Geometric Langlands and Bridgeland stabilities

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1 Setup

The subject of this talk is the geometry of symlpectic resolutions. Suppose $X \xrightarrow{\pi} Y$ is a symplectic resolution of singularities, so (X, ω) is algebraic symplectic and π is a resolution (i.e. birational).

Assume further that there is a $\mathbb{G}_m$-action on X and Y, and π is equivariant for this action.

Suppose that ω has weight $d > 0$, i.e. $t^* \omega = t^d \omega$ ($d = 2$). Assume that $k = \overline{k}$, and sometimes we need characteristic 0.

We are really interested in the category $C = D^b \operatorname{Coh}(X)$. Let C_0 be the full subcategory of sheaves supported on the special fiber, equivalent ot $D^b \operatorname{Coh}_{\pi^{-1}(0)}(X)$. We will see conjectures/Theorems concerning

- the action of $\pi_1(K)$, where K is the "complexified Kähler parameter space," on C and C_0,
- the relation to $QH^*_{\mathbb{C}^*}(X)$
- Bridgeland stabilities,

Example 1.1. Let G be a semisimple algebraic group. Form the flag vartiety G/B. Then T^*G/B is algebraic symplectic, and we can consider the moment map

$$T^*(G/B) \xrightarrow{\pi} \mathcal{N} \subset \mathfrak{g} \cong \mathfrak{g}^*.$$

For future reference, let's recall the explicit description. We can think of

$$T^*(G/B) = \{B, u \in \operatorname{rad}(B)\}$$

and π maps $(B, u) \mapsto u$.

2 Geometric Langlands

Geometric Langlands is about an equivalence between the derived category of D-modules on $\operatorname{Bun}_G(C)$, for a complete curve C, with the derived category of local systems with structure group ${}^L G$ on C.

$$D(D - \mathbf{mod}(\operatorname{Bun}_G(C))) \longleftrightarrow D\operatorname{LocSys}_{{}^L G}(C).$$

This is the “unramified version.”

How do we add ramification? The simplest way is to add ramification at some $x \in C$. You account for this on the left hand side by including the data of a trivialization at x. The corresponding modification of the Galois (right hand) side is to allow local systems to have a regular singularity at x and a ${}^L B$ structure at x such that the residue of the connection is stricly upper triangular: $\mathrm{Res}_\Delta \in \mathrm{rad}(\mathfrak{b})$.

Local situation. Work locally around x, by replacing C by the formal disc around x. Then one gets a local version. The right hand side will be replaced with local systems on $\mathbb{D}$ with regular singularities at x, with a ${}^L B$ structure satisfying the residue condition. The left hand side becomes $\mathrm{Bun}_G(\mathbb{D})$ with a flag at x and trivialization on $\mathbb{D}^*$, i.e. the affine flag variety Fl.

$$D(D-\mathbf{mod}_{\mathrm{Whit}}(\mathrm{Fl})) \longleftrightarrow D\mathrm{LocSys}_{{}^L G}(\mathbb{D}, {}^{\text{reg. sing. at } x}_{B\text{-struc. on residue}})$$

where we have imposed a technical condition on the allowed D-modules, which we don’t want to go into.

This allows us to transport the perverse t-structure on the left hand side to the right hand side, called the Geometric-Langlands (GL) t-structure. There are two aspects: global over $Y := \mathcal{N}$, and local over Y.

Theorem 2.1. *The GL t-structure on $D^b Coh^G(X)$ is compatible under the direct image functor π_* with the perverse coherent sheaves of middle perversity. (To define it, use that G acting on Y has finitely many orbits, of even dimension.)*

Remark 2.2. Schnell proved that under Geometric Langlands for GL(1), holonomic D-modules on $\mathrm{Bun}_{\mathrm{GL}(1)} = \mathrm{Pic}(C)$ go to perverse coherent sheaves of middle perversity.

The Geometric Langlands correspondence also induces a t-structure on $D^b(\mathrm{Coh}_{\pi^{-1}(y)}(X))$ for any $y \in Y$.

3 Enter Bridgeland Stabilities

3.1 Local operators (functors)

The set $\{(E, F), (\mathcal{E}', \mathcal{F}'), \varphi\colon \mathcal{E}|_{C-x} \cong \mathcal{E}'|_{C-x}\}$ turns out to be isomorphic to W_{aff}. ♠♠♠ TONY: [um??] The data corresponding to $w \in W_{\mathrm{aff}}$ can be thought of as a pair of bundles with “relative position w.”

If we consider stuff with a *fixed* relative position $w \in W_{\mathrm{aff}}$, this we have obvious projection maps to $\mathrm{Bun}_G(C, x)$:

$$\begin{array}{ccc} & \{(\mathcal{E},\mathcal{F}),(\mathcal{E}',\mathcal{F}'),\varphi\colon \mathcal{E}|_{C-x}\cong\mathcal{E}'|_{C-x}\}_w & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \mathrm{Bun}_G(C,x) & & \mathrm{Bun}_G(C,x)\end{array}$$

Let $R_w = p_{2*}p_1^*$. Then $R_{w_1}R_{w_2} = R_{w_1w_2}$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. These are precisely the relations for the affine Braid group B_{aff}, so we have an action of B_{aff} on $\text{Bun}_G(C, x)$.

Corollary 3.1. *B_{aff} acts on $D^b(Coh^G(X))$.*

The action is equivariant, hence extends to an action on the the equivariant derived category.

The action on the Grothendieck group $K(\text{Coh}^{\mathbb{C}^\times}(X))$ is isomorphic to the monodromy of the equivariant quantum connection on $QH^*_{\mathbb{C}^*}(X)$.

Conjecture 3.2. *Something like this works for any symplectic resolution, where B_{aff} is replaced by π_*K and it fits into a structure resembling Bridgeland stability.*

3.2 The affine braid group

Definition 3.3. Fix the inclusion $t^*_{\mathbb{R}} \subset t^*_{\mathbb{C}}$. We have an action of W on Σ, the set of coroot hyperplanes H_α. The affine Weyl group W_{aff} acts on $\widetilde{\Sigma}$, the set of affine coroot hyperplanes. We define the *affine braid group* $B_{\text{aff}} = \pi_1(t^*_{\mathbb{C}} \setminus \bigcup_{H\in\widetilde{\Sigma}} H/W_{\text{aff}})$.

Set $(\mathrm{t}^*_{\mathbb{C}})^0 = t^*_{\mathbb{C}} \setminus \bigcup_{H\in\widetilde{\Sigma}} H/W_{\text{aff}}$, so $B_{\text{aff}} = \pi_1((\mathrm{t}^*_{\mathbb{C}})^0)$.

Theorem 3.4. *There exists an open set $U \subset (t^*_{\mathbb{C}})^0$ and a covering $\widetilde{U} \to U$ with an action of B_{aff} and an embedding $\widetilde{U} \hookrightarrow \text{Stab}(C_0)$, the space of Bridgeland stability conditions on C_0.*

Let $X \to Y$ be a symplectic resolution. Consider another symplectic resolution $X' \to Y$. Let $V = H^2(X)$, which can be canonically identified for all such X. A given X is determined by its ample cone $C_X \subset V$. There is a symmetry group W acting on V, and $\bigcup_{X,w} w \cdot C_X$ is dense in V. Let Σ be the collection of the hyperplanes bounding $w(C_x)$ over all w, X.

Conjecture 3.5. *$QH^*_{\mathbb{C}^*}(X)$ has a family of connections on $\mathbb{C}^* \otimes H^2(X, \mathbb{Z})$ with singularities on subtori $\exp(2\pi i H)$ where H is a hyperplane parallel to one in Σ.*

Remark 3.6. This is known for $X \subset T^*(G/B)$, and more recently for X quiver varieties by work of Maulik-Okounkov.

Let $V^0_{\mathbb{C}}$ be the complement to $H_{\mathbb{C}}$ for $H \in \widetilde{\Sigma}$. This has many alcoves, which are components of $V_{\mathbb{R}} \cap V^0_{\mathbb{C}}$, and imaginary cones, which are of the form $V_{\mathbb{R}} + iwC_x$.

Conjecture 3.7. *We can assign to each alcove an abelian category (thought of as a noncommutative resolution of Y) and to each cone $Coh(X)$, and to each class of a path a derived equivalence, so that $\pi_1(V^0_{\mathbb{C}})$ acts on $D^b(\text{Coh}^{\mathbb{C}^*}(X))$ such that the action on the Grothendieck group K is isomorphic to the mondromy of $QH^*_{C^*}(X)$.*