PREPARATION FOR BEYOND ENDOSCOPY

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1. Motivation

Beyond Endoscopy is an idea of Langlands for approaching functoriality, introduced in a 2000 paper (where he calls it a "pipe dream").

Let G, H be reductive groups. Suppose you have a "nice" homomorphism $\psi\colon {}^LH \hookrightarrow {}^LG$. Then there should be a transfer of packets of automorphic representations $\pi_H \dashrightarrow \pi_G$ satisfying various compatibility conditions. Perhaps the most important one is that for any finite-dimensional representation $\rho\colon {}^LG \to \mathrm{GL}(V_\rho)$, we should have

$$L(s, \pi_H, \rho \circ \psi) = L(s, \pi_G, \rho).$$

Remark 1.1. The definition of these L-functions is subtle. At all but finitely many places it is clear how to define them, but of course we want to pin down *all* of the factors.

Why should we have such a transfer? Conjecturally, the automorphic representations are parametrized by Langlands parameters

$$\mathcal{L} \xrightarrow{\varphi_\pi} {}^LG$$

where $\mathcal{L}$ is conjectural "global Langlands group". If we believe in this, then any parameter $\mathcal{L} \to {}^LH$ induces by composition with ψ to a parameter $\mathcal{L} \to {}^LG$.

Now the global Langlands group, and hence the parameters, are all very conjectural. But we could imagine trying to describe the image of the transfer from H to G *without* knowing the φ_π, and this is what Beyond Endoscopy is about.

More precisely, we could try to classify the parameters φ_π according to their images.

- At the top we have parameters φ_π such that $\overline{\varphi_\pi(\mathscr{L})} = {}^LG$.

- For each H such that ${}^L H \hookrightarrow {}^L G$, we could try to identify the representations transferred from H.

The idea of Beyond Endoscopy is to try to detect the automorphic representations of G that "came from" H by transfer, for each H. In other words, we want to stratify $L^2(G(F)\backslash G(\mathbf{A}_F))$ according to the images of the associated parameters, to get

$$L^2(G(F)\backslash G(\mathbf{A}_F)) = \bigoplus_{({}^L H, \sigma \colon {}^L H \hookrightarrow {}^L G)} \bigoplus_{\pi \colon \overline{\mathrm{Im}\,(\varphi_\pi)} \sim \sigma({}^L H)} m_\pi \cdot \pi$$

The aim of Beyond Endoscopy is to establish a candidate stratification of this kind, *without* any reference to the parameters φ_π, and instead using the trace formula.

So we want to detect "transfers from H". This should be detected by L-functions. More precisely, the pole at $s = 1$ of $L(s, \pi_G, \rho)$ for some ρ should detect whether π_G came from some H. Why? The principle that for Artin L-functions, the order of the pole at $s = 1$ is the multiplicity of the trivial representation.

Consider the conjectural picture

$$\begin{tikzcd} \mathscr{L} \arrow[rr, "\varphi_\pi"] \arrow[dr, "\varphi_{\pi_H}"'] & & {}^L G \\ & {}^L H \arrow[ur, "\psi"'] & \end{tikzcd}$$

Suppose $\overline{\mathrm{Im}\ \varphi_\pi} = {}^L H$. So we expect that if $\overline{\varphi_\pi(\mathscr{L})} = {}^L H$, then $m_\pi(\rho)$, the multiplicity of the pole of $L(s, \pi_H, \rho \circ \psi) = L(s, \pi_G, \rho)$ is $m(\mathbf{1}, \rho|_{\psi({}^L H)})$. By varying ρ, one can try to detect whether π_G comes from π_H.

Langlands proposed a way to use the trace formula to isolate the representations π such that $L(s, \pi_G, \rho)$ has a pole at $s = 1$, taking multiplicity into account. We want a formula for

$$\sum_\pi \mathrm{Tr}\, \pi(f) \cdot m_\pi(\rho)$$

where $m_\pi(\rho)$ is the multiplicity of the pole of $L(s, \pi, \rho)$ at $s = 1$.

Fix a finite set of places S, including the infinite ones (because of issues of defining local L-factors). We want a "trace-like" formula

$$\sum_\pi \mathrm{Tr}(\pi_S(f_S)) m_\pi(\rho)$$

where $m_\pi(\rho)$ is the multiplicity of the pole of the *partial* L-function $L^S(s, \pi, \rho)$. The goal of next lectures will be to develop some sort of "computable" (in terms of orbital integrals) expression for this. By playing with f_S, we can then try to isolate π, and hence $m_\pi(\rho)$.

The strategy for getting such a formula is to use the stable trace formula and then perform a Poisson summation on the geometric side. In these lectures, we will just develop the geometric side until it looks like we could use Poisson summation. (More precisely, we follow [FLN] and our goal is to explain (3.31) of [FLN], which converts the geometric side of the trace formula into something that looks like a sum over a lattice in an affine space.)

1.1. **The geometric side.** The geometric side of the trace formula roughly looks like

$$\sum_{\gamma \in G(F)/\sim_{\mathrm{st}}} \mathrm{vol}(\gamma) O_\gamma^{\mathrm{st}}(f)$$

where

- $\sim_{\mathrm{st}}$ is the equivalence relation of stable conjugacy and O_γ^{st} is a stable orbital integral,
- f is a test function,
- $\mathrm{vol}(\gamma) = \mathrm{vol}(Z(\mathbf{A})G_\gamma(F)\backslash G_\gamma(\mathbf{A}))$.

We promised to make this look like a sum over lattices in an affine space. This affine space is the *Steinberg-Hitchin base*, and we'll explain it. After that, we'll discuss choices of measures for the orbits. The point is that the orbital integrals are singular, which makes them not amenable to Poisson summation, but there is a way to change the measures to remove this problem.

2. The Steinberg-Hitchin base

2.1. **The Chevalley map.** We want a space that parametrizes stable conjugacy classes in G.

2.1.1. *The case of* GL_n. For GL_n, stable conjugacy classes of regular semisimple elements are in bijection with characteristic polynomials with nonvanishing discriminant. The map

$$g \mapsto \text{coefficients of } \mathrm{CharPoly}(g)$$

induces a morphism

$$\mathrm{GL}_n \to \mathbf{A}^n.$$

The fibers over an open dense subset of points are stable conjugacy classes.

2.1.2. *Chevalley map for Lie algebras.* Assume for the moment that we are over an algebraically closed field $\overline{F}$. The case of interest is where F is a non-archimedean local field.

Let G be a reductive group over F and $\mathfrak{g} := \mathrm{Lie}(G)$. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$.

The G-orbits on $\mathfrak{g}$ form an affine variety Spec $\mathcal{O}(\mathfrak{g})^G$. We denote by $\overline{F}[t]^W$ the W-invariant regular functions on $\mathfrak{t}$. In the case $G = \mathrm{GL}_n$, this is the algebra of symmetric polynomials in n variables, which we know is the free algebra on elementary symmetric polynomials. In general, one thinks of $\overline{F}[t]^W$ as the algebra of "elementary W-invariant symmetric polynomials"; it turns out to be free:

$$\overline{F}[t]^W \cong \overline{F}[a_1, \dots, a_r].$$

In other words,

$$\mathfrak{t}//W \cong \mathrm{Spec}\ \overline{F}[a_1, \dots, a_r] \cong \mathbf{A}^r.$$

The map $\mathfrak{t} \to \mathfrak{t}//W$ is

$$(t_1, \dots, t_r) \mapsto (a_1(t_1, \dots, t_r), \dots, a_r(t_1, \dots, t_r)).$$

Example 2.1. For $G = \mathrm{GL}_n$,

$$\begin{aligned} a_1(t_1, \ldots, t_r) &= t_1 + \ldots + t_r \\ \vdots \quad & \quad\quad \vdots \\ a_r(t_1, \ldots, t_r) &= t_1 \ldots t_r \end{aligned}$$

Theorem 2.2 (Chevalley restriction theorem)**.** *The map $P \mapsto P|_{\mathfrak{t}}$ induces*

$$\overline{F}[\mathfrak{g}]^G \cong \overline{F}[\mathfrak{t}]^W.$$

The map $\overline{F}[\mathfrak{g}]^G \hookrightarrow \overline{F}[\mathfrak{g}]$ induces

$$\mathfrak{g} \to \mathfrak{t}/W \simeq \mathbf{A}^r$$

sending X to the unique W-orbit in $\mathfrak{t}$ consisting of elements conjugate to the semisimple part of X.

The map $\mathfrak{g} \to \mathfrak{t}/W$ is defined over F. What does the image look like?

Example 2.3. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$. The Chevalley map sends

$$X \sim \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} \in \mathfrak{sl}_2(\mathbf{R})$$

to $\det X = -x^2 - (y^2 - z^2) = z^2 - (x^2 + y^2)$.

The fibers are hyperboloids, with 1 or 2 components. The fibers are stable orbits, but the rational orbits are the components.

Let T be a split maximal torus over F. We use T to form $\mathbf{A}_G := \mathfrak{t}/W$. Recall that tori in $\mathrm{GL}_2(F)$ (up to F-conjugacy) correspond bijectively to degree 2 algebras E/F (possibly $E = F \oplus F$). Our space $\mathbf{A}_G(F)$, up to a set of positive codimension, are a disjoint union of images of representatives of conjugacy classes of tori. Explicitly, for $(a, b) \in \mathbf{A}_G$ (with coordinates such that $X \mapsto (\mathrm{Tr}(X), \det(X)) =: a, b)$, if $a^2 - 4b$ is a square then the point is in the image of a split torus, and if it is non-square then it comes from the torus $F(\sqrt{a^2 - 4b})$.

2.2. **The group.** Let G be split over F, and assume that G_{der} is simply connected.

For now, assume also that G is *semisimple*, so $G = G_{\mathrm{der}}$ and hence is simply connected under our assumptions.

We briefly remind some basics of algebraic groups. Given a choose of split maximal torus $T \subset G$, we can form a root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$. Let Λ be the root lattice $\mathbf{Z} \cdot \Phi \subset X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Let P be the weight lattice

$$P := \{\mu \in X^*(T) \otimes_{\mathbf{Z}} \mathbf{R} \colon \langle \mu, \lambda \rangle := 2\frac{(\mu, \lambda)}{(\lambda, \lambda)} \in \mathbf{Z}\}.$$

We have $P \supset X^*(T) \supset \Lambda$.

Definition 2.1. For semisimple G, we say G is *simply connected* if $P = X^*(T)$.

Example 2.2. For $G = \mathrm{SL}_2$, the root is

$$\alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t^2.$$

Hence $\Lambda = 2\mathbf{Z}$ and $P = \mathbf{Z} = X^*(T)$.

Recall that the Chevalley map is given by "elementary symmetric functions". These can be thought of as traces of rational representations of G. To make this precise, let $\{\mu_i\}$ be the fundamental weights for G, a $\mathbf{Z}$-basis for P. For each μ_i, we have an algebraic representation $\rho_i \colon G \to \mathrm{GL}(V)$ over F, of highest weight μ_i.

Example 2.3. For SL_2, $\mu_1 = 1 \in \mathbf{Z}$ and ρ_1 is the standard representation (on a 2-dimensional space over F). The representation of highest weight n is $\mathrm{Sym}^n \rho_1$.

More generally, for SL_n the representations corresponding to the fundamental weights are the $\wedge^i(\rho_{\mathrm{std}})$. Their traces recover the elementary symmetric polynomials, which we met before.

We want $\mathrm{Tr}\,\rho_i$ to generate the whole algebra of W-invariant polynomials on T. If the μ_i span $X^*(T)$ (i.e. the simply connected case) then this happens.

The upshot is that if G is semisimple and simply connected, then we get a map $G \to T/W$ defined over F,

$$g \mapsto (\mathrm{Tr}\,\rho_i(g)).$$

This is called the *Steinberg map*.

Remark 2.4. Note that for SL_2, the Steinberg map is Tr, while the Chevalley map is det.

When G is reductive, and G_{der} is simply connected, there is a sequence

$$1 \to A := Z \cap G_{\mathrm{der}} \to Z \times G_{\mathrm{der}} \to G \to 1$$

and there is a map $Z \times G_{\mathrm{der}} \to Z \times \mathbf{A}_{G_{\mathrm{der}}}$ which we view as the Steinberg map in that case. For defining measures on stable orbits, write $\gamma \in \mathrm{GL}_n$ as $\gamma = \gamma' \cdot z$ with $\gamma' \in G_{\mathrm{der}}$ and $z \in Z$, and use the Steinberg map. (It doesn't matter much that this expression isn't unique.)

Note: for a classical group, e.g. $G = \mathrm{GL}_n$, you still have a map $G \to Z \times \mathbf{A}_{G_{\mathrm{der}}}$, but it will induce a slightly different measure on the base.

For reductive split G, and G_{der} simply connected, the Steinberg-Hitchin base is $\mathfrak{A} := Z \times \mathbf{A}_{G_{\mathrm{der}}}$. ([FLN] call $\mathfrak{B} := \mathbf{A}_{G_{\mathrm{der}}}$.)[1]

3. Measures on orbits

Let γ be regular semisimple. The *orbital integral* of f over γ is

$$O_\gamma(f) := \int_{T_\gamma \backslash G} f(g^{-1}\gamma g)\,\frac{dg}{dt}.$$

We need to explain the normalization of measures.

Note: a stable orbit is a finite union of rational orbits, and the stabilizers are all isomorphic, so it is equivalent to define a measure on stable orbits or rational orbits.

[1] If not simply connected, the base would be a quotient, which could be singular.

3.1. **The approach in the trace formula.** There are finitely many conjugacy classes of tori. You can pick a measure dt on a representative of each conjugacy class. Then you take the quotient measure. We need to discuss this more carefully.

How do you pick a Haar measure on a group? There are two natural ways.

(1) Start with an invariant differential form ω. (We'll explain shortly how this gives a measure.)
(2) Pick a compact open subgroup and specify its volume. Since Haar measures are unique up to scalar, this pins down the Haar measure.

A lot of arithmetic information is hiding in the conversion between these two ways.

To start, pin down a normalization of a measure dx on $\mathbf{A}^1$. (From this choice, other normalizations will be built canonically.) There are two natural ways normalize this:

(1) Demand $\mathrm{vol}(\mathcal{O}_F) = 1$. This is what we will do.
(2) ([FLN]) Fix global field L and a place v such that $L_v = F$. Fix a character of L. Normalize all local measures $|dx|$ on $\mathbf{A}^1$ so that Fourier inversion holds.

These are different, e.g. if ψ has conductor 0, the difference is a factor of $\sqrt{q}$.

As explained by Weil [Adeles and algebraic groups], any volume form (meaning top-degree, everywhere non-vanishing) gives a measure$|d\omega|_v$: writing

$$\omega = f(x_1, \ldots, x_d) dx_1 \wedge \ldots \wedge dx_d$$

locally corresponds to the measure

$$\int |f(x_1, \ldots, x_d)|_v |dx_1 \wedge \ldots \wedge dx_d|_v.$$

Fact 3.1. Let F be a local field. Weil explained that if X is a smooth scheme over $\mathcal{O}_F$, and ω is a volume form on X, then

$$\int_{X(\mathcal{O}_F)} |\omega|_F$$

is canonical (since the volume form is unique up to $\mathcal{O}_F^\times$), and is equal to

$$\frac{\# X_k(\mathbf{F}_q)}{q^{\dim X}}$$

Proof sketch. This works for $\mathbf{A}^1$ by explicit computation, hence also for $\mathbf{A}^d$. Since X is smooth over $\mathcal{O}_F$, the reduction modulo the uniformizer ϖ map on X looks locally the same as on $\mathbf{A}^d$. (We are using Hensel's Lemma here.) But the volume of the fiber over every point in $\mathbf{A}^d/\mathbf{F}_q$ is q^{-d}, so the same holds for X. □

Next we discuss the dg on G. We choose an invariant differential form ω on G.

Example 3.2. For $G = \mathbf{G}_m$, the invariant form is $\frac{dx}{x}$, and

$$\mathrm{vol}(\mathbf{G}_m(\mathcal{O}_F)) = \frac{q-1}{q}.$$

Example 3.3. For $G = \mathrm{GL}_2$, we can take

$$\omega = \frac{dg^+}{|\det g|^2}.$$

Because G is smooth,

$$\mathrm{vol}\, G(\mathcal{O}_{F_v}) = \frac{\#\,\mathrm{GL}_2(\mathbf{F}_q)}{q^4}$$

3.2. **Measure on tori.** For T_γ which is not split, it is a non-trivial issue to make sense of its integral points, since it does not come with a natural integral model.

To get ω, we need (local) coordinates. There is an obvious choice of T is split, but what if it's not? The point is that there is still a natural choice.

Over $\overline{F}$, we can split T and choose a basis $\chi_1, \ldots, \chi_r$ of $X^*(T)$. Then we get a volume form "$\omega = d\chi_1 \wedge \ldots d\chi_r$" on T over $\overline{F}$.

Example 3.1. A torus $T \subset \mathrm{GL}_2(F)$ corresponds to a quadratic extension E/F. Suppose $E = F(\sqrt{\epsilon})$. Then

$$T(F) = \begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix}.$$

We can choose χ_1 to map to $x + \sqrt{\epsilon} y$ and χ_2 to map to $x - \sqrt{\epsilon} y$. Then

$$(dx + \sqrt{\epsilon} dy) \wedge (dx - \sqrt{\epsilon} dy) = -2\sqrt{\epsilon} dx \wedge dy.$$

Since we're using $\mathbf{G}_m$, we should divide by $\chi_1\chi_2$, so we get $\frac{-2\sqrt{\epsilon} dx \wedge dy}{x^2 - \epsilon y^2}$. So we end up with a differential form, but it is not defined over F.

Weil proposed a better solution. We'll just say it in the special case of tori. If $T = \mathrm{Res}_{E/F}\,\mathbf{G}_m$, Weil builds in the discriminant of E/F as well. In the global situation, one would put in $|\Delta|^{1/2 \dim T}$. If E/F is unramified and $p \neq 2$, then $|2\sqrt{\epsilon}|_p = 1$, so it doesn't affect anything (and also $|\Delta|_p = 1$). The $\sqrt{\epsilon}$ matters when the local extension is ramified, and $|\Delta|_p = \frac{1}{\sqrt{p}}$ which agrees with $|2\sqrt{\epsilon}|_p$ if $p \neq 2$.

Question: what if T is not a restriction $\mathrm{Res}_{E/F}\,\mathbf{G}_m$? We don't know a general answer.

Unlike a non-commutative G, T has a unique maximal compact subgroup, which can be described as

$$\{t \in T(F) \colon |\chi(t)| = 1 \text{ for all } \chi \colon T \to \mathbf{G}_m \text{ defined over } F\}.$$

Example 3.2. If $T \subset \mathrm{SL}_2$ is the norm 1 torus, then there are no characters and $T(F) = E^1$ is already compact.

Example 3.3. In GL_2, $T(F) = E^\times$ and $T^0 = \{e \in E^\times \colon \mathrm{Nm}_{E/F}(E) \in \mathcal{O}_F^\times\}$.

The volume of T^0 with respect to our measure ω (the *honest* pull-back, without tweaking by discriminant) is

$$|2\sqrt{\epsilon}| \int_{\mathcal{O}_E^\times} \frac{dxdy}{xy}.$$

The factor $|2\sqrt{\epsilon}|$ is 1 if E is unramified and $1/\sqrt{q}$ if E is ramified, and by Fact 3.1 the second factor is $1/q^2 \cdot \#\mathcal{O}_E^\times \pmod{\varpi_E}$, which is $q^2 - 1$ in the unramified case and $(q-1)q$ in the ramified case. (Imagine solving for x^2-py^2 to be a unit: this forces x to be a unit, and y can be anything.)

Summary:

$$\mathrm{vol}_{|d\omega|}(T^0) = \begin{cases} \frac{q-1}{q}\frac{q+1}{q} & E/F \text{ unramified,} \\ \frac{q-1}{q}\frac{q}{q}\frac{1}{\sqrt{q}} & E/F \text{ ramified,} \\ \left(\frac{q-1}{q}\right)^2 & T \text{ split.} \end{cases}$$

Suppose this situation came from $T = \mathrm{Res}_{K/L}\,\mathbf{G}_m$ for a global extension K/L, and we use Weil's pullback (to get rid of $\frac{1}{\sqrt{q}}$). Let χ_K be the quadratic character associated to the extension K/L

$$\chi_K(v) = \begin{cases} 0 & v \text{ ramified,} \\ -1 & v \text{ inert,} \\ 1 & v \text{ split.} \end{cases}$$

Then we can write the answer as follows:

$$\mathrm{vol}(T^0) = \left(1-\frac{1}{q}\right)\left(1-\frac{\chi_E(v)}{q_v}\right) \tag{3.1}$$

This suggests a connection to L-functions.

Remark 3.4. We want to take a product over primes eventually. The factor

$$1-\frac{1}{q_v}$$

will cause the product to diverge. So Weil introduced convergence factors, which for will be $(1-\frac{1}{q_v})^{-1}$ (which can be seen as coming from the "$\mathbf{G}_m$-part of T").

In general, we have a representation σ_T of $\mathrm{Gal}(K/L)$ on $X^*(T)$, and it restricts trivially to the subspace of rational characters, by definition. The convergence factor is $L(1,\sigma_T)$. This kills the pole, but leaves the residue of the L-function at $s=1$. When working with $T = \mathrm{Res}_{E/\mathbf{Q}_p}\,\mathbf{G}_m$, you get the residue of the L-function.

This expresses $\mathrm{vol}_\omega(T^0)$ as the value at 1 of an Artin L-function. This is almost true in general, but if T is ramified then it is true if you replace T^0 with a subgroup of finite index.

Now we have a quotient measure dg/dt on G/G_γ, so we have defined the orbital integral

$$O_\gamma(f) = \int_{G/G_\gamma} f(g^{-1}\gamma g)\frac{dg}{dt}.$$

We next want to explain how to change the measure.

3.3. **Another measure on orbits.** Recall that stable orbits are fibers of the Steinberg map

$$c\colon G \to Z \times \mathbf{A}_{G_{\mathrm{der}}}).$$

Once we specify the measure dg on G and $Z\times\mathbf{A}_{G_{\mathrm{der}}}$ (on which we take dx/x on Z and dx normalized so that $\mathcal{O}$ has volume 1 on the affine part), this should induce a quotient measure on stable orbits.

Each stable orbit gets a measure ω_a, which differs from the old one, by a constant *depending on the orbit*. We can express this as a "quotient form". Splitting (locally)

$$T(G) = T(O_\gamma) \oplus T(Z \times \mathbf{A}_{G_{\mathrm{der}}}$$

(here O_γ is the orbit through γ) we have chosen volume forms on G and $Z \times A_{G_{\mathrm{der}}}$, so we get a volume form ω_a on TO_γ

$$\omega_G = \omega_a \wedge da.$$

Another way to think about this is that ω_a is characterized by the property that for $f \in C_c^\infty(G)$,

$$\int_G f(g)\, dg = \int_{Z \times \mathbf{A}_{G_{\mathrm{der}}}} \left(\int_{c^{-1}(a)} f d\omega_a \right) da.$$

How does ω_a relate to dg/dt? The latter depends only on the stable conjugacy class (because it depends only on the centralizer), while ω_a depends on γ.

Let's consider the Lie algebra, because the Chevalley map there is a little easier to think about. Recall the Weyl integration formula

$$\int_{\mathfrak{g}} f(Y) dY = \sum_{T/\sim_{\mathrm{conj}}} |W_T|^{-1} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| f(g^{-1}Xg) \frac{dg}{dt} dX. \tag{3.2}$$

Here $D(X)$ is the *Weyl discriminant*

$$D(X) = \prod_{\alpha \in \Phi} \alpha(X) = \det(\mathrm{ad}(X), \mathfrak{g}/\mathfrak{t})$$

where

- $\mathfrak{t} := \mathrm{Lie}(Z_G(X))$,
- dX is the affine space measure on $\mathfrak{t}$,
- dY is the affine space measure on $\mathfrak{g}$, and
- dg is related to dY by the exponential map.

The formula (3.2) comes from the fact that the map

$$G/T \times \mathfrak{t} \to \mathfrak{g}$$

sending

$$(g, X) \mapsto g^{-1}Xg$$

has non-vanishing Jacobian.

We want to replace the integral over $\mathfrak{t}_{\mathrm{reg}}$ with an integral over $\mathfrak{t}_{\mathrm{reg}}/W$, which looks like the Steinberg-Hitchin base. We saw that $\mathfrak{t}^{\mathrm{split}}/W$ was a disjoint union of images of tori. For each of them, the map $\mathfrak{t}^{\mathrm{split}} \to \mathfrak{t}^{\mathrm{split}}/W$ is $|W_T| : 1$. But its Jacobian is non-trivial. But it has the form $c_T \prod_{\alpha > 0} \alpha(X)$. Summing it up, we get

$$\frac{1}{|W|} |D(X)| dX \wedge \frac{dg}{dt} = \frac{1}{|W|} d\omega_a \wedge (da = |D(X)|^{1/2} dX)$$

So it looks like we're getting (for the Lie algebra)

$$d\omega_a \sim \frac{dg}{dt} |D(X)|^{1/2}. \tag{3.3}$$

3.4. **Summary.** We briefly summarize what we have discussed so far. We have introduced three different choices of measure.

(1) ("Canonical normalization") In the trace formula, the measure used is $\frac{dg}{dt}$ where dg is normalized so that $\mathrm{vol}(G(\mathcal{O})) = 1$ and $\mathrm{vol}(\widetilde{T}(\mathcal{O}_v)) = 1$.

This normalization is useful for the trace formula, because with this normalization almost all finite places have local orbital integrals equal to 1, and so the products that appear are really just finite products.

(2) On the other hand, we explained that one could take $\frac{d\omega_G}{d\omega_T}$, the quotient of a measure on G by a measure on T, with both measures induced by invariant volume forms.

(3) ("Geometric measures") Finally, the "geometric measures" introduced in [FLN] are of the form $\frac{d\omega_G}{d\omega_{\mathbf{A}}}$. The idea behind this measures is to treat stable orbits as fibers of the Steinberg map (the $\omega_{\mathbf{A}}$ is a measured induced by an invariant form on the Steinberg-Hitchin base).

The conversion between (1) and (2) is $G(\mathbf{F}_q)q^{-\dim G}$ from the volume of $G(\mathcal{O})$, and $\overline{T}(\mathbf{F}_q)q^{-\dim T}$ from the volume of $T(\mathcal{O})$. In other words,

$$\frac{d\omega_G}{d\omega_T} = \frac{G(\mathbf{F}_q)q^{-\dim G}}{\overline{T}(\mathbf{F}_q)q^{-\dim T}}\frac{dg}{dt}.$$

The ratio can be written as a ratio of Artin L-factors evaluated at 1. In particular, the conversion factor doesn't depend on the element.

To go from (2) to (3), by (3.3) we find pick up a factor of $|D(\gamma)|^{1/2}$ where $\gamma \in G(F)$ and $D(\gamma)$ is the Weyl discriminant.

4. Examples

4.1. $G = \mathrm{GL}_2$**.** In this case one can use the Bruhat-Tits tree to compute orbital integrals. This is explained by Kottwitz; we will just give the answer. Let $\gamma \in G$ be regular. If γ is split or unramified, then

$$d_\gamma = \mathrm{val}(1 - \frac{e_1}{e_2}) \tag{4.1}$$

where e_1, e_2 are the eigenvalues of γ if γ. If γ is ramified, then

$$d_\gamma = \sup\{\mathrm{val}_E(\gamma - a) \colon a \in \mathcal{O}_F^\times\}.$$

Perhaps a better way to think about d_γ is that it's basically half the valuation of $D(\gamma)$, but rounded to an integer, i.e.

$$|D(\gamma)| = \begin{cases} q^{-2d_\gamma} & \text{unramified or split,} \\ q^{-2d_\gamma - 1} & \text{ramified.} \end{cases}$$

Remark 4.1. The way that d_γ really comes up in the calculation is that it's the integer such that the set of fixed vertices under the action by γ on the tree equals the number of vertices whose distance to the standard apartment is $\leq d_\gamma$.

Assume $\gamma \in \mathrm{GL}_2(\mathcal{O}_v)$ (if it is not conjugate to such an element, its orbital integrals are 0). Then

$$O_\gamma^{\mathrm{can}}(\mathbf{I}_{G(\mathcal{O}_v)}) = \begin{cases} q^{d_\gamma} & \gamma \text{ split}, \\ 1+(q+1)\frac{q^{d_\gamma}-1}{q-1} & \gamma \text{ unramified}, \\ 2\frac{q^{d_\gamma+1}-1}{q-1} & \gamma \text{ ramified}. \end{cases}$$

Here we're using the canonical measure, with a caveat. The factor of 2 in the ramified case is unpleasant. In fact Kottwitz actually doesn't quite say which measure he's using in the ramified case. I believe that the 2 disappears if you use what we're calling the canonical measure.

Remark 4.2. This example illustrates the principle that orbital integrals diverge as γ approaches non-regular elements. Indeed, if d_γ is large, i.e. γ is close to being non-regular, then $O_\gamma(\mathbf{I}_{G(\mathcal{O}_v)}) \sim |D(\gamma)|^{-1/2}$, and $|D(\gamma)| \to 0$.

What happens if you use the geometric normalization? By §3.4, if E/F is the quadratic extension corresponding to T then

$$O_\gamma^{\mathrm{geom}}(\mathbf{I}_{G(\mathcal{O}_v)}) = |D(\gamma)|^{1/2}\frac{\#G(\mathbf{F}_q)q^{-\dim G}}{\#T(\mathbf{F}_q)q^{-\dim T}}|\Delta_E|_F^{1/2} O_\gamma^{\mathrm{can}}(\mathbf{I}_{G(\mathcal{O}_v)}).$$

This comes out to

$$\begin{cases} 1+\frac{1}{q} & \gamma \text{ split} \\ (1-\frac{1}{q})(q^{-d_\gamma}+\frac{q+1}{q-1}(1-q^{-d_\gamma}) & \gamma \text{ unramified} \\ \frac{q^2-1}{q^2}q^{-d_\gamma}\frac{q^{d_\gamma+1}-1}{q-1} & \gamma \text{ ramified} \end{cases}$$

The thing to note here is what happens when $d_\gamma = 0$, which is the case for almost all primes. In that case one gets $1+\frac{1}{q}$ if γ is split, and $1-\frac{1}{q}$ if γ is unramified. So this looks like an L-value, and one has to worry about convergence.

4.2. **Analytic class number formula.** Next we want to give a simple illustration of how some of these factors encode arithmetic invariants, namely the class number formula, which says

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r+t}\pi^t R_K h_K}{\omega_K|\Delta_K|^{1/2}}.$$

Here

- K is a number field,
- Δ_K is the discriminant of $K/\mathbf{Q}$,
- h_K is the class number,
- ω_K is the number of roots of unity in K,
- r is the number of real embeddings,
- $2t$ is the number of complex embeddings,
- R_K is the regulator.
- ζ_K is the Dedekind zeta function.

We focus on the simplest case, which is associated to imaginary quadratic extensions of $\mathbf{Q}$. In that case, the formula specializes to

$$L(1,\chi_K) = \frac{2\pi h_K}{|\Delta_K|^{1/2}\omega_K}.$$

We're going to explain that this formula can be re-interpreted as a volume calculation, and specifically that both sides calculate the same volume.

Recall that weak approximation says $\mathbf{Q}^\times\backslash(\mathbf{A}_f)^\times/\prod_p \mathbf{Z}_p^\times = 1$ because the class number of $\mathbf{Q}$ is 1. In general this kind of double coset space is almost the class group, but one has to be careful with units. What one has is the following short exact sequence:

$$1 \to (K^\times \cap \prod_v \mathcal{O}_v^\times)\backslash \prod \mathcal{O}_v^\times \to K^\times\backslash\mathbf{A}_f^\times \to \mathrm{Cl}(K) \to 1.$$

If we assume $\mathrm{vol}(\mathcal{O}_v^\times) = 1$, i.e. what we have been calling the "canonical measures", then we find that the (canonically normalized) volume of $K^\times\backslash\mathbf{A}_f^\times$ is $\frac{h_K}{\omega_K}$.

We may interpret $K^\times = \mathrm{Res}_{K/\mathbf{Q}}(\mathbf{G}_m)(\mathbf{Q})$. In §3.2 we found that

$$\mathrm{vol}_{\frac{dx}{x}}(\mathcal{O}_v^\times) = (1-\frac{1}{p})L_v(1,\chi)^{-1}|\Delta_K|_v^{1/2}. \tag{4.2}$$

The factors $1-\frac{1}{p}$ (which come from the split part $\mathbf{G}_m \subset T$) are killed when passing to the Tamagawa measure τ.

Next we need to know some facts about the *Tamagawa number*, which is the volume of $K^\times\backslash\mathbf{A}_K^\times$ for a measure coming from a differential form, but adjusted by volume factors for convergence.

Fact 4.1. We have $\tau(\mathbf{G}_m) = 1$, and τ is preserved by restriction of scalars, hence $\tau(R_{K/\mathbf{Q}}(\mathbf{G}_m)) = 1$.

Now we compute

$$1 = \tau(T(\mathbf{Q})\backslash T(\mathbf{A}))$$

in another way. We have

$$T(\mathbf{Q})\backslash T(\mathbf{A}_\mathbf{Q}) = K^\times\backslash\mathbf{A}_{K,f}^\times \cdot T(\mathbf{R})$$

and $T(\mathbf{R})$ is a circle after killing the $\mathbf{G}_m$, so the natural contribution from it is 2π. Hence we have $2\pi\tau(K^\times\backslash\mathbf{A}_{K,f}^\times) = 1$, and by (4.2) and the fact that $\mathrm{vol}^{\mathrm{can}}(K^\times\backslash\mathbf{A}_f^\times) = \frac{h_K}{\omega_K}$ we get

$$\tau(K^\times\backslash\mathbf{A}_{K,f}^\times) = L(1,\chi)^{-1}|\Delta_K|^{1/2}.$$

Putting everything together then gives

$$\prod_{v<\infty} L_v(1,\chi)^{-1}\cdot|\Delta_K|_v^{1/2}\cdot 2\pi\cdot\underbrace{\frac{h_K}{\omega_K}}_{\mathrm{vol}^{\mathrm{can}}} = 1.$$

5. Eichler-Selberg trace formula

The Eichler-Selberg trace formula is a formula for the trace of a Hecke operator T_n on S_k, the space of cusp forms of weight k and (for simplicity) level 1.

Think of this as a special case of the Arthur-Selberg trace formula for GL_2, applied to a specific test function. We will choose a test function that gives T_n on S_k. The trace formula reads

$$\begin{aligned}\mathrm{Tr}(T_{n,k}) = \frac{k-1}{12}\mathbf{I}_{\square}(n)n^{k/2-1} \\ - \frac{1}{2}\sum_{t^2<4n} \frac{\rho^{k+1}-\overline{\rho}^{k-1}}{\rho-\overline{\rho}} \sum_m h_w\left(\frac{t^2-4n}{m^2}\right) \\ - \frac{1}{2}\sum_{d\mid n} \min(d,n/d)^{k-1}\end{aligned}$$

where

- We define
$$\mathbf{I}_{\square}(n) := \begin{cases} 1 & n = \text{ perfect square} \\ 0 & \text{otherwise}\end{cases}$$
- $\rho, \overline{\rho}$ are the roots of $X^2 - tX + n$,
- $h_w(\frac{t^2-4n}{m^2})$ is the "(weighted) class number of the order in $\mathbf{Q}(\rho)$ which has discriminant $\frac{t^2-4n}{m^2}$", weighted by the size of its automorphism group.

The term $\frac{k-1}{12}\mathbf{I}_{\square}(n)n^{k/2-1}$ is the contribution of the (orbital integral over) the volume.

The term $\frac{1}{2}\sum_{t^2<4n}\frac{\rho^{k+1}-\overline{\rho}^{k-1}}{\rho-\overline{\rho}}\sum_m h_w(\frac{t^2-4n}{m^2})$ is the contribution from the elliptic part.

The term $\frac{1}{2}\sum_{d\mid n}\min(d,n/d)^{k-1}$ is the contribution from the hyperbolic and unipotent conjugacy classes.

A book by Knightly-Li explains how to deduce this from the Arthur-Selberg trace formula. We obviously don't have time to explain this, so we will just make a couple remarks.

First of all, what's the test function that one plugs into the Arthur-Selberg trace formula? Answer: $f = f_\infty \prod f_p$ where

- f_∞ is the complex conjugate of the matrix coefficient of discrete series with highest weight k (this kills off everything except the weight k cusp forms),
- $f_p = \mathbf{I}_{G(\mathbf{Z}_p)} \cdot \mathbf{I}_{Z(\mathbf{Q}_p)}$ if $p \nmid n$,
- $\mathbf{I}_{M(n)} \cdot \mathbf{I}_{Z(\mathbf{Q}_p)}$ if $p \mid n$. This means the characteristic function of matrices in $M(\mathbf{Z}_p)$ whose determinant has valuation n.

The sum $\sum_{t^2<4n}$ is already a sum of traces. We can view (t,n) as a point in the Steinberg-Hitchin base. The factor $\frac{\rho^{k+1}-\overline{\rho}^{k-1}}{\rho-\overline{\rho}}$ comes from $O_\gamma(f_\infty)$. What about $\sum_m h_w(\frac{t^2-4n}{m^2})$? We have basically already seen that

$$O_\gamma \cdot \mathrm{vol}(T(\mathbf{Q})\backslash T(\mathbf{A})) = h(\mathbf{Z}[\rho])$$

when the discriminant of $X^2 - tX + n$ has valuation 0. Zhiwei Yun generalized this kind of calculation to GL_n (namely, expressing orbital integrals in terms of class numbers).

References

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