THE BRAVERMAN-KAZHDAN-NGO APPROACH TO L-FUNCTIONS

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Contents

1. Local Godement-Jacquet-Tamagawa theory

Let $G = \mathrm{GL}_n$ and let F be a p-adic field. (There are analogous results for archimedean local fields and local function fields.) The goal is to establish an analytic theory for standard L-functions of GL_n, via integral representations of L-functions.

1.1. **Preliminaries.** There are two main ingredients: Schwartz space, and Fourier transform, and we'll discuss each of these in turn.

1.1.1. *Schwartz space.* We have an embedding $G \hookrightarrow M_n$, the space of $n \times n$-matrices. It is important that this is a $G \times G$-equivariant embedding (with the action by left and right translations). Aside: this is an *affine spherical embedding.*

Remark 1.1. M_n is a reductive monoid for G. (In fact, it is the only smooth one.)

Definition 1.2. The space of *Schwartz functions* for G is the space of functions obtained by restriction of $C_c^\infty(M_n)$ to G.

1.1.2. *Fourier transform.* There is a Fourier transform

$$\mathcal{F} \colon C_c^\infty(M_n) \to C_c^\infty(M_n)$$

given by sending $f \in C_c^\infty(M_n)$ to

$$\mathcal{F}(f)(x) = \int_{M_n} \psi(\mathrm{Tr}(xy)) f(y) dy^+$$

where

- dy^+ is a Haar measure on M_n,
- $\psi \colon F \to \mathbf{C}$ is self-dual with respect to $\mathcal{F}$, so

$$\mathcal{F}(\mathcal{F}(f))(x) = f(-x).$$

We will sometimes write $\widehat{f} := \mathcal{F}(f)$.

1.2. **Main results.** Let $\pi \in \mathrm{Irr}(G)$. Let $\mathcal{C}(\pi)$ be the space of matrix coefficients for π. Define

$$\mathscr{Z}(s,f,\varphi)=\int_G f(g)\varphi(g)|\det g|^{s+\frac{n-1}{2}}\,dg$$

for $f \in C_c^\infty(M_n)$ and $\varphi \in C(\pi)$.

Theorem 1.1. *We have the following facts.*

1. $\mathscr{Z}(s,f,\varphi)$ *is absolutely convergent for* $\mathrm{Re}\, s \gg 0$.
2. $\mathscr{Z}(s,f,\varphi)$ *is a rational function in* q^{-s} *(where* q *is the cardinality of the residue field of* F*). Moreover, the family of rational functions*
$$I(\pi):=\{\mathscr{Z}(s,f,\varphi)\colon f\in C_c^\infty(M_n),\varphi\in C(\pi)\},$$
viewed as functions on G *by restriction, admits a greatest common denominator, denoted* $\mathscr{L}(s,\pi)$.
3. *(Functional equation) There exists a rational function* $\gamma(s,\pi,\psi)\in \mathbf{C}(q^{-s})$ *such that*
$$\mathscr{Z}(1-s,\mathcal{F}(f),\varphi^\vee)=\gamma(s,\pi,\psi)\mathscr{Z}(s,f,\varphi)$$
with
$$\varphi^\vee(g)=\varphi(g^{-1})\in C(\widetilde{\pi}).$$

1.2.1. *The L-function.* Assuming Theorem 1.1, we can define $\mathscr{L}(s,\pi)$ as follows. For all $f\in C_c^\infty(M_n)$ and $\varphi \in C(\pi)$, for $h \in G$ we define new functions f_1, φ_1 by $f_1(g) := f(gh)$ and $\varphi_1(g) := \varphi(gh)$. Then one can compute that

$$\mathscr{Z}(s,f,\varphi_1)=\int_G f_1(g)\varphi_1(g)|\det g|^{s+\frac{n-1}{2}}\,dg=|\det h|^{-s-\frac{n-1}{2}}|\mathscr{Z}(s,f,\varphi).$$

This implies that $I(\pi)$ is a $\mathbf{C}[q^{\pm s}]$-module in $\mathbf{C}(q^{-s})$, i.e. a fractional ideal. For $0\neq\varphi\ni C(\pi)$ with $\varphi(e)\neq 0$ and $\varphi^{K_0}=\varphi$, we have that $\mathscr{Z}(s,\mathbf{I}_{K_0},\varphi)$ is constant. In particular,

$$I(\pi)\supset \mathbf{C}[q^s,q^{-s}].$$

Since $\mathbf{C}[q^{\pm s}]$ is a PID, $I(\pi)$ has a generator of the form $P(q^{-s})^{-1}$ for $P(X)\in\mathbf{C}[X]$ with $P(0)=1$. We define

$$\mathscr{L}(s,\pi):=P(q^{-s})^{-1}.$$

Remark 1.1. Under the local Langlands correspondence, the Godement-Jacquet L-function coincides with the Langlands L-function associated to the corresponding Weil-Deligne representation.

1.2.2. *The ϵ-factor.* The functional equation then gives an expression for the γ-factor in terms of $\mathscr{L}(s,\pi)$. We define the *ϵ-factor*

$$\epsilon(s,\pi,\psi)=\gamma(s,\pi,\psi)\cdot\frac{\mathscr{L}(s,\pi)}{\mathscr{L}(1-s,\pi^\vee)}.$$

The functional equation can then be reformulated as

$$\frac{\mathscr{Z}(1-s,\mathcal{F}(f),\varphi^\vee)}{\mathscr{L}(1-s,\pi^\vee)}=\epsilon(s,\pi,\psi)\frac{\mathscr{Z}(s,f,\varphi)}{\mathscr{L}(s,\pi)}.$$

Since the ratio $\mathscr{Z}(s,f,\varphi)/\mathscr{L}(s,\pi) \in \mathbf{C}[q^{\pm s}]$, we deduce that $\epsilon(s,\pi,\psi)$ is a unit in $\mathbf{C}[q^{\pm s}]$. Hence it is a monomial in q^{-s}.

Corollary 1.1. *We have*

$$\gamma(s,\pi,\psi)\cdot\gamma(1-s,\pi^\vee,\psi) = \omega_\pi(-1)$$

where ω_π is the central character of π.

Proof sketch. The main input is that $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$. Applying the functional equation twice, we get on one hand

$$\begin{aligned}\mathscr{Z}(s,\mathcal{F}(\mathcal{F}(f)),\varphi) &= \gamma(1-s,\pi^\vee,\psi)\mathscr{Z}(1-s,\mathcal{F}(f),\varphi^\vee)\\ &= \gamma(1-s,\pi^\vee,\psi)\gamma(s,\pi,\psi)\mathscr{Z}(s,f,\varphi),\end{aligned}$$

but on the other hand the LHS can also be identified with $\omega_\pi(-1)\mathscr{Z}(s,f,\varphi)$. □

1.3. **Relation with parabolic induction.** By the work of Jacquet and Harish-Chandra, we know that for any $\pi \in \mathrm{Irr}(G)$, we can find a parabolic subgroup $P \subset G$ with Levi decomposition $P = MN$, and a supercuspidal representation τ of M, such that

$$\pi \hookrightarrow \mathrm{Ind}_P^G \tau.$$

For convenience we restrict ourselves to the case where P is a maximal parabolic. So let P be the standard maximal parabolic subgroup in G of type (m_1, m_2) with $m_1 + m_2 = n$,

$$P = \begin{pmatrix} \boxed{\mathrm{GL}_{m_1}} & N \\ & \boxed{\mathrm{GL}_{m_2}} \end{pmatrix}.$$

So $M \cong \mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}$ and $\tau = \sigma_1 \otimes \sigma_2$, where σ_i is a supercuspidal representation of $\mathrm{GL}(m_i)$.

We recall some properties of $\mathrm{Ind}_P^G \tau$.

(1) By definition,

$$\mathrm{Ind}_P^G \tau = \{\text{smooth functions } F\colon G \to V_\tau \colon F(pg) = \delta_P(p)^{1/2}\tau(p)F(g)\}.$$

(2) $\widetilde{\mathrm{Ind}_P^G \tau} \cong \mathrm{Ind}_P^G \widetilde{\tau}$.

(3) For $F \in \mathrm{Ind}_P^G \tau$ and $\widetilde{F} \in \mathrm{Ind}_P^G \widetilde{\tau}$,

$$\varphi(g) := \langle \widetilde{F}, (\mathrm{Ind}_P^G \tau)(g)F\rangle_{\mathrm{Ind}_P^G \tau}$$

can be computed in terms of matrix coefficients of τ:

$$\varphi(g) = \int_K \langle \widetilde{F}(k), F(kg)\rangle_\tau, dk$$

where $K = \mathrm{GL}_n(\mathcal{O}_F)$ is our fixed maximal compact subgroup.

1.3.1. *Zeta integrals.* Our immediate is to reduce the theory of zeta integrals for such π to those of τ.

For $f \in C_c^\infty(M_n)$ and $\varphi \in C(\pi) \subset C(\mathrm{Ind}_P^G \tau)$ as above,

$$\begin{aligned}\mathscr{Z}(s,f,\varphi) &= \int_G f(g)\varphi(g)|\det g|^{s+\frac{n-1}{2}}\,dg \\ &= \int_G \int_K f(g)\langle \widetilde{F}(k), F(kg)\rangle_\tau \,dk|\det g|^{s+\frac{n-1}{2}}\,dg\end{aligned}$$

After the change of variables $g \mapsto k^{-1}g$, this becomes

$$\mathscr{Z}(s,f,\varphi) = \int_K \int_G f(k^{-1}g)\langle \widetilde{F}(k), F(g)\rangle_\tau |\det g|^{s+\frac{n-1}{2}}\,dgdk \tag{1.1}$$

Using the Iwasawa decomposition $G = PK$, we can break up the integral over G into separate integrals over P and K. Writing $g = pk'$ with respect to this decomposition, we get

$$(1.1) = \int_K \int_K \int_P f(k^{-1}pk')\delta_P(p)^{1/2}\langle \widetilde{F}(k), \tau(p)F(k)\rangle_\tau |\det p|^{s+\frac{n-1}{2}}\,dpdkdk' \tag{1.2}$$

Writing $p = \begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix}$, with $g_i \in \mathrm{GL}_{m_i}$ and $u \in M_{m_1\times m_2}$, and noting that $\delta_P(p) = |\det g_1|^{m_1}|\det g_2|^{-m_2}$, and using the explicit expression for Haar measure

$$dp = \frac{1}{|\det g_1|^{m_i}}\,dg_1dg_2du$$

the integral becomes

$$\begin{aligned}(1.2) = \int_K \int_K \int_M \int_N f\left(k^{-1}\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix}k'\right)\left\langle \widetilde{F}(k), \tau\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}F(k')\right\rangle_\tau \\ \cdot |\det g_1|^{s+\frac{m_1-1}{2}}|\det g_2|^{s+\frac{m_2-1}{2}}dg_1dg_2dudkdk'\end{aligned} \tag{1.3}$$

Now,

$$\left\langle \widetilde{F}(k), \tau\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}F(k')\right\rangle_\tau \in C(\sigma_1)\otimes C(\sigma_2) \tag{1.4}$$

so it is a valid test function φ for the zeta integral for $\mathrm{GL}_{m_1}\times\mathrm{GL}_{m_2}$. It only remains to see that

$$\int_N f\left(k^{-1}\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix}k'\right)\,du \in C_c^\infty(M_{m_1})\otimes C_c^\infty(M_{m_2}).$$

Lemma 1.1. *Let* $f \in C_c^\infty(M_n)$. *Then*

$$\int_N f\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix}du \in C_c^\infty(M_{m_1})\otimes C_c^\infty(M_{m_2}).$$

Proof. Decompose $C_c^\infty(M_m) \cong \otimes_{n^2} C_c^\infty(F)$. Then the statement is obvious. □

Using K-finiteness of f and (1.4) gives: $\mathscr{Z}(s,f,\varphi)$ is a finite linear combination of terms $\mathscr{Z}(s,f_1,\varphi_1)\mathscr{Z}(s,f_2,\varphi_2)$ with $f_i \in C_c^\infty(M_{m_1})$ with $\varphi_i \in C(\sigma_i)$. Hence if the GCD property holds for $\sigma_1\otimes\sigma_2$, it also holds for $\pi \hookrightarrow \mathrm{Ind}_P^G\tau = \sigma_1\otimes\sigma_2$.

Remark 1.2. The L-factor for π is not always exactly the product of the L-factors for σ_1 and σ_2. What we find is that

$$\frac{\mathscr{L}(s,\pi)}{\mathscr{L}(s,\sigma_1)\mathscr{L}(s,\sigma_2)} \in \mathbf{C}[q^{\pm s}].$$

An example where this fails is $G = \mathrm{GL}_2$, and $\pi = \mathrm{St} \hookrightarrow \mathrm{Ind}_{B_2}^{\mathrm{GL}_2}(\delta_{B_2}^{1/2})$.

1.3.2. *Functional equation.* What about the functional equation? We want to show that the functional equation for the Levi factor implies it for the induced representation. This comes from an interaction of parabolic induction and Fourier transform.

Lemma 1.1. *Let $f \in C_c^\infty(M_n)$, $g_i \in M_{m_i}$, and define*

$$r(f)\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} := \int_N f\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix} du.$$

Then

$$\widehat{r(f)} = r(\widehat{f})$$

where the LHS Fourier transform is on $C_c^\infty(M_{m_1}) \otimes C_c^\infty(M_{m_2})$ and the RHS Fourier transform is on $C_c^\infty(M_m)$.

Proof. By definition,

$$\begin{aligned} r(\widehat{f})\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} &= \int_N \widehat{f}\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix} du \\ &= \int_N \int_{M_n} f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi\left(\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix}\right]\right) du da db dc dd \end{aligned}$$

An easy computation shows that the trace is just $\mathrm{Tr}(ag_1 + cu + dg_2)$, so

$$r(\widehat{f})\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} = \int_N \int_{a,b,c,d} f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi(\mathrm{Tr}(ag_1 + cu + dg_2)) du da db dc dd. \tag{1.5}$$

Now apply Fourier inversion for the variables c, u and you get

$$(1.5) = \int_{a,b,d} f\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \psi(\mathrm{Tr}(am_1 + dm_2))\, da db dd = \widehat{r(f)}\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}.$$

∎

We will now show that if the functional equation holds for τ, then it holds for any $\pi \hookrightarrow \mathrm{Ind}_P^G \tau$. As in (1.3), the zeta integral is

$$\begin{aligned} \mathscr{Z}(1-s, \mathcal{F}(f), \varphi^\vee) = \int_K \int_K \int_M \int_N \mathcal{F}(f)\left((k')^{-1}\begin{pmatrix} g_1 & u \\ & g_2 \end{pmatrix} k\right) \left\langle F(k'), \widetilde{\tau}\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \widetilde{F}(k)\right\rangle_\tau \\ \cdot |\det g_1|^{1-s+\frac{m_1-1}{2}} |\det g_2|^{1-s+\frac{m_2-1}{2}} dg_1 dg_2 du dk dk' \end{aligned} \tag{1.6}$$

We set

$$\xi\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} = \left\langle \widetilde{F}(k), \tau\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} F(k')\right\rangle_\tau$$

so that

$$\xi^\vee \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} = \left\langle \widetilde{F}(k), \tau \begin{pmatrix} g_1^{-1} & \\ & g_2^{-1} \end{pmatrix} F(k') \right\rangle_\tau = \left\langle F(k'), \widetilde{\tau} \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \widetilde{F}(k) \right\rangle_{\widetilde{\tau}}.$$

Moving the $\tau \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}^{-1}$ to the other side, one gets that $\xi^\vee$ is the matrix coefficient appearing earlier.

Defining

$${}^{k^{-1}}f^{k'}(g) = f(k^{-1}gk'),$$

we have

$$r(\widehat{{}^{k^{-1}}f^{k'}}) = r(\widehat{{}^{k^{-1}}f^{k'}}).$$

It follows that FE for $\sigma_1 \otimes \sigma_2$ implies it for π.

Remark 1.2. All subrepresentations π of $\mathrm{Ind}_P^G \tau$ have the same γ-factor, which is $\gamma = \gamma(s, \sigma_1, \psi)\gamma(s, \sigma_2, \psi)$. This is a very important fact – it implies that the γ-factor defines a rational function on the *Bernstein variety* $\Omega(G)$.

Definition 1.3. For a reductive group G, the *Bernstein center* $\mathfrak{z}(G)$ is the ring of conjugation-invariant, essentially compact distributions on G, which means that for $\Phi \in \mathfrak{z}(G)$, $\Phi * C_c^\infty(G) \subset C_c^\infty(G)$, e.g. the δ-distribution. It was proved by Bernstein that Plancherel transform induces an isomorphism between $\mathfrak{Z}(G)$ and the regular functions on $\Omega(G)$. This is a countable disjoint union of finite-dimensional complex varieties.

$$\Omega(G) = \coprod_{[M,\sigma]} [M, \sigma]_G$$

where $[M, \sigma]$ runs over Levi subgroups and σ runs over their supercuspidal representations, up to G-conjugation.

The set $\{[M, \chi\sigma] \colon \chi \in \Psi(M)\}$ where $\Psi(M)$ is the set of unramified characters of M, is a connected component in $\Omega(G)$. You can show that $\Psi(M)/\operatorname{Stab}(\sigma) \to \{[M, \chi\sigma] \colon \chi \in \Psi(M)\}$ is surjective with finite fibers, which gives a $\mathbf{C}$-algebraic variety structure on the latter.

The γ-function is not a regular function but a *rational* function on $\Omega(G)$.

1.4. **The supercuspidal case.** Suppose π is supercuspidal.

1.4.1. *Absolute convergence.* Recall that by definition, matrix coefficients of supercuspidal representations are compactly supported mod center: for all $\varphi \in C(\pi)$, $Z(G)\backslash \mathrm{supp}(\varphi)$ is compact.

This implies that

$$\mathscr{Z}(s, f, \varphi) \le C \int_{M_n} |f(g)| |\det g|^{s + \frac{n-1}{2}} \, dg$$

Recall that the Haar measures on M_n and G are related by

$$dg = \frac{dg^+}{|\det g|^n}$$

where dg^+ is the Haar measure on M_n and dg is the Haar measure on G. Hence we can write this estimate as

$$\mathscr{Z}(s,f,\varphi) \leq C \int_{M_n} |f(g)||\det g|^{s+\frac{n-1}{2}} g \leq C \int_{M_n} |f(g)||\det g|^{s+\frac{n-1}{2}-n}\, dg^+$$

1.4.2. *Calculation of $\mathscr{L}(s,\pi)$.* We next want to show that $\mathscr{L}(s,\pi)=1$. By Bushnell-Kutzko's type theory for supercuspidal π, there exists $0 \neq \varphi_\pi^0 \in C(\pi)$ and an idempotent $e_\pi \in C_c^\infty(G)$ with support in $K = \mathrm{GL}_n(\mathcal{O}_F)$ such that $e_\pi^\vee * \varphi_\pi^0 * e_\pi^\vee = \varphi_\pi^0$, and moreover,

$$e_\pi * C_c^\infty(M_n) * e_\pi = C_c^\infty(G).$$

Consider $\mathscr{L}(s,f,\varphi)$. We have $\mathscr{C}(\pi) \cong V^\vee \otimes_{\mathbf{C}} V$ as a $G\times G$-representation. By irreducibility (since the $G\times G$-translates of φ_π^0 span $C(\pi)$, and we can move this translation to the f part), we only need to consider zeta integrals of the form $\mathscr{Z}(s,f,\varphi_\pi^0) = \mathscr{Z}(s,f,e_\pi^\vee * \varphi_\pi^0 * e_\pi^\vee)$.

We claim that

$$\mathscr{Z}(s,f,e_\pi^\vee * \varphi_\pi^0 * e_\pi^\vee) = \mathscr{Z}(s, e_\pi * f * e_\pi, \varphi_\pi^0). \tag{1.7}$$

Hence we only need test functions of $f \in C_c^\infty(G)$, for which the zeta integral is always convergent (no denominator).

Next we prove the claim (1.7). Let's just expand out the definitions.

$$\mathscr{Z}(s,f,e_\pi^\vee * \varphi_\pi^0 * e_\pi^\vee) = \int_G f(g)(e_\pi^\vee * \varphi_\pi^0 * e_\pi^\vee)(g)|\det g|^{s+\frac{n-1}{2}}\, dg.$$

$$e_\pi^\vee * \varphi_\pi^0 * e_\pi^\vee(g) = \int_{G\times G} e_\pi^\vee(a)\varphi_\pi^0(b)e_\pi^\vee(b^{-1}a^{-1}g)\, dadb.$$

Substituting the second formula into the first, the zeta integral becomes

$$\begin{aligned} &\int_{G\times G\times G} f(g)e_\pi^\vee(a)\varphi_\pi^0(b)e_\pi^\vee(b^{-1}a^{-1}g)|\det g|^{s+\frac{n-1}{2}} dadbdg \qquad (1.8)\\ =&\int_{G\times G\times G} f(g)e_\pi(a^{-1})\varphi_\pi^0(b)e_\pi(g^{-1}ab)\,|\det g|^{s+\frac{n-1}{2}} dadbdg\end{aligned}$$

Recalling $\operatorname{supp} e_\pi \subset K_0$, we have $|\det a| = |\det g^{-1}ab| = 1$. Hence $|\det g| = |\det(ga^{-1}g^{-1}ab)| = |\det b|$. So we can rewrite (1.8) as

$$\int_{G^3} f(g)e_\pi(a^{-1})\varphi_\pi^0(b)e_\pi(g^{-1}ab)|\det b|^{s+\frac{n-1}{2}} dadbdg.$$

and we have

$$\int_{G^2} f(g)e_\pi(g^{-1}ab)e_\pi(a^{-1})dadg = e_\pi * f * e_\pi(b) \quad \square$$

In particular, $I(\pi) = \{\mathscr{Z}(s,f,\varphi)\colon f\in C_c^\infty(G), \varphi\in C(\pi)\}$ and $\mathscr{Z}(s,f,\varphi)$ is holomorphic in s, so $\mathscr{L}(s,\pi)=1$.

1.4.3. *The functional equation.* Next, the functional equation. Consider $G\times G$ acting on $C_c^\infty(M_n)$ by

$$(g,h)\cdot f(x) = f(g^{-1}xh).$$

One easily computes that

$$\widehat{(g,h)\cdot f} = |\det gh^{-1}|^n (h,g)\widehat{f}.$$

Then

$$\begin{aligned}\mathscr{Z}((g,h)\cdot f,(g,h)\cdot\varphi,s) &= \int_G f(g^{-1}xh)\varphi(g^{-1}xh)|\det x|^{s+\frac{n-1}{2}}\,dx\\ &= |\det gh^{-1}|^{s+\frac{n-1}{2}}\mathscr{Z}(s,f,\varphi)\end{aligned}$$

We define an action of $G\times G$ on $\mathbf{C}(q^{-s})$ by $(g,h)\cdot t = |\det gh^{-1}|^{s+\frac{n-1}{2}}t$. Then this shows that $\mathscr{L}(s,-,-)$ can be interpreted as an element of

$$\mathrm{Hom}_{G\times G}(C_c^\infty(M_n)\otimes_{\mathbf{C}} C(\pi),\mathbf{C}(q^{-s})).$$

We want to show that $\mathscr{Z}(1-s,\mathcal{F}(-),(-)^\vee)\in\mathrm{Hom}_{G\times G}(C_c^\infty(M_n)\otimes_{\mathbf{C}} C(\pi),\mathbf{C}(q^{-s}))$ as well. This will be an explicit calculation.

$$\begin{aligned}\mathscr{Z}(1-s,\widehat{(g,h)f},(g,h)\varphi^\vee) &= \int_G |\det gh^{-1}|^n(h,g)\cdot\widehat{f}(x)[(h,g)\cdot\varphi]^\vee(x)|\det x|^{1-s+\frac{n-1}{2}}\,dx\\ &= \underbrace{|\det gh^{-1}|^n\cdot|\det hg^{-1}|^{1-s+\frac{n-1}{2}}}_{=|\det gh^{-1}|^{s+\frac{n-1}{2}}}\mathscr{L}(1-s,\widehat{f},\varphi^\vee)\end{aligned}$$

To get the functional equation it only remains to establish that

$$\dim_{\mathbf{C}(q^{-s})}\mathrm{Hom}_{G\times G}(C_c^\infty(M_n)\otimes_{\mathbf{C}}\mathbf{C}(\pi),\mathbf{C}(q^{-s}))\le 1.$$

Let $\ell\in\mathrm{Hom}_{G\times G}(C_c^\infty(M_n)\otimes_{\mathbf{C}} C(\pi),\mathbf{C}(q^{-s}))$. By the same argument as before, we know that ℓ is determined by $\ell|_{C_c^\infty(M_n)\otimes_{\mathbf{C}}\mathbf{C}\varphi_\pi^0}$. Then

$$\ell(C_c^\infty(M_n)\otimes_{\mathbf{C}}\mathbf{C}\varphi_\pi^0) = \ell(C_c^\infty(M_n)\otimes_{\mathbf{C}} e_\pi^\vee*\varphi_\pi^0*e_\pi^\vee) = \ell(e_\pi*C_c^\infty(M_n)*e_\pi,\varphi_\pi^0)$$

so ℓ is determined by its restriction to $C_c^\infty(G)\otimes_{\mathbf{C}}\mathbf{C}\varphi_\pi^0$. So we reduce to showing that

$$\dim_{\mathbf{C}(q^{-s})}\mathrm{Hom}_{G\times G}(C_c^\infty(G)\otimes_{\mathbf{C}} C(\pi),\mathbf{C}(q^{-s}))\le 1.$$

The Hom space is isomorphic to

$$\mathrm{Hom}_{G\times G}(C(\pi)=V^\vee\otimes V,(C_c^\infty(G))^*\otimes_{\mathbf{C}}\mathbf{C}(q^{-s})).$$

Since $C(\pi)$ is smooth, this lands in the smooth vectors, so it's the same as

$$\mathrm{Hom}_{G\times G}(C(\pi)=V^\vee\otimes V,(C_c^\infty(G))^\vee\otimes_{\mathbf{C}}\mathbf{C}(q^{-s})).$$

Now we'll give an explicit description for $C_c^\infty(G)$. Let $H=G$, viewed as the diagonal subgroup of $G\times G$. Then $C_c^\infty(G)=c-\mathrm{Ind}_H^{G\times G}(\mathbf{1})$, so $(C_c^\infty)^\vee\cong\mathrm{Ind}_H^{G\times G}(\mathbf{1})$.

So what we're interested in is

$$\mathrm{Hom}_{G\times G}(C(\pi),\mathrm{Ind}_H^{G\times G}(\mathbf{1})\otimes_{\mathbf{C}}\mathbf{C}(q^{-s})).$$

As H acts on $\mathbf{C}(q^{-s})$ trivially, there is a natural embedding

$$\mathrm{Ind}_H^{G\times G}(\mathbf{1}) \otimes_{\mathbf{C}} \mathbf{C}(q^{-s}) \hookrightarrow \mathrm{Ind}_H^{G\times G}(\mathbf{C}(q^{-s})),$$

and the Hom space of interest has an embedding into $\mathrm{Hom}_{G\times G}(C(\pi), \mathrm{Ind}_H^{G\times G}\mathbf{C}(q^{-s}))$. By Frobenius reciprocity, this is the same as

$$\mathrm{Hom}_H(C(\pi), \mathbf{C}(q^{-s})) = \mathrm{Hom}_H(C(\pi), \mathbf{C}) \otimes_{\mathbf{C}} \mathbf{C}(q^{-s}).$$

By irreducibility, we win.

1.5. **Spherical case.** Let π be a spherical representation of G, so $\dim_{\mathbf{C}} \pi^K = 1$.

Recall: we have a zonal spherical function $\Gamma(g)$ associated to π, satisfying

- $\pi(g)v = \Gamma(g)v$ for $v \in V^K$,
- $\int_K \Gamma(hkg)dk = \Gamma(h)\Gamma(g)$
- The zonal spherical function for $\widetilde{\pi}$ is $\Gamma_{\widetilde{\pi}}(g) = \Gamma_\pi(g^{-1})$.

For the standard Borel $B \subset G$ we have

$$B/U \cong T \cong (F^\times)^n.$$

Fix χ_i unramified characters of $F^\times$. We can form the unramified character χ for T, and then $\mathrm{Ind}_B^G(\chi)$.

Theorem 1.1 (Borel, Matsumoto, Casselman)**.** $\mathrm{Ind}_B^G(\chi)$ *contains a unique vector* ϕ *such that* $\phi(bk) = \delta_B(b)^{1/2}\chi(b)$.

The vector ϕ generates a spherical representation π_0.

The contragredient is $\widetilde{\mathrm{Ind}_B^G(\chi)} = \mathrm{Ind}_B^G \chi^{-1}$. It has a corresponding vector $\widetilde{\phi} \in \mathrm{Ind}_B^G \widetilde{\chi}$, and

$$\Gamma_\chi(g) = \int_K \phi(kg)\widetilde{\phi}(k)dk = \int_K \phi(gk)\, dk$$

is a matrix coefficient.

Theorem 1.2. *We have*

$$\mathscr{L}(s, \pi_0) = \prod_{i=1}^n \mathscr{L}(s, \chi_i).$$

and

$$\epsilon(s, \pi_0, \psi) = 1.$$

Proof. We compute $\mathscr{Z}(s, f, \Gamma_\chi)$ from the definition. We can assume $f \in C_c^\infty(M_n)^{K\times K}$ because Γ_χ is bi-K-invariant. So this is

$$\int_{G\times K} f(g)\phi(kg)|\det g|^{s+\frac{n-1}{2}}\, dgdk.$$

By Iwasawa decomposition $g = bk$ for $b \in B, k \in K$ and we can rewrite this as

$$= \int_{K\times B} f(bk)\phi(bk)|\det b|^{s+\frac{n-1}{2}}\, db = \int_B f(b)\delta_B(b^{1/2})\chi(b)|\det b|^{s+\frac{n-1}{2}}\, db$$

The answer is a finite linear combination of expressions of the form $\prod_{i=1}^n \mathscr{L}(s, f_i, \chi_i)$ for $f_i \in C_c^\infty(F)$, and we can use Tate's thesis.

On the other hand, for $f = \mathbf{I}_{M_n(\mathcal{O}_F)}$, we can compute $\mathcal{F}(f) = f$, and this implies that $\epsilon(s, \pi, \psi) = 1$. □

2. Global Godement-Jacquet theory

We first fix some notation. Let

- $G = \mathrm{GL}_n$ over a number field F,
- $\mathbf{A}_F$ be the ring of adeles,
- $[G] = G(F)\backslash G(\mathbf{A}_F)$.

Our goal is to establish the analytic theory of the standard L-function of GL_n, following the framework of Tate's thesis.

2.1. Cuspidal automorphic forms.

Fix a Hecke character $\omega_i \colon [\mathrm{GL}_1] \to \mathbf{C}^\times$. Let $L^2([G], \omega)$ be the space of square-integrable $f \colon [G] \to \mathbf{C}$ with central character ω: $f(zg) = \omega(z) f(g)$ for $z \in Z(\mathbf{A}_F)$, $g \in G(\mathbf{A}_F)$.

We say $f \in L^2([G], \omega)$ is a *cusp form* if

$$\int_{[N]} f(ng)\, dn = 0$$

for any unipotent radical $N \subset G$, for almost all $g \in G(\mathbf{A}_F)$. Let $L^2_0([G], \omega)$ be the space of cusp forms. One can show that it is a closed subspace of $L^2([G], \omega)$ and is invariant under the action of $G(\mathbf{A}_F)$. Furthermore, it has a discrete decomposition

$$L^2_0([G], \omega) = \bigoplus \pi \tag{2.1}$$

with each π being an irreducible $G(\mathbf{A}_F)$-representation.

Definition 2.1. Call a π appearing in (2.1) a *cuspidal automorphic representation* of G.

2.2. Global zeta integrals.

For each cuspidal automorphic representation π, define

$$C(\pi) := \left\{ \varphi := \left(g \mapsto \int_{Z(\mathbf{A}_F)G(F)\backslash G(\mathbf{A}_F)} \beta_1(hg)\overline{\beta_2(h)} dh \right) : \beta_1, \beta_2 \in \pi \right\}.$$

Then define the *global Schwartz space*

$$\mathscr{S}(M_n(\mathbf{A}_F)) := \bigotimes_{p \in |F|}{}' \mathscr{S}(M_n(F_p))$$

where

- If p is non-archimedean,

$$\mathscr{S}(M_n(F_p)) := C_c^\infty(M_n(F_p))$$

- If p is archimedean,

$$\mathscr{S}(M_n(F_p)) := \mathbf{C}[M_n(F_p)]G_p$$

where

$$G_p(x) = \begin{cases} \exp(-\pi \operatorname{Tr}(x^T x)) & F_p \cong \mathbf{R}, \\ \exp(-2\pi \operatorname{Tr}(\overline{x}^T x)) & F_p \cong \mathbf{C}. \end{cases}$$

The restricted tensor product is with respect to $\mathbf{I}_{M_n(\mathcal{O}_{F_p})}$ at the non-archimedean places.

Remark 2.1. When p is archimedean, $M_n(F_p)$ is viewed as a real algebraic variety.

Definition 2.2. For $f \in \mathscr{S}(M_n(\mathbf{A}_F))$ and $\varphi \in C(\pi)$, define

$$\mathscr{Z}(s, f, \varphi) = \int_{G(\mathbf{A}_F)} f(g)\varphi(g)|\det g|^{s+\frac{n-1}{2}} dg.$$

Theorem 2.3 (Tate for GL_1, Godement-Jacquet for GL_n)**.** *We have the following facts.*

(1) $\mathscr{Z}(s, f, \varphi)$ is absolutely convergent for $\operatorname{Re} s \gg 0$.
(2) $\mathscr{Z}(s, f, \varphi)$ is entire.
(3) $\mathscr{Z}(1-s, \mathcal{F}(f), \varphi^\vee) = \mathscr{Z}(s, f, \varphi)$.

Remark 2.4. We can write π as a restricted tensor product

$$\pi \cong \bigotimes{}' \pi_p,$$

where π_p is a unitary representation of $\mathrm{GL}_n(F_p)$ if F_p is non-archimedean, such that π_p is unramified almost everywhere. Furthermore, this induces a restricted tensor product structure on $C(\pi)$:

$$C(\pi) \cong \bigotimes_{p \in |F|}{}' C(\pi_p)$$

with respect to Γ_p (which is taken to be the zonal spherical function of π_p whenever π_p is unramified).

Let $f = \bigotimes_p f_p$ and $\varphi = \bigotimes_p \varphi_p$. Then

$$\mathscr{Z}(s, f, \varphi) = \prod_p \mathscr{Z}(s, f_p, \varphi_p).$$

2.2.1. *Absolute convergence.* We know that $f_p = \mathbf{I}_{M_n(\mathcal{O}_{F_p})}$ and $\varphi_p = \Gamma_p$ almost everywhere. For such places, the local factor is

$$\mathscr{Z}(s, f_p, \varphi_p) = \mathscr{Z}(s, \mathbf{I}_{M_n(\mathcal{O}_{F_p})}, \Gamma_p) = \mathscr{L}(s, \pi_p).$$

If the Satake parameter of π_p is

$$\alpha(\pi_p) = \begin{pmatrix} \alpha_1(\pi_p) & & \\ & \ddots & \\ & & \alpha_n(\pi_p) \end{pmatrix}$$

then $q^{-1/2} \leq |\alpha_i(\pi_p)| \leq q^{1/2}$. Since

$$\mathscr{L}(s, \pi_p) = \det(I_n - \alpha(\pi_p) q^{-s})^{-1}$$

the convergence then follows from the convergence of the Riemann ζ function.

2.2.2. *Meromorphic continuation and functional equation.*

Theorem 2.1 (Poisson summation)**.** *Let $f \in \mathscr{S}(M_n(\mathbf{A}_F))$ and $g, h \in G(\mathbf{A}_F)$. Then*

$$\sum_{\gamma \in M_n(F)} f(h^{-1}\gamma g) = \sum_{\gamma \in M_n(F)} |\det gh^{-1}|_{\mathbf{A}}^{n} \mathcal{F}(f)(g^{-1}\gamma h).$$

Proof. The proof goes by Poisson summation. It is essentially the same argument as in Tate's thesis, iterated n^2 times. ☐

The Poisson summation implies the meromorphic continuation without much trouble, so we turn our attention to the functional equation. We work in the region Re $s > 3/2$, where the zeta integral converges absolutely. (Also, we elide some analytic issues.) Define

$$G^1 := \{g \in G(\mathbf{A}_F) \colon |\det g| = 1\}.$$

Then $G^1 Z(\mathbf{A}_F) = G(\mathbf{A}_F)$, so

$$G(F)Z(\mathbf{A}_F)\backslash G(\mathbf{A}_F) = G(F)(Z(\mathbf{A}_F) \cap G^1)\backslash G^1 = G(F)\backslash \overline{G}^1$$

Taking

$$\varphi(g) = \int_{G(F)\backslash \overline{G^1}} \beta_1(hg)\overline{\beta_2(h)} dh$$

the zeta integral becomes

$$\begin{aligned} \mathscr{Z}(s, f, \varphi) &= \int_{G(\mathbf{A}_F)} f(g)\varphi(g)|\det g|^{s+\frac{n-1}{2}}\, dg \\ &= \int_{G(\mathbf{A}_F)} f(g) \int_{G(F)\backslash \overline{G^1}} \beta_1(hg)\overline{\beta}_2(h) dh |\det g|^{s+\frac{n-1}{2}}\, dg \\ (g \rightsquigarrow h^{-1}g) &= \int_{G(\mathbf{A}_F)} \int_{G(F)\backslash \overline{G^1}} f(h^{-1}g)\beta_1(g)\overline{\beta}_2(h)|\det g|^{s+\frac{n-1}{2}}\, dh dg \end{aligned}$$

Now we're going to apply Poisson summation and unfolding.

Since β_1 is automorphic, we can rewrite this as

$$\sum_{g \in [G]} \int_{h \in G(F)\backslash \overline{G^1}} \left[\sum_{\gamma \in G(F)} f(h^{-1}\gamma g) \right] \beta_1(g)\overline{\beta}_2(h)|\det g|^{s+\frac{n-1}{2}}\, dg dh.$$

We can apply Poisson summation to the bracketed term. Up to boundary terms, this gives

$$\sum_{\gamma \in G(F)} f(h^{-1}\gamma g) = \sum_{\gamma \in G(F)} \mathcal{F}(f)(g^{-1}\gamma^{-1}h)|\det g|^{-n} + (\text{boundary terms}).$$

In fact the boundary terms will vanish due to cuspidality of β_1, β_2. We can insert this above to get

$$\begin{aligned}
&= \int_{g\in[G]} \int_{h\in G(F)\backslash \overline{G^1}} \sum_{\gamma\in G(F)} \mathcal{F}(f)(g^{-1}\gamma^{-1}h)\beta_1(g)\overline{\beta_2(h)}|\det g|^{-n}|\det g|^{s+\frac{n-1}{2}}\, dhdg \\
&= \int_{g\in[G]} \int_{h\in G(F)\backslash \overline{G^1}} \sum_{\gamma\in G(F)} \mathcal{F}(f)^\vee(h^{-1}\gamma g)\beta_1(g)\overline{\beta_2(h)}|\det g|^{s-\frac{n+1}{2}}\, dhdg \\
&= \int_{g\in[G]} \int_{h\in G(F)\backslash \overline{G^1}} \mathcal{F}(f)^\vee(h^{-1}g)\beta_1(g)\overline{\beta_2(h)}|\det g|^{s-\frac{n+1}{2}}\, dgdh \\
(g \rightsquigarrow hg) &= \int_{g\in G(\mathbf{A}_F)} \int_{h\in G(F)\backslash \overline{G^1}} \mathcal{F}(f)^\vee(g)\beta_1(hg)\overline{\beta_2(h)}|\det g|^{s-\frac{n+1}{2}}\, dhdg \\
&= \int_{g\in G(\mathbf{A}_F)} \mathcal{F}(f)^\vee(g)\varphi(g)|\det g|^{s-\frac{n+1}{2}}\, dg \\
(g \rightsquigarrow g^{-1}) &= \int_{g\in G(\mathbf{A}_F)} \mathcal{F}(f)(g)\varphi^\vee(g)|\det g|^{1-s+\frac{n-1}{2}}\, dg \\
&= \mathscr{Z}(1-s, \mathcal{F}(f), \varphi^\vee).
\end{aligned}$$

We now handle the boundary terms which we previously ignored.

Lemma 2.2. *For* $0 \le m < n$*,*

$$\int_{g\in[G]} \int_{h\in G(F)\backslash G^1} \sum_{\substack{\gamma\in M_n(F)\\ \operatorname{rank}\gamma = m}} f(h^{-1}\gamma g)\beta_1(g)\overline{\beta_2(h)}|\det g|^{s+\frac{n-1}{2}} dhdg = 0.$$

Proof. The point is that we want to break this up into integrals over unipotent subgroups, which will be 0 by cuspidality. For convenience, assume first that $m > 0$. Let

$$R_m = \{\gamma \in M_n(F) \colon \operatorname{rank}\gamma = m\}.$$

This has an action of $G(F)$ by right translation. We claim that the representatives for each orbit are

$$\gamma \cdot \begin{pmatrix} \mathrm{Id}_m & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma \in \mathrm{GL}_n(F) \tag{2.2}$$

and its stabilizer is

$$P_m := \left\{ \begin{pmatrix} \mathrm{Id}_m & \\ * & * \end{pmatrix} \right\} \subset G(F)$$

The subgroup

$$U_m := \left\{ \begin{pmatrix} \mathrm{Id}_m & 0 \\ * & \mathrm{Id}_{n-m} \end{pmatrix} \right\}$$

is a unipotent radical in G.

For each orbit with representative (2.2), the contribution to the integral is

$$\int_{g\in[G]}\int_{h\in G(F)\backslash\overline{G^1}}\sum_{\xi\in P_m(F)\backslash G(F)} f\left(h^{-1}\gamma\begin{pmatrix}\mathrm{Id}_m & 0\\ 0 & 0\end{pmatrix}\xi g\right)\beta_1(g)\overline{\beta_2(h)}|\det g|^{s+\frac{n-1}{2}}\,dhdg$$

$$=\int_{g\in P_m(F)\backslash G(\mathbf{A}_F)}\int_{h\in G(F)\backslash\overline{G^1}} f\left(h^{-1}\gamma\begin{pmatrix}\mathrm{Id}_m & 0\\ 0 & 0\end{pmatrix} g\right)\beta_1(g)\overline{\beta_2(h)}|\det g|^{s+\frac{n-1}{2}}\,dhdg$$

We then break up $P_m(F)\backslash G(\mathbf{A}_F) = U_m(F)\backslash U_m(\mathbf{A}_F) \times P_m(F)U_m(\mathbf{A}_F)\backslash G(\mathbf{A}_F)$. Using that for all $n \in U_m(\mathbf{A})$,

$$\begin{pmatrix}\mathrm{Id}_m & 0\\ 0 & 0\end{pmatrix}\cdot n = \begin{pmatrix}\mathrm{Id}_m & 0\\ 0 & 0\end{pmatrix}$$

we can rewrite this integral as

$$=\int_{g\in P_m(F)U_m(\mathbf{A}_F)\backslash G(\mathbf{A}_F)}\int_{h\in G(F)\backslash\overline{G^1}} f\left(h^{-1}\gamma\begin{pmatrix}\mathrm{Id}_m & 0\\ 0 & 0\end{pmatrix} g\right)$$
$$\cdot\int_{[U]}\beta_1(ng)dn\;\overline{\beta_2(h)}|\det g|^{s+\frac{n-1}{2}}\,dgdh$$

Now the inner term $\int_{[U]}\beta_1(ng)dn = 0$ by cuspidality of β_1.

That was all for $m > 0$. When $m = 0$, we get

$$f(0)\int_{g\in[G]}\int_{h\in G(F)\backslash\overline{G^1}}\beta_1(g)\overline{\beta_2(h)}|\det g|^{s+\frac{n-1}{2}}dhdg$$

Taking the h integration, and using that cusp forms are orthogonal to constant functions, we also win. □

3. The Braverman-Kazhdan program

3.1. **The conjectures.** Let G be a split, connected reductive algebraic group over a local field F. For convenience, assume that F is p-adic.

Fix a maximal torus $T \subset G$, which gives rise to a root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$. We then form the reductive group $\widehat{G}/\mathbf{C}$ with swapped roots and coroots. The *L-group* is ${}^L G := \widehat{G} \times W_F$.

Let $\rho\colon {}^L G \to \mathrm{GL}(V_\rho)$, and set $n = \dim V_\rho$. Let $\pi \in \mathrm{Irr}(G)$. We are interested in defining an L-factor $L(s,\pi,\rho)$. If the Local Langlands Correspondence is known for G, then we can define $L(s,\pi,\rho)$ by lifting to GL_n, and using Langlands' definition of the L-function for a Galois representation. However, a definition of this form is usually not useful for global purposes, e.g. proving analytic continuation and functional equation.

Braverman-Kazhdan (2000) proposed a generalization of Godement-Jacquet theory, with the goal of obtaining integral representations for L-functions for other G. We have seen that Godement-Jacquet theory has two main ingredients of local harmonic analysis: a definition of a *Schwartz space* $\mathscr{S}_\rho(G)$, and a theory of the *Fourier transform* $\mathcal{F}_\rho$. To have generalizations of these, we need to impose some assumptions on G.

(1) We assume that there is a short exact sequence

$$1 \to G_0 \to G \xrightarrow{\sigma} \mathbf{G}_m \to 1$$

where G_0 is a split semi-simple, simply connected algebraic group over F, and σ plays the role of the determinant for GL_n.

(2) ρ is a faithful representation such that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{G}_m & \xrightarrow{\widehat{\sigma}} & {}^L G & \longrightarrow & {}^L G_0 & \longrightarrow & 1 \\ & & \| & & \downarrow \rho & & \downarrow \overline{\rho} & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathrm{GL}(V_\rho) & \longrightarrow & \mathrm{PGL}(V_\rho) & \longrightarrow & 1 \end{array}$$

and $\rho|_{\mathbf{G}_m}(z) = z\,\mathrm{Id}_{V_\rho}$.

Remark 3.1. These may seem like strong conditions, but Ngô pointed out that they are often satisfied. Indeed, start with any split semisimple $\widehat{G_0}$ and fix a representation $\overline{\rho}$ of $\widehat{G_0}$. Then, by Vinberg's theory of reductive monoids, there always exists a short exact sequence

$$1 \to \mathbf{G}_m \to \widehat{G} \xrightarrow{\sigma} \widehat{G_0} \to 1$$

giving rise to these assumptions. Note that $\widehat{G}$ may depend on $\overline{\rho}$.

Example 3.2. Let $G_0 = \mathrm{SL}_2$, $\overline{\rho} = \mathrm{Sym}^n$ as a (projective) representation of $\widehat{G_0}$. Then

$$G = \begin{cases} \mathrm{SL}_2 \times \mathbf{G}_m & n = 2k, \\ \mathrm{GL}_2 & n = 2k+1. \end{cases}$$

Conjecturally, there exists a *Schwartz space* $S_\rho(G) \subset C^\infty(G)$ such that

$$\mathscr{Z}(s, f, \varphi) = \int_G f(g)\varphi(g)|\sigma(g)|^{s+\frac{\ell}{2}}\, dg$$

with analogous properties to the Godement-Jacquet case, e.g. meromorphic continuation, a Fourier transform $\mathcal{F}_\rho$, and a functional equation

$$\mathscr{Z}(1-s, \mathcal{F}_\rho(f), \varphi^\vee) = \gamma(s, \pi, \rho, \psi)\mathscr{Z}(s, f, \rho).$$

Remark 3.3. The number $\ell \in \mathbf{C}$ is not important for analytic purposes. A different normalization gives an unramified shift of $S_\rho(G)$ and $\mathcal{F}_\rho$. But it is important for *geometric* reasons. It was first pointed out in work of Bouthier-Ngô-Sakellaridis that ℓ should be of the form

$$\ell := 2\langle \rho_B, \lambda \rangle \tag{3.1}$$

where ρ_B is the half-sum of the positive roots with respect to the Borel $B \supset T$, and λ is the highest weight of ρ.

Example 3.4. For GL_n and $\rho = \mathrm{std}$, $\rho_B = (\frac{n-1}{2}, \frac{n-3}{2}, \dots)$ and $\lambda_{\mathrm{std}} = (1, 0, \dots, 0)$, (3.1) gives $\rho = n-1$, as in Godement-Jacquet theory.

3.2. **The Schwartz space.** There are three approaches/properties for defining the Schwartz space $\mathscr{S}_\rho(G) \subset C^\infty(G)$.

3.2.1. *Analytic desiderata.* For $f \in \mathscr{S}_\rho(G)$ and $\varphi \in C(\pi)$, the zeta integral $\mathscr{Z}(s, f, \varphi)$ should be absolutely convergent for $\mathrm{Re} s \gg 0$, and a rational function in q^{-s} which admits a common denominator $\mathscr{L}(s, \pi, \rho)$.

3.2.2. $\mathscr{S}_\rho(G)$ should be a $\mathcal{H}(G) \times \mathcal{H}(G)$-module by left and right convolution.

3.2.3. *Geometric property.* To phrase the third (crucial) property, we need to recall the theory of reductive monoids.

For $G = \mathrm{GL}_n$, we took the affine spherical embedding $G \hookrightarrow M_n$, which is $G \times G$-equivariant, whose image is open and dense. We then took the restriction of $C_c^\infty(M_n)$ to G as our space of Schwartz functions.

In general, given (G, ρ) we have a $G \times G$-equivariant affine spherical embedding $G \hookrightarrow M_\rho$ with open dense image, where M_ρ is a reductive monoid. Note that in almost all cases *except* the Godement-Jacquet case $(\mathrm{GL}_n, \mathrm{std})$, M_ρ is *singular*. So $C_c^\infty(M_\rho)$ is *not* the right object to use.

Example 3.1. An example is the case of $G = \mathrm{GSp}(4)$, $\rho =$ Std representation of $\widehat{\mathrm{GSp}(4)}$ (by the exceptional isomorphism $\mathrm{GSp}(4) \cong \mathrm{GO}(5)$. Then $M_\rho = \mathrm{MSp}(4)$ is a cone (matrices m such that $m^T J m = \lambda J$, possibly with $\lambda = 0$). One can show that $C_c^\infty(\mathrm{MSp}(4))$ does not give the standard L-factors. It turns out that one needs to allow moderate growth near the singularities, rather than constancy.

We want to define $\mathscr{S}_\rho(G) = \Gamma_c(M_\rho, \widetilde{\mathscr{S}_\rho})$ where $\widetilde{\mathscr{S}_\rho}$ is the "Schwartz sheaf", where cohomology is taken with respect to the p-adic topology on M_ρ. The idea is that $S_\rho(G)$ should be related to a "conjectural theory of perverse sheaves on M_ρ".

Example 3.2 (Bouthier-Ngô-Sakellaridis)**.** Let $F = \mathbf{F}_q((t))$ be the trace of Frobenius of "$\mathrm{IC}_{M_\rho(\mathcal{O}_F)}$" (the trace is well-defined) then you get the basic function $\mathbf{L}_\rho$.

Let's review the concept of the basic function. This should be $\mathbf{L}_\rho \in \mathscr{S}_\rho(G)^{K \times K}$ where K is a fixed maximal compact open subgroup of G. There is a spectral characterization of $\mathbf{L}_\rho$: for an unramified irerducible representation π of G, one should have

$$\int_G \mathbf{L}_\rho(g) \Gamma_\pi(g) |\sigma(g)|^{s+\frac{\ell}{2}} \, dg = \mathscr{L}(s, \pi, \rho).$$

In particular, one can show that

$$\int_G \mathbf{L}_\rho(g) \Gamma_\pi(g) |\sigma(g)|^{s+\frac{\ell}{2}} \, dg = \mathrm{Sat}(\mathbf{L}_g)(\alpha(\pi_{s+\frac{\ell}{2}}))$$

where $\alpha(\pi_{s+\frac{\ell}{2}})$ is the Satake parameter of $\pi_{s+\frac{\ell}{2}}$, and $\pi_{s+\frac{\ell}{2}} = \pi \otimes |\sigma(-)|^{s+\frac{\ell}{2}}$.

Example 3.3. For $(\mathrm{GL}_n, \mathrm{std})$, $\mathbf{L}_\rho = \mathbf{I}_{M_n(\mathcal{O}_{F_p})}$. M_n is smooth, so we can take IC to be the constant sheaf.

Remark 3.4. $\mathbf{L}_\rho$ is important for global purposes.

The global Schwartz space should be restricted tensor product

$$\mathscr{S}_\rho(G(\mathbf{A}_F)) = \bigotimes_p{}' \mathscr{S}_p(G(F_p))$$

with respect to the basic functions $\mathbf{L}_{\rho,p}$.

3.3. **Fourier transform.** Braverman-Kazhdan conjectured that $\mathcal{F}_\rho$ can be given by convolution against a stable distribution $\Phi_{\rho,\psi}$ on G.

Example 3.1. Let's see this in the standard case $(\mathrm{GL}_n, \mathrm{std})$. We have $G \hookrightarrow M_n$ and

$$dg = \frac{dg^+}{|\det g|^n}.$$

Then

$$\begin{aligned}\mathcal{F}(f)(x) &= \int_{M_n} \psi(\mathrm{Tr}(xy)) f(y) dy^+ \\ &= \int_G \psi(\mathrm{Tr}(xy)) f(y) dy^+ \\ &= \int_G \psi(\mathrm{Tr}(xy)) |\det y|^n f(y)\, dy \\ &= |\det x|^{-n} \int_{\mathrm{GL}_n} \psi(\mathrm{Tr}(xy)) |\det xy|^n f(y)\, dy \\ &= |\det x|^{-n} (\Phi_{\mathrm{std},\psi} * f^\vee)(x)\end{aligned}$$

where

$$\Phi_{\mathrm{std},\psi}(x) = \psi(\mathrm{Tr}(x))\ |\det x|^n.$$

To generalize this, we will construct $\Phi_{\rho,\psi}$, a stable distribution on G, and define

$$\mathcal{F}_\rho(f)(g) = |\sigma(g)|^{-\ell-1} (\Phi_{\rho,\psi} * f^\vee)(g).$$

It was conjectured by Braverman-Kazhdan that the action of $\Phi_{\rho,\psi,s}$ on a representation $\pi \in \mathrm{Irr}(G)$ is given by

$$\pi(\Phi_{\rho,\psi,s}) = \gamma(s, \pi, \rho, \psi)\,\mathrm{Id}_\pi$$

in a generic sense, because $\Phi_{\rho,\psi,s}$ is just a rational function on the Bernstein center (it may have poles).

We need to make sense of the action $\pi(\Phi_{\rho,\psi,s})$, since $\Phi_{\rho,\psi,s}$ is a distribution. Furthermore, it turns out that we need to modify things in order to get the functional equation

$$\mathscr{L}(1-s, \mathcal{F}_\rho(f), \varphi^\vee) = \gamma(s, \pi, \rho, \psi) \mathscr{L}(s, f, \varphi).$$

Luo found that to get this functional equation, the action should instead be given by

$$\pi(\Phi_{\rho,\psi,s}) = \gamma(-s - \frac{\ell}{2}, \pi^\vee, \rho, \psi)\,\mathrm{Id}_\pi\,. \tag{3.2}$$

Let's try to give a formal derivation of the functional equation from (3.2). We assume $\operatorname{Re} s \ll 0$, and ignore analytic issues.

$$\begin{aligned}
\mathscr{Z}(1-s, \mathcal{F}_\rho(f), \varphi^\vee) &= \int |\sigma(g)|^{-s-\frac{\ell}{2}} (\Phi_{\rho,\psi} * f^\vee)(g) \varphi^\vee(g)\, dg \\
(g \rightsquigarrow g^{-1}) &= \int_G |\sigma(g)|^{s+\frac{\ell}{2}} (\Phi_{\rho,\psi} * f^\vee)(g^{-1}) \varphi(g)\, dg \\
&= \int_G \int_G \Phi_{\rho,\psi}(y) f(gy) dy \varphi(g) |\sigma(g)|^{s+\frac{\ell}{2}}\, dg \\
(g \rightsquigarrow gy^{-1}) &= \int_{G\times G} \Phi_{\rho,\psi}(y) f(g) \varphi(gy^{-1} |\sigma(gy^{-1})|^{s+\frac{\ell}{2}}\, dy dg
\end{aligned}$$

Focus on the integration over y:

$$\begin{aligned}
\int_G \Phi_{\rho,\psi}(y) |\sigma(y^{-1})|^{s+\frac{\ell}{2}} \varphi(gy^{-1}) dy &= \int_G \Phi_{\rho,\psi}(y) |\sigma(y)|^{-(s+\frac{\ell}{2})} \varphi^\vee(yg^{-1}) dy \\
&= \pi^\vee(\Phi_{\rho,\psi,-s-\frac{\ell}{2}}) \cdot \underbrace{\varphi^\vee(g^{-1})}_{\varphi(g)}
\end{aligned}$$

Inserting this above, we get

$$\pi^\vee(\Phi_{\rho,\psi,-s-\frac{\ell}{2}}) \int_G f(g)\varphi(g) |\sigma(g)|^{s+\frac{\ell}{2}}\, dg = \mathscr{Z}(s, f, \varphi).$$

Remark 3.2. The distribution $\Phi_{\rho,\psi}$ is closely related to the distribution in the Bernstein center $\mathfrak{z}(G)$. Originally, $\mathfrak{z}(G) = \operatorname{End}_{\operatorname{Rep}(G)}(\operatorname{Id})$. This turns out to be isomorphic to conjugation-invariant essentially compactly supported distributions on G, where "essentially compact" means $\Phi * C_c^\infty(G) \subset C_c^\infty(G)$. Bernstein showed that the Plancherel formula identifies this with regular functions on $\Omega(G) := \coprod_{(M,\sigma)} [M, \sigma]_G$.

We explain this notation: M is a Levi subgroup of G, and σ is a supercuspidal representation of M (for $M = \mathrm{GL}_1$, this means a quasi-character). $[M, \sigma]_G$ is the G-conjugation equivalence classes of (M, σ). The subset $\{[M, \chi\sigma]_G \colon \chi \in \Psi(M)\}$ is a connected component in $\Omega(G)$.

3.4. **Braverman-Kazhdan conjectural construction for $\Phi_{\rho,\psi}$.** First we discuss Braverman-Kazhdan's conjectural description. Since we assumed that G is split, we can pick a maximal split torus $T \subset G$. We have

$$\widehat{T} \overset{\iota}{\hookrightarrow} \widehat{G} \overset{\rho}{\hookrightarrow} \mathrm{GL}(V_\rho).$$

By conjugating, we may assume that $\rho \circ \iota$ on $\widehat{T}$ factors through the standard $\widehat{T}_n \subset \mathrm{GL}(V_\rho)$. (Here $n = \dim V_\rho$.) Hence we get a group homomorphism

$$\widehat{T} \xrightarrow{\rho\circ\iota} \widehat{T}_n.$$

This induces $T_n \xrightarrow{\rho^\vee} T$. On T_n we have a standard Fourier transform distribution,

$$\psi(t_1 + \ldots + t_n) |t_1 \cdot \ldots \cdot t_n|\ |dt_1 \ldots dt_n|.$$

We can construct a distribution on T by pushing forward:

$$\Phi_{\rho\circ\iota,\psi}(t) = \rho^\vee_!(\psi(t_1 + \ldots + t_n)|t_1 \cdot \ldots \cdot t_n|\ |dt_1 \ldots dt_n|).$$

Remark 3.1. A regularization is needed for the pushforward $\rho^\vee_!$.

How do we get a distribution on G? We use the Chevalley map

$$c\colon G^{\mathrm{reg}} \to T/W$$

where W is the Weyl group of (G, T). If we can define a W-equivariant structure on $\Phi_{\rho\circ\iota,\psi}$ then we can pull it back to G^{reg} via the Chevalley map c, and extend it to G.

Conjecture 3.2 (Braverman-Kazhdan)**.** *This construction gives the correct distribution.*

Remark 3.3. The W-action is non-trivial. In particular, in the finite field case and the $\mathscr{D}$-module setting, there are parallel constructions. Ngô-Cheng confirmed the conjecture for (GL_n, ρ) over finite fields, and Chen confirmed it for (G, ρ) in the $\mathscr{D}$-module setting.

3.5. **Ngô's conjectural construction.** Ngô emphasized that the stable distribution $\Phi_{\rho,\psi}$ should be locally integrable and smooth on the regular semisimple points. Moreover, $\Phi_{\rho,\psi}|_T$ should agree with a canonical construction on T for all maximal tori $T \subset G$.

Remark 3.1. Let $T \subset G$ be a split torus, and $\rho^\vee\colon T_n \to T$ be as before. We had the function $\psi(t_1 + \ldots + t_n)|t_1 \ldots t_n|$ on T_n. Let $U = \ker(\rho^\vee) \subset T_n$. This is also a torus, so it has a natural Haar measure du. We define

$$\Phi_{\rho\circ\iota,\psi}(t) = \int_{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} =: a \in (\rho^\vee)^{-1}(t)} \psi(a_1 + \ldots + a_n)|a_1 \ldots a_n| d_t u$$

Example 3.2. Let $\pi \in \mathrm{Irr}_{\mathrm{un}}(G)$, Luo showed that for $K \leq G$ a maximal open compact,

$$\mathscr{S}_\rho(G)^{K\times K} = \mathbf{L}_\rho * \mathcal{H}(G, K).$$

Also,

$$\Phi^K_{\rho,\psi} = \mathbf{L}_{\rho,1+\ell} * \mathrm{Sat}^{-1}\left(\frac{1}{\mathscr{L}(-\frac{\ell}{2}, \pi, \rho^\vee)}\right)$$

where $\rho^\vee$ is the contragredient of ρ, and $\Phi^K_{\rho,\psi}$ is the projection of $\Phi_{\rho,\psi}$ to the unramified component. The analytic issues are fine, and

$$\mathscr{F}^K_\rho(\mathbf{L}_\rho) = \mathbf{L}_\rho$$

and

$$\mathscr{F}^K_\rho(\mathscr{S}_\rho(G)^{K\times K}) = \mathscr{S}_\rho(G)^{K\times K}$$

and similarly $\mathscr{F}^K_\rho$ preserves $L^2(G, K, |\sigma|^{\ell+1}\, dg)$.

A parallel theory exists for archimedean local fields, and you can define $\mathbf{L}_\rho$ for archimedean local fields via the Harish-Chandra transform. In particular, $\mathbf{L}_{\rho,s}$ can be plugged into the Arthur-Selberg trace formula for $\operatorname{Re} s \gg 0$. This gives something like

$$\sum_\pi m_\pi \operatorname{Tr} \pi(f) = \sum_\gamma \operatorname{vol}(\gamma) O_\gamma(f)$$

The spectral side picks up everywhere unramified representations, and we expect that $O_\gamma(\mathbf{L}_{\rho,s})$ can tell us information about $\mathscr{L}(s,\pi,\rho)$.

3.6. **Global theory.** Let F be a number field. We have a global Fourier transform $\mathcal{F}_\rho := \bigotimes_p \mathcal{F}_{\rho,p}$ on $\mathscr{S}_\rho(G(\mathbf{A}_F)) := \bigotimes_p' \mathscr{S}_\rho(G(F_p))$.

For $f = \bigotimes_p f_p$, assume that there are $p_0, p_1 \in |F|$ such that $f_{p_0} \in C_c^\infty(G(F_{p_0}))$ and $\mathcal{F}_{\rho,p_1}(f_{p_1}) \in C_c^\infty(G(F_{p_1}))$.

Conjecture 3.1 (Poisson summation)**.** *Under these assumptions, we have*

$$\sum_{\gamma \in G(F)} f(\gamma) = \sum_{\gamma \in G(F)} \mathcal{F}_\rho(f)(\gamma).$$

Remark 3.2. In general there should be "boundary terms" in Conjecture 3.1, but the assumptions should imply that they are 0.

Remark 3.3. By the converse theorem of Piatetskii-Shapiro–Cogdell, we expect that a Poisson summation formula should imply functorial lifting. Laurent Lafforgue has defined kernels giving functorial liftings from the PSF.