ_n} by the cartesian diagrams

$$\begin{array}{ccc} M^{\lambda_n} & \longrightarrow & M^+ \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^+} \\ \mathbf{G}_a & \xrightarrow{\lambda_n} & \mathbf{G}_a^r \end{array}$$

Explicitly,

$$M^{\lambda_n} = \{(a, m) \in \mathbf{G}_a \times M_{2\times 2} \colon \pi^+(m) = \lambda_n(a), \text{ i.e. } \det(m) = a^n\}.$$

- If $n = 2m$ is even, then we have

  $$\mathrm{GL}_1 \times \mathrm{SL}_2 \xrightarrow{\sim} G(M^{\lambda_n})$$

  by

  $$(a, g) \mapsto (a, a^m g).$$

- If $n = 2m + 1$ is odd, then

  $$\mathrm{GL}_2 \xrightarrow{\sim} G(M^{\lambda_n})$$

  by

  $$g \mapsto (\det g, (\det g)^m g).$$

The dual groups are then $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$. We take $\rho = \mathrm{Sym}^n$. This is always well defined.

Exercise 4.1. Check that $\mathrm{supp}(\mathbf{L}_{\rho,s}) \subset M^{\lambda_n}(\mathcal{O}) \cap G^{\lambda_n}(F)$.